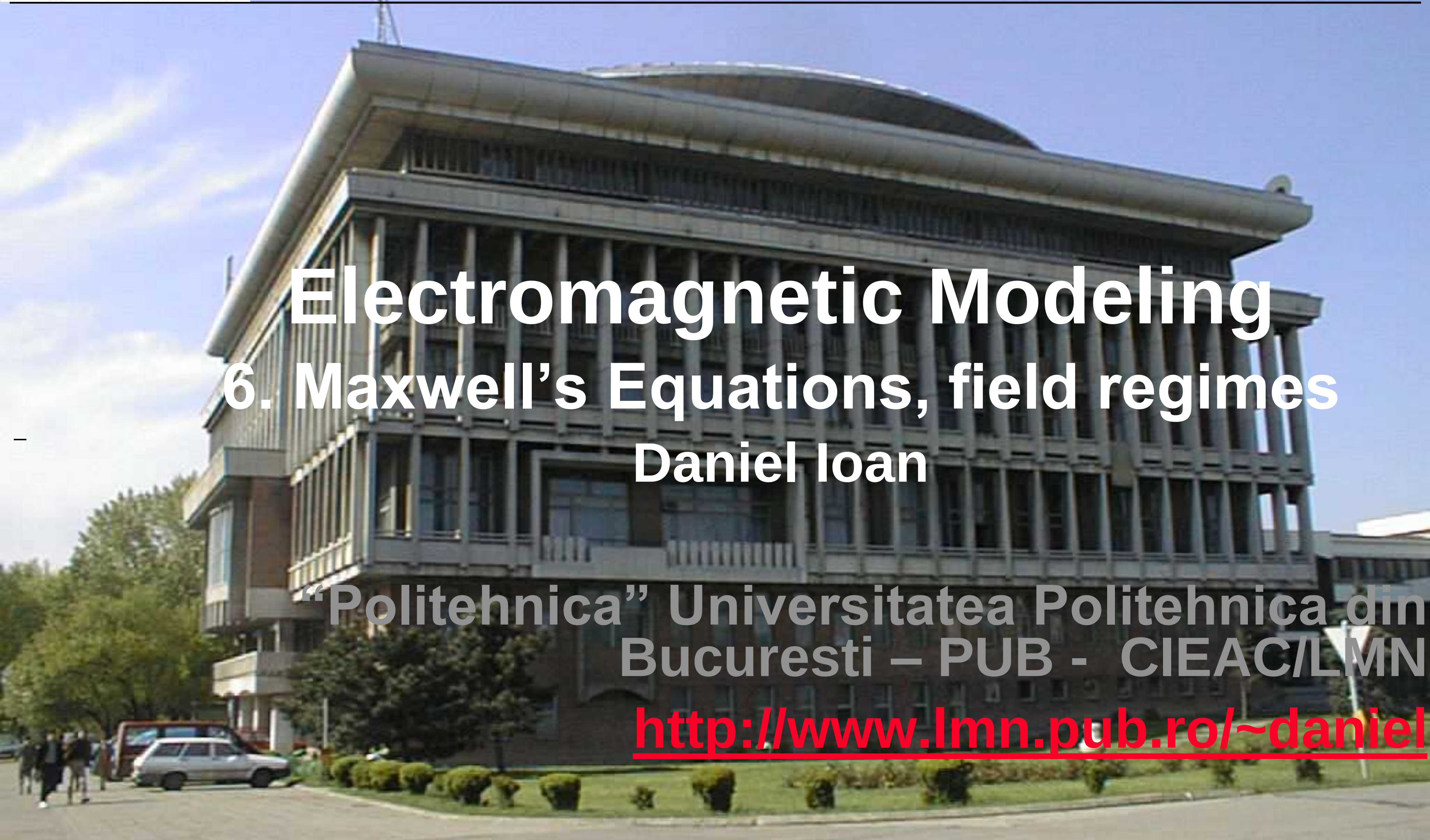


Electromagnetic Modeling

6. Maxwell's Equations, field regimes

Daniel Ioan

"Politehnica" Universitatea Politehnica din Bucuresti – PUB - CIEAC/LMN

http://www.lmn.pub.ro/~daniel

Maxwell's equations

1. They are the differential form of the EM field laws in non-moving media:

$$\nabla \cdot \mathbf{D} = \rho$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

