Electromagnetic Modeling

7. Fundamental theorems:

Charge, EM energy and impulse conservation, problems

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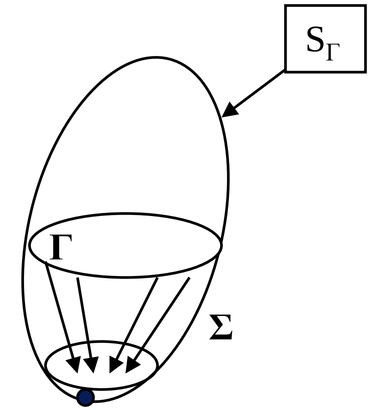
Contents

1. Charge conservation

1. The integral form of the theorem:

$$i_\Sigma = -\frac{dq_{D_\Sigma}}{dt} \Leftrightarrow \oint_{\Sigma=\partial D_\Sigma} \boldsymbol{J} d\boldsymbol{A} = -\frac{d}{dt}\int_{D_\Sigma} \rho dv$$



2. Poof:

$$u_{m_\Gamma} = i_{s_\Gamma} + \frac{d\psi_{S_\Gamma}}{dt} \rightarrow 0 = i_\Sigma + \frac{d\psi_\Sigma}{dt}$$

$$u_{m_\Gamma} = H_{tave} l_\Gamma \rightarrow 0, \psi_{S_\Gamma} \rightarrow \psi_\Sigma = q_{D_\Sigma}$$

3. The local form of the theorem:

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} + \rho \boldsymbol{v} + \nabla \times (\boldsymbol{D} \times \boldsymbol{v}) \Rightarrow$$

$$\nabla \cdot \nabla \times \boldsymbol{H} = \nabla(\boldsymbol{J} + \rho \boldsymbol{v}) + \frac{\partial \nabla \boldsymbol{D}}{\partial t} +$$

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$$\nabla \cdot \nabla \times (\boldsymbol{D} \times \boldsymbol{v}) \Rightarrow \nabla(\boldsymbol{J} + \rho \boldsymbol{v}) + \frac{\partial \rho}{\partial t} = 0$$

4. The extended integral form:

$$\oint_{\Sigma=\partial D_\Sigma} (\boldsymbol{J} + \rho \boldsymbol{v}) d\boldsymbol{A} = -\int_{D_\Sigma(t)} \frac{\partial \rho}{\partial t} dv$$

5. In motionless body:

$$div \boldsymbol{J} = -\frac{\partial \rho}{\partial t}, \quad \oint_{\Sigma=\partial D_\Sigma} \boldsymbol{J} d\boldsymbol{A} = -\int_{D_\Sigma} \frac{\partial \rho}{\partial t} dv$$

Consequences of charge conservation theorem

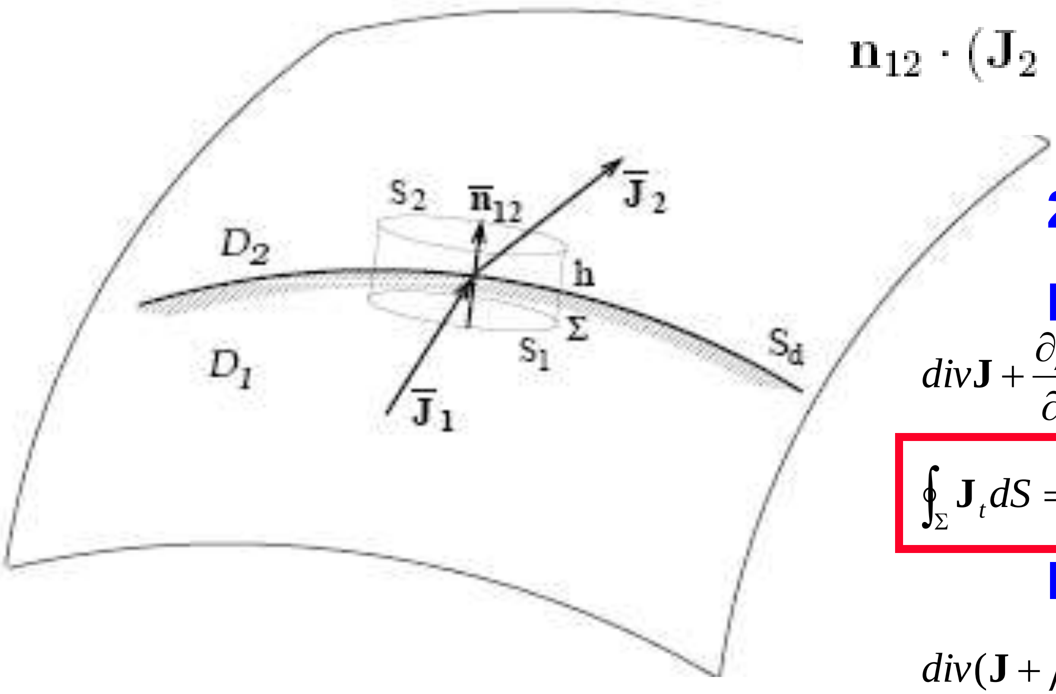
1. On the interfaces/ boundaries:

$$i_\Sigma = \int_\Sigma \mathbf{J} d\mathbf{A} = J_{nmed} hl - J_{1nmed} A + J_{2nmed} A,$$

$$\int_{D_\Sigma} \rho dv = \rho_{vmed} Ah + \rho_{smed} A.$$

$$J_{nmed} \frac{hl}{A} + J_{2nmed} - J_{nmed} = \frac{d}{dt} \rho_{vmed} h - \frac{d\rho_{smed}}{dt}$$

$$\mathbf{n}_{12} \cdot (\mathbf{J}_2 - \mathbf{J}_1) = -\frac{\partial \rho_s}{\partial t}, \qquad div_s \mathbf{J} = -\frac{\partial \rho_s}{\partial t}$$



2. Conservation of total current:

In motionless media:

$$div\mathbf{J} + \frac{\partial \rho}{\partial t} = div\left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) = div(\mathbf{J} + \mathbf{J}_d) = div\mathbf{J}_t = 0$$

$$\oint_\Sigma \mathbf{J}_t dS = 0, \quad \text{where } \mathbf{J}_t = \mathbf{J} + \mathbf{J}_d, \mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t}$$

In moving media:

$$div(\mathbf{J} + \rho\mathbf{v}) + \frac{\partial \rho}{\partial t} = 0 \Rightarrow \mathbf{J}_t = \mathbf{J} + \mathbf{J}_d + \mathbf{J}_v, \mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t}, \mathbf{J}_v = \rho\mathbf{v}$$

Charge relaxation

2. In linear, homogeneous body:

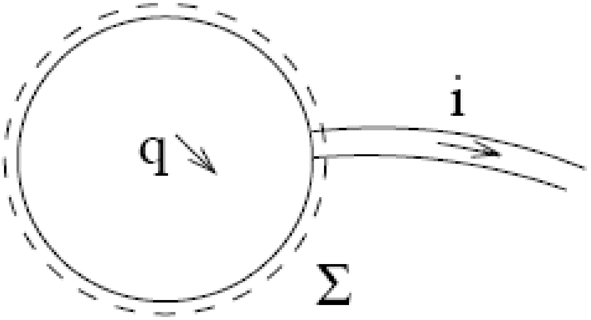
$$\frac{\partial \rho}{\partial t} = -\nabla \mathbf{J} = -\nabla(\sigma \mathbf{E}) = -\sigma \nabla \mathbf{E} = -\frac{\sigma}{\varepsilon} \nabla \mathbf{D} = -\frac{\sigma}{\varepsilon} \rho$$

$$\boxed{\tau = \frac{\varepsilon}{\sigma}} \Rightarrow \frac{\partial \rho}{\partial t} + \rho / \tau = 0 \Rightarrow \boxed{\rho = \rho(0) e^{-t/\tau}}$$

3. In static regimes → current conservation/ continuity:

$$i_\Sigma = -\frac{dq_{D_\Sigma}}{dt} = 0 \Rightarrow$$

$$\boxed{\oint_{\Sigma = \partial D_\Sigma} \mathbf{J} d\mathbf{A} = 0 \Rightarrow \nabla \mathbf{J} = 0}$$

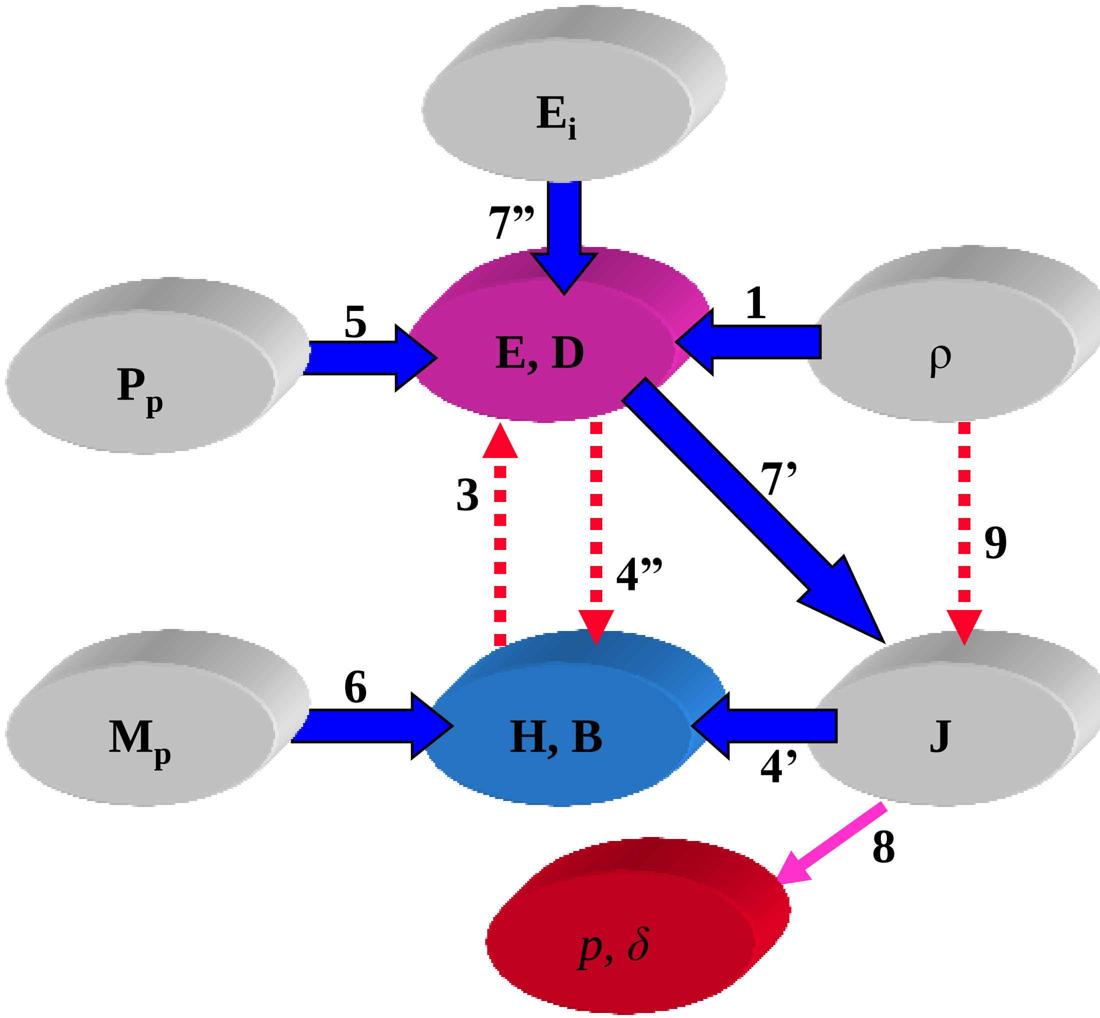


4. Charge conservation of the insulated systems:

$$0 = i_\Sigma = -\frac{dq_{D_\Sigma}}{dt} \Rightarrow \boxed{q_{D_\Sigma} = ct.}$$

