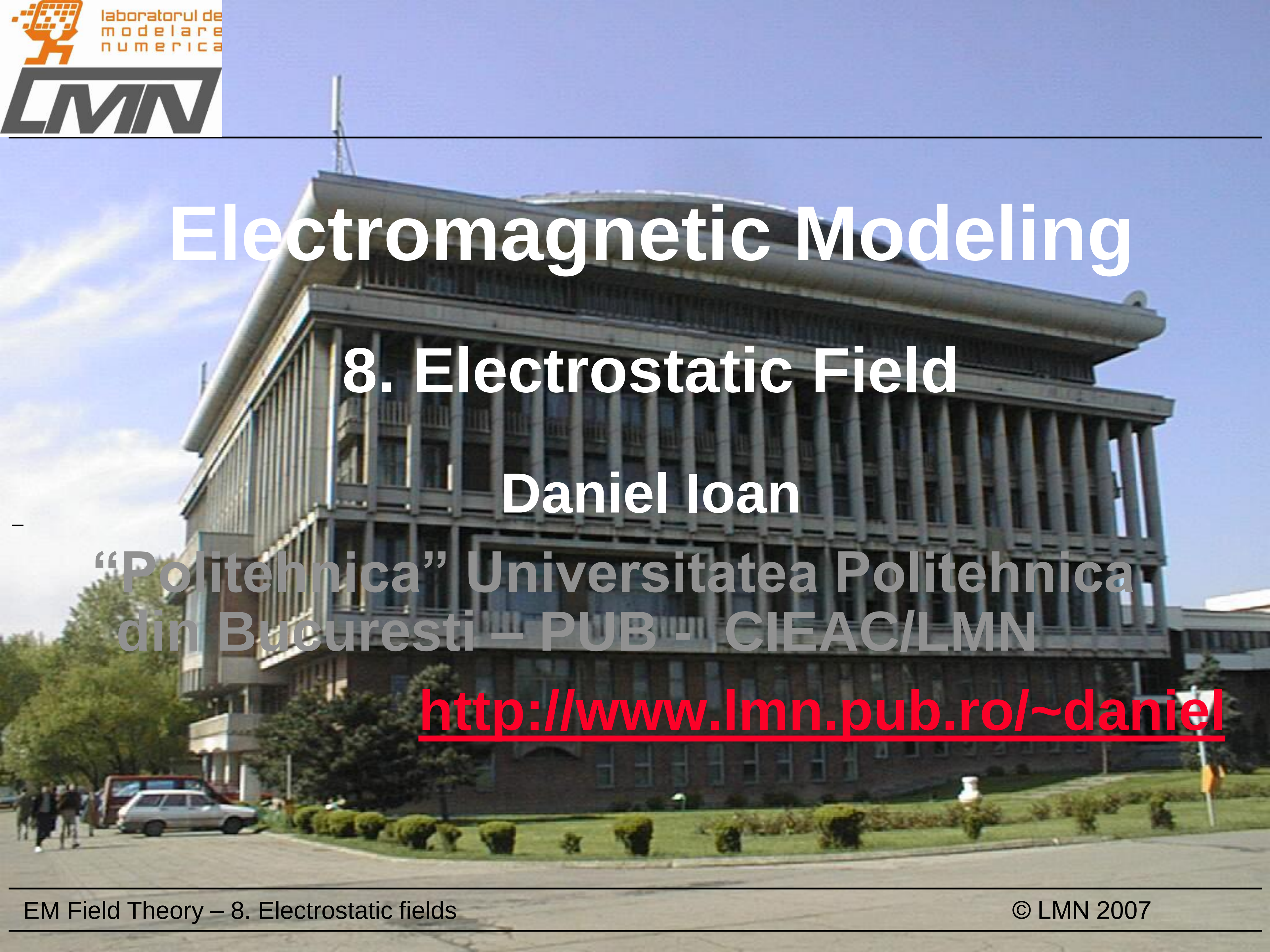

laboratorul de modelare numerica

LMN

Electromagnetic Modeling

8. Electrostatic Field

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Contents

EM field regimes

Regime: defined by simplifying hypothesis (some effects are neglected)

- **General – field propagation described by**

  Hyperbolic PDE

- **Quasi-static – field diffusion**

  Parabolic PDE

- **Steady-state – time invariant field/current distribution**
- **Static – invariant field distribution – no power losses**

  Elliptic PDE (no time)

Static regimes

$$1.\ \nabla \cdot \mathbf{D} = \rho$$

$$2.\ \nabla \cdot \mathbf{B} = 0$$

$$3.\ \nabla \times \mathbf{E} = \cancel{-\frac{\partial \mathbf{B}}{\partial t}}$$

$$4.\ \nabla \times \mathbf{H} = \cancel{\mathbf{J}} + \cancel{\frac{\partial \mathbf{D}}{\partial t}}$$

$$5.\ \mathbf{D} = \varepsilon\, \mathbf{E} + \mathbf{P}_p(\mathbf{E})$$

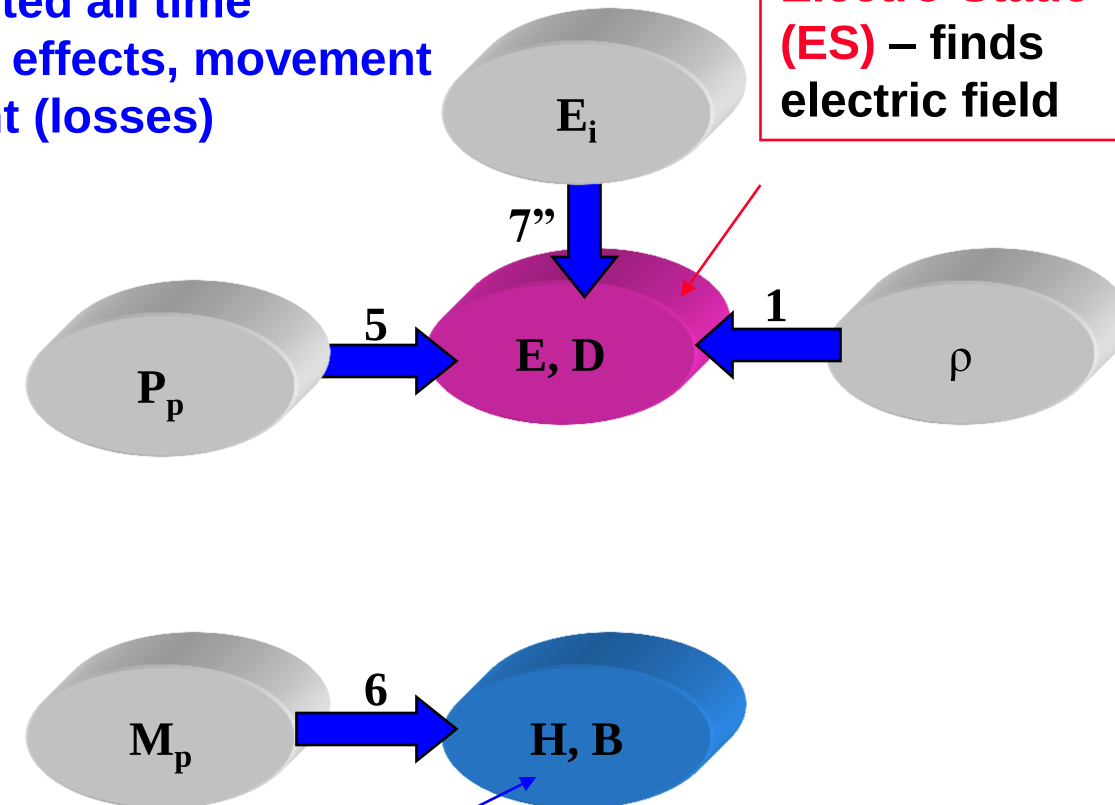
$$6.\ \mathbf{B} = \mu\, (\mathbf{H} + \mathbf{M}_p(\mathbf{H}))$$

$$7.\ \cancel{\mathbf{J}} = \sigma(\mathbf{E} + \mathbf{E}_i(\mathbf{E}))$$

$$8.\ p = \cancel{\mathbf{EJ}}$$

$$9.\ \cancel{\nabla \cdot \mathbf{J}} = \cancel{-\frac{\partial \rho}{\partial t}}$$

Are neglected all time dependent effects, movement and current (losses)



Electro-Static (ES) – finds electric field

Magneto- Static (MS) – finds magnetic field distribution

Electro-Static regime

- **Hypothesis:**
  - no movement
  - no time variation
  - no losses (current)
  - no magnetic field of interest

- **Fundamental Equations:**
- **Gauss' theorem**
- **Theorem of ES potential**
- **Electric constitutive relation**
- **ES field balance in conductors**

- **Field sources:**
  - Charge
  - Permanent polarization
  - Intrinsic field in conductors

$$\begin{cases} \psi_\Sigma = q_{D_\Sigma} \Leftrightarrow \oint_\Sigma \mathbf{D}d\mathbf{A} = \int_{D_\Sigma} \rho dv \\ div\mathbf{D} = \rho \\ \mathbf{n}_{12} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \rho_s \Leftrightarrow div_s\mathbf{D} = \rho_s \end{cases}$$

$$\begin{cases} u_\Gamma = 0 \Leftrightarrow \oint_\Gamma \boldsymbol{E}d\boldsymbol{r} = 0 \\ \mathbf{curl}\mathbf{E} = 0 \Rightarrow \mathbf{E} = -\mathbf{grad}V \\ \mathbf{n}_{12} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0, \Leftrightarrow \mathbf{E}_{t2} = \mathbf{E}_{t1} \end{cases}$$

$$\mathbf{D} = \mathbf{f}(\mathbf{E}) \Rightarrow \mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} \Rightarrow \mathbf{D} = \overline{\overline{\varepsilon}}\mathbf{E} + \mathbf{P}_p$$

$$\overline{\overline{\sigma}}(\boldsymbol{E} + \boldsymbol{E}_i) = 0 \Rightarrow \boldsymbol{E} = -\boldsymbol{E}_i \text{ in conductors}$$

Second order equation for scalar potential

$$\begin{cases} div\mathbf{D} = \rho \Rightarrow -div(\bar{\bar{\varepsilon}}\mathbf{grad}V + \mathbf{P}_p) = \rho \\ \mathbf{curl}\mathbf{E} = 0 \Rightarrow \mathbf{E} = -\mathbf{grad}V \\ \mathbf{D} = \bar{\bar{\varepsilon}}\mathbf{E} + \mathbf{P}_p \Rightarrow \mathbf{D} = -\bar{\bar{\varepsilon}}\mathbf{grad}V + \mathbf{P}_p \end{cases} \Rightarrow \boxed{-div(\bar{\bar{\varepsilon}}\mathbf{grad}V) = \rho_t}$$

where $\rho_t = \rho + \rho_p$,

Polarization charge density: $\boxed{\rho_p = -div\mathbf{P}_p}$

Particular cases:

- Linear homogeneous isotropic media (Poisson equation):

$$-div(\mathbf{grad}V) = \rho/\varepsilon \Rightarrow \boxed{\Delta V = -\rho/\varepsilon}$$

- No internal ES field sources (Laplace equation):

$$div(\bar{\bar{\varepsilon}}\mathbf{grad}V) = 0 \Rightarrow div(\mathbf{grad}V) = 0 \Leftrightarrow \boxed{\Delta V = 0}$$

Boundary conditions are necessary for a unique solution. They can be:

- Dirichlet b.c. or Neumann b,c. (no both in same P)

$$\boxed{V(P) = f_D(P),} \text{ on } S_D \neq \emptyset \qquad \boxed{\frac{\partial V}{\partial n} = f_N(P)} \text{ on } S_N = \Sigma - S_D$$

The fundamental ES problem

Input (known) data:

– Computational domain D bounded by Σ

– (CM) Material characteristics ε(**r**)>0 in D

– (CD) Internal field sources ρ(**r**) and **P**p(r) in D

– (CΣ) Boundary conditions (external sources) V(**r**) = f_D(**r**) or dV/dn=f_N(**r**) on Σ

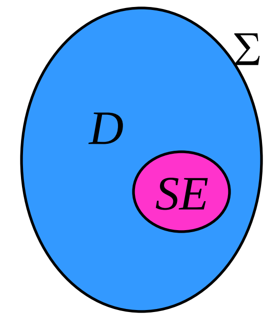
In terms of fields: Boundary conditions (field comp.)

(CΣ'): $\mathbf{E}_t(\mathbf{r})$ on S_E connected and $D_n(\mathbf{r})$ on $S_D=\Sigma\text{-}S_E$

Output data (solution):

E(r) and **D**(r) in D

Equations:



$$\begin{cases} div\mathbf{D} = \rho \\ \mathbf{curl}\mathbf{E} = 0 \\ \mathbf{D} = \bar{\bar{\varepsilon}}\mathbf{E} + \mathbf{P}_p \end{cases}$$

Output data (solution):

$V(\mathbf{r})$ in D ➔ $\mathbf{E}=\text{-}gradV,\ \mathbf{D}=f(\mathbf{E})$

Equations:

$$V = f_D(P) = \int_{PP_0} \mathbf{E}_t d\mathbf{r} \Leftrightarrow \mathbf{E}_t = \mathbf{grad}_s V = \mathbf{grad}_s f_D$$

$$\frac{dV}{dn} = f_N(P) = -\mathbf{E}\cdot\mathbf{n} = -\mathbf{n}\cdot\bar{\bar{\varepsilon}}^{-1}(\mathbf{D}-\mathbf{P}_p) \Rightarrow$$

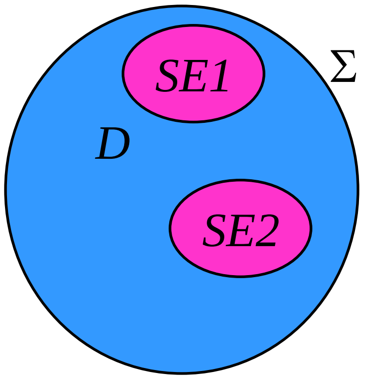
$$f_N(P) = -\mathbf{D}\mathbf{n}/\varepsilon \quad \text{if } \bar{\bar{\varepsilon}} = \varepsilon\bar{\bar{1}} \text{ and } \mathbf{P}_p = 0$$

$$-div(\bar{\bar{\varepsilon}}\mathbf{grad}V) = \rho_t$$

$$\text{where} \quad \rho_t = \rho_t - div\mathbf{P}_p$$

Non-connected Dirichlet surfaces
Solution uniqueness

$$\begin{cases} \mathbf{E}_t = \mathbf{grad}_s V = \mathbf{grad}_s f_D \\ V = f_D(P) \neq \int_{PP_0} \mathbf{E}_t d\mathbf{r} \end{cases}$$



in addition, for each $S_{Ek}, k = 1,2,.., n\text{-}1$ has to be known

(CΣ"): $U_k = \int_{PkP_0} \mathbf{E}_t d\mathbf{r}$ or $\Psi_k = \int_{S_{Ek}} D_n dS$ and $U_n = 0$

Uniqueness is based on the lemma of the ES trivial solution:

– **Any ES problem with zero internal and external sources has only zero solution:**

$$div(V\boldsymbol{D}) = \boldsymbol{D}\cdot \boldsymbol{grad}V + Vdiv(\boldsymbol{D}) = -\boldsymbol{D}\cdot\boldsymbol{E} + \rho V \Rightarrow$$

$$\int_D \boldsymbol{D}\cdot\boldsymbol{E}dv = \int_D (\varepsilon \boldsymbol{E}^2 + \boldsymbol{P}_p\cdot\boldsymbol{E})dv = \int_D \rho V dv - \oint_\Sigma V\boldsymbol{D}\cdot\boldsymbol{n}dS$$

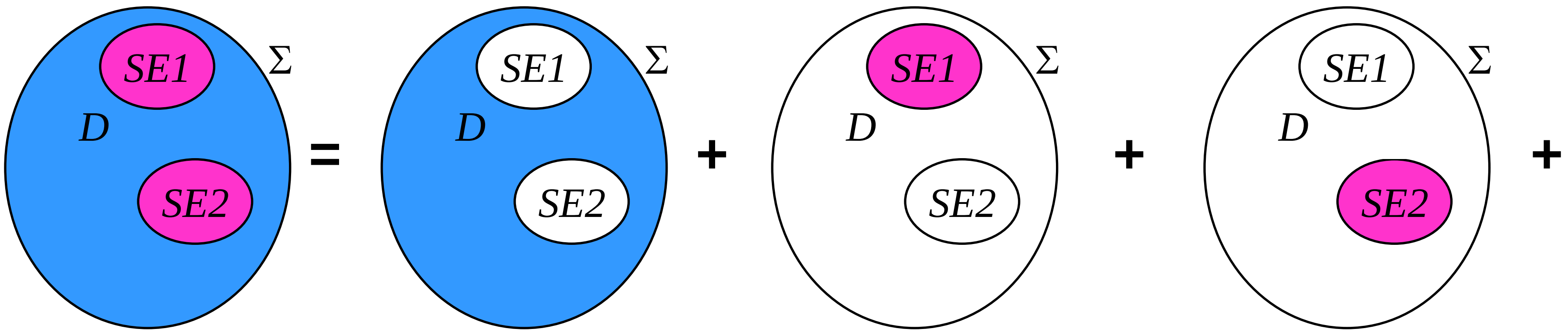
⇒ CΣ = CΣ' + CΣ"

$$\oint_\Sigma V\boldsymbol{D}\cdot\boldsymbol{n}dS = \int_{S_D} VD_n dS + \sum_{k=1}^{n}\int_{S_{Ek}} VD_n dS = 0, \quad U_k = 0$$

$$V(P) = \int_{PP_0} \boldsymbol{E}_t d\boldsymbol{r} = \int_{PP_k\subset S_{Ek}} \boldsymbol{E}_t d\boldsymbol{r} + \int_{P_kP_0\subset S_D} \boldsymbol{E}_t d\boldsymbol{r}, \qquad \int_{S_{Ek}} VD_n dS = U_k\int_{S_{Ek}} D_n dS = U_k\Psi_k$$

$$\int_D \boldsymbol{E}^2 dv = 0 \Rightarrow \boldsymbol{E} = 0 \text{ almost avrywhere in } D \Rightarrow V(P) = \int_{PP_0\subset D} \boldsymbol{E}\, d\boldsymbol{r} = 0$$

ES fields superposition



In linear media, the solutions **F = [E, D, V]** of well formulated ES fundamental problems in *D* with boundary Σ= *SE* + *SD* are linked by its field sources **C = [**CD, CΣ**]** by a linear operator: *S*: **C→ F**

$$S\left(\sum_{k=1}^{n}\lambda_k \boldsymbol{C}_k\right)=\sum_{k=1}^{n}\lambda_k S(\boldsymbol{C}_k)$$

The superposition can not be applied in nonlinear media. However the uniqueness is still valid if the material characteristic D=f(E) is monotonic:

[f(E2) - f(E1)] (E2 – E1) > 0

May be superposed **E, D,** *V* but not *E* = |**E**| or *D* = |**D**|

Superposition may be applied to (CD), (CΣ) but not to (CM) or *D*.

