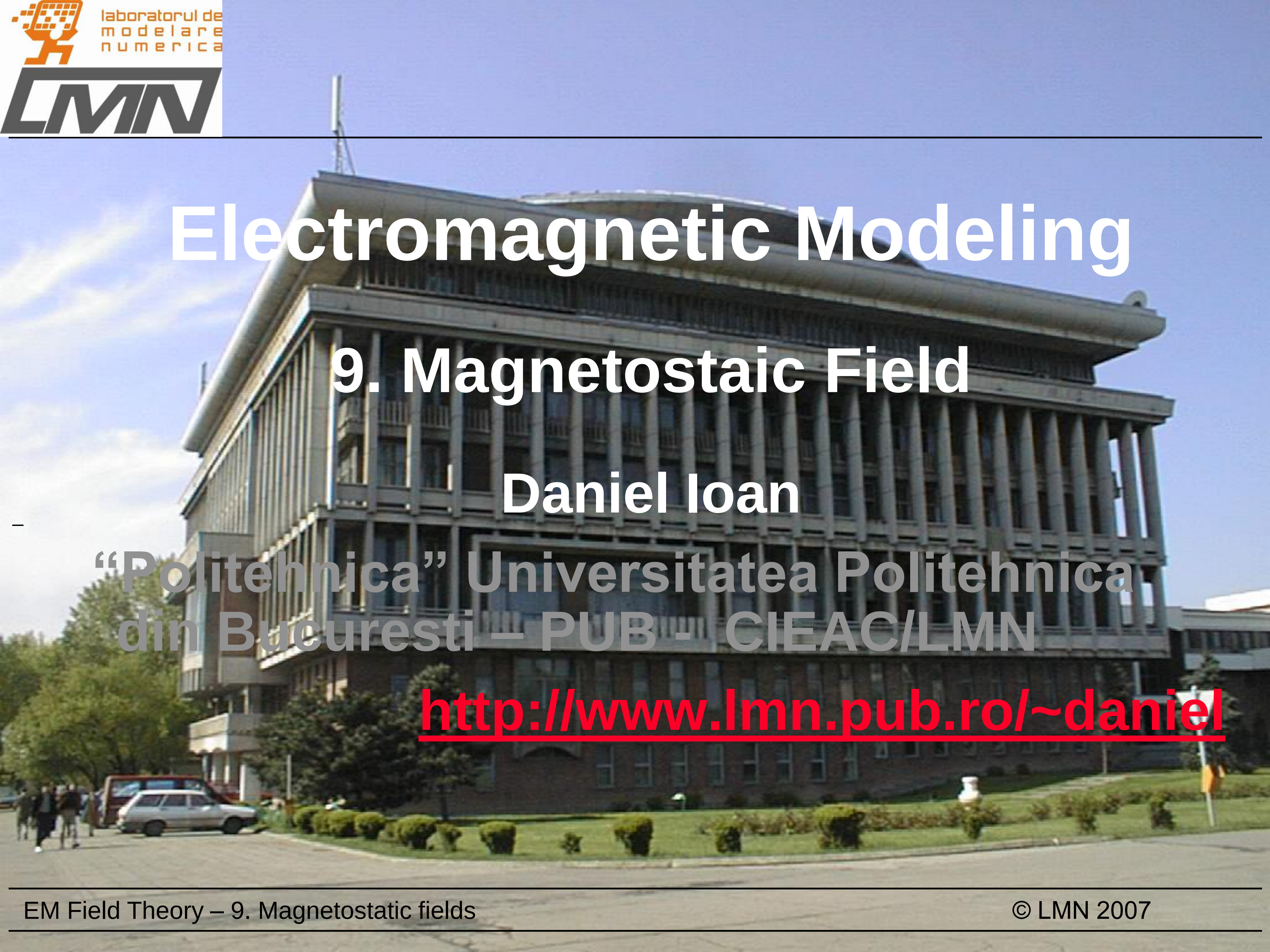

Electromagnetic Modeling

9. Magnetostaic Field

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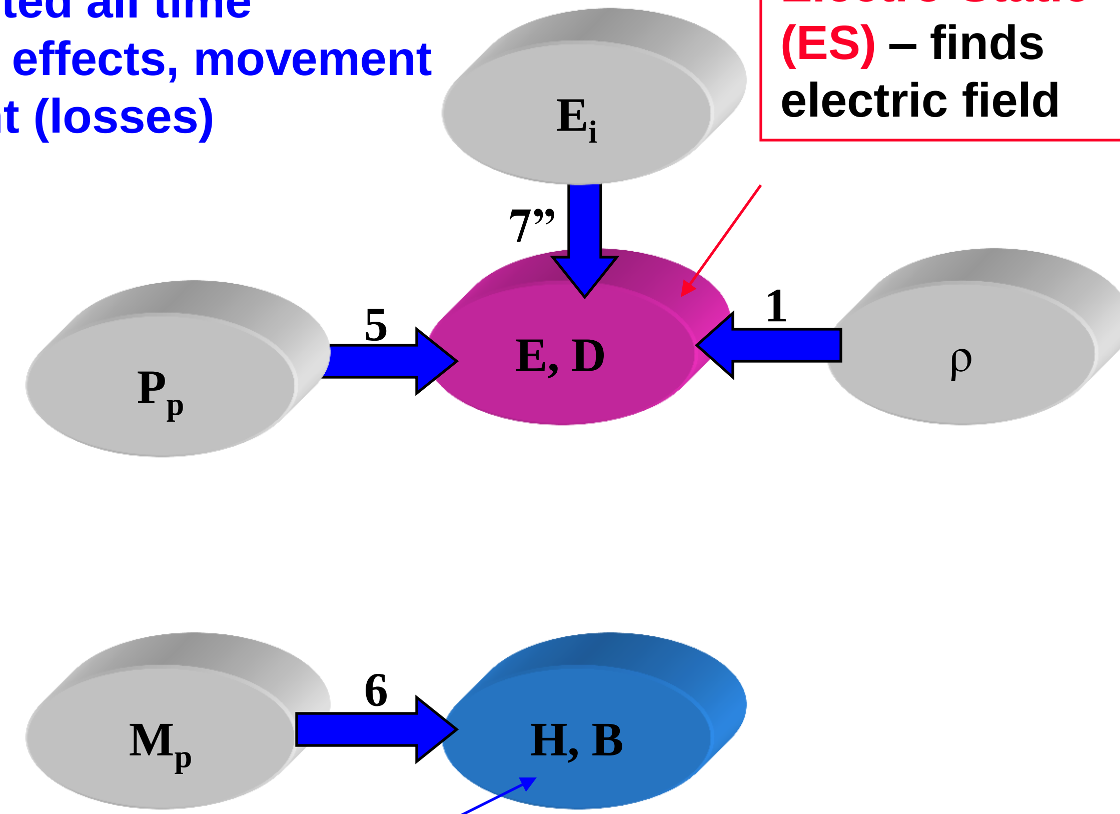
Contents

Static regimes

1. $\nabla \cdot \mathbf{D} = \rho$
2. $\nabla \cdot \mathbf{B} = 0$
3. $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
4. $\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$
5. $\mathbf{D} = \varepsilon\, \mathbf{E} + \mathbf{P}_p(\mathbf{E})$
6. $\mathbf{B} = \mu\, (\mathbf{H} + \mathbf{M}_p(\mathbf{H}))$
7. $\mathbf{J} = \sigma(\mathbf{E} + \mathbf{E}_i(\mathbf{E}))$
8. $p = \mathbf{EJ}$
9. $\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$

Are neglected all time dependent effects, movement and current (losses)



Electro-Static (ES) – finds electric field

Magneto- Static (MS) – finds magnetic field distribution

Magneto-Static regime

- **Hypothesis:**
  - no movement
  - no time variation
  - no losses (current)
  - no electric field of interest

$$\Phi_\Sigma = 0 \Leftrightarrow \oint_\Sigma \mathbf{B} d\mathbf{A} = 0$$
$$div\mathbf{B} = 0 \Rightarrow \mathbf{B} = curl\mathbf{A}$$
$$\mathbf{n}_{12} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0 \Leftrightarrow div_s\mathbf{B} = 0$$

- **Fundamental Equations:**
- **Gauss' theorem**
- **Theorem of MS circulation**
- **Magnetic constitutive relation**

$$u_{m\Gamma} = 0 \Leftrightarrow \oint_\Gamma \mathbf{H} d\mathbf{r} = 0$$
$$\mathbf{curl}\mathbf{H} = 0 \Rightarrow \mathbf{H} = -\mathbf{grad}V$$
$$\mathbf{n}_{12} \times (\mathbf{H}_2 - \mathbf{H}_1) = 0, \Leftrightarrow \mathbf{H}_{t2} = \mathbf{H}_{t1}$$

- **Field sources:**
  - Permanent magnetization

$$\mathbf{B} = \mathbf{f}(\mathbf{H}) \Rightarrow \mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \Rightarrow \mathbf{B} = \overline{\overline{\mu}}\mathbf{H} + \mu_0\mathbf{M}_p$$

- **MS field is similar to ES field in uncharged domains:**

ES:	$\mathbf{E}$	$\mathbf{D}$	$\mathbf{P}$	ε	V	ψ
MS:	$\mathbf{H}$	$\mathbf{B}$	$\mu_0\mathbf{M}$	μ	U_m	Φ

Second order equation for the scalar potential

$$\begin{cases} div\mathbf{B} = 0 \Rightarrow -div(\mathbf{grad}V_m - \mathbf{I}_p) = 0 \\ \mathbf{curl}\,\mathbf{H} = 0 \Rightarrow \mathbf{H} = -\mathbf{grad}V_m \\ \mathbf{B} = \overline{\overline{\mu}}\mathbf{H} + \mathbf{I}_p \Rightarrow \mathbf{B} = -\overline{\overline{\mu}}\mathbf{grad}V_m + \mathbf{I}_p,\ \mathbf{I}_p =_{def} \mu_0 \mathbf{M}_p \end{cases}$$

$$\boxed{-div(\overline{\overline{\mu}}\mathbf{grad}V_m) = \rho_m}$$

$$\rho_m = -div\mathbf{I}_p = -\mu_0 div\mathbf{M}_p$$

Magnetization charge

Magnetic polarization

Magnetization

Particular cases:

- **Linear homogeneousizotropic media (Poisson equation):**

$$-div(\mathbf{grad}V_m) = \rho_m / \mu \Rightarrow \boxed{\Delta V = -\nu\rho_m,} \quad \nu =_{def} \mu^{-1}$$

- **No internal ES field sources (Laplace equation):**

$$div(\mu\mathbf{grad}V_m) = 0 \Rightarrow div(\mathbf{grad}V_m) = 0 \Leftrightarrow \boxed{\Delta V_m = 0}$$

Boundary conditions are necessary for a unique solution. They can be:

•Dirichlet b.c. **or Neumann b,c. (no both in same P)**

$$\boxed{V_m(P) = f_D(P),} \quad \text{on } S_D \neq \emptyset \qquad \boxed{\frac{\partial V_m}{\partial n} = f_N(P)} \quad \text{on } S_N = \Sigma - S_D$$

Second order equation for the vector potential

$$\begin{cases} div\mathbf{B}=0 \Rightarrow \quad \mathbf{B}=\mathbf{curl\,A} \\ \mathbf{curl\,H}=0 \Rightarrow \mathbf{curl}\left[\overline{\overline{\nu}}(\mathbf{curl\,A}-\mathbf{I}_p)\right]=0 \\ \mathbf{B}=\overline{\overline{\mu}}\mathbf{H}+\mathbf{I}_p \Rightarrow \mathbf{H}=\overline{\overline{\nu}}(\mathbf{B}-\mathbf{I}_p) \end{cases}$$

$$\mathbf{curl}\left[\overline{\overline{\nu}}\,\mathbf{curl\,A}\right]=\mathbf{J}_m$$

$$\mathbf{J}_m=\mathbf{curl}(\overline{\overline{\nu}}\mathbf{I}_p)$$

Magnetization current density

Particular cases:

- **Linear homogeneous isotropic media (Poisson vector equation):**

$$\mathbf{curl}\left[\mathbf{curl\,A}\right]=\mu\mathbf{J}_m \Rightarrow \mathbf{grad}(div\mathbf{A})-\Delta\mathbf{A}=\mu\mathbf{J}_m \Rightarrow \boxed{\Delta\mathbf{A}=-\mu\mathbf{J}_m}$$

Is added to

- **No internal ES field sources (Laplace vector equation):**

$$\mathbf{curl}\left[\mathbf{curl\,A}\right]=0 \Rightarrow \boxed{\Delta\mathbf{A}=0}$$

with Coulomb gauge condition: $div\mathbf{A}=0$

$\mathbf{A}$ still has an arbitrary component $\mathbf{A}_0=grad\varphi$, with φ harmonic $(\Delta\varphi=0 \text{ in D})$

so that $\mathbf{B}=curl\ \mathbf{A}=\ curl(\mathbf{A}+\mathbf{A}_0)$ and $div\ \mathbf{A}=div(\mathbf{A}+\mathbf{A}_0)=0.$

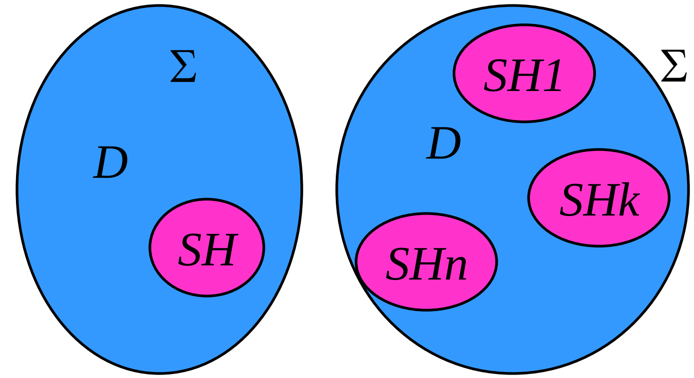
Vector boundary conditions are necessary for a unique solution (at least for B).
They may be: Dirichlet b.c. for A or Neumann b,c. for A (no both in same P)

$\boxed{\mathbf{A}_t(P)=\mathbf{f}_D(P)}$, on $S_D\neq\emptyset$ $\qquad$ $\boxed{\mathbf{n}\times(\mathrm{curl}\mathbf{A}\times\mathbf{n})=\mathbf{f}_N(P)}$, on $S_N=\Sigma-S_D$

The fundamental MS problem in terms of fields

Input (known) data:

- Computational domain D bounded by Σ
- (CM) Material characteristics $\mu(\mathbf{r})>0$ in D
- (CD) Internal field sources $\mathbf{M}p(\mathrm{r})$ in D
- (CΣ') Boundary conditions (external sources), the invariant field components:



Ht(r) on *SH* connected and **Bn(r)** on *SB=Σ-SH*

Output data (solution): H(r), B(r) in D

Equations:

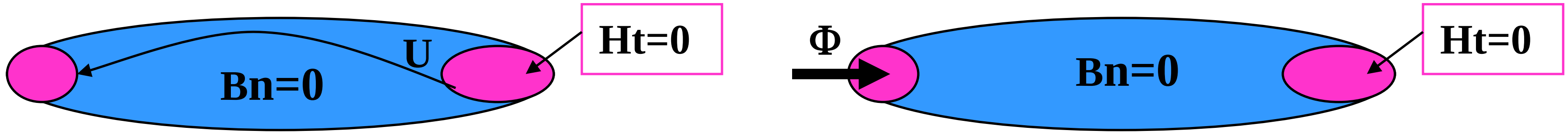
$$\begin{cases} div\mathbf{B}=0 \\ \mathbf{curl}\,\mathbf{H}=0 \\ \mathbf{B}=\overline{\overline{\mu}}\mathbf{H}+\mathbf{I}_p \end{cases}$$

For non-connected Dirichlet surfaces $S_H=\bigcup_{k=1}^{n} S_{Hk}, S_{Hk}\cap S_{Hj}=\emptyset$

according to ES-MS similitude in addition to (CΣ') solution uniqueness requires :

(CΣ'') $U_k=\int_{PkP_0}\mathbf{H}_t d\mathbf{r}$ or $\Phi_k=\int_{S_{Fk}}\mathbf{B}_n dS$, for $k=1,2,..,n\text{-}1$, and $U_n=0$.

Examples: dipolar elements of magnetic circuit, excited in "voltage" and in flux



MS boundary conditions in terms of potentials

- (CΣ) **for scalar potential** :

  V(r) = fDV(r) on SDV and dVm/dn=fNV(r) on SNV=Σ- SDV

- (CΣ')+ (CΣ") **for field components** (SH=SDV=SNA, SB=SNV=SDA):

  Ht(r) on *SH* and **Bn(r)** on *SB=Σ-SH*

$$U_k = \int_{PkP_0} \mathbf{H}_t d\mathbf{r} \text{ or } \Phi_k = \int_{S_{Ek}} \mathbf{B}_n dS, \quad \text{for } k = 1,2,.., n\text{-}1, \text{and } U_n = 0.$$

- **(CΣ''') for vector potential:**

  At(r) = fDA(r) on SDA and **nx(curlAxn)=fNA(**r**)** on SNA=Σ–SDA

$$\mathbf{B}_n = \mathbf{n}\cdot\mathbf{curl}\mathbf{A} = \mathbf{curl}\mathbf{A}_t = \mathbf{curl}(\mathbf{f}_{DA}), \quad \mathbf{H}_t = \mathbf{n}\times\overline{\overline{\nu}}(\mathbf{B}-\mathbf{I}_p)\times\mathbf{n} \Rightarrow \mathbf{H}_t = \nu\,\mathbf{f}_{NA}$$

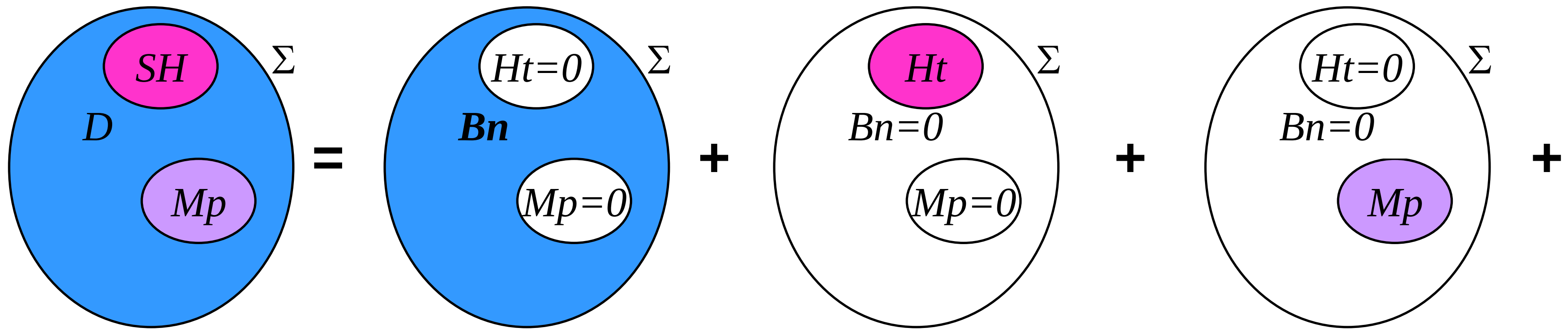
$$\Phi_k = \int_{S_{Hk}} \mathbf{B}_n dS = \int_{S_{Hk}} (curl\mathbf{A})\mathbf{n} dS = \oint_{\partial S_{Hk}} \mathbf{A} d\mathbf{r} = \oint_{\partial S_{Hk}} \mathbf{f}_{DA} d\mathbf{r}$$

