Positive Real Balancing for Nonlinear Systems

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***Abstract*—Electrical circuits belong to the important class of passive (positive real) systems. In a physical sense, positive realness means that the energy produced by the system can never exceed the energy received by it. For linear passive system several model reduction methods that preserve this essential property have been developed. Among these, positive real balanced truncation is an important one. We extend this method to the nonlinear case.**

Keywords—positive real, energy functions, Hamilton-Jacobi equations, nonlinear balancing, truncation

I. Introduction

The nonlinear systems we treat are given in the state space representation as:

$$x = f(x) + g(x)u, \ y = h(x) + d(x)u, \tag{1}$$

where $x \in R^n, u \in R^m, y \in R^p$, with $m = p$. x is called the state vector, u is the input and y is the output of the system. f, g, h are smooth nonlinear vectorfields depending on the state vector x. n is called the dimension of system (1).

Often n is large and it is difficult to deal with it from both analysis and control design point of view. That is why a model order reduction problem can be formulated as follows: Given a system (1) find another system

$$\dot{\hat{x}} = \hat{f}(\hat{x}) + \hat{g}(\hat{x})u, \ \hat{y} = \hat{h}(\hat{x}) + \hat{d}(\hat{x})u$$

such that: $\dim \hat{x} < n$, the response characteristics are similar to those of the original system and certain properties (e.g. passivity) are preserved. The reduced order system might be used to replace the original one for analysis or design.

We work the assumption that the system is reachable and zero-state observable.

II. Passive Systems and Energy Functions

Definition 1. ([11], [7]) A system (1) is called passive (positive real), if there exists a storage function $S : R^n \to R$, with the following properties:

1. $S \geq 0$;
2. $S(x_0) + \int_{t_0}^{t_1} u^T y \geq S(x_1)$, $x_0 = x(t_0)$, $x_1 = x(t_1)$.

Property 2 can also be written in a differential form as:

$$\frac{\partial S(x)}{\partial x}(f(x) + g(x)u) \leq u^T h(x) + u^T d(x)u \tag{2}$$

For our purpose two particular types of storage functions are of interest: the available storage function and the required supply function. They represent the pair of energy functions which are going to be balanced, giving us information with respect to the importance of a state component of system (1). Based on this information we truncate the system by removing the states that have a less energetic meaning. The reduced system obtained in this way will be also passive, thus the property being preserved.

Definition 2. ([11]) The available storage function of a system (1) is the energy function:

$$S_a(x_0) = -\min_u \int_0^\infty u^T y \ dt, \ x(0) = x_0, \ x(\infty) = 0 \tag{3}$$

It represents the maximal amount of energy that can be extracted from the terminals of the system when starting at the initial state x_0.

Definition 3. ([11]) The required supply function of system (1) is the energy function:

$$S_r(x_0) = \min_u \int_{-\infty}^0 u^T y \ dt, \ x(0) = x_0, \ x(-\infty) = 0 \tag{4}$$

It represents the minimal amount of energy required to be supplied to the system in order to reach x_0 from the equilibrium. The reachability from x_0 is a condition for the nonnegativity of the energy functions defined above.

Lemma 4. *[10] Let system (1) be passive as in Definition 1 and reachable from the state x_0. Then, the energy functions S_a and S_r as in Definition 2, 3 exist and are nonnegative. Moreover, $S_a \leq S_r$.*

These two energy functions will be brought into a form such that the information given represents a measure of importance of each state component. First we will briefly show the procedure for linear systems and then try to extend this in the case of nonlinear systems.

III. Linear Systems Case

A linear system is given as: $\dot{x} = Ax+Bu$, $y = Cx+Du$, where A, B, C, D are constant matrices of appropriate dimensions. The system is assumed to be reachable and observable (minimal) and $R = D + D^T > 0$. The energy functions are quadratic.