External problems. Coulomb integrals

External problem = unbounded domain D

- Boundary conditions (CΣ) are substituted by asymptotic conditions (CA):

$$\begin{cases} |V(\mathbf{r})| \le C/r, \quad |E(\mathbf{r})| \le C/r^2 & \text{in 3D} \\ |V(\mathbf{r})| \le C\ln(r), \quad |E(\mathbf{r})| \le C/r & \text{in 2D} \end{cases}$$

- Substance has a limited extension (vacuum in rest)
- The simplest (still important) examples:
- Small charged body in vacuum (3D)
- Strait thin charged wire (2D)

$$\begin{cases} V(\boldsymbol{r}) = \dfrac{q}{4\pi\varepsilon_0 R}, \boldsymbol{E}(\boldsymbol{r}) = \dfrac{q\boldsymbol{R}}{4\pi\varepsilon_0 R^3}, \boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}_0 \\ V(\boldsymbol{r}) = \dfrac{\rho_l}{2\pi\varepsilon_0 R}\ln(R/R_0), \boldsymbol{E}(\boldsymbol{r}) = \dfrac{\rho_l \boldsymbol{R}}{2\pi\varepsilon_0 R^2}, \boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}_0 \end{cases}$$

- By superposition: field of arbitrary charge distribution in vacuum (Coulomb integrals):

$$\begin{cases} V(\boldsymbol{r}) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int_{R^3} \dfrac{\rho(\boldsymbol{r}_0)dv}{R}, \quad \boldsymbol{E}(\boldsymbol{r}) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int_{R^3} \dfrac{\rho(\boldsymbol{r}_0)\boldsymbol{R}dv}{R^3}, \\ V(\boldsymbol{r}) = \dfrac{1}{2\pi\varepsilon_0}\displaystyle\int_{R^3} \rho(\boldsymbol{r}_0)\ln R dS, \quad \boldsymbol{E}(\boldsymbol{r}) = \dfrac{1}{2\pi\varepsilon_0}\displaystyle\int_{R^3} \dfrac{\rho(\boldsymbol{r}_0)\boldsymbol{R}dS}{R^2}, \end{cases}$$

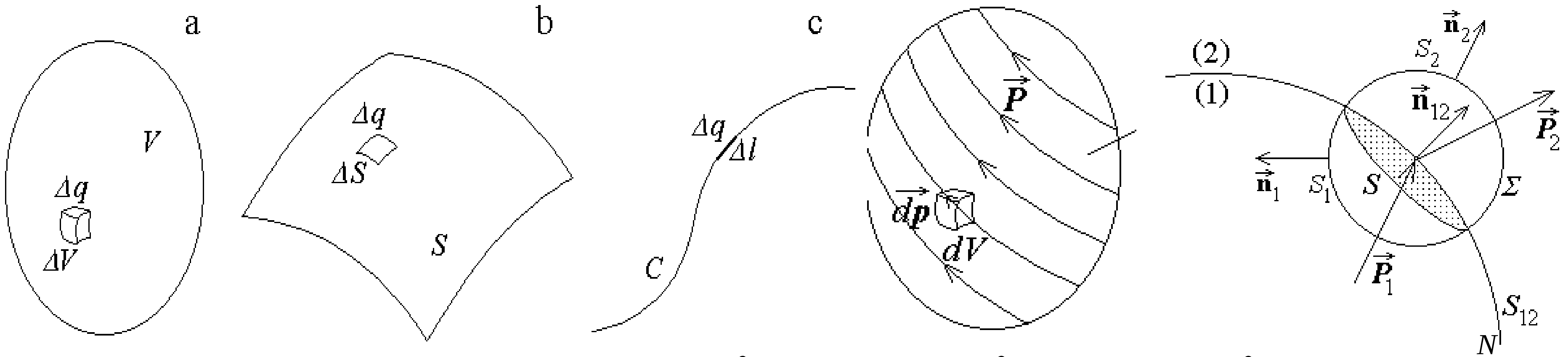
- Field of polarized bodies:
- Integral equation in linear dielectrics (with induced polarization):

$$\mathbf{P}(\mathbf{r}_0) = \chi\mathbf{E}(\mathbf{r}_0) = -\chi\,\mathbf{grad}V(\mathbf{r}_0)$$

$$\chi = \varepsilon_r - 1 = (\varepsilon - \varepsilon 0)/\varepsilon_0$$

$$\begin{cases} V(\mathbf{r}) = \dfrac{-1}{4\pi\varepsilon_0}\displaystyle\int_{R^3} \dfrac{div\mathbf{P}dv}{R}, \quad \mathbf{E}(\mathbf{r}) = \dfrac{-1}{4\pi\varepsilon_0}\displaystyle\int_{R^3} \dfrac{\mathbf{R}div\mathbf{P}dv}{R^3}, \\ V(\mathbf{r}) = \dfrac{-1}{2\pi\varepsilon_0}\displaystyle\int_{R^3} div\mathbf{P}\cdot ln\,R\cdot dS, \quad \mathbf{E}(\mathbf{r}) = \dfrac{-1}{2\pi\varepsilon_0}\displaystyle\int_{R^3} \dfrac{\mathbf{R}div\mathbf{P}\cdot dS}{R^2}, \end{cases}$$

Several charge distributions



$$q = \int_{V\subset D_\Sigma} \rho_V dV + \int_{S\subset D_\Sigma} \rho_S dS + \int_{C\subset D_\Sigma} \rho_l dl + \sum_{k=1}^{N} q_k$$

- **Charge may be distributed on**
- **volume (3D)**
- **film (surface)**
- **wire (curve)**
- **particle**

$$\rho_V = \lim_{\Delta V\to 0} \frac{\Delta q}{\Delta V} \,[C/m^3]$$

$$\rho_S = \lim_{\Delta S\to 0} \frac{\Delta q}{\Delta S} \,[C/m^2]$$

$$\rho_l = \lim_{\Delta l\to 0} \frac{\Delta q}{\Delta l} \,[C/m]$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\int_V \frac{\rho_V\, dV'}{R} + \int_S \frac{\rho_S\, dS'}{R} + \int_C \frac{\rho_l\, dl'}{R} + \sum_{k=1}^{N} \frac{q_k}{R_k}\right)$$

or using "distributions"

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho\, dV'}{R}$$

- **Polarization charge:**
- **In volume (3D)**
- **On body surface**

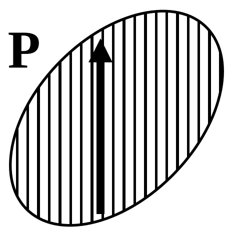
$$\begin{cases} \rho_{VP} = -div\,\mathbf{P}, \\ \rho_{SP} = -div_S\,\mathbf{P} = -\mathbf{n}_{12}\cdot(\mathbf{P}_2 - \mathbf{P}_1)\big|_\Sigma = -\mathbf{n}\cdot(0-\mathbf{P})\big|_\Sigma = \mathbf{P}\cdot\mathbf{n}\big|_\Sigma \end{cases}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\int_{V_P} \frac{(-div'\mathrm{P})dV'}{R} + \int_{S_P} \frac{(-\mathbf{n}_{12})\cdot(\mathrm{P}_2 - \mathrm{P}_1)\big|_{S_{12}}\, dS'}{R}\right]$$

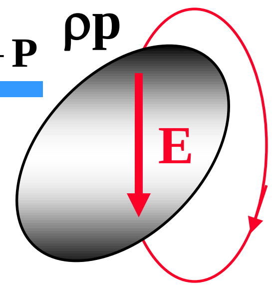
Complementar approaches for polarized bodies

- **Coulombian model: same E, different D**

$$\begin{cases} div\mathbf{D} = 0 \\ curl\mathbf{E} = 0 \\ \mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} \end{cases}$$



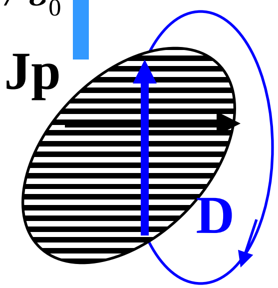
$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$



$$\begin{cases} div(\varepsilon_0\mathbf{E}) = -div\mathbf{P} \\ curl\mathbf{E} = 0 \\ \mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} \end{cases} \Rightarrow \begin{cases} div\,\varepsilon_0\mathbf{E} = \rho_p \\ \mathbf{E} = -gradV \\ \rho_p = -div\mathbf{P} \end{cases}$$

- **Amperian model: same D, different E**

$\mathbf{E} = (\mathbf{D} - \mathbf{P}) / \varepsilon_0$



$$\begin{cases} div\mathbf{D} = 0 \\ curl\mathbf{D} = curl\mathbf{P} \\ \mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} \end{cases} \Rightarrow \begin{cases} \mathbf{D} = curl\mathbf{T} \\ curl\mathbf{D} = \mathbf{J}_p \\ \mathbf{J}_p = curl\mathbf{P} \end{cases}$$

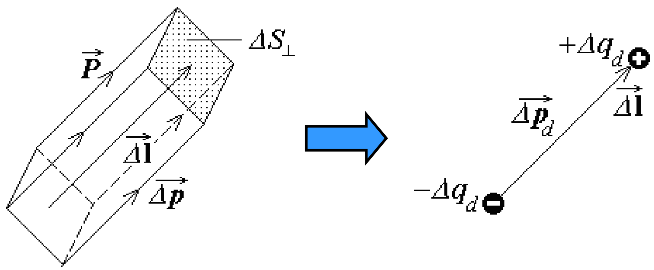
Dual approach: use of scalar V and vector **T** potentials to solve these complementary problems

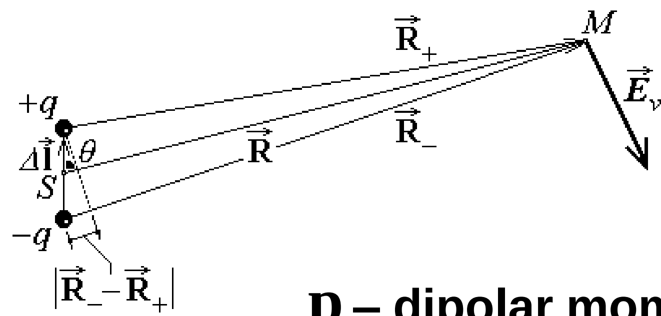
E has open lines. It internally is opposite to P (de-polarizing).

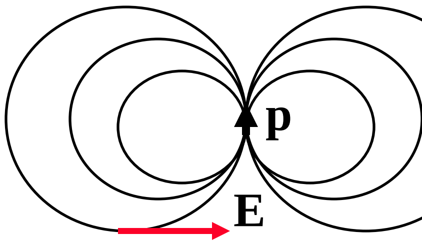
D has closed lines. It has the same sense as P.

P=Pp is independent of E,D and it is dependent of E in dielectrics, e.g. $\mathbf{P} = \varepsilon_0\chi_e\mathbf{E}$

Far field of a small polarized particle







$$\boldsymbol{p} = q \cdot \boldsymbol{\Delta l} = \boldsymbol{P} \cdot \Delta v = \boldsymbol{P} \cdot S\Delta l$$

p – dipolar momentum

$$V(\mathbf{r}) = \frac{-1}{4\pi\varepsilon_0}\int_{R^3}\frac{div\boldsymbol{P}dv}{R} = \frac{-1}{4\pi\varepsilon_0}\int_{\Sigma}\frac{div_s\boldsymbol{P}dS}{R} = \frac{1}{4\pi\varepsilon_0}\int_{\Sigma}\frac{\boldsymbol{P}\boldsymbol{n}dS}{R} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{R_+}+\frac{-q}{R_-}\right) = \frac{q}{4\pi\varepsilon_0}\frac{R_- - R_+}{R_+R_-} \approx$$

$$\approx \frac{q}{4\pi\varepsilon_0}\frac{\Delta l\cos\theta}{R^2} = \frac{p\cos\theta}{4\pi\varepsilon_0 R^2} = \frac{1}{4\pi\varepsilon_0 R^2}\cdot\frac{\boldsymbol{pR}}{R}, \quad \Rightarrow \boxed{V = \frac{\boldsymbol{p}\cdot\boldsymbol{R}}{4\pi\varepsilon_0 R^3}}, \; V = (\boldsymbol{p}\cdot\boldsymbol{grad}V_q)/q, \; V_q = \frac{q}{4\pi\varepsilon_0 R}$$

$$\boldsymbol{E}(\mathbf{r}) = -\boldsymbol{grad}\,V = -\boldsymbol{grad}\,\frac{\boldsymbol{p}\cdot\boldsymbol{R}}{4\pi\varepsilon_0 R^3} = -\frac{1}{4\pi\varepsilon_0}\left[\frac{1}{R^3}\boldsymbol{grad}(\boldsymbol{p}\cdot\boldsymbol{R}) + (\boldsymbol{p}\cdot\boldsymbol{R})\boldsymbol{grad}\frac{1}{R^3}\right] \Rightarrow$$

$$\boxed{\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\left[\frac{3(\boldsymbol{p}\cdot\boldsymbol{R})\boldsymbol{R}}{R^5} - \frac{\boldsymbol{p}}{R^3}\right]}$$

By superposition,

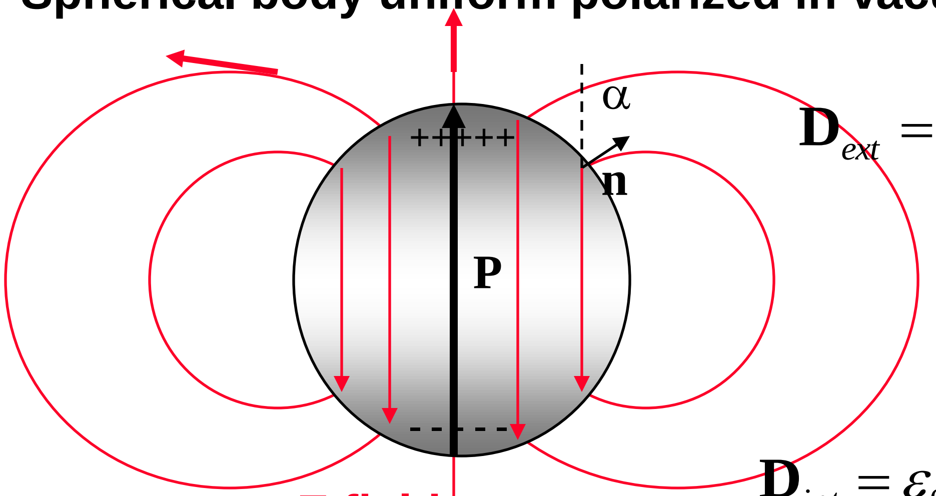
for a large polarized body:

$$V = \frac{1}{4\pi\varepsilon_0}\int_D \frac{\boldsymbol{P}\cdot\boldsymbol{R}dv}{R^3}$$

$$\boldsymbol{E} = -\frac{1}{4\pi\varepsilon_0}\int_D grad\frac{\boldsymbol{P}\cdot\boldsymbol{R}}{R^3}dv$$

Field inside polarized bodies

- **Spherical body uniform polarized in vacuum, polar momentum** $\mathbf{p} = \mathbf{P}V = \mathbf{P}\frac{4\pi a^3}{3}$

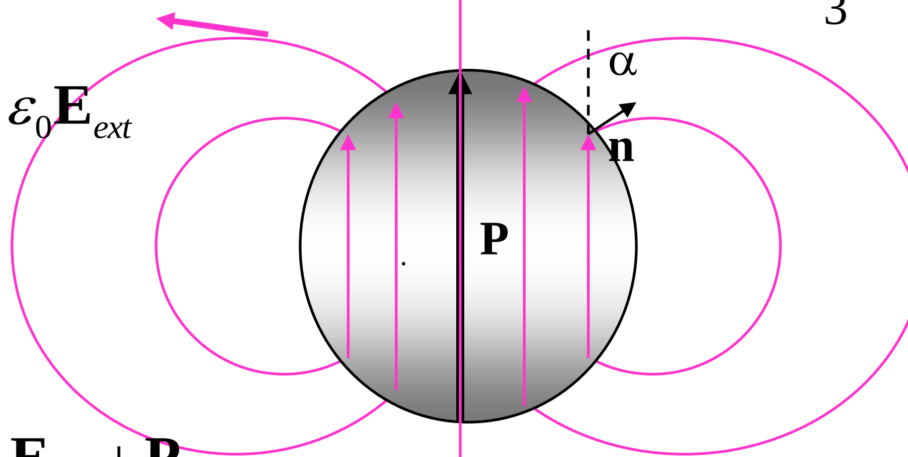


E field

$$curl\mathbf{E} = 0 \Leftrightarrow \oint_{\Gamma} \mathbf{E}d\mathbf{r} = 0$$

$$\mathbf{D}_{ext} = \varepsilon_0 \mathbf{E}_{ext}$$

$$\mathbf{D}_{int} = \varepsilon_0 \mathbf{E}_{int} + \mathbf{P}$$



D field

$$div\mathbf{D} = 0 \Leftrightarrow \oint_{\Sigma} \mathbf{D}\mathbf{n}dS = 0$$

External field:

$$\boldsymbol{E}_{ext} = \frac{1}{4\pi\varepsilon_0}\left[\frac{3(\boldsymbol{p}\cdot\boldsymbol{R})\boldsymbol{R}}{R^5} - \frac{\boldsymbol{p}}{R^3}\right], \quad \Rightarrow E_r = \frac{p\cos\alpha}{2\pi\varepsilon_0 R^3} = \frac{2Pa^3\cos\alpha}{3\varepsilon_0 R^3}, E_\alpha = \frac{-p\sin\alpha}{4\pi\varepsilon_0 R^3} = \frac{-Pa^3\sin\alpha}{3\varepsilon_0 R^3}$$

$$Equator: E_{\text{int}} = E_{ext} \Rightarrow E_{\text{int}} = P/(3\varepsilon_0)$$

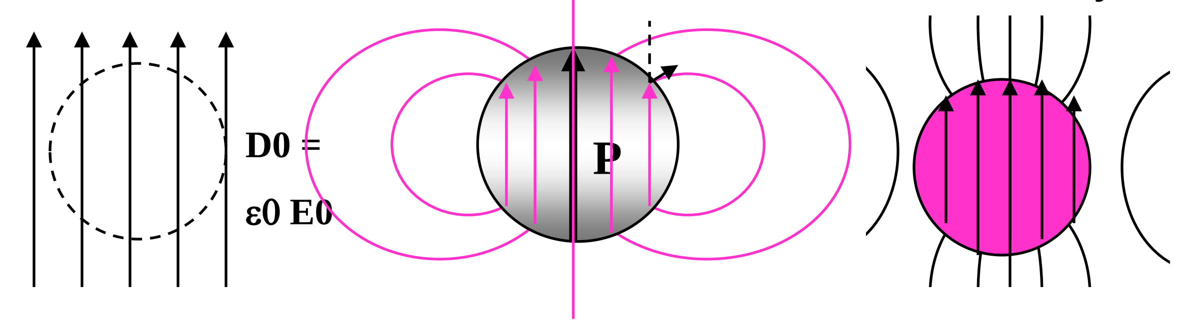
$$\boldsymbol{E}_{\text{int}} = -\frac{\boldsymbol{P}}{3\varepsilon_0}, \quad \boldsymbol{D}_{\text{int}} = \frac{2\boldsymbol{P}}{3}$$

$$Poles: D_{\text{int}} = D_{ext} \Rightarrow -\varepsilon_0 E_{\text{int}} + P = 2P/3$$

Internal field is constant

ES field perturbation due to dielectric bodies

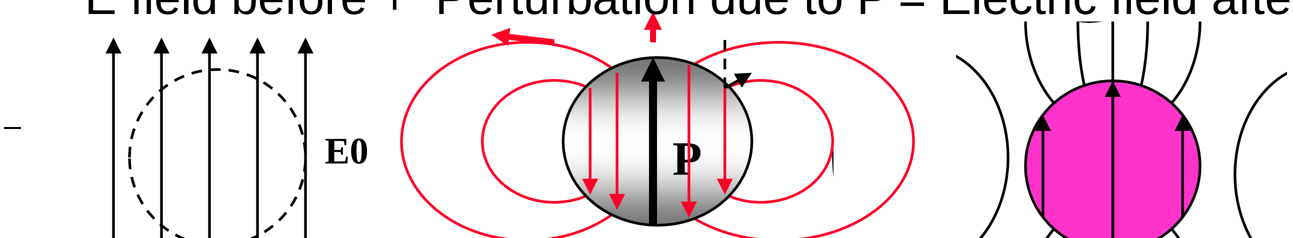
D field before + Perturbation due to P = Flux density after