Diagram of fundamental EM phenomena (causal relations)

1. $\nabla \cdot \mathbf{D} = \rho$
2. $\nabla \cdot \mathbf{B} = 0$
3. $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$
4. $\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$
5. $\mathbf{D} = \varepsilon\, \mathbf{E} + \mathbf{P}_p(\mathbf{E})$
6. $\mathbf{B} = \mu\, (\mathbf{H} + \mathbf{M}_p(\mathbf{H}))$
7. $\mathbf{J} = \sigma(\mathbf{E} + \mathbf{E}_i(\mathbf{E}))$
8. $p = \mathbf{EJ}$
9. $\nabla \cdot \mathbf{J} = -\dfrac{\partial \rho}{\partial t}$



2. Poynting's theorem of EM energy

1. The local form of the theorem in linear motionless media:

$$rot\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \textbf{-H}$$

$$rot\mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad \textbf{E}$$

$$\mathbf{E}rot\mathbf{H} - \mathbf{H}rot\mathbf{E} = \mathbf{J}E + \mathbf{E}\frac{\partial \mathbf{D}}{\partial t} + \mathbf{H}\frac{\partial \mathbf{B}}{\partial t}.$$

$$div\left(\mathbf{E}\times\mathbf{H}\right) = \nabla\left(\mathbf{E}\times\mathbf{H}\right) = \nabla\left(\overset{\downarrow}{\mathbf{E}}\times\mathbf{H}\right) + \nabla\left(\mathbf{E}\times\overset{\downarrow}{\mathbf{H}}\right) = \mathbf{H}\left(\nabla\times\mathbf{E}\right) -$$

$$-\mathbf{E}\left(\nabla\times\mathbf{H}\right) = \mathbf{H}rot\mathbf{E} - \mathbf{E}rot\mathbf{H}$$

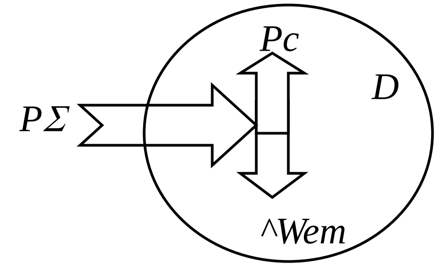
$$\mathbf{E}\frac{\partial \mathbf{D}}{\partial t} = \mathbf{E}\varepsilon\frac{\partial \mathbf{E}}{\partial t} = \frac{\varepsilon}{2}\frac{\partial \mathbf{E}^2}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\mathbf{DE}}{2}\right),$$

$$\mathbf{H}\frac{\partial \mathbf{B}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\mathbf{BH}}{2}\right)$$

$$-div\left(\mathbf{E}\times\mathbf{H}\right) = \mathbf{JE} + \frac{\partial}{\partial t}\left(\frac{\mathbf{DE}}{2} + \frac{\mathbf{BH}}{2}\right),$$

The integral form of the EM-energy conservation theorem

- p = **JE** - power density [W/m^3]
- **S = E x H** – Poynting vector [W/m^2]
- $w_e = \mathbf{DE}/2$ – density of electric energy [J/m^3]
- $w_m = \mathbf{BH}/2$ – density of magn. energy [J/m^3]
- $w_{em} = w_e + w_m$ density of el-mg energy [J/m^3]



$$P_\Sigma = -\int_{\partial D} \mathbf{S}d\mathbf{A} \ [W]$$

$$P_c = \int_D p dv \ [W]$$

$$W_{em} = \int_D w_{em} dv \ [J]$$

$$-div\mathbf{S} = p + \frac{\partial w_{em}}{\partial t} \Rightarrow P_\Sigma = P_c + \frac{dW_{em}}{dt}$$

$$-\int_D div\mathbf{S}dv = -\int_{\partial D} \mathbf{S}d\mathbf{A} = \int_D \left(p + \frac{\partial w_{em}}{\partial t} \right) dv = \int_D p dv + \frac{d}{dt}\int_D w_{em} dv$$

EM energy in nonlinear motionless media

Energy and co-energy

$$rot\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \mathbf{-H}$$

$$rot\mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad \mathbf{E}$$

$$\mathbf{E}\,rot\mathbf{H} - \mathbf{H}\,rot\mathbf{E} = \mathbf{J}E + \mathbf{E}\frac{\partial \mathbf{D}}{\partial t} + \mathbf{H}\frac{\partial \mathbf{B}}{\partial t}.$$

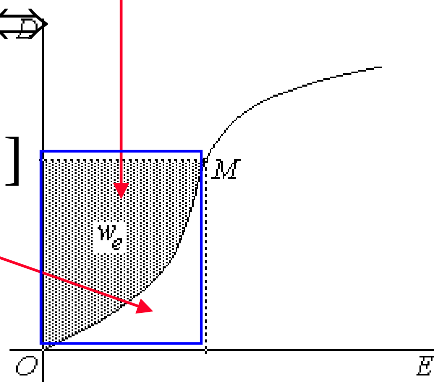
$$\Rightarrow \; div\mathbf{S} = p + \frac{\partial w_e}{\partial t} + \frac{\partial w_m}{\partial t}$$

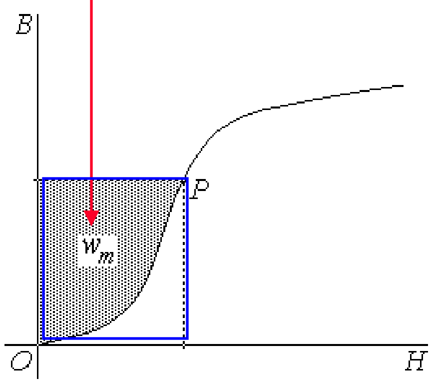
$$- \quad \frac{\partial w_e}{\partial t} = \mathbf{E}\frac{\partial \mathbf{D}}{\partial t} \Rightarrow dw_e = \mathbf{E}\cdot d\mathbf{D} \Rightarrow w_e = \int_0^{\mathrm{D}} \mathbf{E}\cdot d\mathbf{D} \quad , \quad w_m = \int_0^{\mathrm{B}} \mathbf{H}\cdot d\mathbf{B}$$

$$d(\mathrm{E}\cdot\mathrm{D}) = d\mathrm{E}\cdot\mathrm{D} + \mathrm{E}\cdot d\mathrm{D} = dw_e + dw_e^* \Leftrightarrow$$

$$dw_e^* = \mathrm{D}\cdot d\mathrm{E} \Leftrightarrow \quad w_e^* = \int_0^{\mathrm{E}} \mathrm{D}\cdot d\mathrm{E} \;\; [J/m^3]$$

$$\mathbf{D}\cdot\mathbf{E} = w_e + w_e^*, \;\; \mathbf{B}\cdot\mathbf{H} = w_m + w_m^*$$



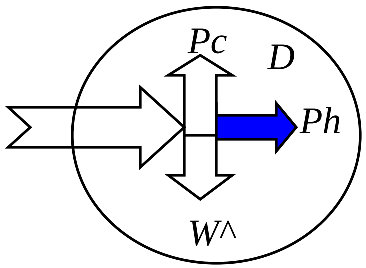


EM energy in nonlinear hysteretic media. Warburg's theorem

Power density of hysteresis losses

$$rot\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \mathbf{-H}$$

$$rot\mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \mathbf{E}$$



$$\mathbf{E}rot\mathbf{H} - \mathbf{H}rot\mathbf{E} = \mathbf{J}E + \mathbf{E}\frac{\partial \mathbf{D}}{\partial t} + \mathbf{H}\frac{\partial \mathbf{B}}{\partial t}.$$

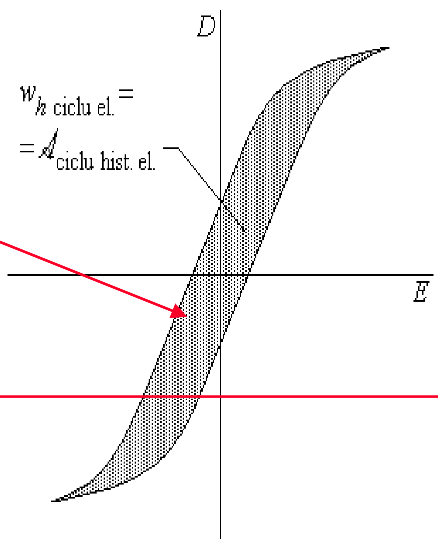
$$div\mathbf{S} = p + \frac{\partial w_e}{\partial t} + \frac{\partial w_m}{\partial t} + p_h$$