Uniqueness of A for a given B: if SNA is simply connected, then the Neumann b.c. may be substituted by An, because following system has only one solution:

$\mathbf{curl}\mathbf{A} = \mathbf{B};$	$div\mathbf{A} = 0;$	$\mathbf{n}\times\mathbf{A} = 0 \ \ on\ S_B;$	$\mathrm{n}\cdot\mathbf{A} = 0 \ \ on\ S_H$
Definition of A	**Coulomb gauge cond.**	**"Dirichlet" b.c.**	**"Neumann b.c."**

Acc. to fundamental MG problem

MS fields superposition. Integral MS solutions in R3



- **In linear media, between field sources C = [CD, CS] and solutions F = [B, H] it is a linear relationship: S: C→ F**

$$S\left(\sum_{k=1}^{n}\lambda_k \mathbf{C}_k\right)=\sum_{k=1}^{n}\lambda_k S(\mathbf{C}_k)$$

Coulomb integrals: solutions in vacuum extended in R3: $\Delta V=-\rho_m/\mu_0 \Rightarrow$

$$V(\mathbf{r})=\frac{1}{4\pi\mu_0}\int_{R^3}\frac{\rho_m(\mathbf{r}_0)dv}{R}=-\frac{1}{4\pi}\int_{R^3}\frac{div\mathbf{M}_p dv}{R},\quad \mathbf{H}(\mathbf{r})=-gradV_m=-\frac{1}{4\pi}\int_{R^3}\frac{\mathbf{R}div\mathbf{M}_p dv}{R^3},$$

Biot-Savart-Laplace integrals $\Delta\mathbf{A}=-\mu_0\mathbf{J}_m=-\mu_0 curl\mathbf{M},\ \Rightarrow$

$$\mathbf{A}(\mathbf{r})=\frac{\mu_0}{4\pi}\int_{R^3}\frac{\mathbf{J}_m(\mathbf{r}_0)dv}{R}=\frac{\mu_0}{4\pi}\int_{R^3}\frac{curl\mathbf{M}dv}{R},\quad \mathbf{B}(\mathbf{r})=curl\mathbf{A}=\frac{\mu_0}{4\pi}\int_{R^3}\frac{curl\mathbf{M}\times\mathbf{R}dv}{R^3},$$

Actually it is an integral equation in H: $\mathbf{B}=\mu_0(\mathbf{H}+\mathbf{M}),\quad \mathbf{M}=\chi_m\mathbf{H}+\mathbf{M}_p \Rightarrow$

$$4\pi\left((\chi_m+1)\mathbf{H}+\mathbf{M}_p\right)-\int_{R^3}\left(curl(\chi_m\mathbf{H}+\mathbf{M}_p)\times\mathbf{R}/R^3\right)dv=0$$

Depolarization factor

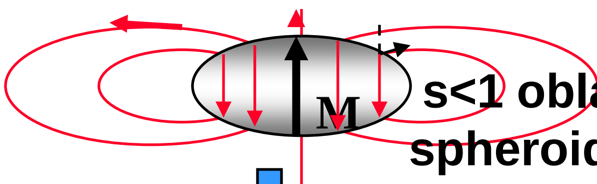
In a perm. polarized body, depolarization factor D is defined by:

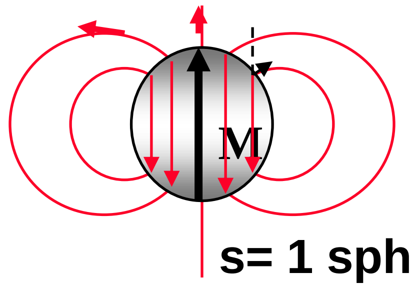
$$D =_{def} \frac{H_{int}}{M_p}$$

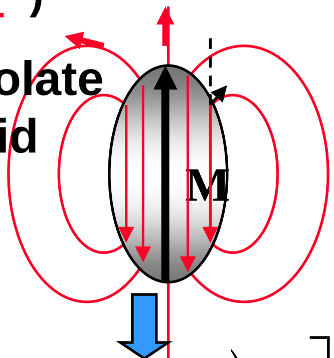
D=1/3 for a sphere, but In general it is a shape dependent tensor

- **An ellipsoid body has an uniform internal field, → D=ct** (it is solely acc. http://media.wiley.com/product_data/excerpt/9X/07803602/078036029X.pdf)

a=length, b= diameter,
s= a/b shape anisotropy factor







$$D_y = \frac{1}{1-s^2}\left[1-\frac{s}{\sqrt{1-s^2}}\cos^{-1} s\right],\ D_x = D_z = \frac{1-D_y}{2},\ D_y = \frac{1}{s^2-1}\left[\frac{s}{\sqrt{s^2-1}}\ln\left(s+\sqrt{s^2-1}\right)-1\right]$$

In a linear polarized body, $\mathbf{M} = \mathbf{B}/\mu_0 - \mathbf{H} = \chi_m \mathbf{H}$ **Internal field:**

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_M = \mathbf{H}_0 - \mathrm{D}\mathbf{M} \Rightarrow H_0 = H + D\chi_m H = (1+D\chi_m)H \Rightarrow H_{int} = (1+D\chi_m)^{-1} H_0$$

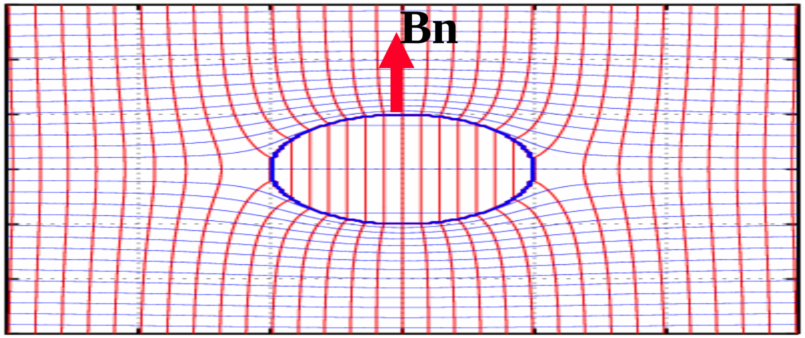
$$B_{int} = \mu H_{int},\ M = \chi_m H_{int} = \chi_m (1+D\chi_m)^{-1} H_0 \Rightarrow m = MV = V\chi_m (1+D\chi_m)^{-1} H_0 \Rightarrow$$

$$\mathbf{H}_{int} = \mathbf{H}_0 - \overline{\overline{D}}\mathbf{M} \Rightarrow D = (H_0 - H_{int})/M$$

Ext. field: $\mathbf{H}_{ext} = \mathbf{H}_0 + \frac{1}{4\pi}\left[\frac{3(\mathbf{m}\cdot\mathbf{R})\mathbf{R}}{R^5} - \frac{\mathbf{m}}{R^3}\right]$

Degenerate cases

Oblate spheroid



Thin disk when s → 0

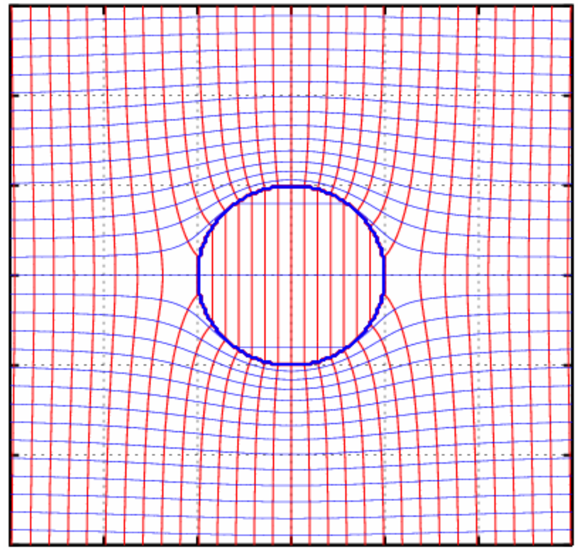
$$-D_y = 1, D_x = D_z = 0$$

$$H_{My} = -M, H_{Mx} = 0$$

$$H_y = H_{0y} / \mu_r, H_x = H_{0x}$$

$$\Rightarrow \mu_0 H_{0y} = \mu H_y \Leftrightarrow B_{next} = B_{nint}$$

Sphere (s=1)

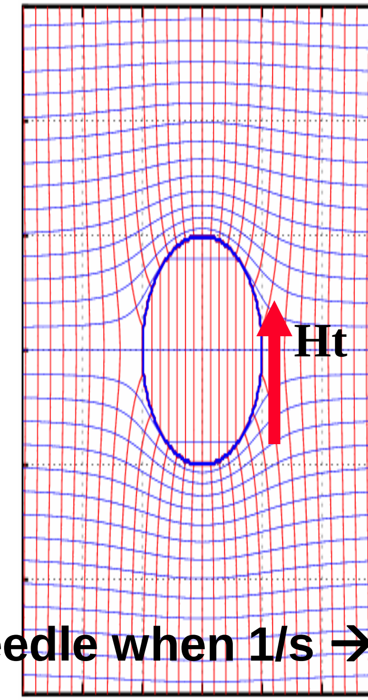


$$D_x = D_y = D_z = 1/3$$

$$H_M = -M/3$$

$$H_{int} = H_0 / (1 + \chi_m / 3)$$

Prolate spheroid



Needle when 1/s → 0

$$D_y = 0, D_x = D_z = 1/2$$

$$H_{My} = 0, H_{Mx} = M/2$$

$$H_y = H_{0y} \Leftrightarrow H_{tint} = H_{tex}$$

$$H_x = 2H_{0x} / (1 + \mu_r)$$

Similar relations in the case of ellipsoidal cavities. ➔

$$\Phi = \int_S \mathbf{B} \cdot \mathbf{n} dS = \int_S \mathbf{B}_v \cdot \mathbf{n} dS, \quad U_m = \int_C \mathbf{H} \cdot d\mathbf{r} = \int_C \mathbf{H}_v \cdot d\mathbf{r}$$

MS field of a set of small particles

A permanent and uniform magnetized sphere M=Mp, m=MV, ES→MS:

$$\mathbf{E} \to \mathbf{H},\ \varepsilon_0 \to \mu_0,\ \ \mathbf{P} \to \mu_0 \mathbf{M} \ \Rightarrow\ \mathbf{H}_{int} = -\frac{\mathbf{M}}{3},\ \mathbf{H}_{ext} = \frac{1}{4\pi}\left[\frac{3(\mathbf{m}\cdot\mathbf{R})\mathbf{R}}{R^5} - \frac{\mathbf{m}}{R^3}\right]$$

A temporal magnetized sphere in uniform field H0. ES→MS:

$$\mathbf{H}_{int} = \mathbf{H}_0 /(1+\chi_m/3)\quad \mathbf{m} = V\mathbf{M} = 4\pi a^3 \mathbf{H}_0 \chi_m /(\chi_m + 3),$$

$$\mathbf{B}_{int} = \mu \mathbf{H}_{int}, \mathbf{M} = \chi_m \mathbf{H} \qquad \mathbf{H}_{ext} = \mathbf{H}_0 + \frac{1}{4\pi}\left[\frac{3(\mathbf{m}\cdot\mathbf{R})\mathbf{R}}{R^5} - \frac{\mathbf{m}}{R^3}\right]$$

A set of n small particles (compared to distances between them, having several shapes, and being permanent and/or temporal magnetized.