- **Dielectrics confines and direct D lines, increasing electric flux**

$$\lim_{\varepsilon \to \infty} \frac{D}{D_0} = 3$$

E field before + Perturbation due to P = Electric field after



- **Electric field E is decreased, as well as voltage**

$$\lim_{\varepsilon \to \infty} \frac{E}{E_0} = 0$$

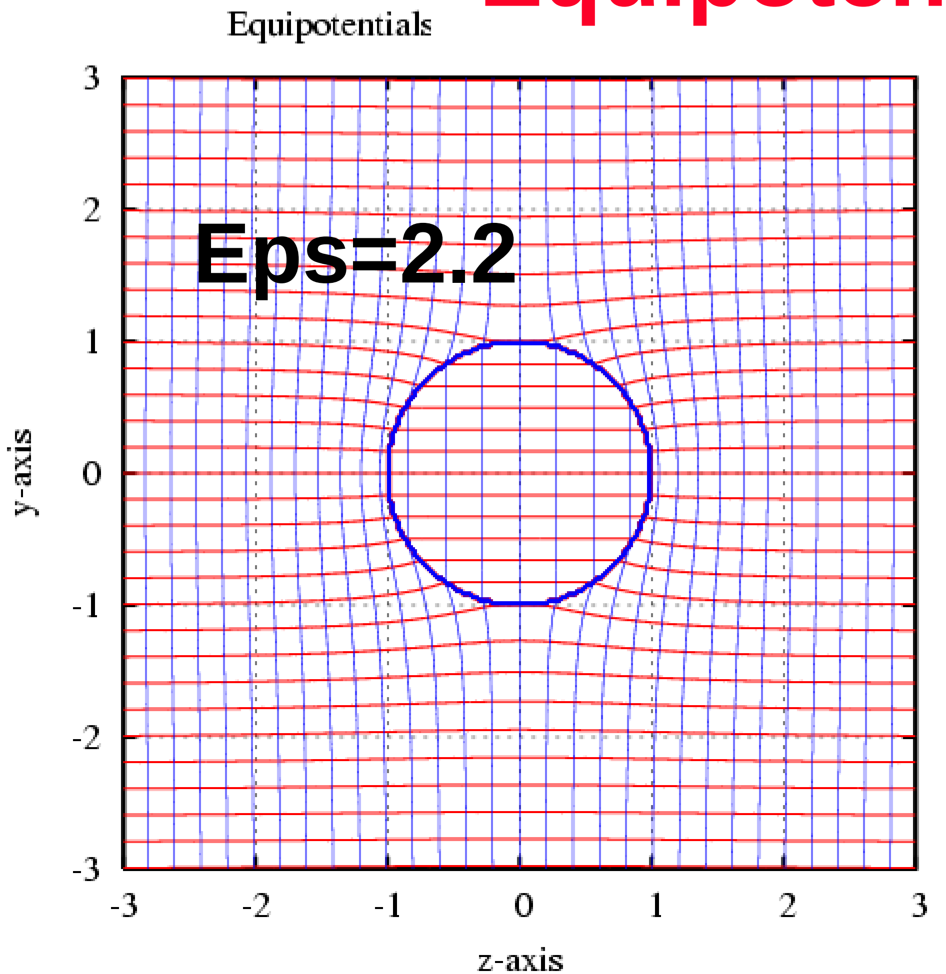
$$P = D - \varepsilon_0 E = (\varepsilon - \varepsilon_0)E = \chi \varepsilon_0 E$$

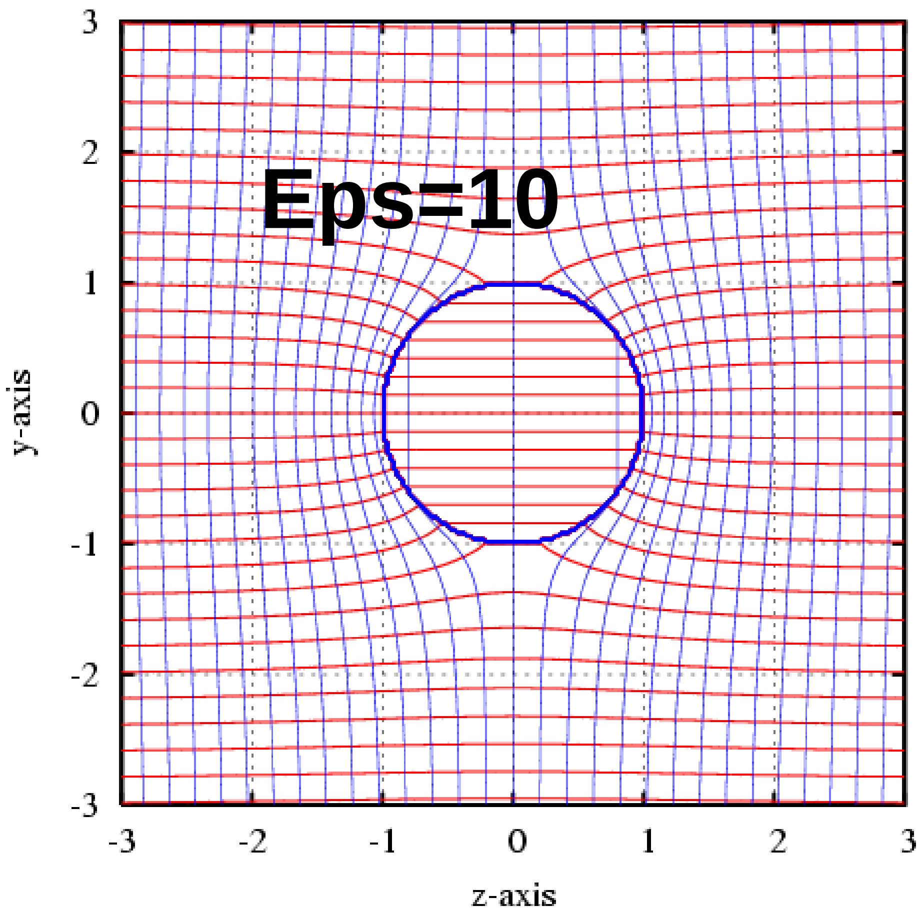
$$\boldsymbol{E} = \boldsymbol{E}_0 + \boldsymbol{E}_{\text{int}} = \boldsymbol{E}_0 - \frac{\boldsymbol{P}}{3\varepsilon_0} \Rightarrow E_0 = E + \chi E/3 = (1+\chi/3)E \Rightarrow E_{\text{int}} = E_0/(1+\chi/3)$$

$$D_{\text{int}} = \varepsilon E_{\text{int}} = \varepsilon E_0/(1+\chi/3) = D_0(1+\chi)/(1+\chi/3) \Rightarrow p \boxed{= \varepsilon_0 E_0 4\pi a^3 \chi/(\chi+3)}$$

$$\mathbf{E}_{ext} = \frac{1}{4\pi\varepsilon_0}\left[\frac{3(\mathbf{p}\cdot\mathbf{R})\mathbf{R}}{R^5} - \frac{\mathbf{p}}{R^3}\right], \quad \Rightarrow E_r = \frac{p\cos\alpha}{2\pi\varepsilon_0 R^3} = \frac{2Pa^3\cos\alpha}{3\varepsilon_0 R^3}, E_\alpha = \frac{-p\sin\alpha}{4\pi\varepsilon_0 R^3} = \frac{-Pa^3\sin\alpha}{3\varepsilon_0 R^3}$$

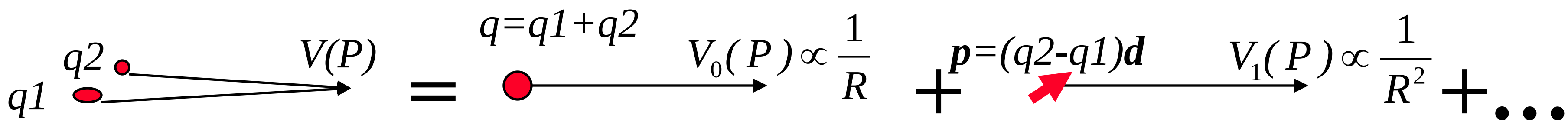
Equipotentials and flux lines



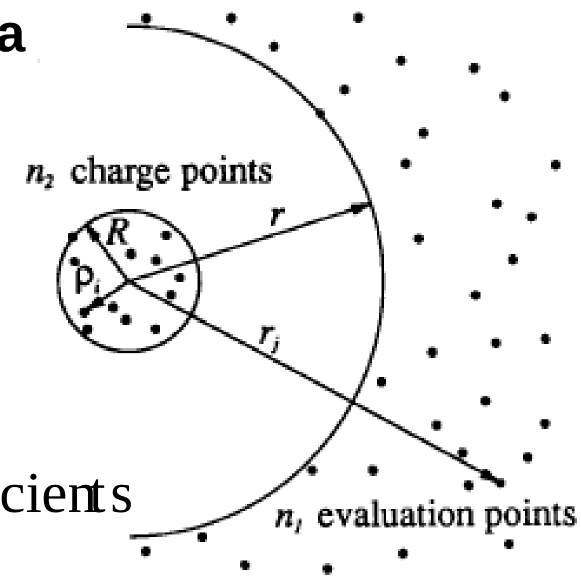


Download from http://wiki.4hv.org/index.php/Dielectric_Sphere_in_Electric_Field

Far field. Multipole expansion



Any arbitrary charge distribution (q1, q2, …, qn2) **inside a sphere can be approximated outside by a truncated multipole expansion, expressed as a sum of harmonics, in spherical coordinates:**



$$V(r,\theta,\varphi) = \sum_{n=0}^{l} \sum_{m=-n}^{n} \frac{M_n^m}{r^{n+1}} Y_n^m(\theta,\varphi),$$ with

$$M_n^m = \sum_{i=1}^{n2} q_i r_i^{\,n} Y_n^{-m}(\theta_i,\varphi_i)$$ are the multipole coefficients

with an error bounded by : $err < K(R/r)^{l+1}$

Details in: J. D. Jackson – *Classical Electrodynamics,* Wiley 75 and

K. Nabors – *FastCap,* IEEE Trans on CAD, nov 91 (FMM – Fast Multipole Method)

The (deceptive) three potentials formula

$$div(f\,\mathbf{grad}\,g) = \mathbf{grad}f \cdot \mathbf{grad}g + f\ div\,(\mathbf{grad}\,g) = \mathbf{grad}f \cdot \mathbf{grad}g + f\,\Delta g$$

$$div(f\,\mathbf{grad}\,g) - div(g\,\mathbf{grad}\,f) = f\,\Delta g - g\,\Delta f \qquad \int_D (f\,\Delta g - g\,\Delta f)\,d\Omega = \oint_{\partial D}\left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right)dS$$

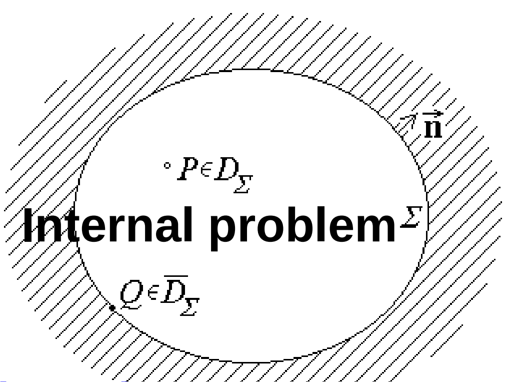
$$\int_{D_\Sigma \setminus D_\sigma}(f\,\Delta g - g\,\Delta f)\,d\Omega' = \oint_\Sigma\left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right)dS' + \oint_\sigma\left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right)dS', \qquad \boxed{g(\mathbf{r}) = \frac{1}{R'} \Rightarrow}$$

$$\Delta g = 0, \quad \lim_{R'\to 0}\oint_\sigma f\frac{\partial g}{\partial n}dS' = \lim_{R'\to 0}\frac{f_{\text{med}}}{R'^2}\oint_\sigma dS' = 4\pi \cdot f(P) \qquad \lim_{R'\to 0}\oint_\sigma g\frac{\partial f}{\partial n}dS' = \left(\frac{\partial f}{\partial n}\right)_{\text{med}} \cdot \frac{4\pi R'^2}{R'} \to 0$$

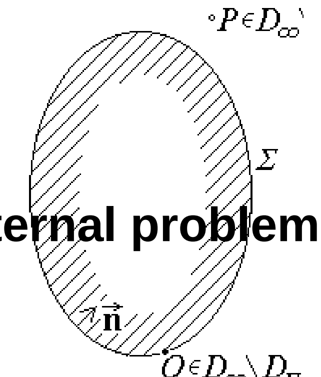
$$f = V \Rightarrow \boxed{V(P) = \frac{-1}{4\pi}\int_{D_\Sigma}\frac{\Delta V|_Q}{R'}d\Omega' - \frac{1}{4\pi}\oint_\Sigma\left[V|_Q\frac{\partial}{\partial n}\left(\frac{1}{R'}\right) - \frac{1}{R'}\frac{\partial V}{\partial n}\bigg|_Q\right]dS'}$$

g - fundamental solution of Laplace equation

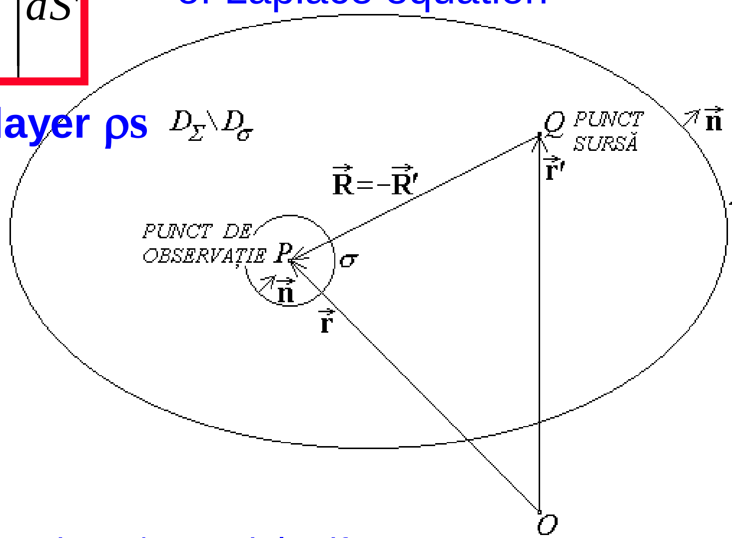
Potentials: 1-volume ρv 2- double layer Ps 3- single layer ρs



Internal problem



External problem



V has a jump on Σ: http://www.stanford.edu/class/math220b/handouts/potential.pdf

Boundary integral equation. Kirchhoff's formula -harmonic potential

- **In 3D: Green function (fundamental solution of Laplace operator):**

$$g(\mathbf{r}) = \frac{1}{R'} \qquad \omega \cdot V(P) = \oint_{\Sigma} \left[\frac{1}{R'} \cdot \frac{\partial V}{\partial n}\bigg|_{Q} - V\big|_{Q} \cdot \frac{\partial}{\partial n}\left(\frac{1}{R'}\right) \right] dS'$$

$$\omega = \begin{cases} 4\pi, & P \text{ internal in } D_{\Sigma} \\ 2\pi, & P \text{ regular on } \Sigma = \partial D_{\Sigma} \\ \omega, & P \text{ corner in } \Sigma = \partial D_{\Sigma}, \omega = \text{solid angle, } D_{\Sigma} \text{ is viewed} \end{cases}$$

- **In 2D: Green function (in free space):**

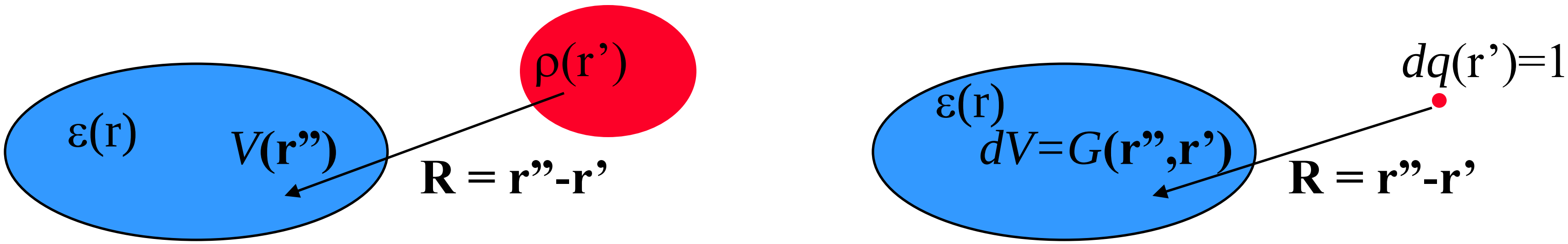
$$g(\boldsymbol{r}) = \ln\frac{1}{R'} \qquad \alpha \cdot V(P) = \oint_{\Gamma} \left[\left(\ln\frac{1}{R'}\right) \cdot \frac{\partial V}{\partial n}\bigg|_{Q} - V\big|_{Q} \cdot \frac{\partial}{\partial n}\left(\ln\frac{1}{R'}\right) \right] dr'$$

$$\alpha = \begin{cases} 4\pi, & P \text{ regular in } S_{\Gamma} \\ 2\pi, & P \text{ regular on } \Gamma \\ \omega, & P \text{ corner of } \Gamma, \ \alpha \text{ is its angle} \end{cases}$$

On conductors: unknown known

- **In non-homogeneous media Green function can be adapted to the problem**

Green function of the entire non-homogeneous space



Green function G is the potential of a punctual unitary charge:

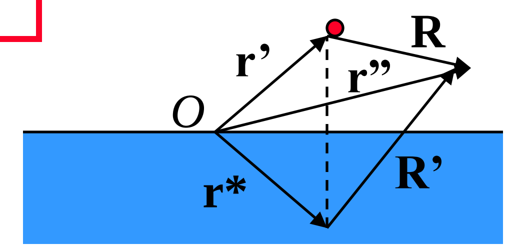
$$-div(\varepsilon\, grad V(\mathbf{r}'')) = \rho(\mathbf{r}') \Rightarrow \boxed{-div(\varepsilon\, grad G(\mathbf{r}'',\mathbf{r}')) = \delta(\mathbf{r}''-\mathbf{r}')}$$

$$\text{Reciprocity} \Rightarrow G(\mathbf{r}'',\mathbf{r}') = G(\mathbf{r}',\mathbf{r}''), \text{ If } \varepsilon = \varepsilon_0 \Rightarrow G(\mathbf{r}'',\mathbf{r}') = \frac{1}{4\pi\varepsilon_0 R}$$

- **By superposition is obtained solution of the generalized Poisson equation:**

$$\boxed{V(\mathbf{r}'') = \int_{R^3} G(\mathbf{r}'',\mathbf{r}')\rho(\mathbf{r}')dv}$$ ➔ Coulomb integral

- **Example: charge above dielectric semi-space**

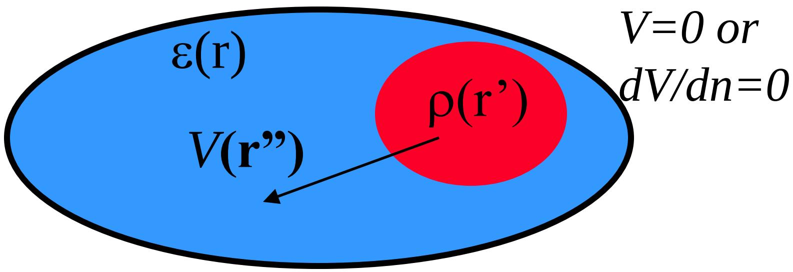


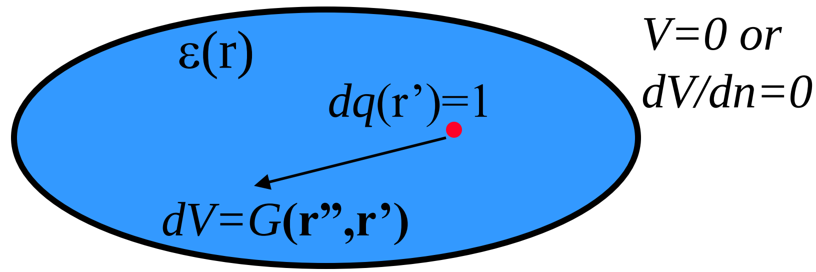
$$G(\mathbf{r}'',\mathbf{r}') = \frac{1}{4\pi\varepsilon_0}\begin{cases}\left[\frac{1}{R} - \frac{\alpha}{R'}\right], & \text{for } z = \mathbf{r}'\cdot\mathbf{k} > 0,\ \alpha = \chi/(\chi+2), \\ \beta/R, & \text{for } z \le 0,\ \beta = 2/(\chi+2)\end{cases}$$

$$R' = \mathbf{r}''-\mathbf{r}^*,\ \mathbf{r}^* = \mathbf{r}'-2\mathbf{k}z$$

See http://farside.ph.utexas.edu/teaching/jk1/lectures/node38.html

Green function of a bounded domain





The Green function G is the potential of a punctual unitary charge in in a domain with zero b.c.:

$$G(\mathbf{r}'',\mathbf{r}') = G(\mathbf{r}',\mathbf{r}''),$$

$$-div(\varepsilon\, grad\, G(\mathbf{r}'',\mathbf{r}')) = \delta(\mathbf{r}''-\mathbf{r}'), \text{ for } \mathbf{r}'' \in D$$

$$G(\mathbf{r}'',\mathbf{r}') = 0, \text{ for } \mathbf{r}'' \in S_D \subset \Sigma = \partial D, S_D \neq \emptyset$$

$$\frac{dG(\mathbf{r}'',\mathbf{r}')}{dn''} = 0, \text{ for } \mathbf{r}'' \in S_N = \Sigma - S_D$$

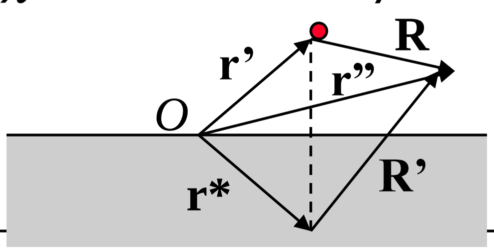
By superposition is obtained the solution of the generalized Poisson equation with zero b.c. of same type:

$$V(\mathbf{r}'') = \int_D G(\mathbf{r}'',\mathbf{r}')\rho(\mathbf{r}')dv$$

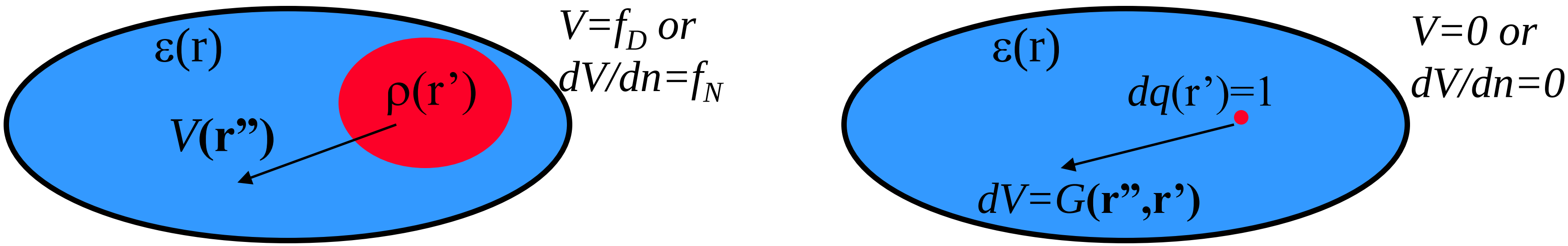
It is the solution of the integral equation "three potentials formula" for zero b.c.