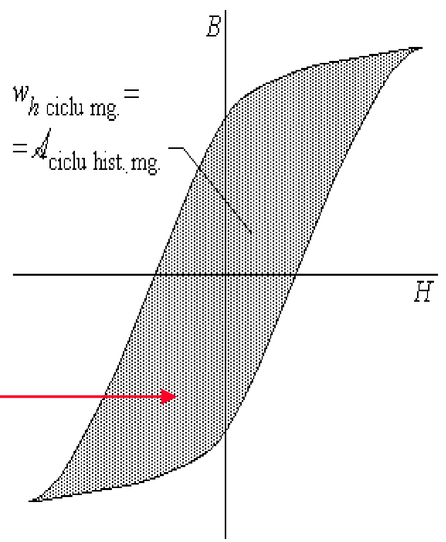
$$\frac{\partial w_e}{\partial t} + p_{he} = \mathbf{E}\frac{\partial \mathbf{D}}{\partial t} \Rightarrow \int_0^T \left(\frac{\partial w_e}{\partial t} + p_{he} \right) dt = w_e(T) - w_e(0) + w_{he} = \int_0^T \mathbf{E} \cdot d\mathbf{D} \Rightarrow$$

$$w_{he} =_{def} \int_0^T p_{he} dt = \int_0^T \mathbf{E} \cdot d\mathbf{D} \ [\mathrm{J/m^3}]$$

$$w_{hm} =_{def} \int_0^T p_{hm} dt = \int_0^T \mathbf{H} \cdot d\mathbf{B} \ [\mathrm{J/m^3}]$$

$$p_h = p_{he} + p_{hm}$$





EM energy in linear moving media

Mechanical power: $$-\frac{dW}{dt} = P_{mec} + P_{cond} - P_{\Sigma} \quad \Leftrightarrow \quad P_{mec} = -\frac{dW}{dt} - P_{cond} + P_{\Sigma} =_{def} \int_D \boldsymbol{f}\cdot\boldsymbol{v}dv$$

$$rot\,\boldsymbol{H} = \boldsymbol{J} + \frac{d_F\boldsymbol{D}}{dt}\quad , \quad \mathrm{div}\,\boldsymbol{B} = 0 \quad , \quad \boldsymbol{B} = \mu\boldsymbol{H} \quad , \quad \boldsymbol{J} = \sigma\boldsymbol{E}, \quad \tau = \text{mass density}$$

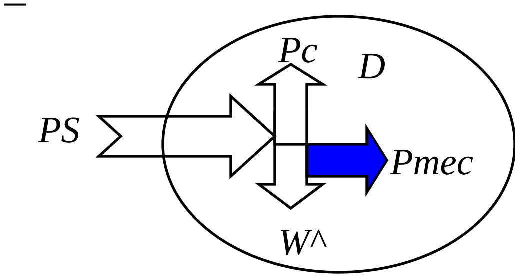
$$rot\,\boldsymbol{E} = -\frac{d_F\boldsymbol{B}}{dt}\quad , \quad \mathrm{div}\,\boldsymbol{D} = \rho_V \quad , \quad \boldsymbol{D} = \varepsilon\boldsymbol{E} \quad \varepsilon = \varepsilon\left(\tau(\mathbf{r},t),T,...\right) \quad , \quad \mu = \mu\left(\tau(\mathbf{r},t),T,..\right.$$

$$P_{mec} = -\frac{dW}{dt} - \int_{D_\Sigma}\boldsymbol{E}\cdot\boldsymbol{J}\,dV - \oint_\Sigma(\boldsymbol{E}\times\boldsymbol{H})\cdot\mathbf{n}\,dS \equiv -\frac{dW}{dt} - \int_{D_\Sigma}(\boldsymbol{E}\cdot\boldsymbol{J} + \boldsymbol{H}\cdot\mathbf{rot}\,\boldsymbol{E} - \boldsymbol{E}\cdot\mathbf{rot}\,\boldsymbol{H})dV$$

$$- \quad = -\frac{dW}{dt} + \int_{D_\Sigma}\left(\boldsymbol{E}\cdot\frac{d_F\boldsymbol{D}}{dt} + \boldsymbol{H}\cdot\frac{d_F\boldsymbol{B}}{dt}\right)dV = -\frac{dW}{dt}\bigg|_{\Psi=\text{const},\,\Phi=\text{const}} \;\; \textbf{Hertz} \;\; = \left[-\frac{d}{dt}\int_{D_\Sigma}\left(\frac{D^2}{2\varepsilon} + \frac{B^2}{2\mu}\right)dV\right]_{\Psi=\text{const},\,\Phi=\text{const}} =$$

$$= \int_{D_\Sigma}\left(-\frac{D}{\varepsilon}\frac{\partial D}{\partial t}\bigg|_{\Psi=\text{const.}} + \frac{D^2}{2\varepsilon^2}\frac{\partial\varepsilon}{\partial t}\bigg|_{\Psi=\text{const.}} - \frac{B}{\mu}\frac{\partial B}{\partial t}\bigg|_{\Phi=\text{const.}} + \frac{B^2}{2\mu^2}\frac{\partial\mu}{\partial t}\bigg|_{\Phi=\text{const.}}\right)dV =$$

$$= \int_{D_\Sigma}\left(-\boldsymbol{E}\cdot\frac{\partial\boldsymbol{D}}{\partial t}\bigg|_{\Psi=\text{const.}} + \frac{E^2}{2}\frac{\partial\varepsilon}{\partial t}\bigg|_{\Psi=\text{const.}} - \boldsymbol{H}\cdot\frac{\partial\boldsymbol{B}}{\partial t}\bigg|_{\Phi=\text{const.}} + \frac{H^2}{2}\frac{\partial\mu}{\partial t}\bigg|_{\Phi=\text{const.}}\right)dV$$



Mechanic effects of EM fields: electric and magn. force density

$$P_{mec} = \int_{D_\Sigma} \left(-\mathrm{E}\cdot\left.\frac{\partial \mathrm{D}}{\partial t}\right|_{\Psi=\text{const.}} + \frac{E^2}{2}\left.\frac{\partial \varepsilon}{\partial t}\right|_{\Psi=\text{const.}} - \mathrm{H}\cdot\left.\frac{\partial \mathrm{B}}{\partial t}\right|_{\Phi=\text{const.}} + \frac{H^2}{2}\left.\frac{\partial \mu}{\partial t}\right|_{\Phi=\text{const.}} \right) dV \Rightarrow$$

$$\int_{D_\Sigma} \boldsymbol{f}\cdot\boldsymbol{v}\, dV = \int_{D_\Sigma} \left\{ -\boldsymbol{E}\cdot\left[-\rho_V \boldsymbol{v} - \mathbf{rot}\left(\boldsymbol{D}\times\boldsymbol{v}\right)\right] + \frac{E^2}{2}\left[-\boldsymbol{v}\cdot\left(\mathbf{grad}\,\varepsilon\right) - \tau\frac{\partial \varepsilon}{\partial \tau}\left(\mathrm{div}\,\boldsymbol{v}\right)\right]\right\} dV +$$

$$+\int_{D_\Sigma} \left(-\boldsymbol{H}\cdot\left[-\mathbf{rot}\left(\boldsymbol{B}\times\boldsymbol{v}\right)\right] + \frac{H^2}{2}\left[-\boldsymbol{v}\cdot\left(\mathbf{grad}\,\mu\right) - \tau\frac{\partial \mu}{\partial \tau}\left(\mathrm{div}\,\boldsymbol{v}\right)\right]\right) dV$$

$$-\quad +\oint_{\Sigma} \left[(\mathrm{D}\times\mathrm{v})\times\mathrm{E} + (\mathrm{B}\times\mathrm{v})\times\mathrm{H} - \left(\frac{E^2}{2}\tau\frac{\partial \varepsilon}{\partial \tau} + \frac{H^2}{2}\tau\frac{\partial \mu}{\partial \tau} \right)\mathrm{v} \right]\cdot \mathbf{n}\, dS$$

$$\Sigma \to \infty \Rightarrow$$

$$\boldsymbol{f} = \boldsymbol{f}_e + \boldsymbol{f}_m = \rho_V \boldsymbol{E} - \frac{E^2}{2}\left(\mathbf{grad}\,\varepsilon\right) + \mathbf{grad}\left(\frac{E^2}{2}\tau\frac{\partial \varepsilon}{\partial \tau}\right) +$$

$$+\boldsymbol{J}\times\boldsymbol{B} - \frac{H^2}{2}\left(grad\,\mu\right) + grad\left(\frac{H^2}{2}\tau\frac{\partial \mu}{\partial \tau}\right)$$

- Coulomb
- Temp. polarization
- Electrostriction
- Laplace
- Temp. manetization
- Magnetostriction

Theorems of generalized forces

Virtual work = (gen.) force x position change: $dL_k = X_k \, dx_k$

In particular:

$$dL = \mathbf{F}\cdot d\mathbf{r} \quad , \quad dL = \mathbf{C}\cdot d\boldsymbol{\alpha} \quad , \quad dL = \mathbf{T}\cdot\mathbf{n}\, dS \quad , \quad dL = p\cdot dV$$

$$P_{mec} =_{def} \frac{d}{dt}\sum_{k=1}^{n} X_k \, dx_k = -\frac{dW}{dt} - P_{cond} - P_{\Sigma} = \int_{D_\Sigma}\left(\mathbf{E}\cdot\frac{d_F \mathbf{D}}{dt} + \mathbf{H}\cdot\frac{d_F \mathbf{B}}{dt}\right)dV - \frac{dW}{dt}$$

$$\Rightarrow X_k = -\left.\frac{\partial W}{\partial x_k}\right|_{\Psi=\text{const.},\,\Phi=\text{const.}} \Rightarrow \quad X_{k\,\text{el}} = -\left.\frac{\partial W_e}{\partial x_k}\right|_{q=\text{const.}} \quad , \quad X_{k\,\text{mg}} = -\left.\frac{\partial W_m}{\partial x_k}\right|_{\Phi=\text{const.}}$$

$$\frac{d_V}{dt}(\mathbf{E}\cdot\mathbf{D}) + \frac{d_V}{dt}(\mathbf{H}\cdot\mathbf{B}) = \left(\mathbf{E}\cdot\frac{d_F \mathbf{D}}{dt} + \mathbf{D}\cdot\frac{d_C \mathbf{E}}{dt}\right) + \left(\mathbf{H}\cdot\frac{d_F \mathbf{B}}{dt} + \mathbf{B}\cdot\frac{d_C \mathbf{H}}{dt}\right) \Rightarrow$$