Theorem 5. *[11] Assume that the linear system is passive. Then $S_a(x) = \frac{1}{2}x^T K_{min} x$ and $S_r(x) = \frac{1}{2}x^T K_{max} x$, where K_{min} and K_{max} are the minimal, respectively maximal solution of the Riccati equation:*

$$KA + A^T K + (KB - C^T)R^{-1}(B^T K - C) = 0 \qquad (5)$$

Definition 6. [1],[2] A passive linear system is called positive real balanced if $K_{min} = (K_{max})^{-1} = \mathrm{diag}(\pi_1 I_{s1}, \pi_2 I_{s_2}, ..., \pi_q I_{s_q})$, where $1 \geq \pi_1 > \pi_2 > ... > \pi_q > 0$, $s_1 + s_2 + ... s_q = n$. A system satisfying this condition is called positive real balanced.
If there exists k such that π_k is much larger than π_{k+1}, then the state vector can be truncated from $k+1$ to n, i.e. $x_l = 0, l = k+1...n$. The main result is as follows:

Theorem 7. *Let the passive linear system be brought into the positive real balanced form (A_b, B_b, C_b, D_b). The reduced system obtained after truncation with dimension k, i.e. $\dim \hat{x} = k$, is minimal and passive.* This can be extended to the nonlinear case.

IV. Nonlinear Systems Case

We consider (1) to be passive, reachable, zero state-observable and satisfying $d(x) + d^T(x) = r(x) > 0$. Then we can state the following: the energy functions defined as in (3) and (4) are the minimal respectively maximal solutions of the following Hamilton-Jacobi equation:

$$\frac{\partial S}{\partial x} f + \frac{1}{2}\left(\frac{\partial S}{\partial x} g - h^T\right) r^{-1} \left(g^T \frac{\partial S^T}{\partial x} - h\right) = 0 \qquad (6)$$

where, f, g, h, r depend on the state vector x.
Consider (1) is in a coordinate chart s.t. $S_a = \frac{1}{2}x^T x$ and $S_r = \frac{1}{2}x^T \mathrm{diag}(v_1(x), ..., v_n(x))x$ (there is always a coordinate transformation to bring the system into a form satisfying this condition) ([6]).
We say that the nonlinear system (1) is brought in positive real balanced form if there exists a coordinate transformation $z = \chi(x)$, such that:

$$S_a = \frac{1}{2} z^T \mathrm{diag}(\pi_1(z_1)^{-1}, ..., \pi_n(z_n)^{-1})z \qquad (7)$$

$$S_r = \frac{1}{2} z^T \mathrm{diag}(\pi_1(z_1)^{-1} v_1(x), ..., \pi_n(z_n)^{-1} v_n(x))z \qquad (8)$$

where $x = \chi^{-1}(z)$. $v_k(x)$ can be called the positive real singular value functions of (1). Applying this coordinate transformation to (1), it becomes: $\dot{z} = \overline{f}(z) + \overline{g}(z)u$, $\overline{y} = \overline{h}(z) + \overline{d}(z)$, being in positive real balanced form.
The energetical properties of a state component can be measured. The available energy extracted at component z_k is given by $S_a(0, ..., z_k, ...0) = \frac{1}{2}z_k^2 \pi_k^{-1}(z_k)$ and the energy supply required to reach component z_i is measured as $S_r(0, ..., z_k, ..., 0) = \frac{1}{2}z_k^2 \pi_k(z_k)$. So, if $v_k(\chi^{-1}(z)) > v_{k+1}(\chi^{-1}(z))$, then $\pi_k^{-1}(z)v_k(\chi^{-1}(z)) > \pi_{k+1}^{-1}(z)v_{k+1}(\chi^{-1}(z))$. This means that to reach state component z_k less supply of energy is required that for the component z_{k+1} and at state component z_k is stored more energy available than at state component z_{k+1}. This makes components $z_1, ..., z_k$ more important from energetic point of view than state components $z_{k+1}, ..., z_n$. It means that we can reduce the system to dimension k. Thus, partitioning the state vector z into $[z_1\ z_2]^T$, for reduction set $z_2 = 0$ (truncation). Then the reduced order system is given as:

$$\dot{z}_1 = f_1(z_1) + g_1(z_1)u,\ y_1 = h_1(z_1) + d_1(z_1)u,$$

where $f_1(z_1) = \overline{f}(z_1, 0)$, $g_1(z_1) = \overline{g}(z_1, 0)$, $h_1(z_1) = \overline{h}(z_1, 0), d_1(z_1) = \overline{d}(z_1, 0)$.
The reduced system obtained in this way, satisfies the following properties:

- it is in positive real balanced form with the positive real singular values $v_1, ..., v_k$;
- it is passive as in property 2.

V. Future Work

The decomposition used in Section IV is not unique. For future, uniqueness as in [3], is to be taken into account.
An important problem to be checked is how to treat the case when $r(x)$ is singular or 0. This arises, for example, in the field of port-Hamiltonian systems (see [7]).

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