- **Example: charge above a conductive semi-space** $(\chi \to \infty \Rightarrow \alpha \to 1, \beta \to 0)$

$$G(\mathbf{r}'',\mathbf{r}') = \frac{1}{4\pi\varepsilon_0}\begin{cases}\left[\dfrac{1}{R} - \dfrac{1}{R'}\right], & \text{for } z = \mathbf{r}'\cdot\mathbf{k} > 0,\\ 0, & \text{for } z \leq 0,\end{cases}$$



Solution in a bounded domain with non-zero b.c.



The definition of Green function is not changed !

$$div(\delta V\varepsilon\, gradV) = \delta V div(\varepsilon\, gradV) + \varepsilon\, gradV \cdot grad\delta V \Rightarrow$$

$$\begin{cases} \int_D (\delta V\rho - \varepsilon\, gradV \cdot grad\delta V)dv - \int_\Sigma \delta V D_n dS = 0 \\ \int_D (V\delta\rho - \varepsilon\, gradV \cdot grad\delta V)dv - \int_\Sigma V\delta D_n dS = 0 \end{cases}$$

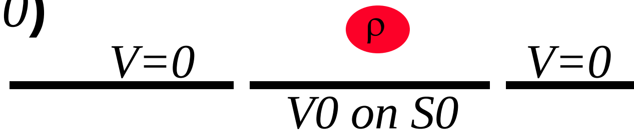
$$\int_D (V\delta\rho - \delta V\rho)dv - \int_\Sigma (V\delta D_n - \delta V D_n)dS = 0$$

$$V(\mathbf{r}'') = \int_D G(\mathbf{r}'',\mathbf{r}')\rho(\mathbf{r}')dv' \quad - \int_{S_D} \varepsilon \frac{dG}{dn'} \cdot f_D(\mathbf{r}')dS' - \int_{S_N} \varepsilon G f_N(\mathbf{r}')dS'$$

Solution of the generalized Poisson equation with non-zero b.c. It is also the solution of the integral equation "three potentials formula" (no longer deceptive).

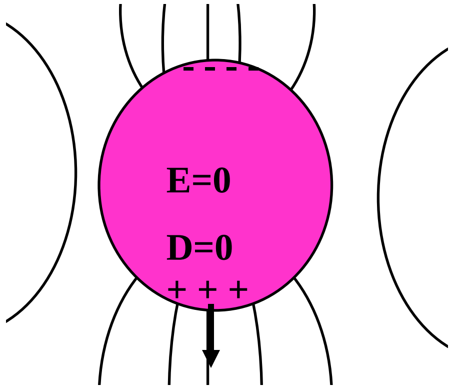
- **Example: charge above plane electrodes (*SN=0*)**

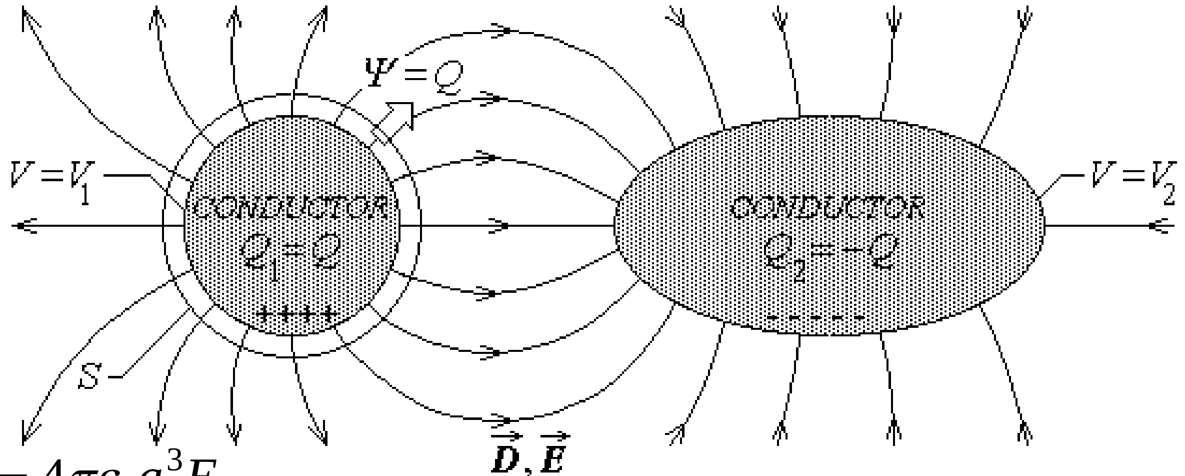
$$V(\mathbf{r}'') = \int_{z>0} G(\mathbf{r}'',\mathbf{r}')\rho(\mathbf{r}')dv' - \varepsilon_0 V_0 \int_S \frac{dG}{dn'} dS'$$



Conductors in ES fields

- **The ES field is zero in conductors with $E_i = 0$, and in general $E = -E_i$**
- V = ct **in conductors $E_t = 0$, on the conductor boundary, hence field lines are perpendicular on it**
- $D = \varepsilon_0 E = 0$ and $\rho_v = \mathrm{div}\, \mathbf{D} = 0$ **in conductors**
- **Conductor charge is distributed only on surface $\rho_s = \mathrm{div}_s \mathbf{D} = \mathbf{n}\mathbf{D}_{ext}$**
- **Apparently conductors in ES have $\varepsilon = D/E \rightarrow$ inf. since $E=0$**





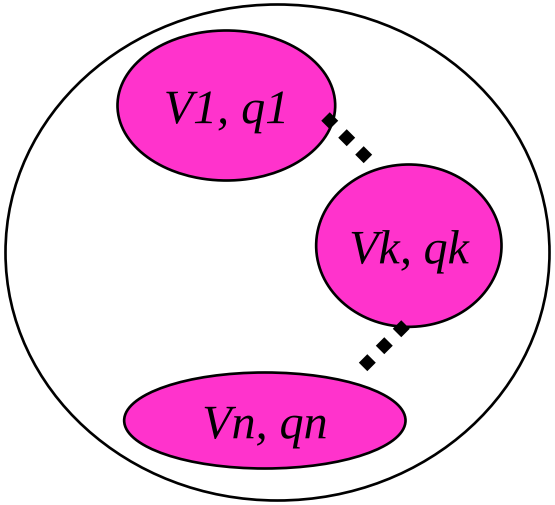
$$E_\alpha = E_0 \sin\alpha - \frac{p \sin\alpha}{4\pi\varepsilon_0 a^3} = 0 \Rightarrow p = 4\pi\varepsilon_0 a^3 E_0,$$

influential charge

$$E_r = E_0 \cos\alpha + \frac{p\cos\alpha}{2\pi\varepsilon_0 a^3} = (E_0 + 2E_0)\cos\alpha = 3E_0\cos\alpha; \Rightarrow \quad \rho_s = D_n = \varepsilon_0 E_r = 3\varepsilon_0 E_0 \cos\alpha$$

Capacitors. Capacitances

Capacitor: one or many conductive bodies separated by a dielectric insulator.



Globally it is characterized by vectors of:

Charges: $\mathbf{q} = [q1, q2, \ldots, qn]^T$

Voltages: $\mathbf{v} = [v1, v2, \ldots, vn]^T$

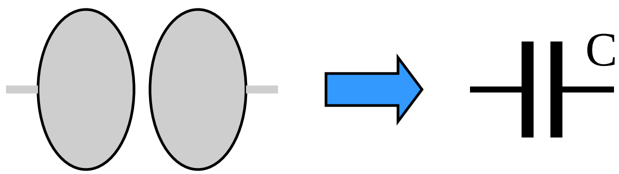
Maxwell theorem of linear capacitors. In linear dielectrics charges are linear combinations of conductor voltages:

$$\mathbf{q} = \mathbf{C}\mathbf{v} \Leftrightarrow \begin{bmatrix} q_1 \\ q_2 \\ \\ q_n \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ \ldots & & & \\ c_{n1} & c_{n2} & \ldots & c_{nn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \\ v_n \end{bmatrix} \Leftrightarrow \begin{cases} q_1 = c_{11} \cdot v_1 + c_{12} \cdot v_2 + \ldots + c_{1n} \cdot v_n \\ q_2 = c_{11} \cdot v_1 + c_{12} \cdot v_2 + \ldots + c_{1n} \cdot v_n \\ \ldots \\ q_n = c_{n1} \cdot v_1 + c_{n2} \cdot v_2 + \ldots + c_{nn} \cdot v_n \end{cases}$$

Prove: ES problem with Dirichlet b.c. v1, v2,…vn on conductors gives V(P) in *D*. Conductor charges $q_k = \int_{\Sigma_k} D_n dS = -\int_{\Sigma_k} \varepsilon \frac{dV}{dn} dS$ may be computed by superposition

Partial and equivalent capacitances

- **Dipolar capacitor:** $q=q1=-q2,\ u = v1-v2,\ q = Cv$, **capacitance:** C [F]

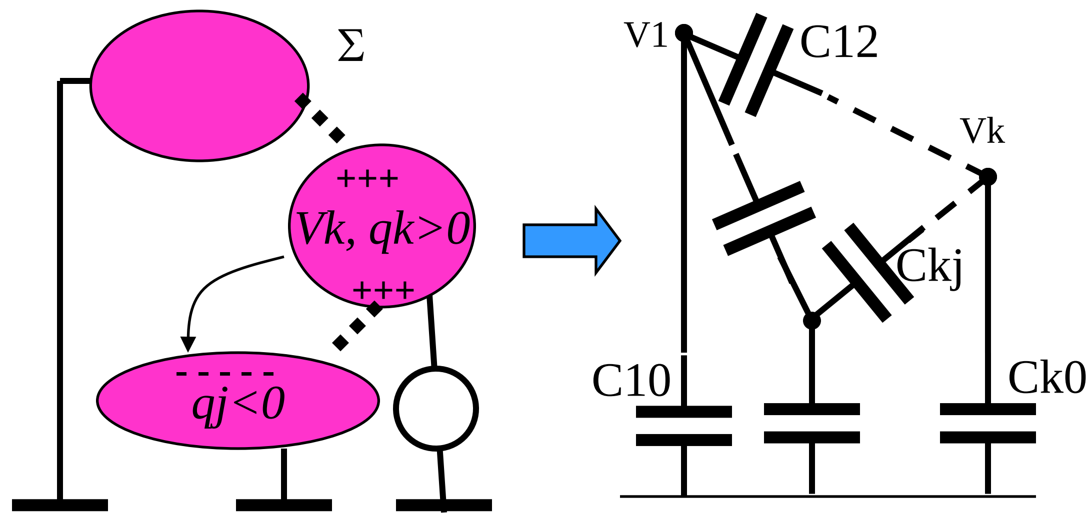


- **Multi-conductors case:**

  Capacitance (nodal) coefficients:

  $c_{kk} > 0,\ c_{kj} = c_{jk} < 0$

  – $\mathbf{C} = \mathbf{C}^{\mathrm{T}}$ due to reciprocity



Partial (branch) capacitances:

$$q_1 = c_{11}\cdot v_1 + c_{12}\cdot v_2 + \ldots + c_{1n}\cdot v_n = C_{10}\cdot v_1 + C_{12}\cdot(v_1 - v_2) + \ldots + C_{1n}\cdot(v_1 - v_n)$$

$$c_{11} = C_{10} + C_{12} + \ldots + C_{1n},\quad c_{12} = -C_{12},\quad \ldots,\quad c_{1n} = -C_{1n} \quad \Rightarrow$$

$$C_{kj} = -c_{kj} > 0, \qquad C_{k0} = c_{k1} + c_{k2} + \ldots + c_{kn} > 0$$

Capacitances extraction

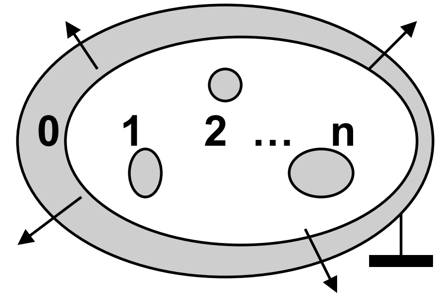
- **Linear approach**

$$q_k = c_{k1}v_1 + \ldots + c_{kj}v_j + \ldots + c_{kn} \cdot v_n \Rightarrow c_{kj} = \left.\frac{q_k}{v_j}\right|_{\substack{v_i=0 \\ i=1..n\neq j}} = -C_{kj}, \text{ for } k \neq j$$

Conductors are excited successively.
ES field is computed n times for several

Dirichlet b.c.: $\mathbf{v}$ = [1; 0;…;0;…;0], $\mathbf{v}$ = [0;1;…;0;…;0] …

Each field solution generates a column in $\mathbf{C}$



- **Energetic approach**

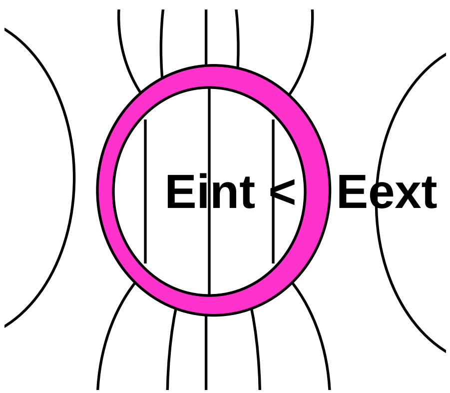
$$W_e = \frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n} c_{kj}v_k v_j \Rightarrow c_{kk} = \left.\frac{2W_e}{v_k^2}\right|_{\substack{v_i=0 \\ i=1..n\neq j}}$$

$$\mathit{If} \quad v_i = 0, \text{ for } i = 1..n \neq k, j \Rightarrow W_e = c_{kk}v_k^2/2 + c_{kj}v_k v_j + c_{jj}v_j^2/2 \Rightarrow c_{kj}$$

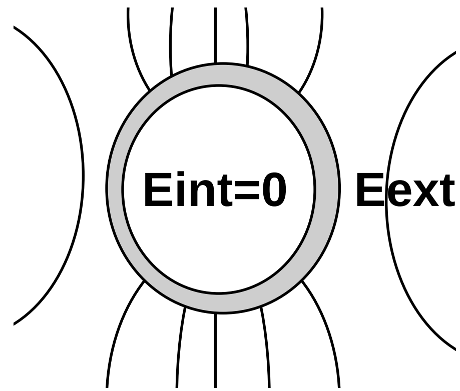
ES field for $\mathbf{v}$ = [0; 0;…;1;…;1;…;0] may be computed by superposition

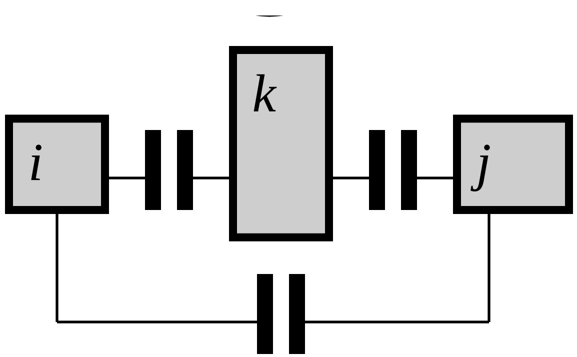
Shielding

Dielectric shielding



Conductive shielding





S matrix

$\mathbf{v} = \mathbf{S}\ \mathbf{q}$

$\mathbf{S}=[s_{ij}]$ **– matrix of the potential coefficients**

$\mathbf{S} = \mathbf{C}^{-1}$

C matrix properties

$c_{ij}=q_j/v_i<0,\ v_k=0$

Shielding property:

$|c_{ij}| << |c_{ik}|,\ |c_{kj}| < c_{ii},\ c_{jj},\ c_{kk}$

C allows sparsification

S matrix properties

$s_{ij}=v_j/q_i>0, \quad q_k=0$

sij **has no shielding property**

$\mathbf{S}=\mathbf{C}^{-1}$ **does not allow sparsification**

Energy of ES field, Tellegen's theorem

Linear dielectrics with Pp: $\boldsymbol{D} = \varepsilon \boldsymbol{E} + \boldsymbol{P}_p$

$$div(V\boldsymbol{D}) = V div(\boldsymbol{D}) + \boldsymbol{D} \cdot \boldsymbol{grad} V = \rho V - \boldsymbol{D} \cdot \boldsymbol{E} \Rightarrow$$

$$\int_D \boldsymbol{D} \cdot \boldsymbol{E} dv = \int_D (\varepsilon E^2 + \boldsymbol{P}_p \cdot \boldsymbol{E}) dv = \int_D \rho V dv - \oint_\Sigma V \boldsymbol{D} \cdot \boldsymbol{n} dS$$

$$w_e = \int_0^D \boldsymbol{E} d\boldsymbol{D} = \boldsymbol{E} \cdot \boldsymbol{D} - \int_0^E \boldsymbol{D} d\boldsymbol{E} = \boldsymbol{E} \cdot \boldsymbol{D} - \int_0^E (\varepsilon \boldsymbol{E} + \boldsymbol{P}_p) d\boldsymbol{E} = \varepsilon E^2 / 2$$

$$W_e = \int_D w_e dv = \frac{1}{2} \int_D \varepsilon E^2 dv = \frac{1}{2} \int_D \rho V dv - \frac{1}{2} \int_D \boldsymbol{P}_p \cdot \boldsymbol{E} dv - \frac{1}{2} \oint_\Sigma V \boldsymbol{D} \cdot \boldsymbol{n} dS$$

Zero b.c. $\Rightarrow$ $< \mathbf{D}, \mathbf{E} > = < \rho, V >$

$$-\oint_\Sigma V \boldsymbol{D} \cdot \boldsymbol{n} dS = \int_{\Sigma_0} V D_n dS + \sum_{k=1}^n \int_{\Sigma_k} V D_n dS = \int_{\Sigma_0} V D_n dS - \sum_{k=1}^n V_k q_k$$

n **conductors in linear dielectrics without sources**

$$\text{If} \quad \rho = 0, \; P_p = 0, \Sigma_0 \to \infty, \quad \Rightarrow \quad W_e = \frac{1}{2} \int_D \boldsymbol{D} \cdot \boldsymbol{E} dv = \frac{1}{2} \sum_{k=1}^n V_k q_k$$

$$\text{Scalar products:} \quad < \boldsymbol{D}, \boldsymbol{E} >_{def} \int_D \boldsymbol{D} \cdot \boldsymbol{E} dv, \quad \boldsymbol{v}^T \cdot \boldsymbol{q} =_{def} \sum_{k=1}^n V_k q_k$$

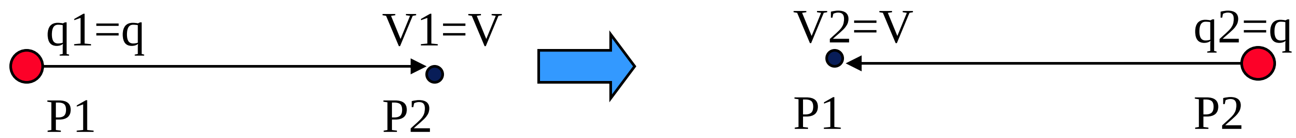
$$\text{Tellegen: if } div\boldsymbol{D}' = 0, \; curl\boldsymbol{E}'' = 0 \Rightarrow < \boldsymbol{D}', \boldsymbol{E}'' > = \boldsymbol{q}'^T \cdot \boldsymbol{v}'' \Rightarrow \boldsymbol{D}' \perp \boldsymbol{E}'' \text{ for zero b.c.}$$