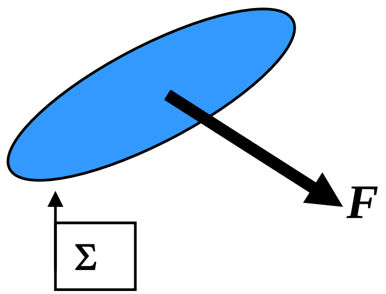
$$\frac{d}{dt}\sum_{k=1}^{n} X_k \, dx_k = \frac{dW^*}{dt} - \int_{D_\Sigma}\left(\boldsymbol{D}\cdot\frac{d_C \boldsymbol{E}}{dt} + \boldsymbol{B}\cdot\frac{d_C \boldsymbol{H}}{dt}\right)dV \qquad X_{k\,\text{el}} = \left.\frac{\partial W_e^*}{\partial x_k}\right|_{u=\text{const.}}$$

$$\Rightarrow X_k dx_k = dW^* \text{ for } \mathbf{E}, \mathbf{H} = \text{ct.} \Rightarrow dX_k = \left.\frac{\partial W^*}{\partial x_k}\right|_{u=\text{const.},\,u_m=\text{const.}} \Rightarrow \quad X_{k\,\text{mg}} = \left.\frac{\partial W_m^*}{\partial x_k}\right|_{i=\text{const.}}$$

Maxwell stress tensor – electric component

EM force:

$$\boldsymbol{F} = \int_{D_\Sigma} \boldsymbol{f}\, dv = \int_{D_\Sigma} \left(\mathrm{div}_d \,\overline{\overline{\boldsymbol{T}}}\right) dv = \oint_\Sigma \overline{\overline{\boldsymbol{T}}} \cdot \boldsymbol{n}\, dA$$



$$\text{If } \frac{\partial \varepsilon}{\partial \tau} = 0 \Rightarrow \boldsymbol{f}_e = \rho_V \boldsymbol{E} - \frac{E^2}{2}(grad\, \varepsilon) =$$

$$= \boldsymbol{E}(\mathrm{div}\, \boldsymbol{D}) - grad\left(\frac{\boldsymbol{D}\cdot\boldsymbol{E}}{2}\right) + (\varepsilon \boldsymbol{E} \cdot grad\,)\boldsymbol{E}$$

$$\int_{D_\Sigma} \boldsymbol{f}_e\, dv = \int_{D_\Sigma} \left[\boldsymbol{E}(\mathrm{div}\, \boldsymbol{D}) + (\boldsymbol{D}\cdot \boldsymbol{grad}\,)\boldsymbol{E} - \boldsymbol{grad}\left(\frac{\boldsymbol{D}\cdot\boldsymbol{E}}{2}\right)\right] dV =$$

$$= \oint_\Sigma \left[\boldsymbol{E}(\boldsymbol{D}\cdot\boldsymbol{n}) - \left(\frac{\boldsymbol{D}\cdot\boldsymbol{E}}{2}\right)\boldsymbol{n}\right] dA = \oint_\Sigma \overline{\overline{\boldsymbol{T}}}_{e\boldsymbol{n}}\, dA = \oint_\Sigma \overline{\overline{\boldsymbol{T}}}_e \cdot \boldsymbol{n}\, dA$$

$$\boldsymbol{T}_{e\boldsymbol{n}} = \overline{\overline{\boldsymbol{T}}}_e \cdot \boldsymbol{n} = \boldsymbol{E}(\boldsymbol{D}\cdot\boldsymbol{n}) - \left(\frac{\boldsymbol{E}\cdot\boldsymbol{D}}{2}\right)\boldsymbol{n}$$

$$\overline{\overline{\boldsymbol{T}}}_e = \boldsymbol{E} \wedge \boldsymbol{D}^T - w_e \overline{\overline{\boldsymbol{I}}}$$

$$\overline{\overline{\boldsymbol{T}}}_e = \begin{bmatrix} E_x D_x - \frac{\mathrm{E}\cdot\mathrm{D}}{2} & E_x D_y & E_x D_z \\ E_y D_x & E_y D_y - \frac{\mathrm{E}\cdot\mathrm{D}}{2} & E_y D_z \\ E_z D_x & E_z D_y & E_z D_z - \frac{\mathrm{E}\cdot\mathrm{D}}{2} \end{bmatrix}$$

Maxwell stress and E/M-striction tensor

Magnetic component:

$$\boldsymbol{f}_m = \boldsymbol{J}\times\boldsymbol{B} - \frac{H^2}{2}(\mathbf{grad}\,\mu) + \mathbf{grad}\left(\frac{H^2}{2}\tau\frac{\partial\mu}{\partial\tau}\right) =$$

$$= (\mathbf{rot}\,\boldsymbol{H})\times\boldsymbol{B} - \mathbf{grad}\left(\mu\frac{H^2}{2}\right) + \mu\left(\mathbf{grad}\frac{H^2}{2}\right) + \mathbf{grad}\left(\frac{H^2}{2}\tau\frac{\partial\mu}{\partial\tau}\right)$$

$$\boldsymbol{f}_m = \boldsymbol{H}(\mathrm{div}\,\boldsymbol{B}) - \mathbf{grad}\left(\frac{\boldsymbol{H}\cdot\boldsymbol{B}}{2}\right) + (\boldsymbol{B}\cdot\mathbf{grad})\boldsymbol{H} + \mathbf{grad}\left(\frac{H^2}{2}\tau\frac{\partial\mu}{\partial\tau}\right)$$

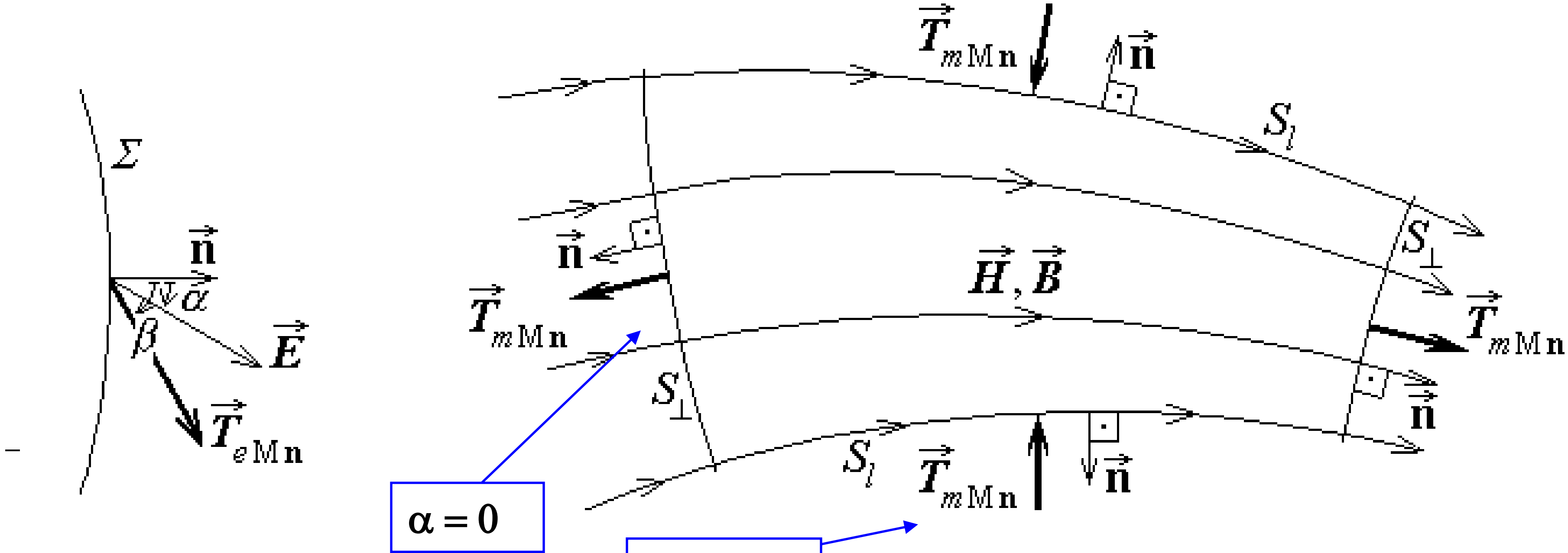
$$\int_{D_\Sigma}\boldsymbol{f}_m\,dV = \int_{D_\Sigma}\left[\boldsymbol{H}(\mathrm{div}\,\boldsymbol{B}) + (\boldsymbol{B}\cdot\mathbf{grad})\boldsymbol{H} - \mathbf{grad}\left(\frac{\boldsymbol{H}\cdot\boldsymbol{B}}{2}\right) + \mathbf{grad}\left(\frac{H^2}{2}\tau\frac{\partial\mu}{\partial\tau}\right)\right]dV =$$

$$= \oint_\Sigma\left[\mathrm{H}(\mathrm{B}\cdot\mathbf{n}) - \left(\frac{\mathrm{H}\cdot\mathrm{B}}{2}\right)\mathbf{n} + \left(\frac{H^2}{2}\tau\frac{\partial\mu}{\partial\tau}\right)\mathbf{n}\right]dS = \oint_\Sigma \mathrm{T}_{m\,\mathbf{n}}\,dS = \oint_\Sigma \mathbf{T}_m\cdot\mathbf{n}\,dS \Rightarrow$$

$$\boldsymbol{T}_{m\,\boldsymbol{n}} = \overline{\overline{\boldsymbol{T}}}_m\cdot\boldsymbol{n} = \boldsymbol{H}(\boldsymbol{B}\cdot\boldsymbol{n}) - \left(\frac{\mathrm{H}\cdot\mathrm{B}}{2}\right)\boldsymbol{n} \Rightarrow \boxed{\overline{\overline{\boldsymbol{T}}}_m = \boldsymbol{H}\wedge\boldsymbol{B}^T - w_m\overline{\overline{\boldsymbol{I}}},\ \overline{\overline{\boldsymbol{T}}}_{ms} = \overline{\overline{\boldsymbol{I}}}\frac{H^2}{2}\tau\frac{\partial\mu}{\partial\tau},\ \overline{\overline{\boldsymbol{T}}}_{es} = \overline{\overline{\boldsymbol{I}}}\frac{E^2}{2}\tau\frac{\partial\varepsilon}{\partial\tau}}$$

$$\overline{\overline{\boldsymbol{T}}}_m = \begin{bmatrix} H_xB_x - \frac{\mathrm{H}\cdot\mathrm{B}}{2} & H_xB_y & H_xB_z \\ H_yB_x & H_yB_y - \frac{\mathrm{H}\cdot\mathrm{B}}{2} & H_yB_z \\ H_zB_x & H_zB_y & H_zB_z - \frac{\mathrm{H}\cdot\mathrm{B}}{2} \end{bmatrix} \qquad \overline{\overline{\boldsymbol{T}}}_{ms} = \begin{bmatrix} \frac{H^2}{2}\tau\frac{\partial\mu}{\partial\tau} & 0 & 0 \\ 0 & \frac{H^2}{2}\tau\frac{\partial\mu}{\partial\tau} & 0 \\ 0 & 0 & \frac{H^2}{2}\tau\frac{\partial\mu}{\partial\tau} \end{bmatrix}$$