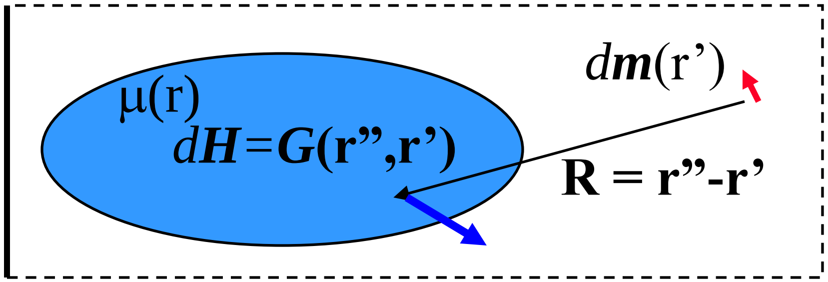
$$\mathbf{m}_j = V_j(\mathbf{M}_j + \mathbf{M}_{pj}) = V_j(\mathbf{M}_{pj} + \chi_{mj}(1+D_j\chi_{mj})^{-1}\mathbf{H}_j),$$

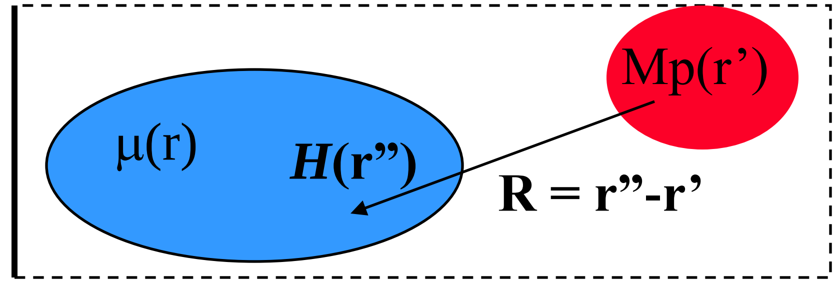
$$\mathbf{H}_j = \frac{1}{4\pi}\sum_{\substack{k=1\\k\neq j}}^{n}\left[\frac{3(\mathbf{m}_k\cdot\mathbf{R})_k \mathbf{R}_k}{R_k^{\ 5}} - \frac{\mathbf{m}_k}{R_k^{\ 3}}\right]$$

The solution is obtained by solving the system with 3n linear equations, projection on x, y, z of:

$$\mathbf{m}_j - \frac{V_j\chi_{mj}(1+D_j\chi_{mj})^{-1}}{4\pi}\sum_{\substack{k=1\\k\neq j}}^{n}\left(\frac{3(\mathbf{m}_k\cdot\mathbf{R}_k)\mathbf{R}_k}{R_k^{\ 5}} - \frac{\mathbf{m}_k}{R_k^{\ 3}}\right) = \mathbf{m}_{pj},\ j = 1,\ldots,n$$

Green function of a non-homogeneous domain





Green function in R3 is the field of a punctual unitary magnetic moment:

$$d\mathbf{H}(\mathbf{r}'') = \overline{\overline{G}}(\mathbf{r}'',\mathbf{r}')d\mathbf{m}(\mathbf{r}'), \overline{\overline{G}} = gradG$$

It is a tensor, because m may have an arbitrary direction m=δ(r’=r’)u, u=i, j, k

By superposition is obtained the magnetic field for an arbitrary distribution of permanent magnetization Mp

$$\mathbf{H}(\mathbf{r}'') = \int_{R^3} \overline{\overline{G}}(\mathbf{r}'',\mathbf{r}')\mathbf{M}_p(\mathbf{r}')dv$$

The Green function G of a bounded domain is the field of a punctual unitary momentum in a domain with zero b.c.: Bn=0 on SN, Ht=0 on SH and Φk=0 or Uk=0

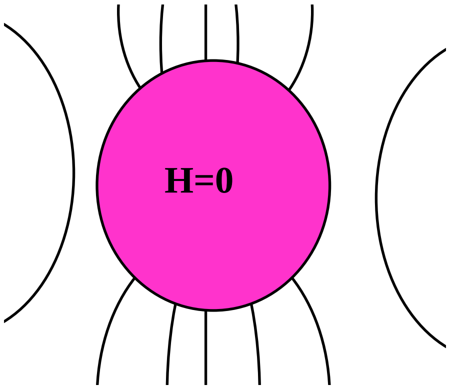
By superposition is obtained the magnetic field for an arbitrary distribution of permanent magnetization M with same zero boundary conditions. Then, have to be superposed the contribution of non-zero b.c. (ES-MS):

$$V(\mathbf{r}'') = -\int_{S_H} \frac{dG}{dn'} \cdot \mu f_{DV}(\mathbf{r}')dS' - \int_{S_B} G(\mathbf{r}'',\mathbf{r}')\mu f_{NV}(\mathbf{r}')dS'$$

$$\mathbf{H}(\mathbf{r}'') = -gradV(\mathbf{r}'') = \int_{S_H} grad\frac{dG}{dn'} \cdot \mu f_{DV}(\mathbf{r}')dS' + \int_{S_B} gradG(\mathbf{r}'',\mathbf{r}')\mu f_{NV}(\mathbf{r}')dS'$$

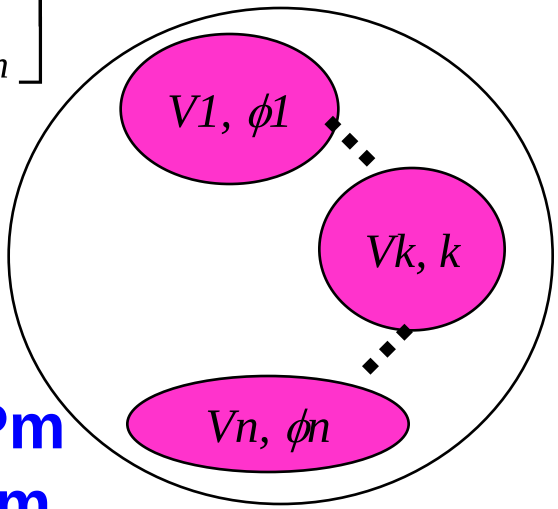
Perfect ferromagnetic bodies. Magnetic reluctances/permeances

- **IF $\mu \rightarrow$ infinity, then H→0 and the body is similar to a conductor in ES.**
- $Vm = ct$, $H_t=0$, **on the boundary, hence ext. field lines are perpendicular on it**
- **By ES→ MS similitude the Maxwell relations for capacitances are transformed in the linear relations for n perfect ferromagnetic bodies :**



$$\varphi = \mathbf{P}_m \mathbf{v}_m \Leftrightarrow \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \cdots & & & \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

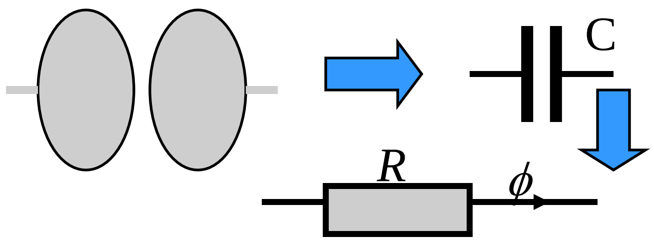
$$\Rightarrow \mathbf{v}_m = \mathbf{R}_m \varphi \Leftrightarrow \mathbf{R}_m = \mathbf{P}_m^{-1}$$

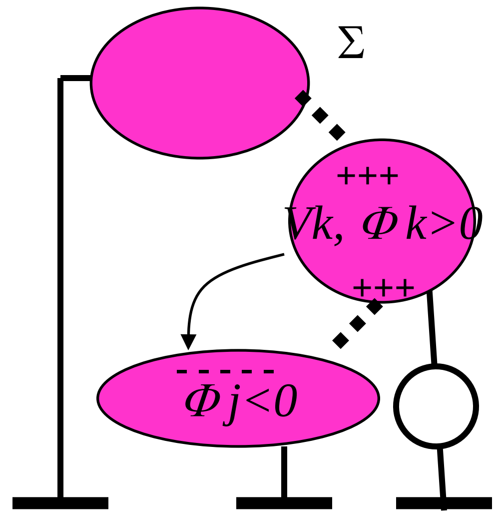


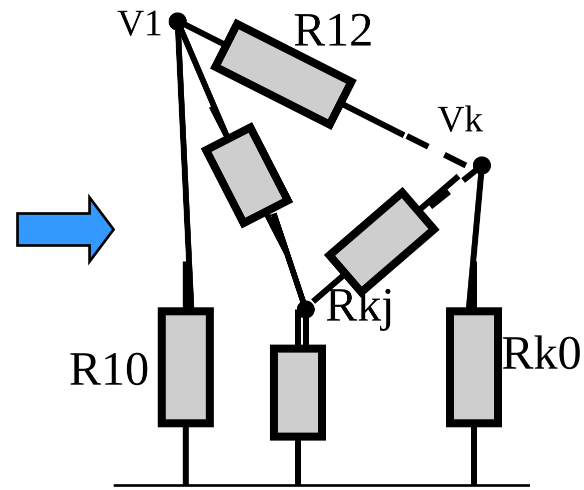
- **Fluxes:** $\phi=[\phi1,\phi2,\ldots,\phi n]$
- **Magnetic voltages:** $\mathbf{vm}=[v1,v2,\ldots,vn]T$
- **Matrix of nodal magnetic permeances Pm**
- **Matrix of nodal magnetic reluctances Rm**