Tellgen theorem of pseudo-energy - regardless f', f'' material relations!

$$W_e = \frac{1}{2} \mathbf{v}^T \cdot \mathbf{q} = \frac{1}{2} \mathbf{v}^T \mathbf{C} \mathbf{v} = \frac{1}{2} \mathbf{q}^T \mathbf{S} \mathbf{q} > 0, \qquad \mathbf{S} = \mathbf{C}^{-1}$$

Reciprocity theorem. Permittivity variation

Reciprocity theorem: in linear reciprocal dielectrics ($\varepsilon = \varepsilon^T$) the relation between sources (charges) and responses (voltages) is symmetric. Moreover it is positive defined when $\varepsilon > 0$.



$$\Sigma \to \infty \text{ or zero b.c.} \Rightarrow \boxed{< V_1, \rho_2 > = < V_2, \rho_1 > \Leftrightarrow < V_1, AV_2 > = < AV_1, V_2 >}$$

$$A\bullet =_{def} -div(\bar{\bar{\varepsilon}} grad \bullet) \Rightarrow AV = \rho, \qquad \forall V \neq 0 \quad \boxed{< V, AV > = < \bar{\bar{\varepsilon}} \mathbf{E}, \mathbf{E} > > 0}$$

$$< V_1, \rho_2 > - < V_2, \rho_1 > = < \boldsymbol{E}_1, \boldsymbol{D}_2 > - < \boldsymbol{D}_1, \boldsymbol{E}_2 > + \int_\Sigma (V_1 D_{n2} - V_2 D_{n1}) dS = \int_{R^3} (\bar{\bar{\varepsilon}} - \bar{\bar{\varepsilon}}^T) \boldsymbol{E}_1 \cdot \boldsymbol{E}_2 dv + 0 = 0$$

$$\text{For } n \text{ conductors: } \boldsymbol{q}'^T \cdot \boldsymbol{v}'' - \boldsymbol{v}'^T \cdot \boldsymbol{q}'' = < \boldsymbol{D}', \boldsymbol{E}'' > - < \boldsymbol{E}', \boldsymbol{D}'' > = 0 \Rightarrow \boxed{\mathbf{C} = \mathbf{C}^T, \ \mathbf{S} = \mathbf{S}^T}$$

Cohn-Vratsanos: Increase of ε or metallization ➔ ΔC>0:

1. Initial problem — $V, q = Cv$; $\mathbf{D} = \varepsilon \mathbf{E}$, V'

2. Perturbed problem — $V, q + \delta q$; $\mathbf{D}' = (\varepsilon + \delta\varepsilon)\mathbf{E}'$; $\rho = -div(\delta \mathbf{P}) = -div(\delta\varepsilon \mathbf{E}') =$

3. Variation effect = 2 - 1 — $V = 0, \delta q$; $\rho = -div(\delta \mathbf{P}) = -div(\delta\varepsilon \mathbf{E}') =$

Reciprocity between 1 and 3:

$$V\delta q = < V', \rho > \Rightarrow V^2 \delta C = - < V', div(\delta\varepsilon \mathbf{E}') > \Rightarrow \boxed{\frac{\delta C}{\delta \varepsilon} = -\frac{< V', \Delta V') >}{V^2} > 0}$$

Variational ES formulation. Minimization and projection

Thomson's theorem: the energy functional is minimal for the exact ES field distribution compared to those of any other distribution which satisfies essential restrictions:

$$\mathbf{E}' = \mathbf{E} + \delta\mathbf{E} \Rightarrow W' = \int_D \frac{\mathbf{D}'\cdot\mathbf{E}'}{2} dv = <\mathbf{E}+\delta\mathbf{E}, \mathbf{D}+\delta\mathbf{D}>/2 = W + <\delta\mathbf{E},\delta\mathbf{D}>/2 > W \Rightarrow W < W'$$

$$\Delta V = 0, \Sigma = \mathrm{S_D} \Rightarrow (\delta\mathbf{E},\mathbf{D}) = (\mathbf{E},\delta\mathbf{D}) = (\rho,\delta V) - \int_\Sigma \delta V\cdot\mathbf{D}\cdot\mathbf{n}dS = 0, \quad \mathbf{v}^{\mathrm{T}}(\mathbf{C'}-\mathbf{C})\mathbf{v} > 0$$

In general the "energy" functional is:

$$F(V) = \frac{1}{2}\int_D \left[\varepsilon(gradV)^2 - \rho V\right]dv + \int_{S_N} VD_n dS < F(V')$$

–

Neumann are natural boundary conditions while Dirichlet are essential boundary conditions. Weak (integral-differential) formulations:

- **Ritz:** $\delta F(V) = 0 \Leftrightarrow \int_D \left[\varepsilon\, gradV\cdot grad\delta V - \rho\delta V\right]dv + \int_{S_N} \delta V\cdot D_n dS = 0$
- **Galerkin:** $\left[div(\varepsilon\, gradV) + \rho\right] = 0 \Rightarrow \int_{\mathrm{D}} \delta V\left[div(\varepsilon\, gradV) + \rho\right]dv = 0$

$$div(\delta V\varepsilon\, gradV) = \delta V div(\varepsilon\, gradV) + \varepsilon\, gradV\cdot grad\delta V \Rightarrow$$

are equivalent

$$\int_D (\delta V\rho - \varepsilon\, gradV\cdot grad\delta V)dv - \int_{S_N} \delta V D_n dS = 0$$

Correct mathematical formulation of ES (div-grad) fundamental problem

- **Known data:**
  - Computational domain: Ω - Lipchitz type
  - Material characteristics ($\boldsymbol{D}=\varepsilon\boldsymbol{E}$), *with* $\varepsilon = f(\mathbf{r}): \Omega \rightarrow IR, \varepsilon>0$
  - Internal sources of field (charge density): $\rho=g(\mathbf{r}): \Omega \rightarrow IR \in L^2(\Omega)$
  - Boundary conditions (ext. sources) $\begin{cases} V = f_D(P), P \in S_D \subset \partial\Omega \\ \varepsilon \dfrac{dV}{dn} = f_N(P), P \in S_N = \partial\Omega - S_D \end{cases}$
- **Solution (unknown result- scalar potential):** $V: \Omega \rightarrow \mathrm{IR}$,
- **Equation (weak formulation):** Find $V \in H_D^1(\Omega)$, s.t.

$$\int_\Omega (\varepsilon\, gradV \cdot gradU - U\rho)dv = \int_{S_N} U f_N dS, \forall U \in H_0^1(\Omega)$$

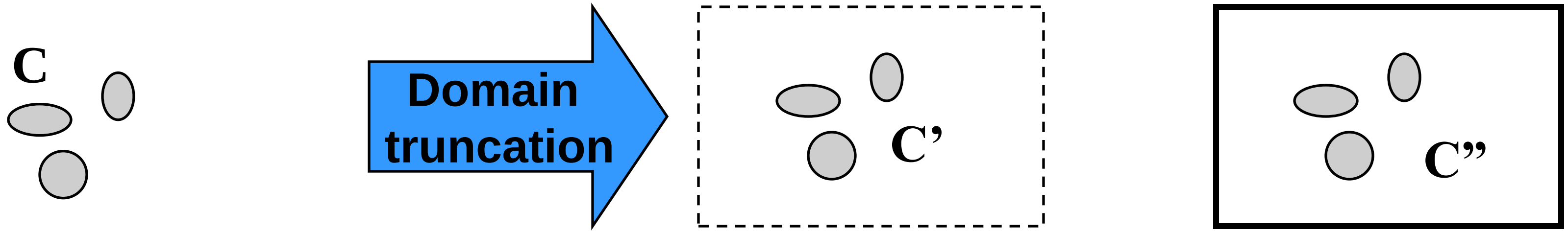
with $H_D^1(\Omega) = \left\{ V \in L^2(\Omega) \mid gradV \in L^2(\Omega), V\big|_{S_D} = f_D \right\}, H_0^1(\Omega) = H_D^1(\Omega)\big|_{f_D=0}$

- **Existence, uniqueness and stability:** Lax-Milgram theorem,

$a(U,V) = \int_\Omega \varepsilon\, gradV \cdot gradU dv$ is bounded and coercive

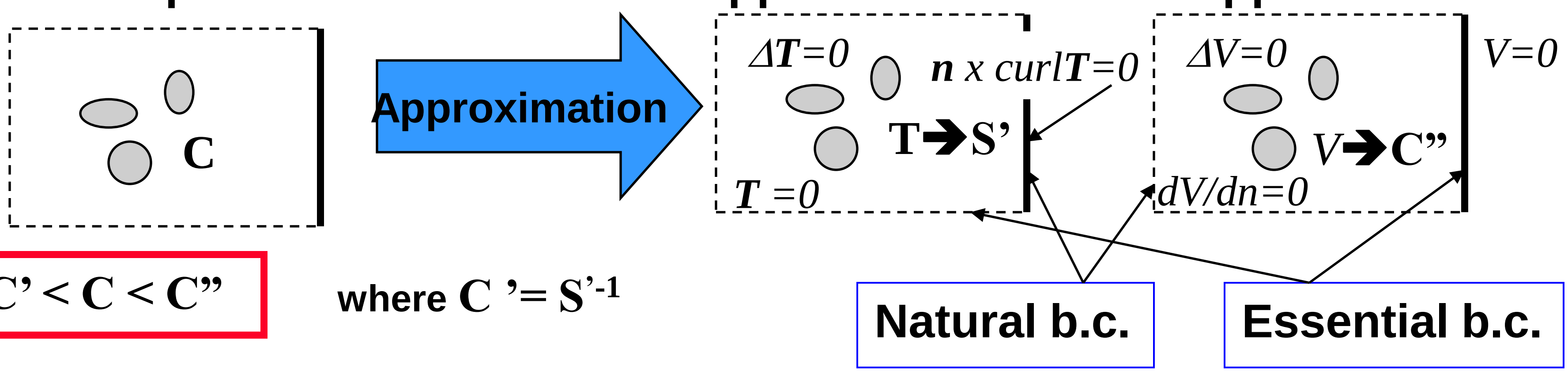
Dual approaches – scissors relations

- Original unbounded problem ➔ Zero Neumann b.c. Zero Dirichlet b.c.



Theorem of the permittivity variation ➔ $C' < C < C''$ where $C > 0 \iff v^T C v > 0$

- Exact solution of the bounded problem
- div conform approx.
- curl conform approx.



$C' < C < C''$ where $C' = S'^{-1}$

- Duality provides complementary bounds of the exact solution

Behavior of ES field and potential

- **In vacuum, V is a harmonic function ($\Delta V=0$):**
  - It is smooth (indefinite derivable function, see http://www.axler.net/HFT.pdf)
  - in any internal point P it has the the average value on any sphere Σ centered in P:

$$V(P)=\frac{1}{4\pi}\oint_\Sigma\left[\frac{1}{R'}\cdot\left.\frac{\partial V}{\partial n}\right|_Q-V\big|_Q\cdot\frac{\partial}{\partial n}\left(\frac{1}{R'}\right)\right]dS'=\frac{1}{4\pi a}\oint_\Sigma\frac{\partial V}{\partial n}dS+\frac{1}{4\pi a^2}\oint_\Sigma VdS$$

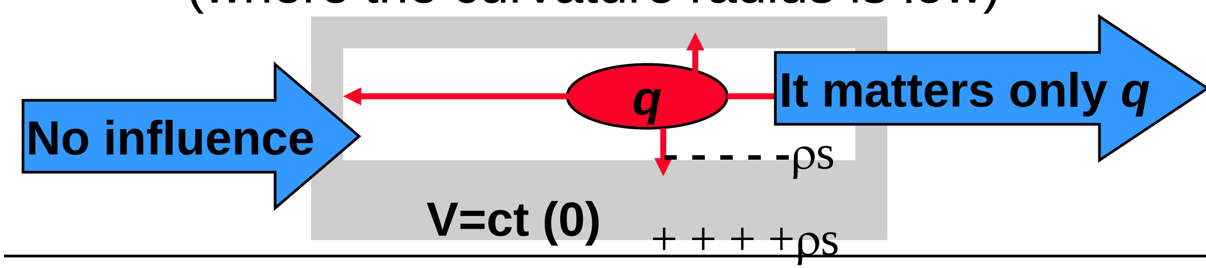
  - It has extreme values (minim and maxim) solely on boundary
  - D and E have same direction
- **Same in linear homogeneous uncharged dielectrics**
- **In charged dielectrics, V may have a local extreme**
  - It is maxim (ρ>0) or minim (ρ<0) e.g. center of uniformly charged sphere
- **In permanent polarized dielectrics (electrets)** , D has same orientation as P, but E is opposite ("depolarized field")
- **In dielectrics** D is confined (increased) but E is decreased (approx χ times)

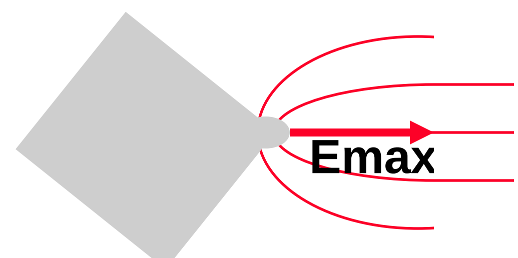
Behavior of ES field and potential (cont.)

- **On uncharged interface between two dielectrics**:
  – field line is refracted so that

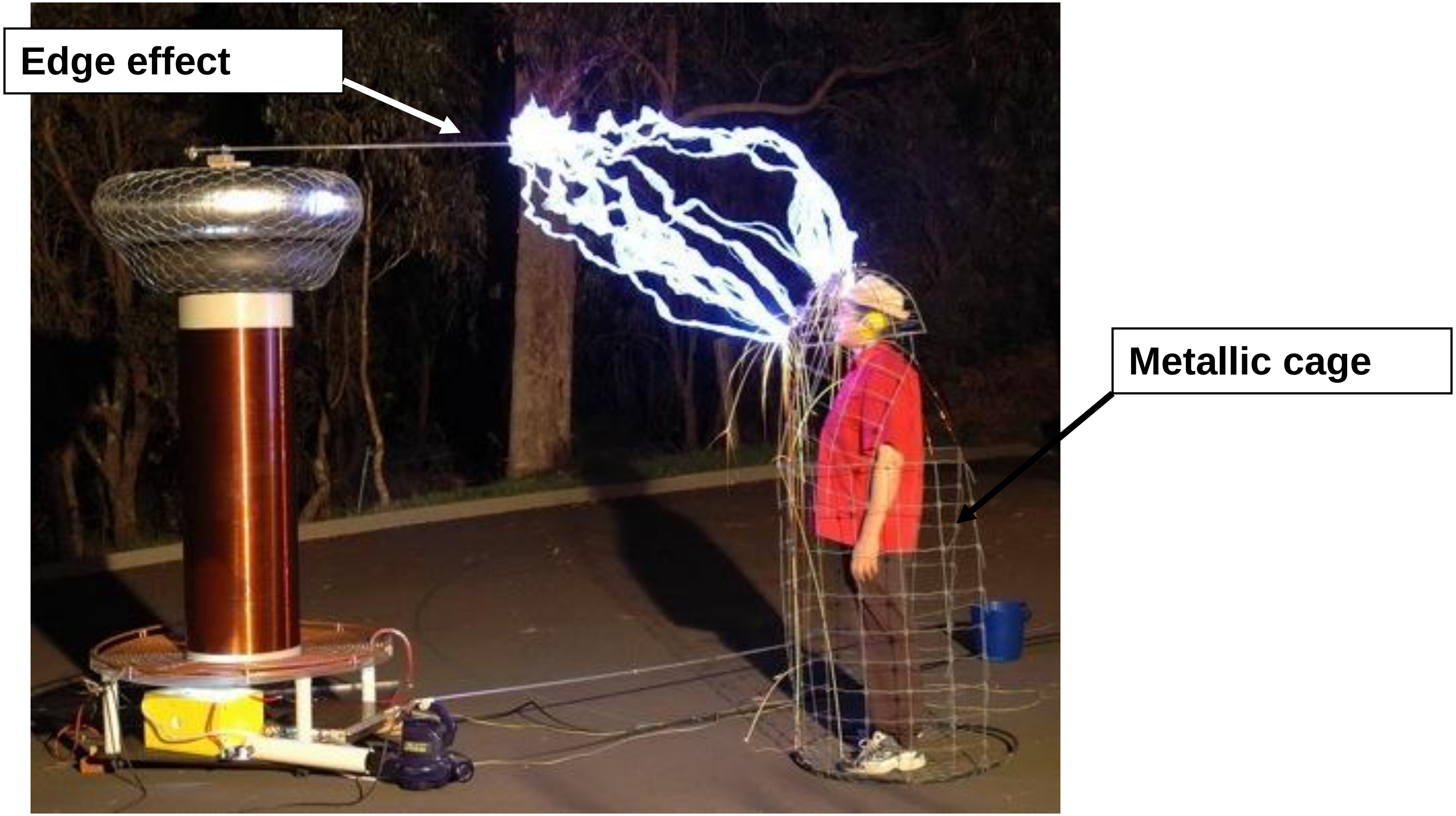
$$\begin{cases} D_{n1} = D_{n2} \\ \mathbf{E}_{t1} = \mathbf{E}_{t2} \end{cases} \Longleftrightarrow \begin{cases} \varepsilon_1 \dfrac{\partial V_1}{\partial n} = \varepsilon_2 \dfrac{\partial V_2}{\partial n} \\ V_1 = V_2 \end{cases}$$

- **In/on conductors:** V=ct, $\mathbf{E}=0$, $\mathbf{D}=0$, $\rho_v=0$,
  – Two conductors may have different potentials even in contacts (double layer, "the electrode potential", superficial Ei)
  – Screening effect: the field inside a box with conductive walls does not depend of external sources or box voltage ("Faraday cage"). It is zero if there are no sources in the box. From outside pov it matters only the total charge, not its internal distribution
  – Edge effect: D_n, ρ_s and E_{ext} have highest values on corners and edges (where the curvature radius is low)



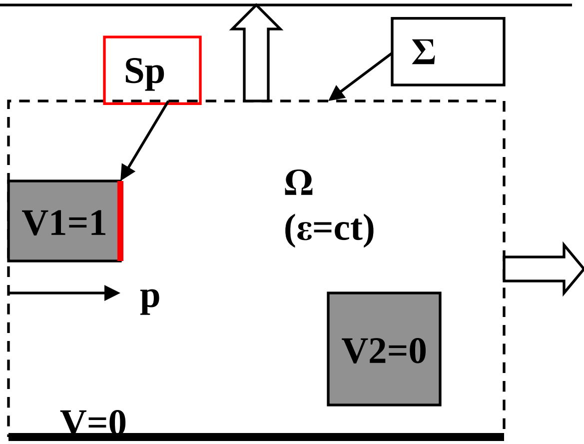


Faraday cage
Do not try at home !!!