Orientation of Maxwell stress and E/M-striction tensors



$$\beta = 2\alpha$$

$$\mathbf{T}_{ms\,n} = \mathbf{n}\tau\frac{\partial\mu}{\partial\tau} \parallel \mathbf{n},\quad \mathbf{T}_{esn} = \mathbf{n}\frac{E^2}{2}\tau\frac{\partial\varepsilon}{\partial\tau} \parallel \mathbf{n}$$

$$\left|\overline{\overline{\boldsymbol{T}}}_{e\ \boldsymbol{n}}\right| = \frac{\mathrm{E}\cdot\mathrm{D}}{2} = w_e$$

$$\left|\overline{\overline{\boldsymbol{T}}}_{m\ \boldsymbol{n}}\right| = \frac{\mathrm{H}\cdot\mathrm{B}}{2} = w_m$$

3. Impulse conservation – local form

$$\begin{cases} \mathbf{rot}\,\mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} & \Big| \times \mathbf{D} \\ \mathbf{rot}\,\mathbf{H} = \mathrm{J} + \dfrac{\partial \mathbf{D}}{\partial t} & \Big| \times \mathbf{B} \end{cases}$$

$$\mathbf{grad}\left(\frac{\mathrm{D}\cdot\mathrm{E}}{2}\right) = \mathbf{grad}\left(\frac{\varepsilon E^2}{2}\right) = \frac{E^2}{2}\,\mathbf{grad}\,\varepsilon + \mathrm{D}\times(\mathbf{rot}\,\mathrm{E}) + (\mathrm{D}\cdot\mathbf{grad})\mathrm{E}$$

$$\mathbf{grad}\left(\frac{\mathrm{B}\cdot\mathrm{H}}{2}\right) = \frac{1}{2}\,\mathbf{grad}\left(\mu H^2\right) = \frac{H^2}{2}\,\mathbf{grad}\,\mu + \mathrm{B}\times(\mathbf{rot}\,\mathrm{H}) + (\mathrm{B}\cdot\mathbf{grad})\mathrm{H}$$

$$-\mathbf{D}\times(\mathbf{rot}\,\mathbf{E}) - \mathbf{B}\times(\mathbf{rot}\,\mathbf{H}) + \mathbf{B}\times\mathbf{J} = \mathbf{D}\times\left(\frac{\partial \mathbf{B}}{\partial t}\right) + \left(\frac{\partial \mathbf{D}}{\partial t}\right)\times\mathbf{B}$$

$$\underbrace{\frac{E^2}{2}\,\mathbf{grad}\,\varepsilon - \mathbf{grad}\left(\frac{\mathrm{D}\cdot\mathrm{E}}{2}\right) + (\mathrm{D}\cdot\mathbf{grad})\mathrm{E}}_{\mathrm{div}(\overline{\overline{T}}_e) - f_e} + \underbrace{\frac{H^2}{2}\,\mathbf{grad}\,\mu - \mathbf{grad}\left(\frac{\mathrm{B}\cdot\mathrm{H}}{2}\right) + (\mathrm{B}\cdot\mathbf{grad})\mathrm{H}}_{\mathrm{div}(\overline{\overline{T}}_m) - f_m} + \mathbf{B}\times\mathbf{J} = \frac{\partial(\mathbf{D}\times\mathbf{B})}{\partial t}$$

$$\boldsymbol{g}_{em} = \boldsymbol{D}\times\boldsymbol{B},$$

EM impulse density [kg/m²s]

$$\frac{\partial \boldsymbol{g}_{em}}{\partial t} = div(\overline{\overline{\boldsymbol{T}}}_{em}) - \boldsymbol{f}_{em},$$

- Laplace force [N/m³]

In static regimes: $\boldsymbol{f}_{em} = div(\overline{\overline{\boldsymbol{T}}}_{em})$

Impulse conservation – global form

The fundamental principle of mechanics:

$$\frac{d\boldsymbol{G}_{mec}}{dt} = \boldsymbol{F}_{tot}, \quad \text{where} \quad \boldsymbol{F}_{tot} = \boldsymbol{F}_{\mathrm{non}em} + \boldsymbol{F}_{em}$$

$$\boldsymbol{G}_{mec} = \int_{D_\Sigma} \boldsymbol{g}_{mec}\, dV = \int_{D_\Sigma} \boldsymbol{v}\, dm$$

Mechanic (mass, linear) impulse [mkg/s]

Impulse density [kg/m²s]

In static regimes:

$$\boldsymbol{F}_{em} = \int_{D_\Sigma} \boldsymbol{f}_{em}\, dV = \oint_\Sigma \overline{\overline{\boldsymbol{T}}}_{em} \cdot \boldsymbol{n}\, dS \Leftrightarrow \boldsymbol{f}_{em} = div\, \overline{\overline{\boldsymbol{T}}}_{em}$$

In general:

$$\frac{\partial \boldsymbol{g}_{em}}{\partial t} = div(\overline{\overline{\boldsymbol{T}}}_{em}) - \boldsymbol{f}_{em}, \quad \text{with} \quad \boldsymbol{g}_{em} = \boldsymbol{D} \times \boldsymbol{B} \Rightarrow$$

(linear) EM impulse [mkg/s]

$$\frac{d\boldsymbol{G}_{em}}{dt} = \oint_\Sigma \overline{\overline{\boldsymbol{T}}}_{em} \cdot \boldsymbol{n}\, dS - \boldsymbol{F}_{em} \quad \text{with} \quad \boldsymbol{G}_{em} = \int_{D_\Sigma} \boldsymbol{g}_{em} dV \Rightarrow \frac{d(\boldsymbol{G}_{mec} + \boldsymbol{G}_{em})}{dt} = \boldsymbol{F}_{non-em} + \oint_\Sigma \overline{\overline{\boldsymbol{T}}}_{em} \cdot \boldsymbol{n}\, dS$$

Theorem of angular impulse conservation:

$$\frac{\partial \boldsymbol{h}_{em}}{\partial t} = div(\boldsymbol{r} \times \overline{\overline{\boldsymbol{T}}}_{em}) - \boldsymbol{m}_{em}, \quad \text{with}\ \boldsymbol{h}_{em} = \boldsymbol{r} \times \boldsymbol{g}_{em} = \boldsymbol{r} \times (\boldsymbol{D} \times \boldsymbol{B}); \quad \boldsymbol{m}_{em} = \boldsymbol{r} \times \boldsymbol{f}_{em} \Rightarrow$$

angular EM impulse [m²kg/s] force's angular momentum their densities

$$\frac{d\boldsymbol{H}_{em}}{dt} = \oint_\Sigma \left(\boldsymbol{r} \times \overline{\overline{\boldsymbol{T}}}_{em}\right) \cdot \boldsymbol{n}\, dS - \boldsymbol{M}_{em} \quad \text{with} \quad \boldsymbol{H}_{em} = \int_{D_\Sigma} \boldsymbol{h}_{em} dV; \quad \boldsymbol{M}_{em} = \boldsymbol{r} \times \boldsymbol{F}_{em}$$

Summary of the conservation theorems

Quantity	Charge	Energy	Impulse
Local form in linear static bodies	$div\boldsymbol{J} = -\frac{\partial\rho}{\partial t}$	$-div\boldsymbol{S} = p + \frac{\partial w_{em}}{\partial t}$; $\boldsymbol{S} = \boldsymbol{E}\times\boldsymbol{H}; w_{em} = (\boldsymbol{ED}+\boldsymbol{BH})/2$	$\boldsymbol{f}_{em} = div(\overline{\overline{\boldsymbol{T}}}_{em}); \overline{\overline{\boldsymbol{T}}}_{em} = \boldsymbol{E}\wedge\boldsymbol{D}^T - w_e\overline{\overline{\boldsymbol{I}}} +$ $\boldsymbol{H}\wedge\boldsymbol{B}^T - w_m\overline{\overline{\boldsymbol{I}}}; w_e = \boldsymbol{E}\cdot\boldsymbol{D}/2; w_m = \boldsymbol{H}\cdot\boldsymbol{B}/2$
Global form in linear static bodies	$i_\Sigma = -\frac{dq_{D_\Sigma}}{dt}$	$P_\Sigma = P_c + \frac{dW_{em}}{dt}$	$\boldsymbol{F}_{em} = \oint_\Sigma \overline{\overline{\boldsymbol{T}}}_{em}\cdot\boldsymbol{n}dS$
Local form in non-linear static bodies	$div\boldsymbol{J} = -\frac{\partial\rho}{\partial t}$	–"–; $w_{em} = \int_0^D E(D')dD' + \int_0^B H(B')dB'$	$\boldsymbol{f}_{em} = div(\overline{\overline{\boldsymbol{T}}}_{em}); \overline{\overline{\boldsymbol{T}}}_{em} = \boldsymbol{E}\wedge\boldsymbol{D}^T - w_e\overline{\overline{\boldsymbol{I}}} +$ $\boldsymbol{H}\wedge\boldsymbol{B}^T - w_m\overline{\overline{\boldsymbol{I}}}$;
Global form in non-linear static bodies	$i_\Sigma = -\frac{dq_{D_\Sigma}}{dt}$	$P_\Sigma = P_c + \frac{dW_{em}}{dt}$	$\boldsymbol{F}_{em} = \oint_\Sigma \overline{\overline{\boldsymbol{T}}}_{em}\cdot\boldsymbol{n}dS$
Local form in linear moving bodies	$\nabla(\boldsymbol{J}+\rho\boldsymbol{v}) + \frac{\partial\rho}{\partial t} = 0$	$\boldsymbol{f} = \boldsymbol{f}_e + \boldsymbol{f}_m = \rho_V\boldsymbol{E} - \frac{E^2}{2}(\boldsymbol{grad}\,\varepsilon)$ $+\boldsymbol{J}\times\boldsymbol{B} - \frac{H^2}{2}(\boldsymbol{grad}\,\mu)$	$\frac{\partial\boldsymbol{g}_{em}}{\partial t} = div(\overline{\overline{\boldsymbol{T}}}_{em}) - \boldsymbol{f}_{em}, \boldsymbol{g}_{em} = \boldsymbol{D}\times\boldsymbol{B}$ $\frac{\partial\boldsymbol{r}\times\boldsymbol{g}_{em}}{\partial t} = div(\boldsymbol{r}\times\overline{\overline{\boldsymbol{T}}}_{em}) - \boldsymbol{r}\times\boldsymbol{f}_{em}$,
Global form in linear moving bodies	$\oint_{\Sigma=\partial D_\Sigma}(\boldsymbol{J}+\rho\boldsymbol{v})d\boldsymbol{A} =$ $-\int_{D_\Sigma(t)}\frac{\partial\rho}{\partial t}dv$	$P_\Sigma = P_{mec} + P + \frac{dW}{dt}$; $P_{mec} = \int_D \boldsymbol{f}\cdot\boldsymbol{v}dv$	$\frac{d\boldsymbol{G}_{em}}{dt} = \oint_\Sigma(\overline{\overline{\boldsymbol{T}}}_{em}\cdot\boldsymbol{n})dS - \boldsymbol{F}_{em}, \boldsymbol{G}_{em} = \int_D \boldsymbol{g}_{em}dv$ $\frac{d(\boldsymbol{r}\times\boldsymbol{G}_{em})}{dt} = \oint_\Sigma \boldsymbol{r}\times(\overline{\overline{\boldsymbol{T}}}_{em}\cdot\boldsymbol{n})dS - \boldsymbol{r}\times\boldsymbol{F}_{em}$