Partial and equivalent reluctances

- **Flux tube:** **area of space between two perfect ferromagnetic bodies, bounded by a field surface** $\phi=\phi 1=-\ \phi 2$, $u = v1\text{-}v2$,







By ES→ MS similitude:

$u=R\phi,$

– R [1/H] **reluctance**

Equivalent (branch) permeances/reluctances:

$$\varphi_1 = p_{11}\cdot v_1 + p_{12}\cdot v_2 + \ldots + p_{1n}\cdot v_n = P_{10}\cdot v_1 + P_{12}\cdot(v_1 - v_2) + \ldots + P_{1n}\cdot(v_1 - v_n)$$

....

$$P_{kj} = -p_{kj} > 0, \qquad P_{k0} = p_{k1} + p_{k2} + \ldots + p_{kn} > 0$$

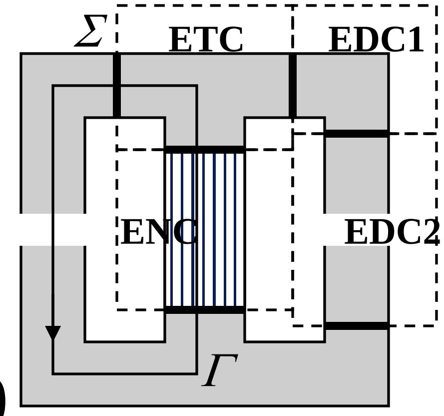
$$R_{kj} = 1/\,P_{kj}\,,\ R_{k0} = 1/P_{k0}$$

The permeance values P may be obtained from C, by ES→ MS similitude by substituting ε→ μ
All ES theorems and methods are still valid.

Magnetic circuits

- **Flux law → KFL:**

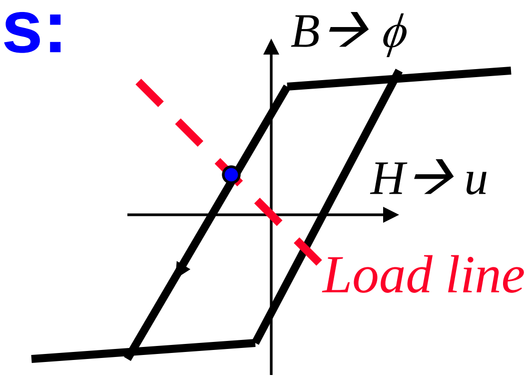
$$\oint_{\Sigma} \mathbf{B}\cdot\mathbf{n}dS = 0 \Rightarrow \sum_{k\in(n)} \varphi_k = 0 \Rightarrow \varphi_1 - \varphi_2 + \varphi_3 = 0$$

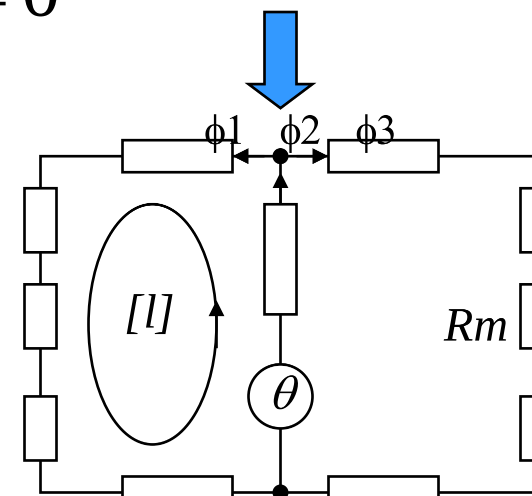


- **Voltage theorem → KVL:**

$$\oint_{\Gamma} \mathbf{H}d\mathbf{r} = 0 \Rightarrow \sum_{k\in[l]} u_k = 0 \Rightarrow u_1 + u_2 + u_3 + \ldots = 0$$

- **Constitutive relations:**
  - ETC – tripolar element
  - EDC1 - dipolar element
  - ENC – nonlinear element (permanent magnet)
  - EDC2 - airgap





$$\mathbf{B} = \mu\mathbf{H} \Rightarrow u_k = R_{mk}\varphi_k, \quad \text{where } u_k = \int_{C_k} \mathbf{H}d\mathbf{r}, \varphi_k = \int_{S_k} \mathbf{B}\cdot\mathbf{n}dS$$

$$\mathbf{B} = \mu\mathbf{H} + \mathbf{I}_p \Rightarrow u_k = R_{mk}\varphi_k + \theta_k \quad \text{where } \theta_k \text{ is m.m.f.}$$

Energy of MS field, Tellegen's and reciprocity theorems

$$W_m = \int_D w_m dv = \frac{1}{2}\int_D \mu \mathbf{H}^2 dv = -\frac{1}{2}\int_D \mathbf{I}_p \cdot \mathbf{H} dv - \frac{1}{2}\oint_\Sigma V\mathbf{B}\cdot\mathbf{n}dS > 0$$

In domains bounded by perfect ferromagnetic bodies or with zero boundary conditions: $\oint_\Sigma V\mathbf{B}\cdot\mathbf{n}dS = -\mathbf{v}^T \cdot \varphi$

Tellegen's theorem: regardless material relations, the total pseudo-energy is zero in zero boundary conditions.

$$\text{If } div\mathbf{B}' = 0,\ curl\mathbf{H}'' = 0 \Rightarrow \boxed{<\mathbf{B}', \mathbf{H}''> - \varphi'^T \cdot \mathbf{v}'' = 0} \Rightarrow \mathbf{B} \perp \mathbf{H}$$

– **Reciprocity theorem:** in linear reciprocal materials ($\mu = \mu^T$) the relation between sources and responses is symmetric. Consequently, the Green function is symmetric:

$$<\mathbf{M}_1, \mathbf{H}_2> - <\mathbf{M}_2, \mathbf{H}_1> = \int_D \int_D (\mathbf{M}_1^T \cdot \overline{\overline{G}}\mathbf{M}_2 - \mathbf{M}_2^T \cdot \overline{\overline{G}}\mathbf{M}_1) dv' dv'' = 0$$

$$\text{If } \mathbf{M}_1 = \mathbf{i}\delta(\mathbf{r}-\mathbf{r}'), \mathbf{M}_2 = \mathbf{j}\delta(\mathbf{r}-\mathbf{r}') \Rightarrow G_{xy}(\mathbf{r}', \mathbf{r}'') = G_{yx}(\mathbf{r}', \mathbf{r}'')$$

$$\text{If } \mathbf{M}_1 = \mathbf{i}\delta(\mathbf{r}-\mathbf{r}'),\ \mathbf{M}_2 = \mathbf{j}\delta(\mathbf{r}-\mathbf{r}'') \Rightarrow G_{xy}(\mathbf{r}', \mathbf{r}'') = G_{yx}(\mathbf{r}'', \mathbf{r}')$$

$$\Rightarrow \boxed{\overline{\overline{G}}(\mathbf{r}', \mathbf{r}'') = \overline{\overline{G}}(\mathbf{r}'', \mathbf{r}') = \overline{\overline{G}}^T(\mathbf{r}', \mathbf{r}'')}$$

Variational MS formulations

- **The MS "energy" functional in terms of scalar potential is similar to the ES one**

$$F(V_m)=\frac{1}{2}\int_D\left[\mu(gradV_m)^2+div(\mathbf{I}_p)V_m\right]dv+\int_{S_N}V_mB_ndS<F(V_m+\delta V)$$

Neumann are natural boundary conditions while Dirichlet are essential boundary conditions. Weak (integral-differential) formulations:

$$\int_D(\mu\, gradV_m\cdot grad\delta V+\delta V div\mathbf{I}_p)dv+\int_{S_{NV}=S_B}\delta VD_ndS=0,\quad \mathrm{f_N}=D_n=-\mu dV_m/dn$$

- **The MS weak formulation in terms of vector potential:**

$$\mathbf{curl}\left[\overline{\overline{\nu}}\mathbf{curl}\mathbf{A}\right]=\mathbf{J}_m,\mathbf{J}_m=\mathbf{curl}(\overline{\overline{\nu}}\mathbf{I}_p)\Rightarrow\int_D\delta\mathbf{A}\cdot\left[\mathbf{curl}\left(\overline{\overline{\nu}}\mathbf{curl}\mathbf{A}\right)-\mathbf{J}_m\right]dv=0$$

$$\nabla\cdot(\delta\mathbf{A}\times\nu\nabla\times\mathbf{A})=\nu\nabla\times\mathbf{A}\cdot\nabla\times\delta\mathbf{A}-\delta\mathbf{A}\cdot\nabla\times(\nu\nabla\times\mathbf{A}),\quad \mathbf{n}\times\delta\mathbf{A}=0\text{ on }S_{DA}\Rightarrow$$

$$\int_D\left[\overline{\overline{\nu}}\mathbf{curl}\,\delta\mathbf{A}\cdot\mathbf{curl}\mathbf{A}-\delta\mathbf{A}\cdot\mathbf{J}_m\right]dv+\int_{S_{NA}=S_H}\delta\mathbf{A}\cdot(\mathbf{n}\times\overline{\overline{\nu}}\mathbf{curl}\mathbf{A})dS=0,\quad \mathbf{f}_{\mathrm{NA}}=\mathbf{n}\times\mathbf{H}$$

Neumann are again natural b. c. and Dirichlet are essential b. c. also for A.