May be a film is more convincing:
http://www.youtube.com/watch?v=mUWxYesR5Wo

Adjoint Field Technique (AFT) to compute C sensitivity



- Tellegen theorem for ES field: $(\mathbf{v},\underline{\mathbf{q}}) = \langle \mathbf{E},\underline{\mathbf{D}} \rangle$
- Tellegen theorem for difference of variations:
  $(\delta\mathbf{v}, \underline{\mathbf{q}}) - (\underline{\mathbf{v}}, \delta\mathbf{q}) = \langle \delta\mathbf{E}, \underline{\mathbf{D}} \rangle - \langle \underline{\mathbf{E}}, \delta\mathbf{D} \rangle$
- Capacitance matrix **C** and its variation due to the variation of the geometric param. p : $\mathbf{q} = \mathbf{C} * \mathbf{v}$ ➔

$\delta\mathbf{q} = \delta\mathbf{C} * \mathbf{v} + \mathbf{C} * \delta\mathbf{v} = \delta\mathbf{C} * \mathbf{v}$, if $\delta\mathbf{v} = 0$ (ct. excitation)➔ for $\mathbf{D} = \varepsilon\,\mathbf{E}$ and $\underline{\mathbf{D}} = \varepsilon\underline{\mathbf{E}}$

$-(\underline{\mathbf{v}}, \delta\mathbf{C} * \mathbf{v}) = \langle \delta\mathbf{E}, \underline{\mathbf{D}} \rangle - \langle \underline{\mathbf{E}}, \delta\mathbf{D} \rangle = \langle \delta\mathbf{E}, \varepsilon\underline{\mathbf{E}} \rangle - \langle \underline{\mathbf{E}}, \delta(\varepsilon\,\mathbf{E}) \rangle = -\langle \underline{\mathbf{E}}, \mathbf{E}\,\delta\varepsilon \rangle$

Choosing $V_k = V_o\,\delta_{ki},\ \underline{V}_k = V_o\,\delta_{kj},\ for\ k = 1,..n$ ➔ $\delta C_{ij} = \langle \delta\varepsilon\,\mathbf{E}, \underline{\mathbf{E}} \rangle / V_o^2$ ➔

Original and adjoint fields are computed only once, independent of *p*

$$\frac{\partial C_{ij}}{\partial p} = \frac{1}{V_0^2}\int_\Omega \frac{\partial\varepsilon}{\partial p}\mathbf{E}\cdot\underline{\mathbf{E}}\,d\Omega = \frac{\varepsilon}{V_0^2}\int_{Sp}\mathbf{E}\cdot\underline{\mathbf{E}}\,dS$$

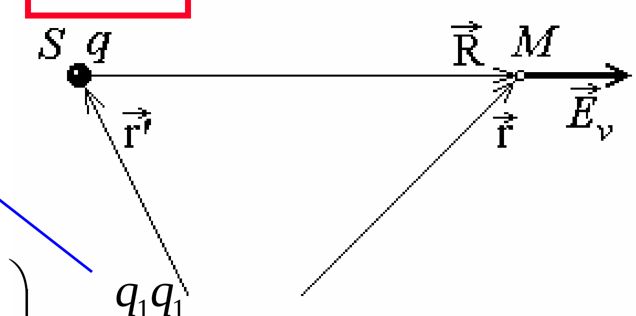
The surface changed by *p*

Forces and torques in ES

- **Coulomb formula**

$$\mathbf{f} = \rho\mathbf{E} \Rightarrow \mathbf{F} = \int_D \rho\mathbf{E}dv = (\rho\mathbf{E})_{ave}V \to \boxed{\mathbf{F} = q\mathbf{E}}$$

$$\mathbf{E} = \frac{Q\mathbf{R}}{4\pi\varepsilon_0 R^3} \Rightarrow \boxed{\mathbf{F} = \frac{q_1 q_2}{4\pi\varepsilon_0 R^2}\frac{\mathbf{R}}{R}}$$



$$W_e = \frac{1}{2}(q_1V_1 + q_2V_2) = (q_1V_{11} + q_1V_{12} + q_2V_{21} + q_2V_{22})/2$$

$$F = -\frac{\partial W_e}{\partial R}\bigg|_q = -\frac{\partial(q_1V_{12} + q_2V_{21})/2}{\partial R} = -q_1\frac{\partial V_{12}}{\partial R} = -q_1\frac{\partial V_{12}}{\partial R}\left(\frac{q_2}{4\pi\varepsilon_0 R}\right) = \frac{q_1q_1}{4\pi\varepsilon_0 R^2}$$

- **Permanent polarized particles**

Polarized particles and dipoles with equal momentum are equivalent from both pov: field and forces.

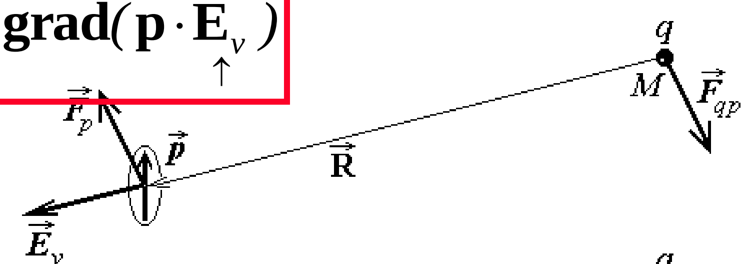
$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} \Rightarrow w_e = \frac{\varepsilon_0\mathbf{E}^2}{2} = \frac{\varepsilon_0\mathbf{E}_v{}^2}{2} + \frac{\mathbf{P}^2}{2\varepsilon_0} - \mathbf{E}_v\mathbf{P} \Rightarrow \quad W_e = \int_{R^3} w_e dv \to W_0 - \mathbf{p}\cdot\mathbf{E}_v$$

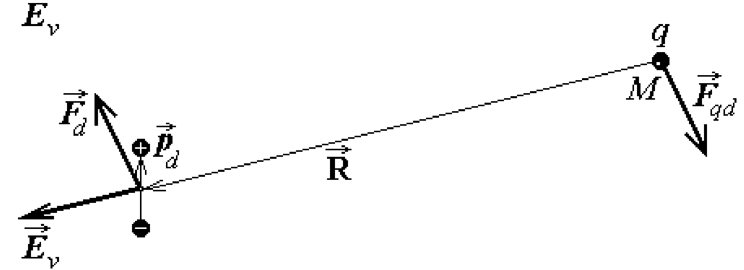
$$X_k = -\frac{\partial W_e}{\partial x_k} = \frac{\partial(\mathbf{p}\cdot\mathbf{E}_v)}{\partial x_k} \Rightarrow F_x = \mathbf{p}\cdot\frac{\partial\mathbf{E}_v(x,y,z)}{\partial x} \Rightarrow \boxed{\mathbf{F}_p = \mathbf{grad}(\mathbf{p}\cdot\underset{\uparrow}{\mathbf{E}_v})}$$

$$T = \frac{\partial(pE_v\cos\alpha)}{\partial\alpha} = -pE_v\sin\alpha \Rightarrow \boxed{\boldsymbol{T}_p = \boldsymbol{p}\times\boldsymbol{E}_v}$$

$$\boldsymbol{F}_d = q\boldsymbol{E}_v(\boldsymbol{r}+) - q\boldsymbol{E}_v(\boldsymbol{r}-) = \boldsymbol{grad}(\boldsymbol{p}_d\cdot\underset{\uparrow}{\boldsymbol{E}_v})$$

$$\boldsymbol{T}_d = \frac{\Delta\boldsymbol{l}}{2}\times q\boldsymbol{E}_v(\boldsymbol{r}+) - \left(-\frac{\Delta\boldsymbol{l}}{2}\right)\times q\boldsymbol{E}_v(\boldsymbol{r}-) = \boldsymbol{p}_d\times\boldsymbol{E}_v$$





Other ES forces

- **Dielectric particles**

Now polarization is induced by field

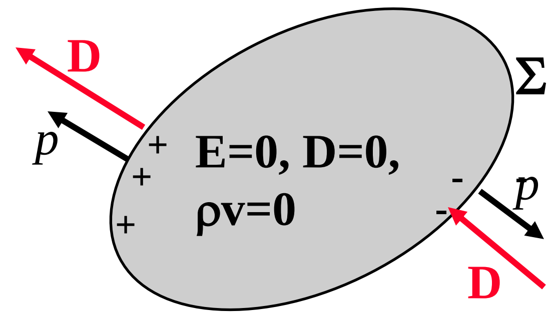
$$\mathbf{f}_p = \mathbf{grad}(\mathbf{P} \cdot \underset{\uparrow}{\mathbf{E}_v}) = \varepsilon_0 \chi \mathbf{grad}(\mathbf{E}_v \cdot \underset{\uparrow}{\mathbf{E}_v}) = \frac{\varepsilon_0 \chi}{2} \mathbf{grad} \mathbf{E}_v^{\ 2} \Rightarrow \boxed{\mathbf{f}_p = \chi \, \mathbf{grad} w_e}$$

$$\mathbf{P} \,||\, \mathbf{E} \Rightarrow T = 0 \ \text{ for spherical particles}$$

- **Conductors**

$$p = \rho_s E_n / 2 = \varepsilon E_n^{\ 2} / 2 = w_e \quad \text{outside conductor}$$

$$\boxed{\boldsymbol{F}_c = \oint_\Sigma p \mathbf{n} dS,} \qquad \boxed{\boldsymbol{T} = \oint_\Sigma p (\mathbf{r} \times \mathbf{n}) dS}$$



- **Capacitors**

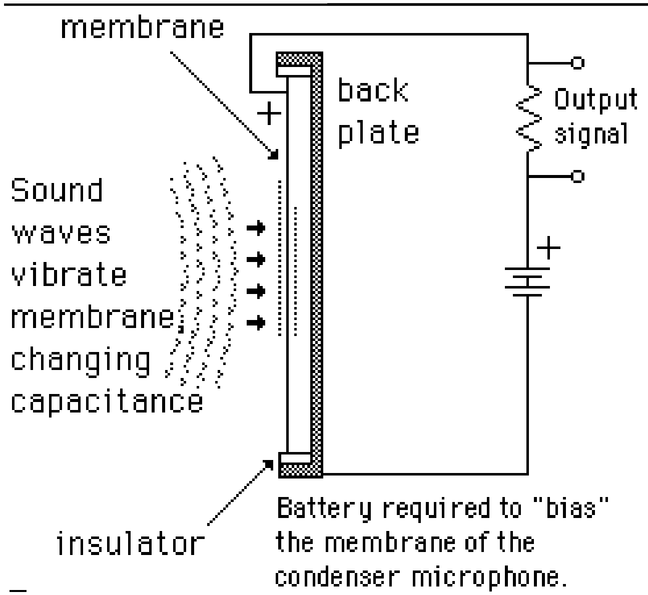
$$W_e = \frac{1}{2} \boldsymbol{q}^T \boldsymbol{S} \boldsymbol{q} \Rightarrow X_k = -\left.\frac{\partial W_e}{\partial x_k}\right|_q \Rightarrow \boxed{X_k = -\frac{1}{2} \boldsymbol{q}^T \frac{\partial \boldsymbol{S}}{\partial x_k} \boldsymbol{q}}; \quad W_e = \frac{1}{2} \boldsymbol{v}^T \boldsymbol{C} \boldsymbol{v} \Rightarrow \boxed{X_k = \frac{1}{2} \boldsymbol{v}^T \frac{\partial \boldsymbol{C}}{\partial x_k} \boldsymbol{v}}$$

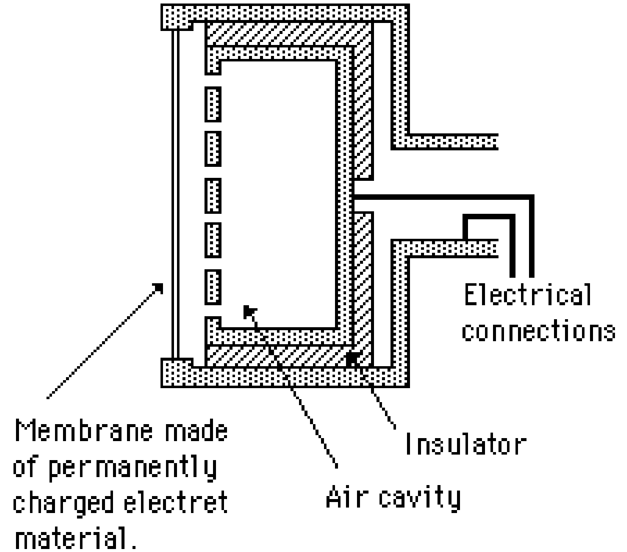
- **Earnshaw's theorem:**

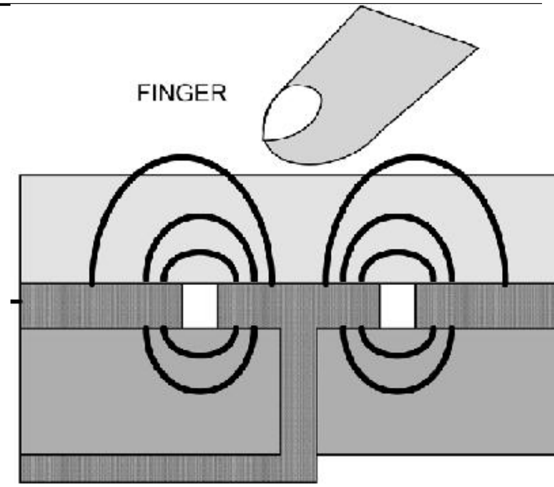
A collection of particles cannot be maintained in a stable stationary equilibrium solely by the ES forces

$$\text{Equillibrium} : div\boldsymbol{F} < 0 \Leftrightarrow div(q\boldsymbol{E}_v) < 0, \text{but} \quad div\boldsymbol{E}_v = 0$$

ES applications

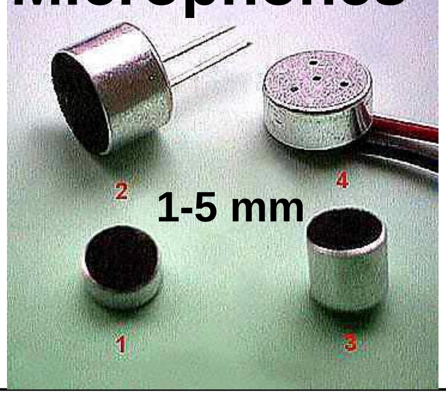






- **Studio Condenser Microphone**

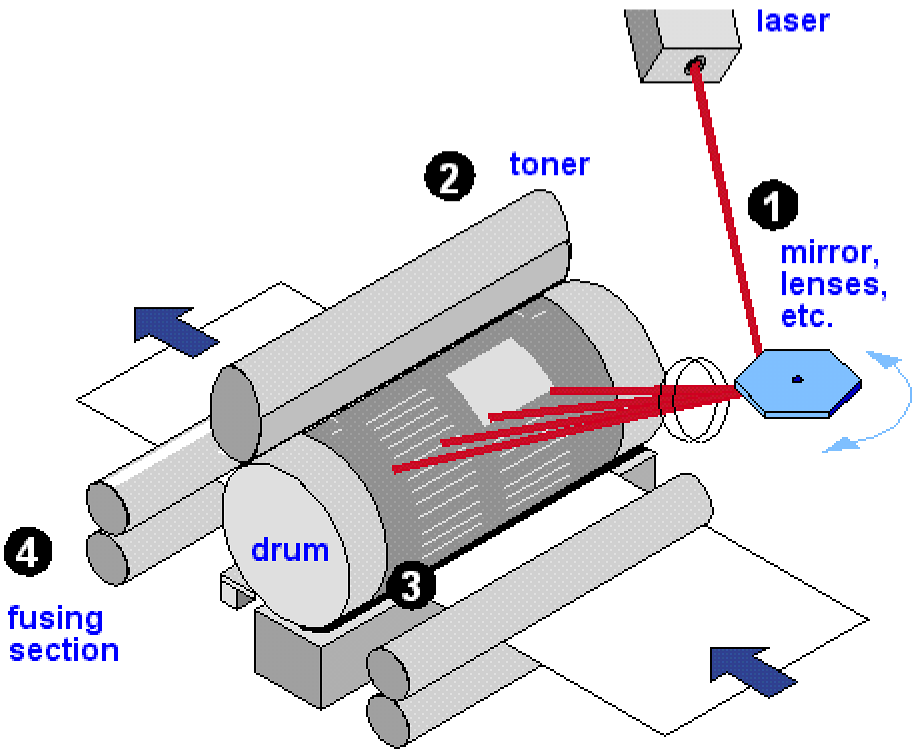
- **Electret Condenser Microphones**

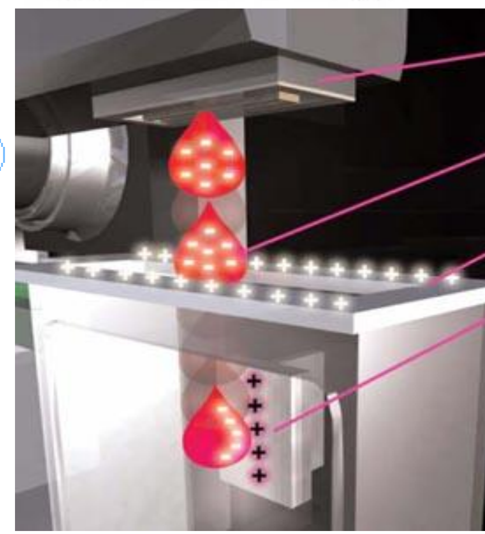


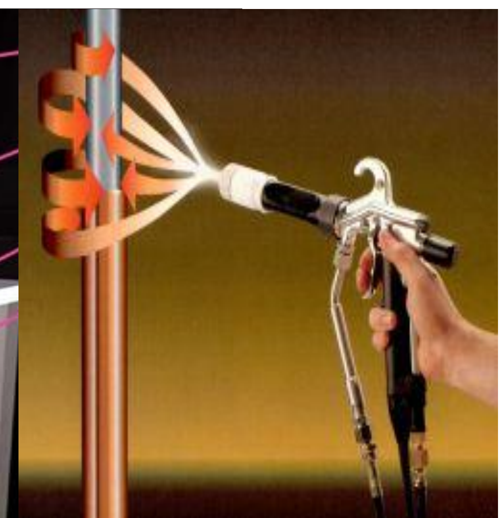
- **Capacitive sensors**

ES applications: printing and painting

- ES image transfer (xerox)
- Laser printer
- Inkjet ES print head
- Electrostatic painting

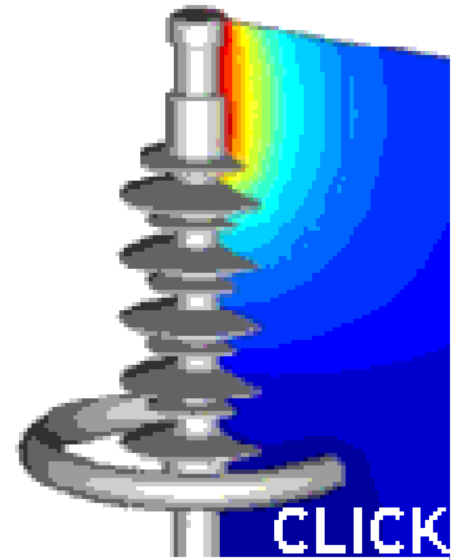


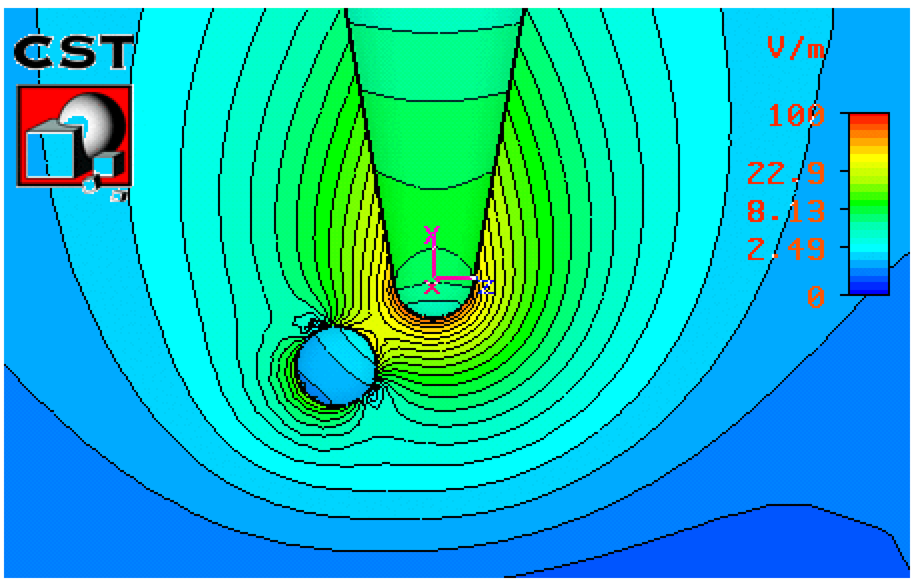


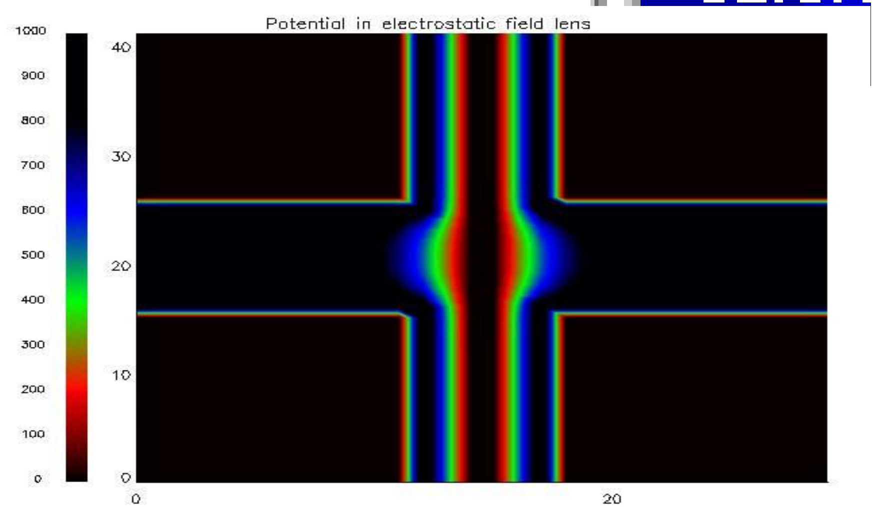


ES CAD

- **Insulation design and optimization- for electric breakdown prevention**
- **Design of ES lens for electron fascicles (Electronic microscopes, particle accelerators, oscilloscopes, etc)**