EM field thermal effects - summary

- **Conduction**

$$p = \mathbf{E} \cdot \mathbf{J} \quad [\mathrm{W/m^3}]$$

- **Electric hysterezis**

$$w_{he} = \int_0^T \mathbf{E} \cdot d\mathbf{D} \ [\mathrm{J/m^3}] \ (\text{cycle area}) \Rightarrow$$

$$\tilde{p}_{he} = w_{he} / T = f w_{he} \quad [\mathrm{W/m^3}]$$

–

- **Magnetic hysterezis**

$$w_{hm} = \int_0^T \mathbf{H} \cdot d\mathbf{B} \ [\mathrm{J/m^3}] \ (\text{cycle area}) \Rightarrow$$

$$\tilde{p}_{hm} = w_{hm} / T = f \ w_{hm} \quad [\mathrm{W/m^3}]$$

EM field mechanic effects - summary

- **Forces densities [N/m³]:**

$$\mathbf{f}_e \begin{cases} \text{– Coulomb} & \mathbf{f}_C = \rho_V \mathrm{E} \\ \text{– Temp. polariz.} & \mathbf{f}_{tp} = -\frac{E^2}{2}(\mathbf{grad}\,\varepsilon) \\ \text{– Electro-striction} & \mathbf{f}_{es} = \mathbf{grad}\left(\frac{E^2}{2}\tau\frac{\partial\varepsilon}{\partial\tau}\right) + \end{cases}$$

$$\mathbf{f}_m \begin{cases} \text{– Laplace} & \mathbf{f}_L = \mathrm{J}\times\mathrm{B} \\ \text{– Temp. magn.} & \mathbf{f}_{tm} = -\frac{H^2}{2}(\mathbf{grad}\,\mu) \\ \text{– Magneto-striction} & \mathbf{f}_{ms} = \mathbf{grad}\left(\frac{H^2}{2}\tau\frac{\partial\mu}{\partial\tau}\right) \end{cases}$$

Generalized forces:

$$X_{k\,\mathrm{el}} = -\left.\frac{\partial W_e}{\partial x_k}\right|_{q=\mathrm{const.}}, \qquad X_{k\,\mathrm{mg}} = -\left.\frac{\partial W_m}{\partial x_k}\right|_{\Phi=\mathrm{const.}}$$

$$X_{k\,\mathrm{el}} = \left.\frac{\partial W_e^*}{\partial x_k}\right|_{u=\mathrm{const.}}, \qquad X_{k\,\mathrm{mg}} = \left.\frac{\partial W_m^*}{\partial x_k}\right|_{i=\mathrm{const.}}$$

$$\mathbf{f}_{em} = \mathbf{f}_e + \mathbf{f}_m = div(\mathbf{T}), \mathbf{T}_{em} = \mathbf{T}_e + \mathbf{T}_m + \mathbf{T}_{es} + \mathbf{T}_{ms}$$

$$\mathbf{T}_e = \mathbf{E}\wedge\mathbf{D}^T - w_e\bar{\bar{\mathbf{I}}}, \quad \mathbf{T}_m = \mathbf{H}\wedge\mathbf{B}^T - w_m\bar{\bar{\mathbf{I}}}, \quad \mathbf{T}_{ms} = \bar{\bar{\mathbf{I}}}\frac{H^2}{2}\tau\frac{\partial\mu}{\partial\tau}, \quad \mathbf{T}_{es} = \bar{\bar{\mathbf{I}}}\frac{E^2}{2}\tau\frac{\partial\varepsilon}{\partial\tau}$$

4. Theorems related to the problem of EM field analysis

Fundamental problem of EM field theory. Known data:

- Geometry (shape and dimensions) of the computational domain
- Material behavior, value of material constants anywhere
- Internal sources of the EM field: permanent polarization, magnetization, intrinsic current, etc. in any point of computational domain
- External sources of the EM field: boundary conditions
- Anterior sources of the EM field: initial conditions

Unknown data (solution of Maxwell equations):

- EM field: **E,D, B, H, J,** ρ in any point of computational domain, for any time t>0

Correct formulation (theorems related to):

- Solution existence
- Solution uniqueness
- Solution continuity

Other theorems are related to:

- Solving methods
- Solution characteristics

Theorems related to the solving methods

Problem formulations:

- EM field formulation based on first order PDE equations
- PDE – strong formulation for potentials
- Weak (Galerkin - projection) formulation
- Variational (Ritz - minimization) formulation
- Formulation based on integral equations

Solving methods:

- Analytical methods
  - Variables separation, Green function, Coulomb/BSL, Complex, etc.
- Numerical methods
  - Finite Element Method (FEM)
  - Finite Difference Methods (FDM), Finite Integration Technique (FIT)
  - Boundary Element methods (BEM), Integral Equations Method (IEM-MoM)

Theorems related to the solution characteristics

Solution dependence w.r.t space, time, domain size, material constants, field sources.

• **Linearity theorem**

In linear media, any local or global quantity which describes the field solution is linear dependent w.r.t. field sources (internal, external or anterior)

• **Reciprocity theorem** http://en.wikipedia.org/wiki/Reciprocity_(electromagnetism)

In linear reciprocal media (with symmetric material tensors), the relation between field sources and solution is a symmetric one

• **Passivity theorem**

In linear passive media (with positive material constants or tensors), the field energy is positive

• **Affinity (monotony) theorem**

In media with affine (monotonic) characteristics, the relation between field sources and solution is an affine (monotonic) one

• **Stability theorem**

In linear passive, dissipative, reciprocal media (with symmetric, positive defined material constants or tensors), the solution due to bounded excitation has bounded time variation

Uniqueness theorem

Maxwell equations in linear media with μ, ε, σ>0 and permanent sources:

$$\begin{cases} rot\,\boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}\ , & div\,\boldsymbol{D} = \rho_V\ , & \boldsymbol{D} = \varepsilon\,\boldsymbol{E} + \boldsymbol{P}_p & \boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{E}_i) \\ rot\,\boldsymbol{H} = \boldsymbol{J} + \dfrac{\partial \boldsymbol{D}}{\partial t}\ , & div\,\boldsymbol{B} = 0\ , & \boldsymbol{B} = \mu\,\boldsymbol{H} + \mu_0 \boldsymbol{M}_p & \end{cases}$$

have a unique solution in *D* bounded by Σ for any $0<t<T$, if there are known:

- Field internal sources (CS): $\boldsymbol{Pp}(\mathbf{r},t)$; $\boldsymbol{Mp}(\mathbf{r},t)$, $\mathbf{r} \in D$, $t \in [0,T]$;
  $\boldsymbol{Ji}(\boldsymbol{r},t) = \sigma\,\boldsymbol{Ei}(\boldsymbol{r},t)$, $\boldsymbol{r} \in D$, $t \in [0,T]$;
- Boundary conditions on Σ (CΣ): $\boldsymbol{E_t}(\boldsymbol{r},t)$, $\boldsymbol{r} \in S_E$; $\boldsymbol{H_t}(\boldsymbol{r},t)$, $\boldsymbol{r} \in S_H = \Sigma - S_E$, $t \in [0,T]$;
- Initial conditions (C0): $\boldsymbol{E}(\boldsymbol{r},0)$; $\boldsymbol{H}(\boldsymbol{r},0)$, $\boldsymbol{r} \in D$, for $t = 0$.

The prove is based on the lemma of trivial solution:

Maxwell equations with zero CS, CΣ, C0 have only zero solution

$$\int_D \sigma \mathrm{E}\cdot\mathrm{E}\,dV + \frac{\partial}{\partial t}\int_{D_\Sigma}\left(\frac{\mu \mathrm{H}\cdot\mathrm{H}}{2} + \frac{\varepsilon \mathrm{E}\cdot\mathrm{E}}{2}\right)dV = \int_D \mathrm{E}\cdot\mathrm{J}_i\,dV - \int_{D_\Sigma}\left(\mu_0 \mathrm{H}\cdot\frac{\partial \mathrm{M}_p}{\partial t} + \mathrm{E}\cdot\frac{\partial \mathrm{P}_p}{\partial t}\right)dV - \oint_\Sigma (\mathrm{E}\times\mathrm{H})\cdot\mathbf{n}\,dS$$

$$\int_D \sigma \mathrm{E}^2 dV + \frac{1}{2}\frac{\partial}{\partial t}\int_{D_\Sigma}\left(\mu \mathrm{H}^2 + \varepsilon \mathrm{E}^2\right)dV = 0 \Rightarrow 0 \le \int_{D_\Sigma}\left(\mu \mathrm{H}^2 + \varepsilon \mathrm{E}^2\right)dV = -2\int_0^t\int_D \sigma \mathrm{E}^2 dV \le 0 \Rightarrow \mathrm{H} = 0, \mathrm{E} = 0 \Rightarrow$$

$$\mathrm{D} = 0, \mathrm{B} = 0, \mathrm{J} = 0, \rho = 0$$

Scalar and vector potentials theorem

According to Helmholtz-Hodge theorem, any vector field **G** can be decomposed in

$$\begin{cases} \boldsymbol{G} = -gradV + rot\boldsymbol{A} + \boldsymbol{h}; \Rightarrow div\boldsymbol{G} = -\Delta V = f; \\ rot\boldsymbol{G} = rotrot\boldsymbol{A} = graddiv\boldsymbol{A} + \Delta\boldsymbol{A} = \boldsymbol{g} \Rightarrow rot\boldsymbol{G} = \Delta\boldsymbol{A} = \boldsymbol{g}; \end{cases}$$