Acc. Preis91-MAG-5 A is unique if to the Galerkin variation formulation are added

$$\int_D\left[\nu\mathbf{curl}\,\delta\mathbf{A}\cdot\mathbf{curl}\mathbf{A}-\delta\mathbf{A}\cdot\mathbf{J}_m\right]dv+\int_{S_{NA}}\delta\mathbf{A}\cdot(\mathbf{n}\times\nu\mathbf{curl}\mathbf{A})dS+$$

$$-\int_D\nu div\,\delta\mathbf{A}div\mathbf{A}dv-\int_{S_{DA}}\delta\mathbf{A}\cdot\nu\mathbf{n}div\mathbf{A}dS=0\Leftrightarrow\quad \mathbf{curl}\left[\nu\mathbf{curl}\mathbf{A}\right]+\mathbf{grad}\left[\nu div\mathbf{A}\right]=\mathbf{J}_m$$

MS applications

Based on the force of the magnets

- Magnetic separators, magnetic holding devices, such as magnetic latches.
- Magnetic torque drives
- Magnetic bearing devices

Conversion of mechanical to electrical energy

- Magnetos
- Generators and alternators
- Eddy current brakes (used widely for watt-hour meter damping).

Conversion of electrical to mechanical energy

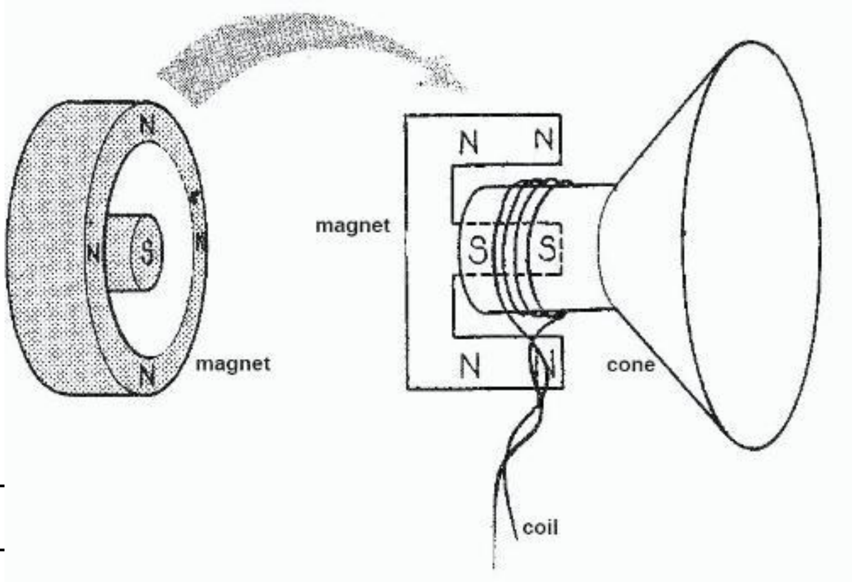
- Motors
- Meters
- Loudspeakers
- Relays
- Actuators, linear, and rotational

Direct, shape and control electron or ion beams

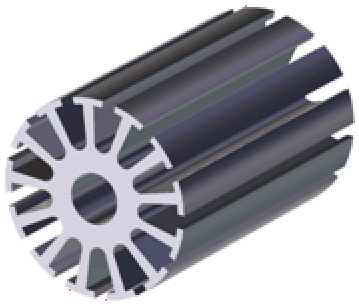
- Magnetic focused cathode-ray tubes
- Traveling Wave Tubes
- Magnetrons, BWO's, Klystrons
- Ion Pumps
- Cyclotrons

Others

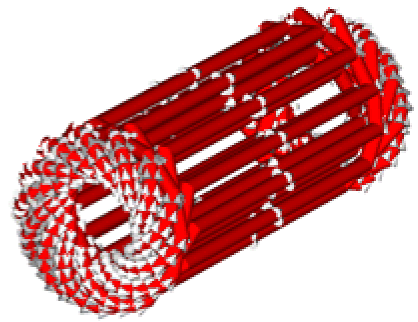
Separators. Laudspeakers



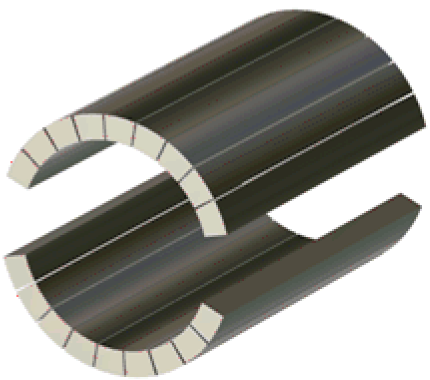
Generators and motors with p.m.