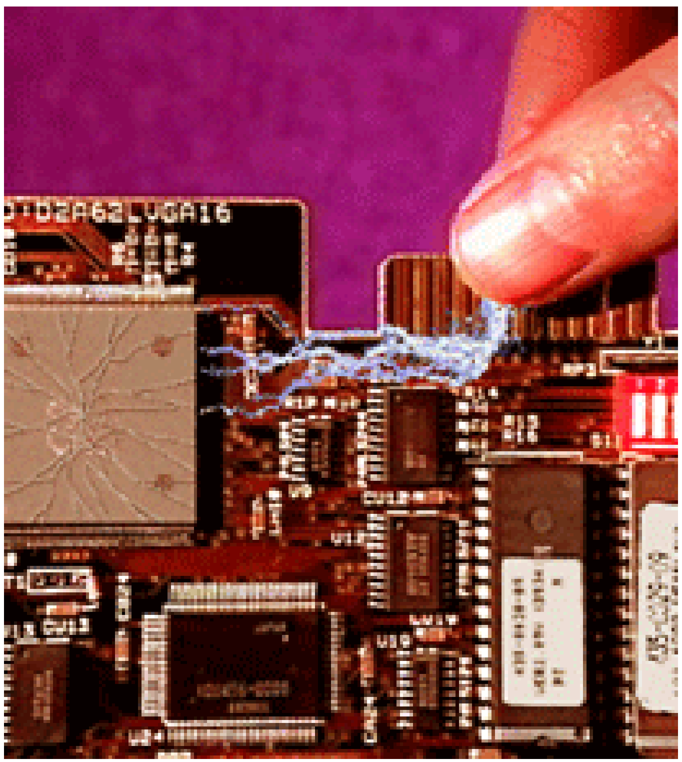






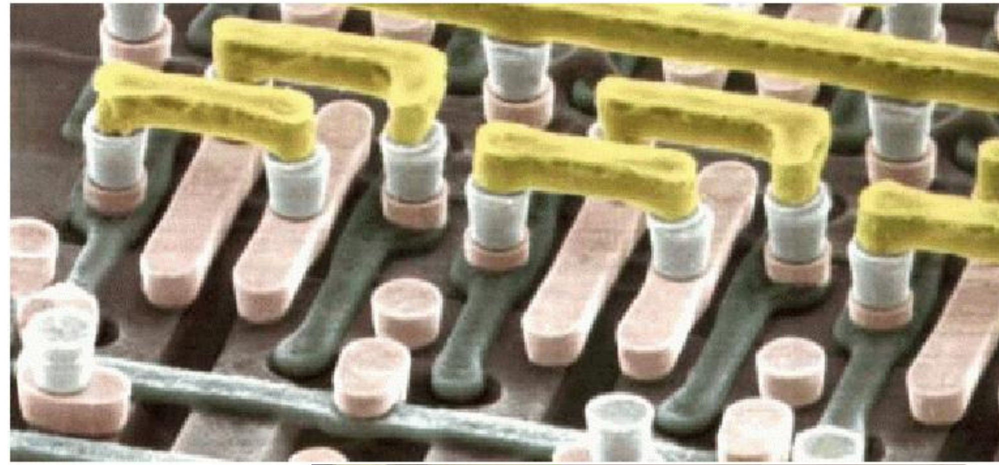
Static charges

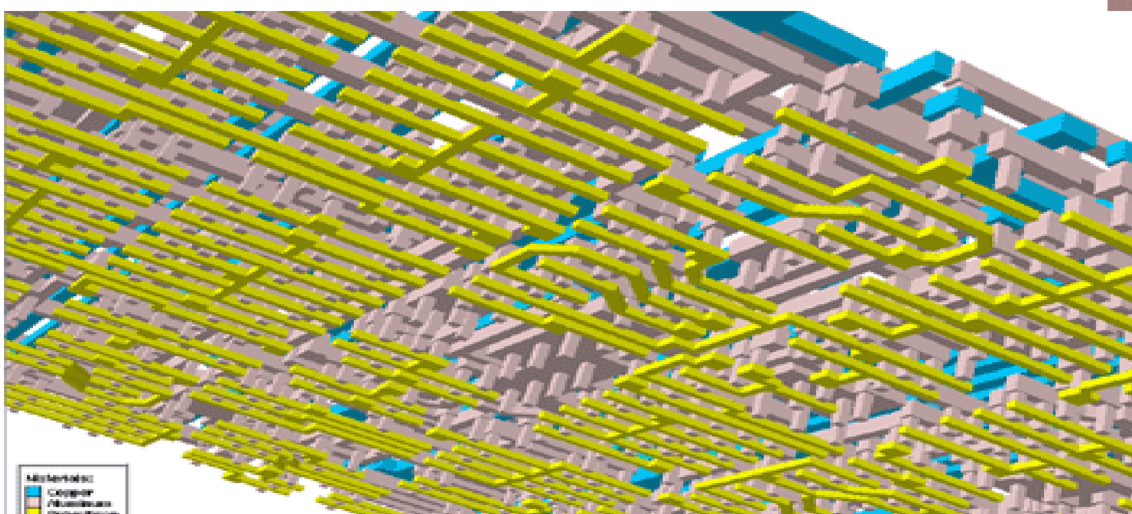
- **Static charges (hair raising)**
- **Electrostatic discharges (ESD)**
- **ESD- electrostatic discharges protection in ICs (design for)**

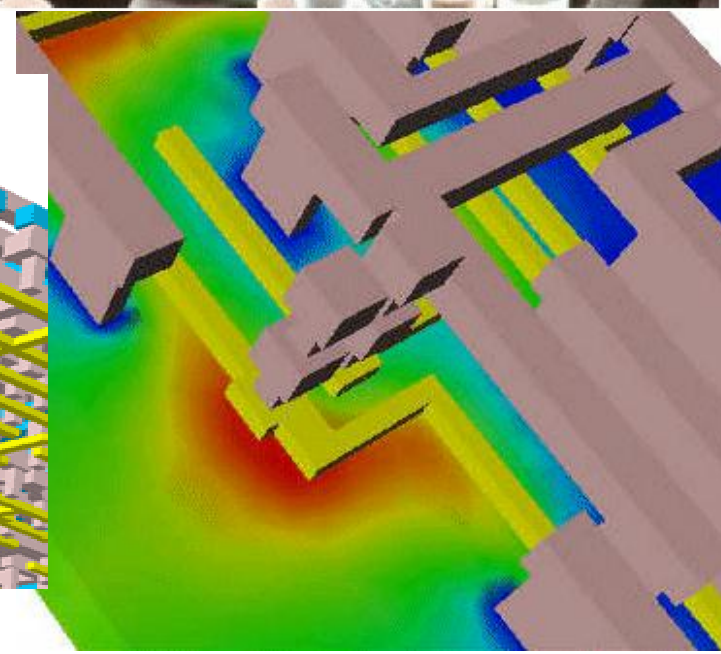


EDA - Parasitic capacitance extraction in integrated circuits

- **Downscaling → C increasing, delay bottleneck**
- **High interconnect density**
- **ES modeling of IC interconnects**

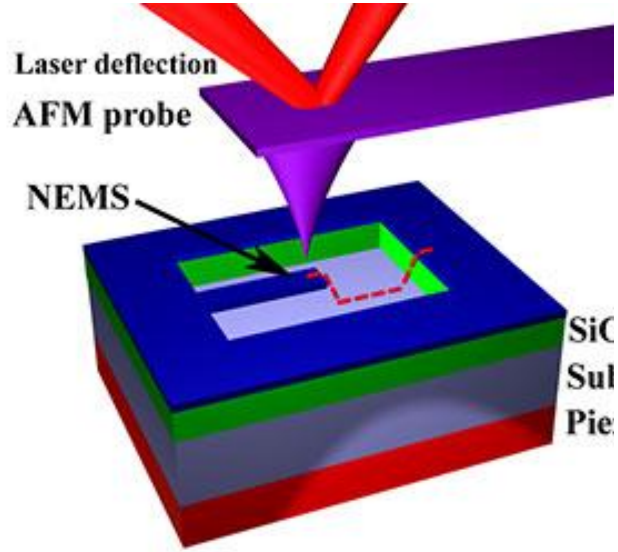


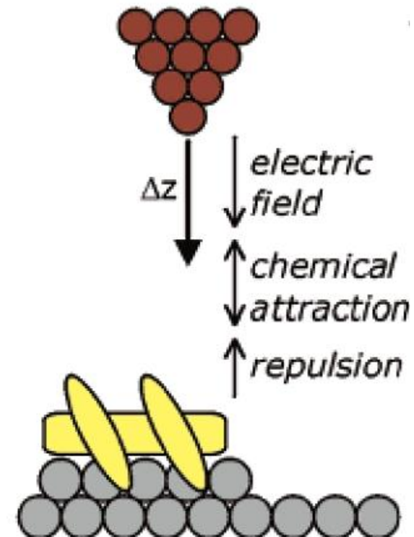


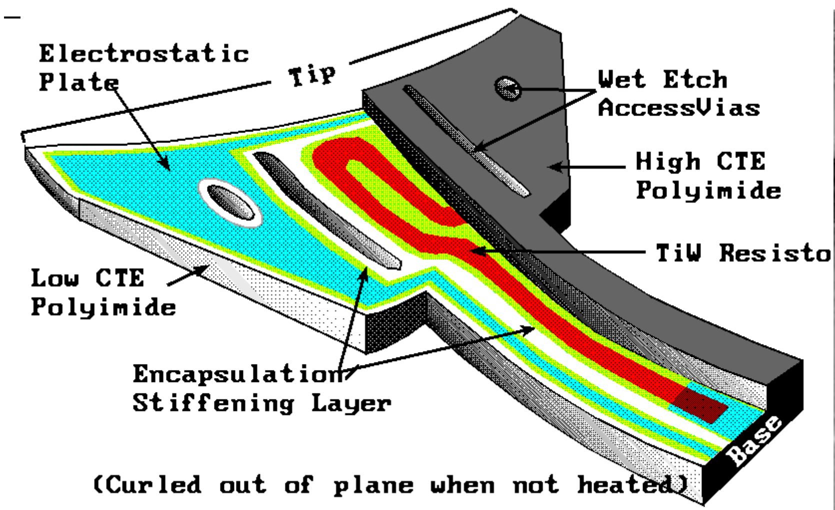


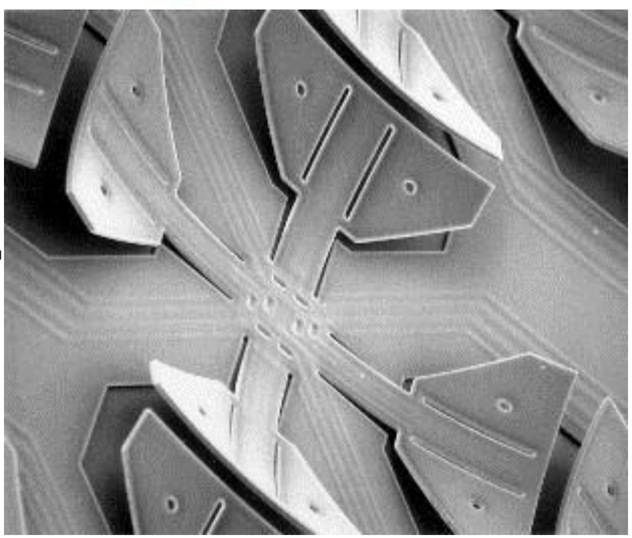
ES applications:

- **AFM –Atomic Force Microscope**
- **ES actuated cilia for lab on a chip**
- **Optical switch (MEMS mirror)**



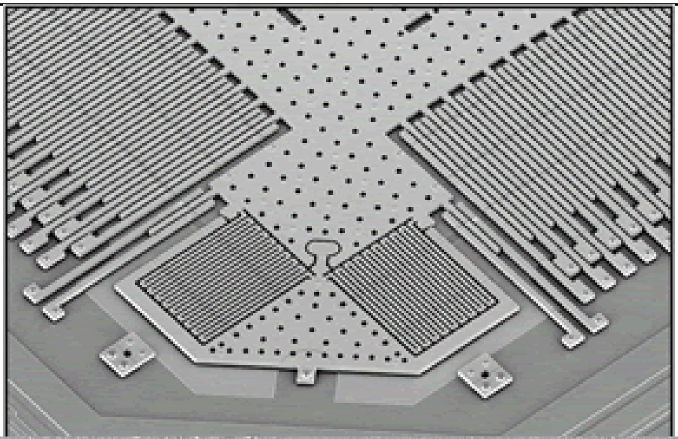






ES applications: MEMS sensors and actuators

- **MEMS comb sensor (force, pressure), accelerometers, gyroscopes for navigation systems and for activating the car airbags**
- **Micro-motors actuated by ES forces**



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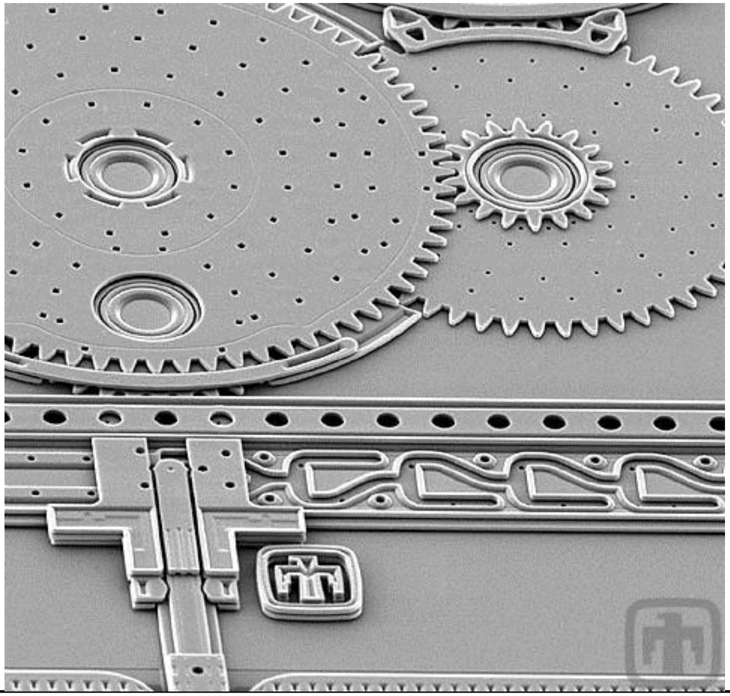
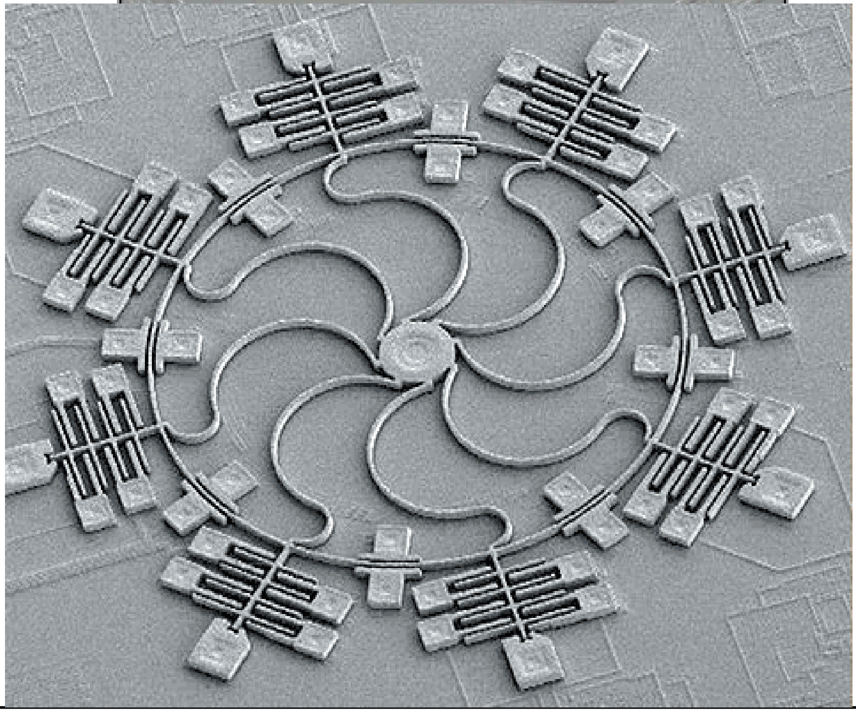




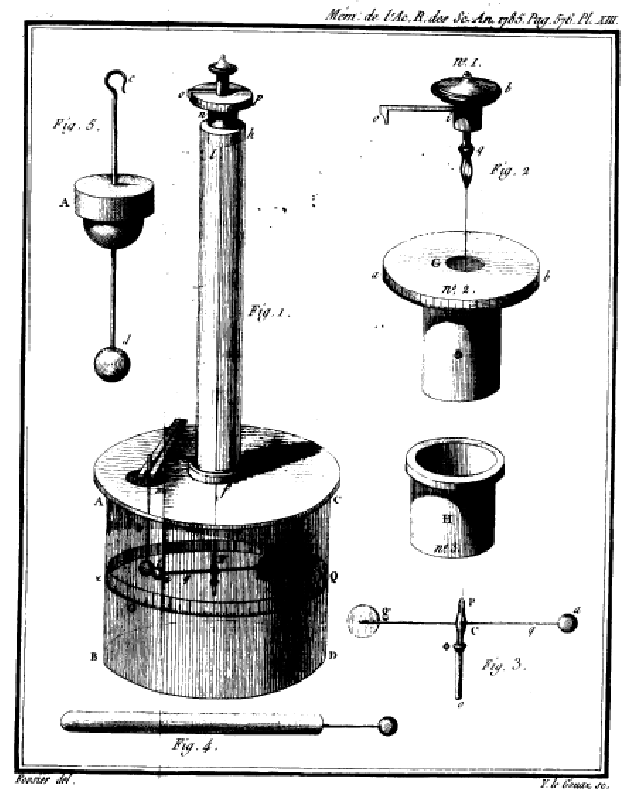
Methods for ES fields computation

- **Analytic methods:**
  - Gauss – elementary method for 1D problems
  - Superposition, Coulomb integrals
  - Method of images
  - Separation of variables
  - Conformal mapping
  - Schwarz-Christoffel
  - Green's function
  - Other
- **Numeric methods:**
  - FDM - Finite difference

  FIT - Finite integral technique (finite volumes)
  - FEM - Finite element methods
  - BEM - Boundary element method
  - Other: Integral equations, Monte Carlo, etc.