a curl-free, a div-free and a harmonic filed (curl and div free). It is unique, if the boundary conditions (V=0, **A** =0) are added. The third component is necessary only in the bounded domains. *So any field G has two potentials: scalar V and vector A, generated by its div and its rot, respectively.* The harmonic component, which can be expressed using an additional scalar or vector potential, describes the external sources (boundary conditions). If **G** is div-free (as **B** is), it has only a vector potential, if it is curl-free (as **E** is in static regimes), it has only a scalar potential. *Potentials may be computed from the field sources, by solving a Poisson equation.*

$$div\boldsymbol{G} = \nabla(-\nabla V + \nabla\times\boldsymbol{A} + \boldsymbol{h}) = f \Rightarrow \Delta V = -f \Rightarrow V(\boldsymbol{x}) = \int_\Omega G(\boldsymbol{x},\boldsymbol{y}) f(\boldsymbol{y}) d\boldsymbol{x} + \int_{\partial\Omega} \frac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial n} V(\boldsymbol{y}) dS,$$

$$curl\boldsymbol{G} = \nabla\times(\nabla V + \nabla\times\boldsymbol{A}) = \boldsymbol{g} \Rightarrow \Delta\boldsymbol{A} = \boldsymbol{g} \Rightarrow \quad \boldsymbol{A}(\boldsymbol{x}) = \int_\Omega G(\boldsymbol{x},\boldsymbol{y}) \boldsymbol{g}(\boldsymbol{y}) d\boldsymbol{x} + \int_{\partial\Omega} \frac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial n} \boldsymbol{A}(\boldsymbol{y}) dS$$

In the whole space R^3, the Green function G(x,y) is

$$\Delta G = \delta(\boldsymbol{x}-\boldsymbol{y}) \Rightarrow G(\boldsymbol{x},\boldsymbol{y}) = \frac{-1}{4\pi\|\boldsymbol{x}-\boldsymbol{y}\|} \Rightarrow V(\boldsymbol{x}) = \int_{IR^3} G(\boldsymbol{x},\boldsymbol{y}) f(\boldsymbol{y}) d\boldsymbol{x}; \;\; \boldsymbol{A}(\boldsymbol{x}) = \int_{IR^3} G(\boldsymbol{x},\boldsymbol{y}) \boldsymbol{g}(\boldsymbol{y}) d\boldsymbol{x}$$

Electro-dynamic potentials theorem

The time varying electromagnetic field (**E, D, B, H**) can be described by two "electro-dynamic" potentials: scalar $V(\mathbf{r},t)$ and vector $\mathbf{A}(\mathbf{r},t)$.

$$div\boldsymbol{B} = 0 \Rightarrow \boxed{\boldsymbol{B} = rot\boldsymbol{A};} \quad rot\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \Rightarrow rot\left(\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t}\right) \Rightarrow \boxed{\boldsymbol{E} = -gradV - \frac{\partial \boldsymbol{A}}{\partial t};}$$

$$\boldsymbol{D} = \varepsilon\,\boldsymbol{E}; \quad \boldsymbol{H} = \boldsymbol{B}/\mu = \nu\boldsymbol{B}; \quad \boldsymbol{J} = \sigma\boldsymbol{E} + \boldsymbol{J}_i$$

They are solutions of hyperbolic (d'Alambert type) PDEs:

$$rot\boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \Rightarrow rot(\nu rot\boldsymbol{A}) + \sigma\frac{\partial \boldsymbol{A}}{\partial t} + \varepsilon\frac{\partial^2 \boldsymbol{A}}{\partial t^2} + \sigma gradV + \varepsilon grad\frac{\partial V}{\partial t} = \boldsymbol{J}_i$$

$$div\boldsymbol{J} = -\frac{\partial \rho}{\partial t} \Rightarrow div\left(\sigma gradV + \sigma\frac{\partial \boldsymbol{A}}{\partial t}\right) = \frac{\partial \rho}{\partial t} \Rightarrow div(\sigma gradV) = \frac{\partial \rho}{\partial t}, \text{ if } div(\sigma\boldsymbol{A}) = 0$$

In vacuum, with Lorenz gauge conditions (for div A)

$$rot(rot\boldsymbol{A}) + \mu_0\varepsilon_0\frac{\partial^2 \boldsymbol{A}}{\partial t^2} + \mu_0\varepsilon_0 grad\frac{\partial V}{\partial t} = \mu_0\boldsymbol{J}_i \Rightarrow \boxed{-\Delta\boldsymbol{A} + \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = \mu_0\boldsymbol{J}_i; c = 1/\sqrt{\mu_0\varepsilon_0}}$$

$$\boxed{div\boldsymbol{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t}}; div\boldsymbol{D} = \rho \Rightarrow \Delta V + div\frac{\partial \boldsymbol{A}}{\partial t} = -\rho/\varepsilon_0 \Rightarrow \boxed{-\Delta V + \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = \rho/\varepsilon_0}$$

Retarded potentials theorem

The solutions of d' Alambert equations of ED potentials in vacuum are

$$-\Delta \boldsymbol{A}+\frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2}=\mu_0 \boldsymbol{J}_i \Rightarrow \boldsymbol{A}(\boldsymbol{r},t)=\frac{\mu_0}{4\pi}\int_{IR^3}\frac{\boldsymbol{J}_i(\boldsymbol{r}',t_r)}{|\boldsymbol{r}-\boldsymbol{r}'|}dv';t_r=t-|\boldsymbol{r}-\boldsymbol{r}'|/c$$

$$-\Delta V+\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}=\rho/\varepsilon_0 \Rightarrow V(\boldsymbol{r},t)=\frac{1}{4\pi\varepsilon_0}\int_{IR^3}\frac{\rho(\boldsymbol{r}',t_r)}{|\boldsymbol{r}-\boldsymbol{r}'|}dv' \qquad G(\boldsymbol{x},\boldsymbol{y},t)=\frac{-\delta(t-\|\boldsymbol{x}-\boldsymbol{y}\|/c)}{4\pi\|\boldsymbol{x}-\boldsymbol{y}\|}$$

Here G is the Green function of d' Alambert operator (zero outside past light cone)

In the static case (elliptic, Poisson type) PDEs (Lorentz ➔ Coulomb gauge: $div\boldsymbol{A}=0$

$$-\Delta \boldsymbol{A}=\mu_0 \boldsymbol{J}_i \Rightarrow \boldsymbol{A}(\boldsymbol{r})=\frac{\mu_0}{4\pi}\int_{IR^3}\frac{\boldsymbol{J}_i(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}dv'; \quad -\Delta V=\rho/\varepsilon_0 \Rightarrow V(\boldsymbol{r})=\frac{1}{4\pi\varepsilon_0}\int_{IR^3}\frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}dv'$$

By using Coulomb gauge in time dependent fields:

$$-\Delta \boldsymbol{A}+\frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2}+\frac{1}{c^2}\nabla\left(\frac{\partial V}{\partial t}\right)=\mu_0 \boldsymbol{J}_i \Rightarrow \boldsymbol{A}(\boldsymbol{r},t)=\frac{1}{4\pi}\int_{IR^3}\frac{\nabla\times\boldsymbol{B}(\boldsymbol{r}',t)}{|\boldsymbol{r}-\boldsymbol{r}'|}dv';$$

$$-\Delta V=\rho/\varepsilon_0 \Rightarrow V(\boldsymbol{r},t)=\frac{1}{4\pi\varepsilon_0}\int_{IR^3}\frac{\rho(\boldsymbol{r}',t)}{|\boldsymbol{r}-\boldsymbol{r}'|}dv'=\frac{1}{4\pi}\int_{IR^3}\frac{\nabla\boldsymbol{E}(\boldsymbol{r}',t)}{|\boldsymbol{r}-\boldsymbol{r}'|}dv$$

Theorem of fields superposition

Fundamental problem of the EM field analysis: find field solution: F = [E, D, B, H, J, ρ] of Maxwell equations in a domain *D* with boundary Σ= *SE + SH,* knowing (internal, external and past) field sources: C = [CS, CS, C0]

In linear media with μ, ε, σ>0 this causal relation is a linear operator.

$$S\left(\sum_{k=1}^{n} \lambda_k \mathbf{C}_k\right) = \sum_{k=1}^{n} \lambda_k S(\mathbf{C}_k) \Rightarrow$$

S: C → F is a well defined operator in any correct formulated field problem.

It is well defined due to the field uniqueness:

– *Any two solutions F1 and F2 of the same problem must be equal, because F=F1-F2 satisfy the same equations with zero sources (C=0), hence F=0 → F1=F2.*

$$\mathbf{C}_k \to \mathbf{F}_k \Rightarrow \sum_{k=1}^{n} \lambda_k \mathbf{C}_k \to \sum_{k=1}^{n} \lambda_k \mathbf{F}_k$$

$$\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2 \to \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$$

$$\lambda\mathbf{C} \to \lambda\mathbf{F}$$

$$S(0) = 0$$

Sum of causes give sum of effects. But only in linear problems.