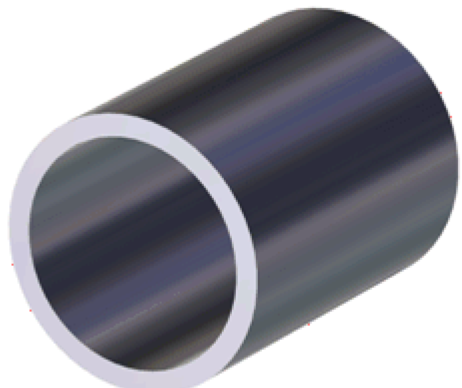
Rotor



Windings

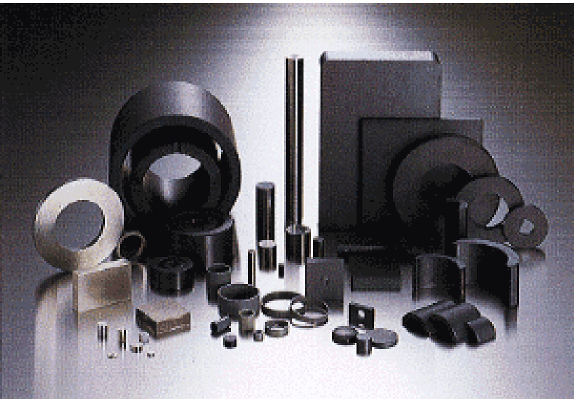


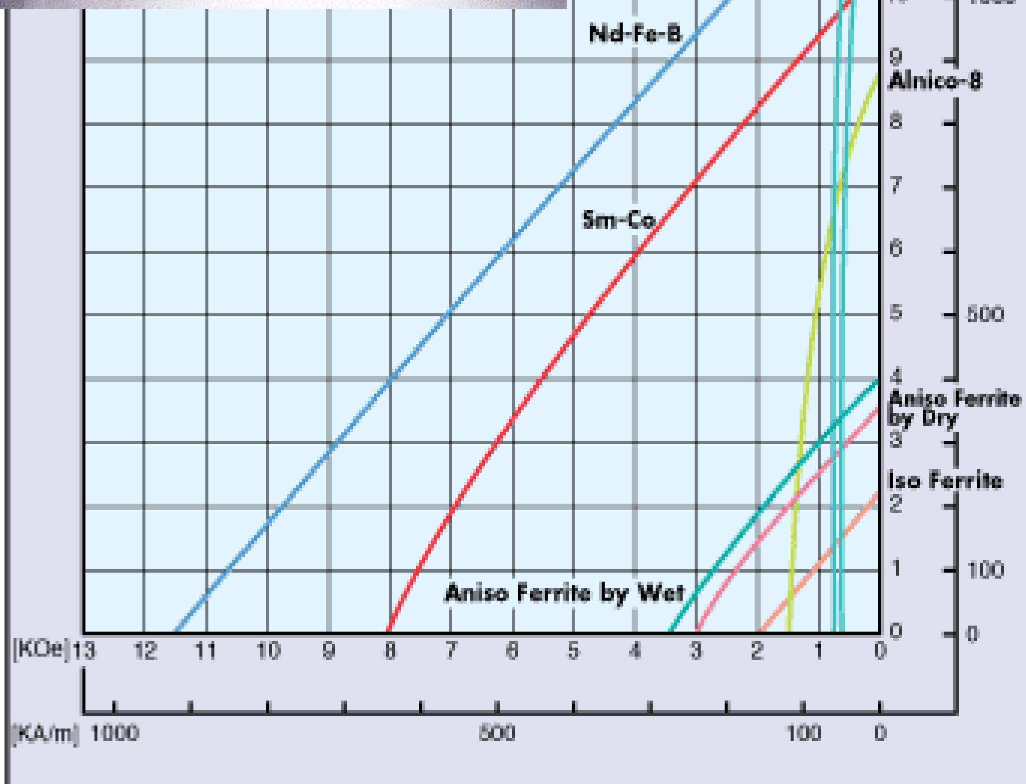
Permanent Magnets



Stator

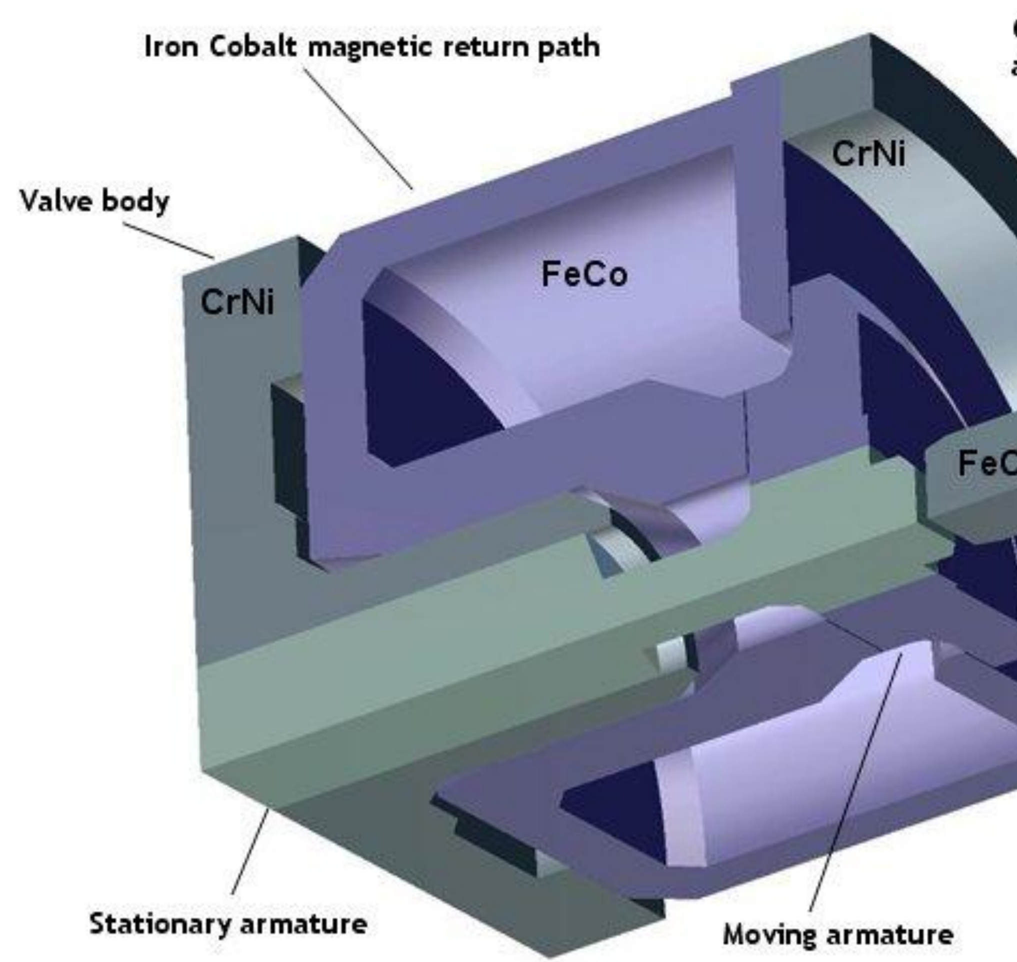
Permanent magnets

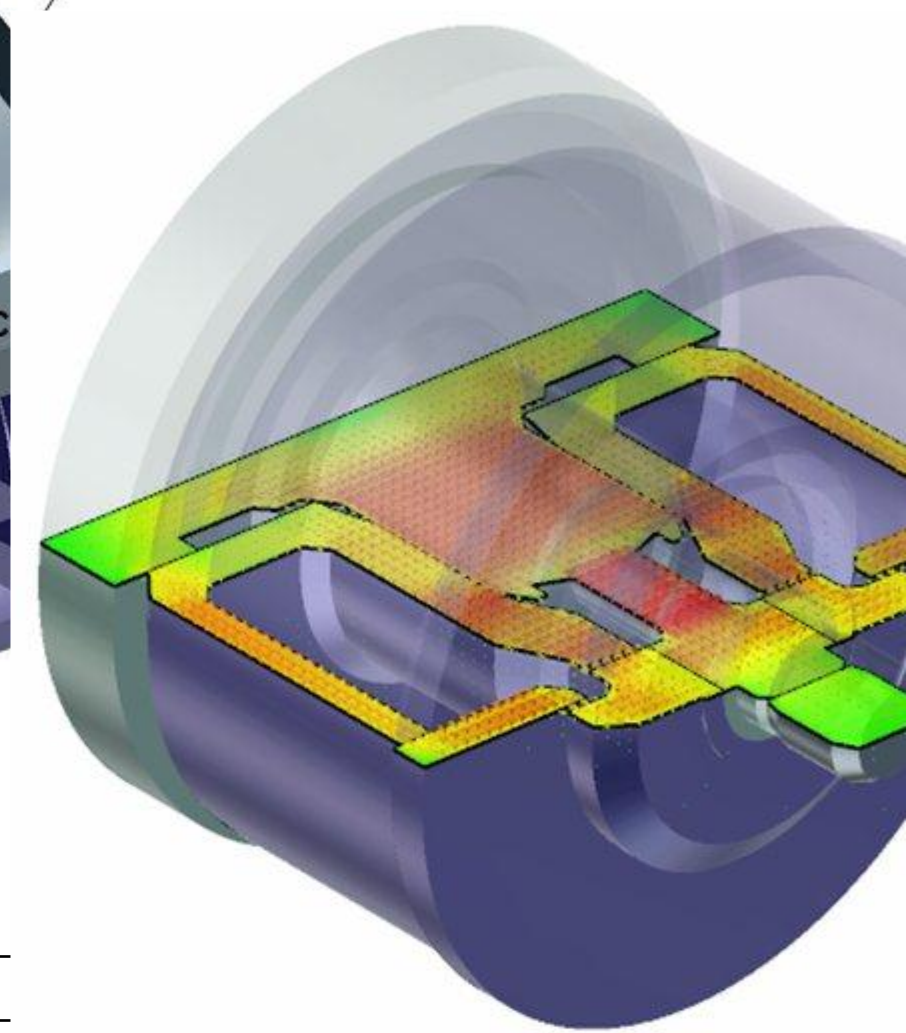




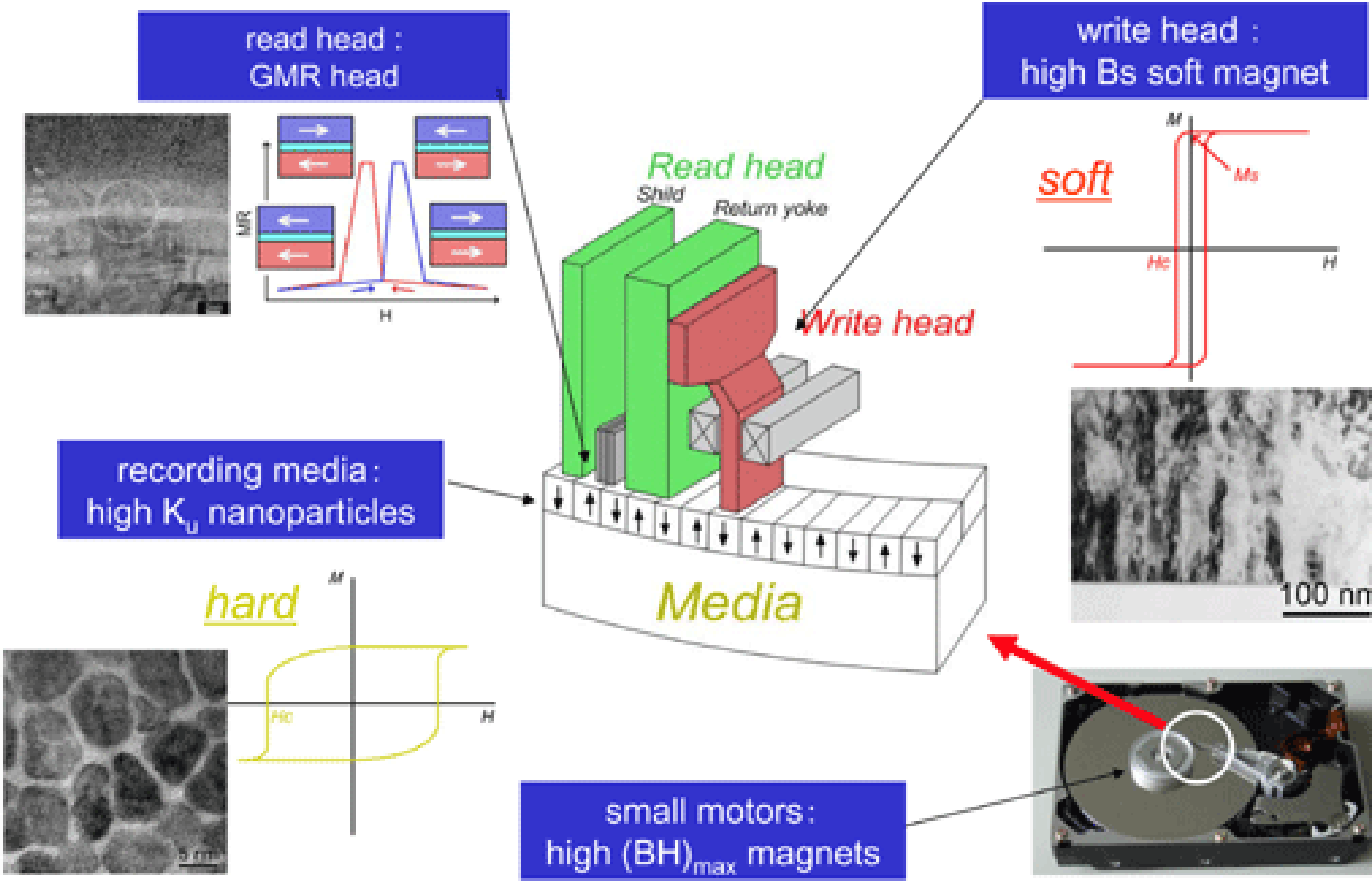
Type	Material	Br [mT]	Hc [kA/m]	Wmx [kJ/m3]
Rare earths	NdFeB	1240	923	294
	SmCo	1050	636	191
Ferrite ceramic	Ba-Ferrite isotropic	220	151	8
	Sr-Ferrite Dry	360	238	24
	SrO-6(Fe2O3)	400	262	30
Metalic	Alnico-5	1.27	51	42
	Alnico-8	880	117	41