Foundation of ES on Coulomb. History of ideas

Coulomb experiment (1780)

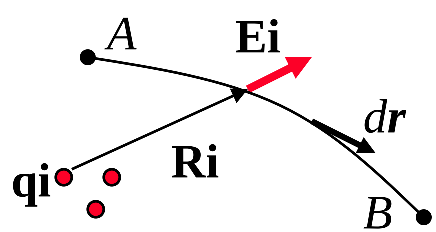


Coulomb law:

$$F = K\frac{q_1 \cdot q_1}{R^2}$$

Electric field (in vacuum):

$$\boldsymbol{E}_v =_{def} \frac{\boldsymbol{F}}{q} \Rightarrow \boldsymbol{E}_v = K\frac{Q}{R^2}\frac{\boldsymbol{R}}{R}$$



By superposition:

$$\boldsymbol{E}_v(\boldsymbol{r}) = K\sum_{i=1}^{n}\frac{q_i \boldsymbol{R}_i}{R_i^3} \Rightarrow \boldsymbol{E}_v = K\int_D \frac{Rdq}{R^3}, \quad dq =_{def} \rho dv$$

$$\boldsymbol{R}_i = \boldsymbol{R}_i - \boldsymbol{r}, \quad R_i = |\boldsymbol{R}_i|, \quad Rdq =_{def} \rho dv$$

Mechanic work:

$$W_{AB} = \int_{C_{AB}} \boldsymbol{F}d\boldsymbol{r} = q\int_{C_{AB}} \boldsymbol{E}_v d\boldsymbol{r} = qU_{AB}, \; U_{AB} =_{def} \int_{C_{AB}} \boldsymbol{E}_v d\boldsymbol{r} = W_{AB}/q$$

Voltage and potential:

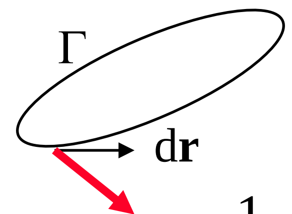
$$U_{AB} = KQ\int_{C_{AB}} \frac{\boldsymbol{R}}{R^3}d\boldsymbol{R} = -\frac{KQ}{R}\Bigg|_A^B =_{\forall C} V_A - V_B, \quad V_P =_{def} U_{PP_\infty} = \frac{kQ}{R} \Rightarrow \quad V = \int_D \frac{dq}{R}$$

Foundation of ES

- **Theorem of the ES potential**

$$V_A =_{def} \int_{C_{AO}} \boldsymbol{E}_v d\boldsymbol{r} \Rightarrow \boldsymbol{E}_v = -gradV, \Leftrightarrow \int_{C_{AB}} \boldsymbol{E}_v d\boldsymbol{r} = -\int_{C_{AB}} gradV d\boldsymbol{r} = -\int_{C_{AB}} dV = V_A - V_B$$

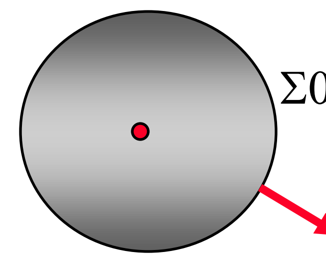
$$curl(gradV) = 0 \Rightarrow curl\boldsymbol{E}_v = 0, \qquad \oint_{\Gamma} \boldsymbol{E}_v d\boldsymbol{r} = \lim_{A \to B} U_{AB} = 0$$



- **Electric flux**

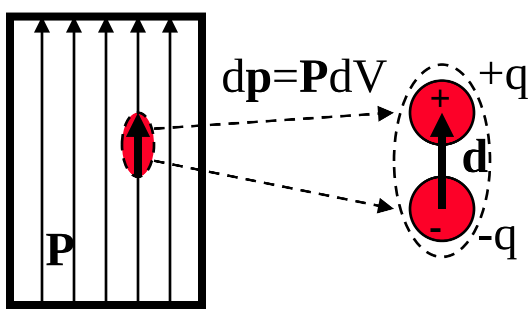
$$\oint_{\Sigma_0} \boldsymbol{E}_v \boldsymbol{n} dS = \frac{KQ}{R^2} \oint_{\Sigma_0} dS \quad 4\pi R^2 \Rightarrow \varepsilon_0 \oint_{\Sigma} \boldsymbol{E}_v \boldsymbol{n} dS = \sum_{i=1}^{n} q_i, \ \varepsilon_0 =_{def} 1/(4\pi K) \Leftrightarrow K = \frac{1}{4\pi\varepsilon_0}$$

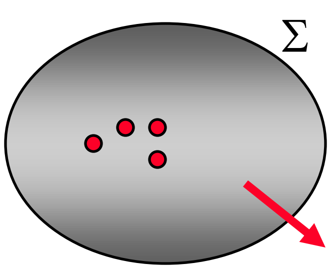
$$div\boldsymbol{E}_v =_{def} \lim_{V_{D_\Sigma} \to 0} \oint_{\Sigma} \boldsymbol{E}_v \boldsymbol{n} dS / V_{D_\Sigma} = \frac{1}{\varepsilon_0} \lim_{V_{D_\Sigma} \to 0} \frac{Q}{V_{D_\Sigma}} = \rho / \varepsilon_0, \quad Q = \sum_{i=1}^{n} q_i$$



- **Polarization**

$$\boldsymbol{p} = \lim_{d \to 0} (q\boldsymbol{d}), \quad \boldsymbol{P} = \lim_{\Delta V \to 0} \frac{\Delta \boldsymbol{p}}{\Delta V} \Leftrightarrow \boldsymbol{p} = \int_D \boldsymbol{P} dv$$





Foundation of ES (cont.)

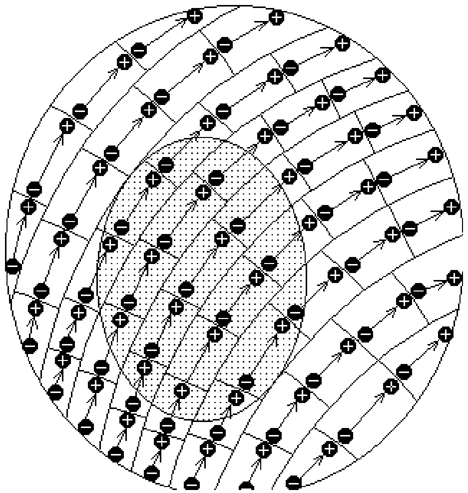
- **Equivalent polarization charge**

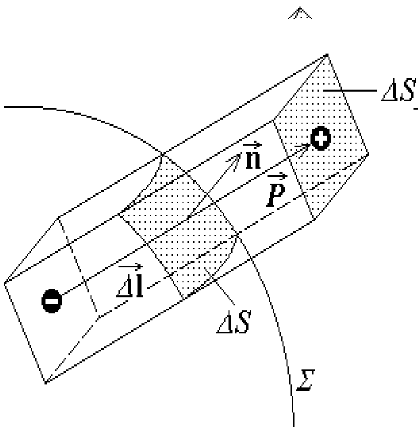
$$\Delta q_d = -P\,\Delta S_\perp = -P\,\Delta S\cos\theta = -\mathrm{P}\cdot\boldsymbol{n}\,\Delta S$$

$$q_{P D_\Sigma} = -\oint_\Sigma \boldsymbol{P}\cdot\boldsymbol{n}\,dS \Rightarrow \boxed{\rho_P = -div\boldsymbol{P}}$$

$$\rho_{S P} = -\boldsymbol{n}_{12}\cdot(\mathrm{P}_2 - \mathrm{P}_1)\big|_{S_{12}} = -\mathrm{div}_S\,\mathrm{P}$$

$$\text{if } P_{ext} = 0 \Rightarrow \int_\Sigma \rho_{S P}\,dS = -q_{PD_\Sigma}$$





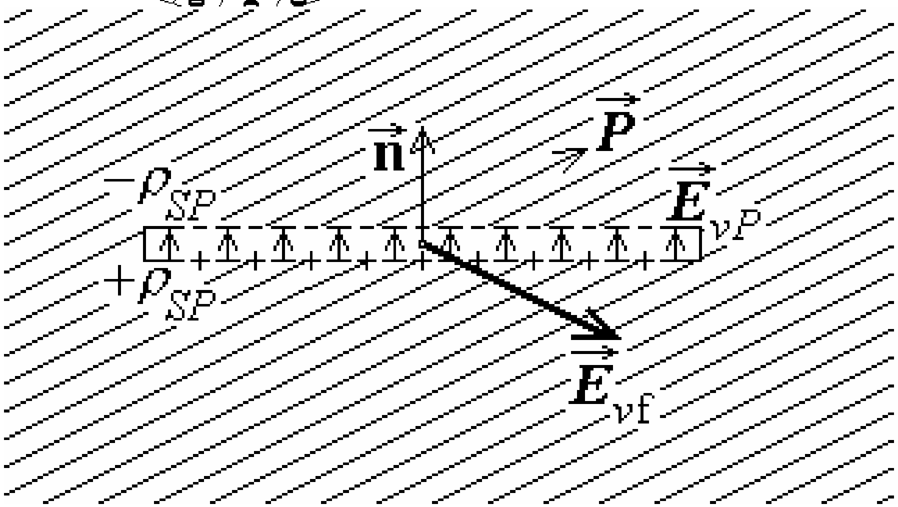
- **Field in substance**

$$\boldsymbol{E}_{vc}(\boldsymbol{n}) = \boldsymbol{E} + \frac{(\boldsymbol{P}\cdot\boldsymbol{n})}{\varepsilon_0}\boldsymbol{n} \Rightarrow \boxed{\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}}$$

$$\mathbf{E} =_{def} \mathbf{E}_{vc}(\mathbf{n}\perp\mathbf{P})$$

$$\mathbf{D} =_{def} \varepsilon_0\,\mathbf{E}_{vc}(\mathbf{n}\,||\,\mathbf{P})$$

$$\Rightarrow \forall\boldsymbol{n},\quad \boldsymbol{E}_{vc}(\boldsymbol{n}) = \boldsymbol{E} + \left[\left(\frac{\boldsymbol{D}}{\varepsilon_0} - \boldsymbol{E}\right)\cdot\boldsymbol{n}\right]\boldsymbol{n}$$



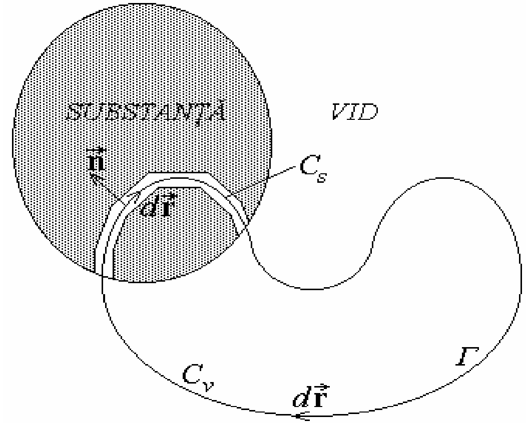
There are necessary and sufficient two vectors E, D to describe field in substance (in an arbitrarily oriented vacuum cavity)

Foundation of ES (cont.)

- **Conservation of field components (Dn, Et)**

$$\mathbf{E}_{vc}(\mathbf{n}) = \mathbf{E} + \left[\left(\frac{\mathbf{D}}{\varepsilon_0} - \mathbf{E}\right)\cdot\mathbf{n}\right]\mathbf{n} \Rightarrow \begin{cases} \mathbf{E}_{vct} = \mathbf{n}\times(\mathbf{E}_{vc}\times\mathbf{n}) = \mathbf{n}\times(\mathbf{E}\times\mathbf{n}) = \mathbf{E}_t \\ D_{vcn} = \varepsilon_0\mathbf{E}_{vc}\cdot\mathbf{n} = \mathbf{D}\cdot\mathbf{n} = D_n \end{cases}$$

- **Electric voltage and flux in substance**

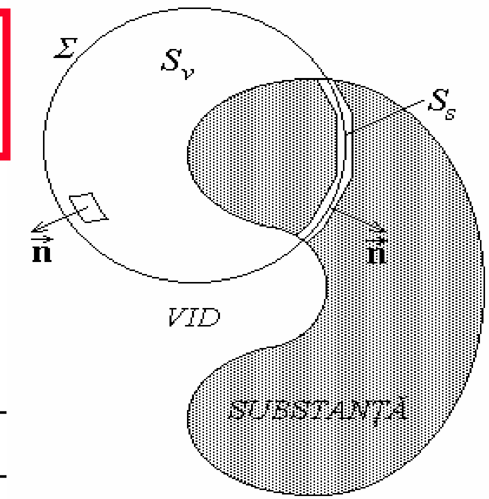


$$U = \int_C \mathbf{E}\cdot d\mathbf{r} = \int_C \mathbf{E}_t\cdot d\mathbf{r} = \int_C \mathbf{E}_v\cdot d\mathbf{r}$$

$$\Rightarrow \oint_\Gamma \mathbf{E}\cdot d\mathbf{r} = \oint_\Gamma \mathbf{E}_v\cdot d\mathbf{r} = 0 \Rightarrow \boxed{\oint_\Gamma \mathbf{E}\cdot d\mathbf{r} = 0, \quad \forall S_\Gamma}$$

$$\Psi_S = \int_S \mathbf{D}\cdot\mathbf{n}\,dS = \int_S D_n dS = \int_S D_{nv} dS = \varepsilon_0\int_S \mathbf{E}_v\cdot\mathbf{n}\,dS$$

$$\oint_\Sigma \mathbf{D}\cdot\mathbf{n}\,dS = \varepsilon_0\oint_\Sigma \mathbf{E}_v\cdot\mathbf{n}\,dS = q \Rightarrow \boxed{\oint_\Sigma \mathbf{D}\cdot\mathbf{n}\,dS = q, \quad \forall D_\Sigma}$$



- **Linear dielectrics**

$$\mathbf{P} = \varepsilon_0\chi\mathbf{E} \Rightarrow \mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon_0\mathbf{E} + \varepsilon_0\chi\mathbf{E} = \varepsilon_0(1+\chi)\mathbf{E} \Rightarrow$$

$$\boxed{\mathbf{D} = \varepsilon\mathbf{E},} \quad \varepsilon = \varepsilon_0(1+\chi)$$

ES summary

- **First and second order equations:**

$$\begin{cases} div\mathbf{D} = \rho \\ \mathbf{curl}\mathbf{E} = 0 \\ \mathbf{D} = \bar{\bar{\varepsilon}}\mathbf{E} + (\mathbf{P}_p) \\ \mathbf{E}(+\mathbf{E}_i) = 0 \text{ in conductors} \end{cases} \Leftrightarrow -div(\bar{\bar{\varepsilon}}\,\mathbf{grad}V) = \rho - div\mathbf{P}_p$$

$$\mathbf{E} = -\mathbf{grad}V \Leftrightarrow V(P) = \int_{PP_0} \mathbf{E}d\mathbf{r}$$

- **Interface conditions:**

$$\begin{cases} D_{n1} = D_{n2} \\ \mathbf{E}_{t1} = \mathbf{E}_{t2} \end{cases} \Leftrightarrow \begin{cases} \varepsilon_1 \dfrac{\partial V_1}{\partial n} = \varepsilon_2 \dfrac{\partial V_2}{\partial n} \\ V_1 = V_2 \end{cases}$$

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- **Boundary conditions:**

$$\begin{cases} \mathbf{E}_t = f_E(P) \quad \text{on } S_E \\ D_n = f_D(P) \quad \text{on } S_D \\ \int_{PkP_0} \mathbf{E}_t d\mathbf{r} = U_k \quad \text{or} \quad \int_{S_{Ek}} D_n dS = \Psi_k, \\ \text{for each } S_{Ek}, k = 1,2,.., n\text{-}1, \text{and } U_n = 0 \end{cases} \Leftrightarrow \begin{cases} V = f_D(P)_D \\ \dfrac{dV}{dn} = f_N(P) \end{cases}$$

- **Capacitors:**

$$\mathbf{q} = \mathbf{C}\mathbf{v}, \quad \mathbf{v} = \mathbf{S}\mathbf{q}, \quad \mathbf{S} = \mathbf{C}^{-1}$$

ES summary – energy

- **Solutions – integral equations**

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\int_D \frac{(\rho(\mathbf{r}_0) - div\mathbf{P})\cdot\mathbf{R}}{R^3} dv + grad...\right]$$

$$V(P) = \frac{1}{4\pi\varepsilon_0}\left[\int_D \frac{\rho(\mathbf{r}_0) - div\mathbf{P}}{R} dv + \oint_\Sigma \left[\frac{1}{R}\frac{\partial V}{\partial n} - V\frac{\partial}{\partial n}\left(\frac{1}{R}\right)\right] dS\right],$$

- **ES energy** $W_e = \int_D w_e dv = \frac{1}{2}\int_D \varepsilon \mathbf{E}^2 dv = \frac{1}{2}\int_D \rho V dv - \frac{1}{2}\int_D \mathbf{P}_p \cdot \mathbf{E} dv - \frac{1}{2}\oint_\Sigma V\mathbf{D}\cdot\mathbf{n} dS$

$$W_e = \frac{1}{2}\mathbf{v}^{\mathrm{T}}\cdot\mathbf{q} = \frac{1}{2}\mathbf{v}^{\mathrm{T}}\mathbf{C}\mathbf{v} = \frac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{S}\,\mathbf{q} > 0,$$

- **Tellegen's theorem** $div\mathbf{D}' = 0,\ curl\mathbf{E}'' = 0 \Rightarrow <\mathbf{D}', \mathbf{E}''> = \mathbf{q}'^{\mathrm{T}}\cdot\mathbf{v}''$
- **Reciprocity** $\int_{R^3} (\rho' V'' - \rho'' V')dv = 0 \Rightarrow \mathbf{q}'^{\mathrm{T}}\cdot\mathbf{v}'' - \mathbf{v}'^{\mathrm{T}}\cdot\mathbf{q}'' = 0$
- **Energy functional:** $F(V) = \frac{1}{2}\int_D \left[\varepsilon(gradV)^2 - \rho V\right] dv + \int_{S_N} V D_n dS < F(V')$
- **Week formulation:** $\int_D (\varepsilon\, gradV \cdot grad\delta V - \rho\delta V)dv + \int_{S_N} D_n \delta V dS = 0$
- **Parameter variations:** $\frac{\partial C_{ij}}{\partial p} = \frac{1}{V_0^2}\int_\Omega \frac{\partial\varepsilon}{\partial p}\mathbf{E}\cdot\underline{\mathbf{E}}d\Omega = \frac{\varepsilon}{V_0^2}\int_{Sp}\mathbf{E}\cdot\underline{\mathbf{E}}dS$

ES forces

- **Coulomb – charged particle** $\mathbf{f} = \rho\mathbf{E} \Rightarrow \mathbf{F} = q\mathbf{E}$ $\mathbf{F} = \frac{q_1 q_2}{4\pi\varepsilon_0 R^2}\frac{\mathbf{R}}{R}$
- **Polarized particle** $\mathbf{F}_p = \mathbf{grad}(\mathbf{p}\cdot\underset{\uparrow}{\mathbf{E}_v})$ $\mathbf{T}_p = \mathbf{p}\times\mathbf{E}_v$
- **Linear dielectric particle** $\mathbf{f}_p = \chi\,\mathbf{grad}\,w_e \Leftrightarrow \mathbf{F}_p = \chi\,V\mathbf{grad}\,w_e$
- **Conductors** $\mathbf{F}_c = \oint_\Sigma w_e \mathbf{n}\,dS,\quad \mathbf{T}_c = \oint_\Sigma w_e(\mathbf{r}\times\mathbf{n})dS$

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- **Capacitors** $X_k = -\frac{1}{2}\mathbf{q}^{\mathrm{T}}\frac{\partial\mathbf{S}}{\partial x_k}\mathbf{q};\quad X_k = \frac{1}{2}\mathbf{v}^{\mathrm{T}}\frac{\partial\mathbf{C}}{\partial x_k}\mathbf{v}$
- **In general** $X_{k\,el} = -\left.\frac{\partial W_e}{\partial x_k}\right|_{q=const.}\quad X_{k\,el} = -\left.\frac{\partial W_e^*}{\partial x_k}\right|_{v=const.}$
- **Maxwell's tensor**

$$\mathbf{f} = \rho_V \mathrm{E} - \frac{E^2}{2}(\mathbf{grad}\,\varepsilon) + \mathbf{grad}\left(\frac{E^2}{2}\tau\frac{\partial\varepsilon}{\partial\tau}\right) = div\left[\mathbf{E}\wedge\mathbf{D}^T + \bar{\bar{\mathbf{I}}}\left(\frac{E^2}{2}\tau\frac{\partial\varepsilon}{\partial\tau} - w_e\right)\right]$$

Please see

- **Electricity and Magnetism: Statics**
- **2-D Electrostatics Applet Demonstrates static electric fields and steady-state current distributions. 2-D Electrostatic Fields Applet Demonstrates electric fields in various 2-D situations; also shows Gauss's law. 3-D Electrostatic Fields Applet Demonstrates electric fields in various 3-D**

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http://www.falstad.com/mathphysics.html

Not so easy questions for curious people

1. Are valid ES equations/methods for slow time variable fields or low speed ?
2. Are valid ES equations/methods in the presence of magnets ?
3. Are valid ES equations/methods for electric field outside d.c. currents ?
4. What about Robin boundary condition (a V+b dV/dn). Correctness and meaning ?
5. What are ES boundary conditions in semi-bounded domains ?
6. Give example of wrong ES problems. What are Hadamard well-posed problems?
7. What about nonlinear dielectrics ? Uniqueness, energy, forces.
8. Are Coulomb integrals convergent ?
9. Is the de-polarizing field always uniform ? What about polarized ellipsoid ?
10. What are the differences between Tellegen and reciprocity theorems ?
11. How may be defined Green function with Neumann b.c.?
12. What space may be used for trial and test functions in the weak ES formulation ?
13. What is the best method for ES field computation ? The best ES-CAD package ?
14. May be chosen $\varepsilon 0=1$. May be chosen the Coulomb constant K=1 ?
15. Interactions of small (or not) particles: charged, polariz. dielctr. conductive