- *J and ρ are here field solutions not sources !*

- *The superposition may be applied only to sources. Domains superposition is mistaken. As well as boundary partitions S= SE + SH.*

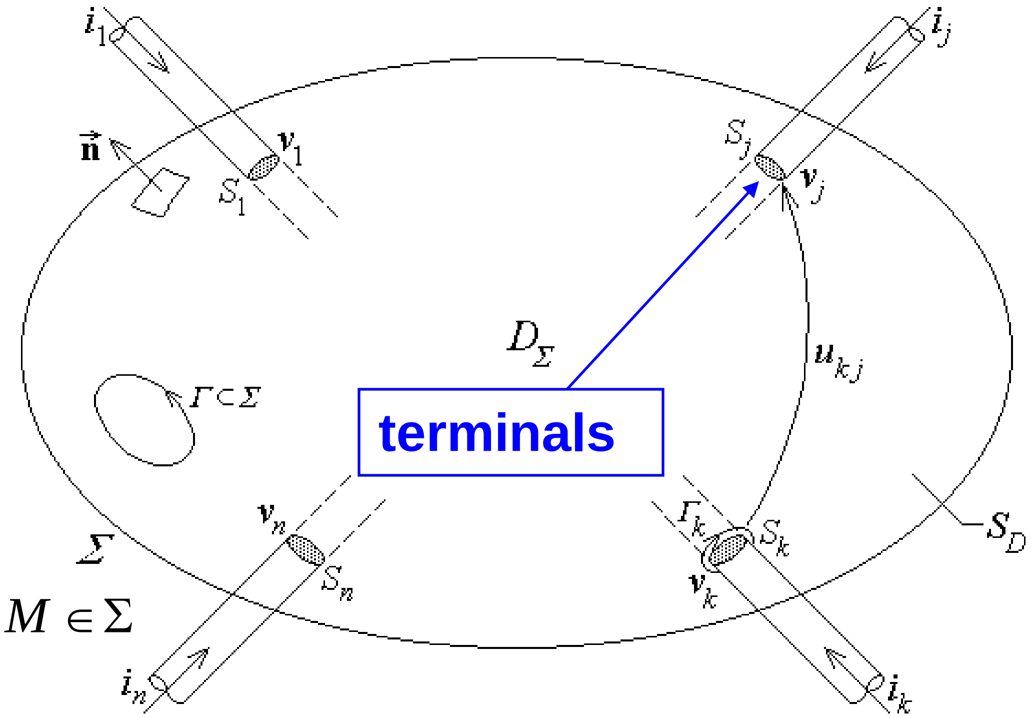
5. Circuit element with multiple terminals

It is defined as a simply connected domain with terminals and b. conditions:

A: no magnetic coupling

B: electric coupling only through terminals

C: eqi-potential terminals



A: $$\boldsymbol{n}\cdot\frac{\partial \boldsymbol{B}(M,t)}{\partial t}=0 \Leftrightarrow \boldsymbol{n}\cdot\nabla\times\boldsymbol{E}=0;\ M\in\Sigma$$

B: $$\boldsymbol{n}\cdot\nabla\times\boldsymbol{H}=0 \quad , \quad M\in S_D=\Sigma\setminus\bigcup_{k=1}^{k=n} S_k$$

$$i_k=-\int_{S_k}\left(\boldsymbol{J}+\frac{\partial \boldsymbol{D}}{\partial t}\right)\cdot\boldsymbol{n}dS$$

$$u_{kj}(t)=\int_{C_{kj}\in\Sigma}\boldsymbol{E}\cdot d\boldsymbol{r}$$

C: $$\boldsymbol{n}\times\mathrm{E}(M,t)=0 \quad , \quad M\in S_k \quad , \quad k=1,2,\ldots,n$$

Circuit's fundamental theorems

On the boundary surface:

- **total current conservation**

$$\oint_{\Sigma}\left(\mathrm{J}+\frac{\partial \mathrm{D}}{\partial t}\right)\cdot \mathbf{n}\,dS=\oint_{\Sigma}(\mathbf{rot}\,\mathrm{H})\cdot \mathbf{n}\,dS=\int_{D_{\Sigma}}[\mathrm{div}(\mathbf{rot}\,\mathrm{H})]\cdot \mathbf{n}\,dS=0$$

- **zero e.m.f. (A: ➔)**

$$\oint_{\Gamma\subset\Sigma}\mathrm{E}\cdot d\mathbf{r}=\int_{S_{\Gamma}}(\mathbf{rot}\,\mathrm{E})\cdot \mathbf{n}\,dS=-\int_{S_{\Gamma}}\frac{\partial \mathbf{B}}{\partial t}\cdot \mathbf{n}\,dS=0$$

Global characteristic quantities:

- **Terminal current:**

$$i_k=_{def}\oint_{\Gamma_k}\mathrm{H}\cdot d\mathbf{r}=-\int_{S_k}(\mathbf{rot}\,\mathrm{H})\cdot \mathbf{n}dS=-\int_{S_k}\left(\mathbf{J}+\frac{\partial \mathbf{D}}{\partial t}\right)\cdot \mathbf{n}dS$$

- **Terminal voltage:**

$$u_{kj}(t)=_{def}\int_{C_{kj}\in\Sigma}\mathrm{E}\cdot d\mathbf{r}=\int_{C_{kj}\in\Sigma}\mathrm{E}_t\cdot d\mathbf{r}=v_k(t)-v_j(t)$$

C: ➔

$$\int_{MN\subset S_k}\mathrm{E}\cdot d\mathbf{r}=\int_{MN\subset S_k}\mathrm{E}_t\cdot d\mathbf{r}=v(M,t)-v(N,t)=0$$

Kirchhoff Laws:

KCL (B:➔)

$$0=\int_{S_D}\left(\mathrm{J}+\frac{\partial \mathrm{D}}{\partial t}\right)\cdot \mathbf{n}\,dS=\int_{S_D}(\mathbf{rot}\,\mathrm{H})\cdot \mathbf{n}\,dS+\sum_{k=1}^{n}\int_{S_k}(\mathbf{rot}\,\mathrm{H})\cdot \mathbf{n}dS=0+\sum_{k=1}^{n}(-i_k)\Rightarrow \boxed{\sum_{b\in\Sigma} i_b=0}$$

KVL (A:➔)

$$\oint_{\Gamma\subset\Sigma}\mathrm{E}\cdot d\mathbf{r}=0\Rightarrow \boxed{\sum_{b\in\Gamma} u_b=0}$$

Theorem of electric power transferred by a multi-polar element

$$\int_{C_{AB}\subset\Sigma} \boldsymbol{E}\cdot d\boldsymbol{r} = \int_{C_{AB}\subset\Sigma} \boldsymbol{E}_t \cdot d\boldsymbol{r} = v(A,t) - v(B,t) \quad \text{independent } of \ \mathrm{C}_{AB} \subset \Sigma \Rightarrow$$

$$(\exists) v : \Sigma \to \mathbf{R},\ \text{s.t.}\ \mathbf{E}_t = -\,\mathbf{grad}\, v \qquad \mathbf{rot}\,(v\mathbf{H}) = (\mathbf{grad}\, v)\times\mathbf{H} + v(\mathbf{rot}\,\mathbf{H})$$

$$P_\Sigma = \oint_\Sigma (\mathbf{E}\times\mathbf{H})\cdot(-\mathbf{n})\,dS = -\oint_\Sigma [(-\mathbf{grad}\, v)\times\mathbf{H}]\cdot(-\mathbf{n})\,dS =$$

$$- \quad \oint_\Sigma [\mathbf{rot}\,(v\mathbf{H})]\cdot\mathbf{n}\,dS - \oint_\Sigma [v(\mathbf{rot}\,\mathbf{H})]\cdot\mathbf{n}\,dS =$$

$$0 + \sum_{k=1}^{n}\int_{S_k} [v(\mathbf{rot}\,\mathbf{H})]\cdot\mathbf{n}\,dS = \sum_{k=1}^{n} v_k \int_{S_k} (\mathbf{rot}\,\mathbf{H})\cdot\mathbf{n}\,dS =$$

$$\sum_{k=1}^{n} v_k \int_{S_k} (\mathbf{rot}\,\mathbf{H})\cdot\mathbf{n}\,dS = \sum_{k=1}^{n} v_k i_k \quad \Rightarrow \quad \boxed{P_\Sigma = \sum_{k=1}^{k=n} v_k\, i_k}$$

P has the conventional sense of i

The constitutive relation of the multi-polar circuit element

The case of voltage excitation

Excitations (input signals): $\int_{C_{kn}\in\Sigma} \boldsymbol{E}_t \cdot d\boldsymbol{r} = v_k(t)$ Known for $\mathrm{k} = 1,2, \ldots, \mathrm{n}\text{-}1$

Responses (output signals): $i_k = \oint_{\Gamma_k} \boldsymbol{H} \cdot d\boldsymbol{r}$ Computed from the field solution, for $\mathrm{k} = 1,2, \ldots, \mathrm{n}$

Let consider D a linear domain without permanent sources (D=εE, B=μH, J=σE) with zero initial field and boundary conditions given by A, B, C and excitations.

The fundamental problem may be simplified: input signals: **v** = [v1, v2, .., vn-1], response – output signals: **i** = [i1, i2, …, in-1].

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The input-output relation **i = Y v** is described by the admittance **Y**. It is a linear, well defined operator due to the solution uniqueness and superposition. These theorems are based on the lemma of the trivial solution for a circuit element: zero excitations produce zero responses. **v** = 0 ➔ **i** = 0:

$$\int_D \sigma \mathrm{E}^2 dV + \frac{1}{2}\frac{\partial}{\partial t}\int_{D_\Sigma} \left(\mu \mathrm{H}^2 + \varepsilon \mathrm{E}^2\right) dV = \oint_\Sigma (\mathrm{E}\times\mathrm{H})\cdot(-\mathbf{n})dS = \sum_{k=1}^{n} v_k i_k 0 \Rightarrow$$

$$0 \le \int_{D_\Sigma} \left(\mu \mathrm{H}^2 + \varepsilon \mathrm{E}^2\right) dV = -2\int_0^t \int_D \sigma \mathrm{E}^2 dV \le 0 \Rightarrow \mathrm{H} = 0 \Rightarrow i_k = 0$$

The dual case of current excitation: **v =Z i**

6. Theorems related to EM modeling and design

Fundamental problem of EM modeling. Known data:

- Geometry (shape and dimensions) of the computational domain
- Material behavior, value of material constants anywhere
- Type of system excitations: boundary conditions and initial conditions

Unknown data:

- Relationship between system response and it excitation

– Solution is represented as a LTI dynamic system (electric circuit), which approximate I/O relationship. The (complexity) order of this system should be as low as possible, keeping an acceptable accuracy and saving other characteristics (e.g. passivity and stability).

Model reduction of EM devices includes extraction of lumped circuit parameters (R, L, C) and equivalent schemes for elements with distributed parameters.

Other theorems are related to:

Optimization and inverse problems in Electromagnetics. Where the problem has as unknown data: geometric or material characteristics of studied EM devices.

Not so easy questions for curious people

1. How looks like the local form of the charge conservation theorem on a surface carrying superficial current J_s ?
2. What form has the charge conservation theorem in several field regimes?
3. Is the Ampere law (in its original form, without displacement current) consistent with the conservation of charge? Is the expression of displacement current a consequence of the charge conservation ?
4. How are the current lines in steady state ?
5. Is the charge destroyed by an atomic explosion?
6. Who is more general, mass or charge conservation ?
7. How is related Kirchhoff law to the charge conservation theorem?
8. In charge relaxation, the initial charge decays to zero. Where is it disappears?
9. Does the charge conservation implies Maxwell's equations ?
10. How looks like the theorem of EM energy conservation in nonlinear media ?
11. How looks like the theorem of EM energy conservation in moving media ?