Magnetic CAD: CST-Magnetostatic simulation of an injection valve

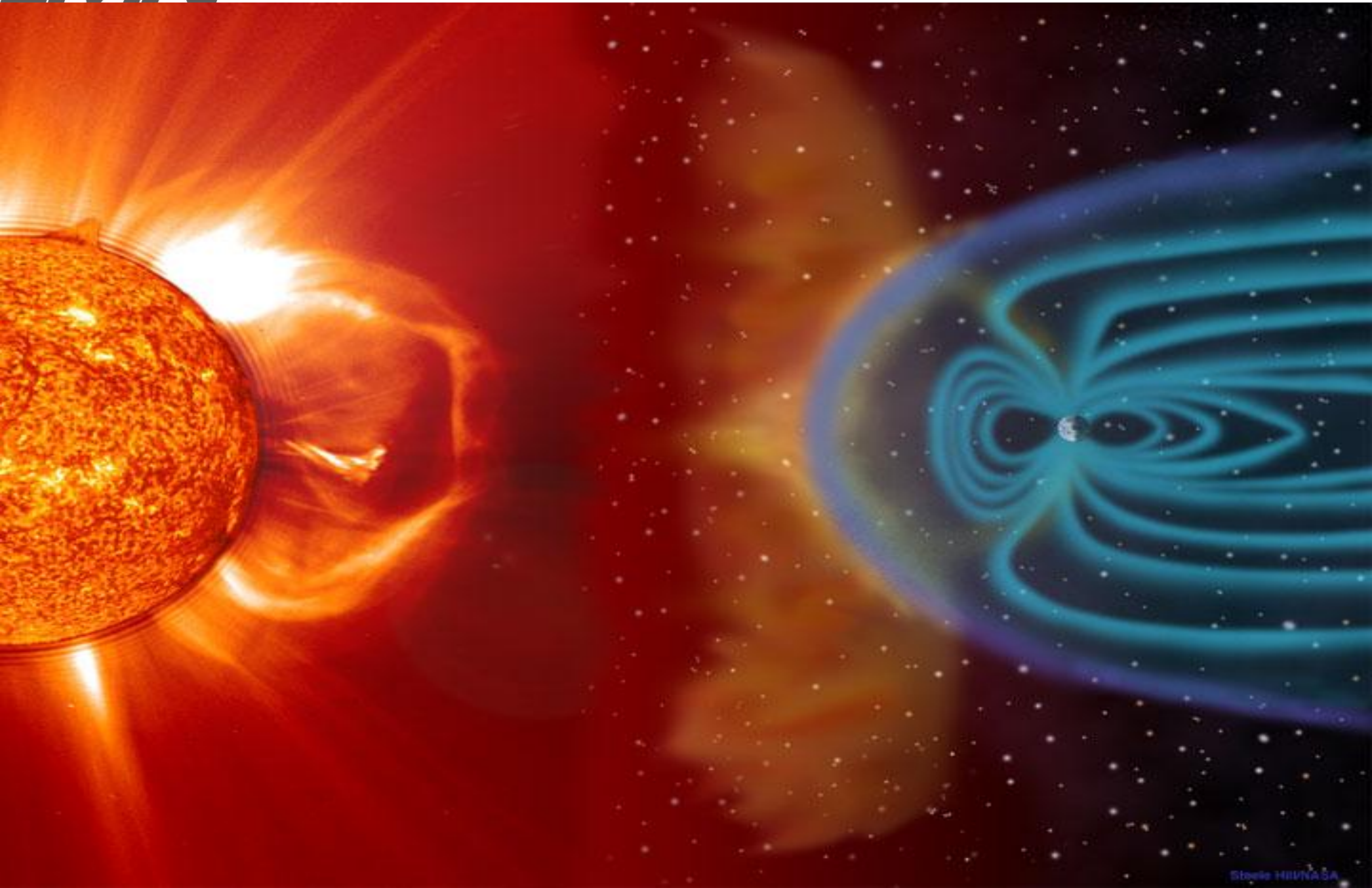




Magnetic recording - HDD



The Sun-Earth Connection
Solar wind - Earth magnetic field



MS summary. Equations, interface and boundary conditions

$$\begin{cases} div\mathbf{B}=0 \\ \mathbf{curl}\mathbf{H}=0 \\ \mathbf{B}=\overline{\overline{\mu}}\mathbf{H}+(\mathbf{I}_p) \end{cases} \Leftrightarrow \begin{cases} -div(\overline{\overline{\mu}}\,\mathbf{grad}V)=\rho_m \\ \rho_m=-div\mathbf{I}_p \\ \mathbf{H}=-\mathbf{grad}V \end{cases} \Leftrightarrow \begin{cases} \mathbf{curl}\left[\overline{\overline{\nu}}\,\mathbf{curl}\mathbf{A}\right]=\mathbf{J}_m \\ \mathbf{J}_m=\mathbf{curl}(\overline{\overline{\nu}}\mathbf{I}_p) \\ \mathbf{B}=curl\mathbf{A},\ div\mathbf{A}=0 \end{cases}$$

$$\begin{cases} B_{n1}=B_{n2} \\ \mathbf{H}_{t1}=\mathbf{H}_{t2} \end{cases} \Leftrightarrow \begin{cases} \mu_1\dfrac{\partial V_1}{\partial n}=\mu_2\dfrac{\partial V_2}{\partial n} \\ V_1=V_2 \end{cases} \Leftrightarrow \begin{cases} \mathbf{A}_1=\mathbf{A}_2 \\ \nu_1\mathbf{n}\times curl\mathbf{A}_1\times\mathbf{n}=\nu_2\mathbf{n}\times curl\mathbf{A}_2\times\mathbf{n} \end{cases}$$

$$\begin{cases} \mathbf{H}_t=\mathbf{f}_H(P)\ \text{ on }\ S_H \\ B_n=f_B(P)\ \text{ on }\ S_B \\ \int_{PkP_0}\mathbf{H}_t d\mathbf{r}=U_k\quad\text{or}\quad\int_{S_{Ek}}B_n dS=\Phi_k, \\ \text{for each }\ S_{Hk},\ k=1,2,..,n\text{-}1,\text{and } U_n=0 \end{cases} \Leftrightarrow \begin{cases} V=f_D(P) \\ \dfrac{dV}{dn}=f_N(P) \end{cases} \Leftrightarrow \begin{cases} \mathbf{n}\times\mathbf{A}=\mathbf{f}_D(P)\ \text{on}\ S_{DA} \\ \mathbf{n}\times curl\mathbf{A}=\mathbf{f}_N(P)\ \text{on}\ S_{NA} \end{cases}$$

- **Circuit parameters:** $\varphi=\mathbf{P}\mathbf{v},\quad \mathbf{v}=\mathbf{R}\varphi,\quad \mathbf{R}=\mathbf{P}^{-1}$

$$\mathbf{R}=\mathbf{R}^T>0,\quad \mathbf{P}=\mathbf{P}^T>0$$

MS summary. Equations, interface and boundary conditions

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$$\begin{cases} B_{n1}=B_{n2} \\ \mathbf{H}_{t1}=\mathbf{H}_{t2} \end{cases} \Leftrightarrow \begin{cases} \mu_1\dfrac{\partial V_1}{\partial n}=\mu_2\dfrac{\partial V_2}{\partial n} \\ V_1=V_2 \end{cases} \Leftrightarrow \begin{cases} \mathbf{A}_1=\mathbf{A}_2 \\ \nu_1\mathbf{n}\times curl\mathbf{A}_1\times\mathbf{n}=\nu_2\mathbf{n}\times curl\mathbf{A}_2\times\mathbf{n} \end{cases}$$

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Circuit parameters:

$$\varphi=\mathbf{P}\mathbf{v},\quad \mathbf{v}=\mathbf{R}\varphi,\quad \mathbf{R}=\mathbf{P}^{-1}$$

$$\mathbf{R}=\mathbf{R}^T>0,\quad \mathbf{P}=\mathbf{P}^T>0$$

MS forces

- **Magnetized particle** $\mathbf{F}_m = \mathbf{grad}(\mathbf{m}\cdot\mathbf{B_v})$ $\quad \mathbf{T}_m = \mathbf{r}\times\mathbf{F}_m + \mathbf{m}\times\mathbf{B}_v$
- **Linear magnetic particle** $\quad \mathbf{m} = V\chi_m(1+\overline{\overline{D}}\chi_m)^{-1}\mathbf{H}_v$
- **Perfect ferromagnetic bodies:**

$$\mathbf{F} = \oint_\Sigma w_m \mathbf{n} dS, \qquad \mathbf{T} = \oint_\Sigma w_m(\mathbf{r}\times\mathbf{n})dS$$

$$X_k = -\frac{1}{2}\varphi^{\mathbf{T}}\frac{\partial \mathbf{R}}{\partial x_k}\varphi; \quad X_k = \frac{1}{2}\mathbf{v}^{\mathbf{T}}\frac{\partial \mathbf{P}}{\partial x_k}\mathbf{v}$$

- **In general**

$$X_{k\,mg} = -\frac{\partial W_m}{\partial x_k}\Bigg|_{\varphi=const.} \qquad X_{k\,mg} = -\frac{\partial W_m^{\;*}}{\partial x_k}\Bigg|_{v=const.}$$

- **Maxwell's tensor**

$$\mathbf{f} = -\frac{H^2}{2}(\mathbf{grad}\,\mu) + \mathbf{grad}\left(\frac{H^2}{2}\tau\frac{\partial\mu}{\partial\tau}\right) = div\left[\mathbf{H}\wedge\mathbf{B}^T + \overline{\overline{\mathbf{I}}}\left(\frac{H^2}{2}\tau\frac{\partial\mu}{\partial\tau} - w_m\right)\right]$$

Not so easy questions for curious people

1. Are valid MS equations/methods for slow time variable fields ?
2. Are valid MS equations/methods for slow moving bodies ?
3. Are valid MS equations/methods in the presence of magnets ?
4. Are valid MS equations/methods for electric field outside d.c. currents ?
5. What about Robin boundary condition (a V+b dV/dn). Correctness and meaning ?
6. What are MS boundary conditions in semi-bounded domains ?
7. Give example of wrong MS problems. What are Hadamard well-posed problems?
8. What about nonlinear magnetic materials? Uniqueness, energy, forces.
9. What are the differences between Tellegen and reciprocity theorems ?
10. How is defined Green function with Neumann b.c.?
11. What space may be used for trial and test functions in weak MS formulation ?
12. What is the best method for MS field computation ?