2. In addition, to obtain a complete system of equations, the constitutive relations should be added

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}(\mathbf{E}) = \varepsilon \mathbf{E} \text{ in linear dielectrics}$$
$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}(\mathbf{H})) = \mu \mathbf{H} \text{ in linear media}$$
$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{E}_i(\mathbf{E})) = \sigma \mathbf{E} \text{ in linear conductors}$$

3. In vacuum, the EM field is described only by E and H:

$$\nabla \cdot \mathbf{D} = 0$$
$$\nabla \cdot \mathbf{H} = 0$$
$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$
$$\nabla \times \mathbf{H} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

They are compatible with the Einstein's theory of relativity, being invariant in inertial moving reference systems (invariant to the Lorentz transform)

Maxwell-Hertz equations

1. They are the differential form of the EM field laws in moving media (in the moving "ether" hypothesis):

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\dfrac{d_f \mathbf{B}}{dt} \\ \nabla \times \mathbf{H} = \mathbf{J} + \dfrac{d_f \mathbf{D}}{dt} \end{cases}$$

3. Expressing the flux derivatives according to their definition:

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{B} \times \mathbf{v}) \\ \nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t} + \rho \mathbf{v} + \nabla \times (\mathbf{D} \times \mathbf{v}) \end{cases}$$

4. They are the best approximation of Einstein's theory of relativity – electro-dynamics for low speeds v << c (invariant to the Galilean transform)

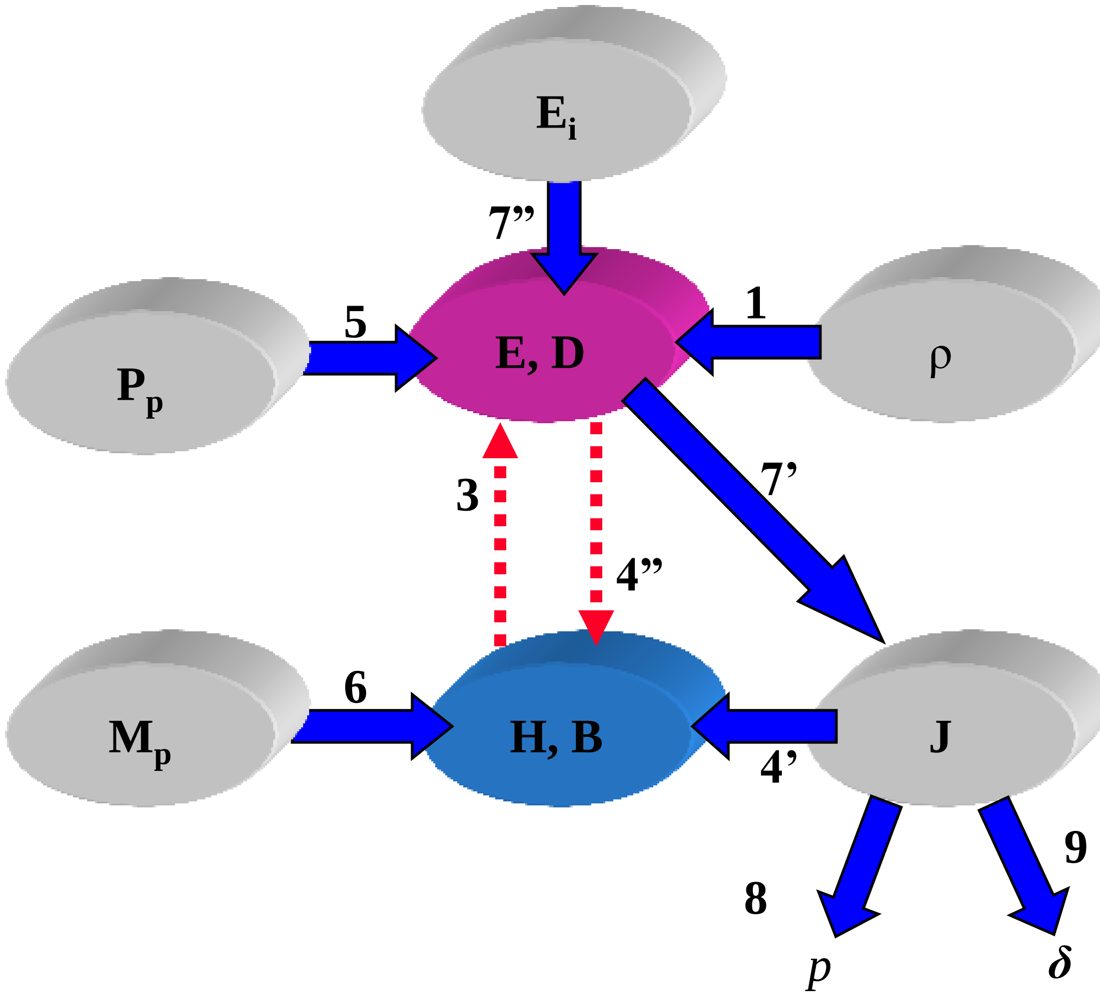
5. Constitutive relations links field quantities defined in the local coordinate system:

$$\begin{cases} \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}(\mathbf{E}) = \varepsilon \mathbf{E} \text{ in linear dielectrics} \\ \mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}(\mathbf{H})) = \mu \mathbf{H} \text{ in linear media} \\ \mathbf{J} = \sigma (\mathbf{E} + \mathbf{E}_i (\mathbf{E})) = \sigma \mathbf{E} \text{ in linear conductors} \end{cases}$$

6. In the 1880s Hertz obtained the experimental evidence of electromagnetic waves. Their existence had been predicted in 1873 by James Clerk Maxwell, on mathematical way

Diagram of fundamental EM phenomena (causal relations)

1. $\nabla \cdot \boldsymbol{D} = \rho$
2. $\nabla \cdot \boldsymbol{B} = 0$
3. $\nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}$
4. $\nabla \times \boldsymbol{H} = \boldsymbol{J} + \dfrac{\partial \boldsymbol{D}}{\partial t}$
5. $\boldsymbol{D} = \varepsilon\, \boldsymbol{E} + \boldsymbol{P}_p(\boldsymbol{E})$
6. $\boldsymbol{B} = \mu\, (\boldsymbol{H} + \boldsymbol{M}_p(\boldsymbol{H}))$
7. $\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{E}_i(\boldsymbol{E}))$
8. $p = \boldsymbol{J} \cdot \boldsymbol{E}$
9. $\boldsymbol{\delta} = k\boldsymbol{J}$



EM field regimes

1. **Static - ST**
   - ES – Electro-Static
   - MS – Magneto-Static
2. **Steady state - SS**
   - EC - Electro-Conductive
   - MG – Magneto- Steady-State
3. **Quasi-static - QS**
   - EQS – Electro-Quasi-Static
   - MQS – Magneto-Quasi-Static
   - EMQS – Electro-Magneto-Quasi-Static
4. **Electro-dynamic - ED**
   - FW – Full-wave, Non-moving
   - LL – Full wave loss-less
5. **QS,ED in Frequency domain**
   - Harmonic time variation
   - Fourier/Laplace transform
6. **General**
   - Moving bodies with known speed
   - Coupled Mechanic-EM field

Diagram of fundamental steady state EM fields

$$1.\ \nabla\cdot\mathbf{D}=\rho$$

$$2.\ \nabla\cdot\mathbf{B}=0$$

$$3.\ \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$$

$$4.\ \nabla\times\mathbf{H}=\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}$$

$$5.\ \mathbf{D}=\varepsilon\,\mathbf{E}+\mathbf{P}_p(\mathbf{E})$$

$$6.\ \mathbf{B}=\mu\,(\mathbf{H}+\mathbf{M}_p(\mathbf{H}))$$

$$7.\ \mathbf{J}=\sigma(\mathbf{E}+\mathbf{E}_i(\mathbf{E}))$$

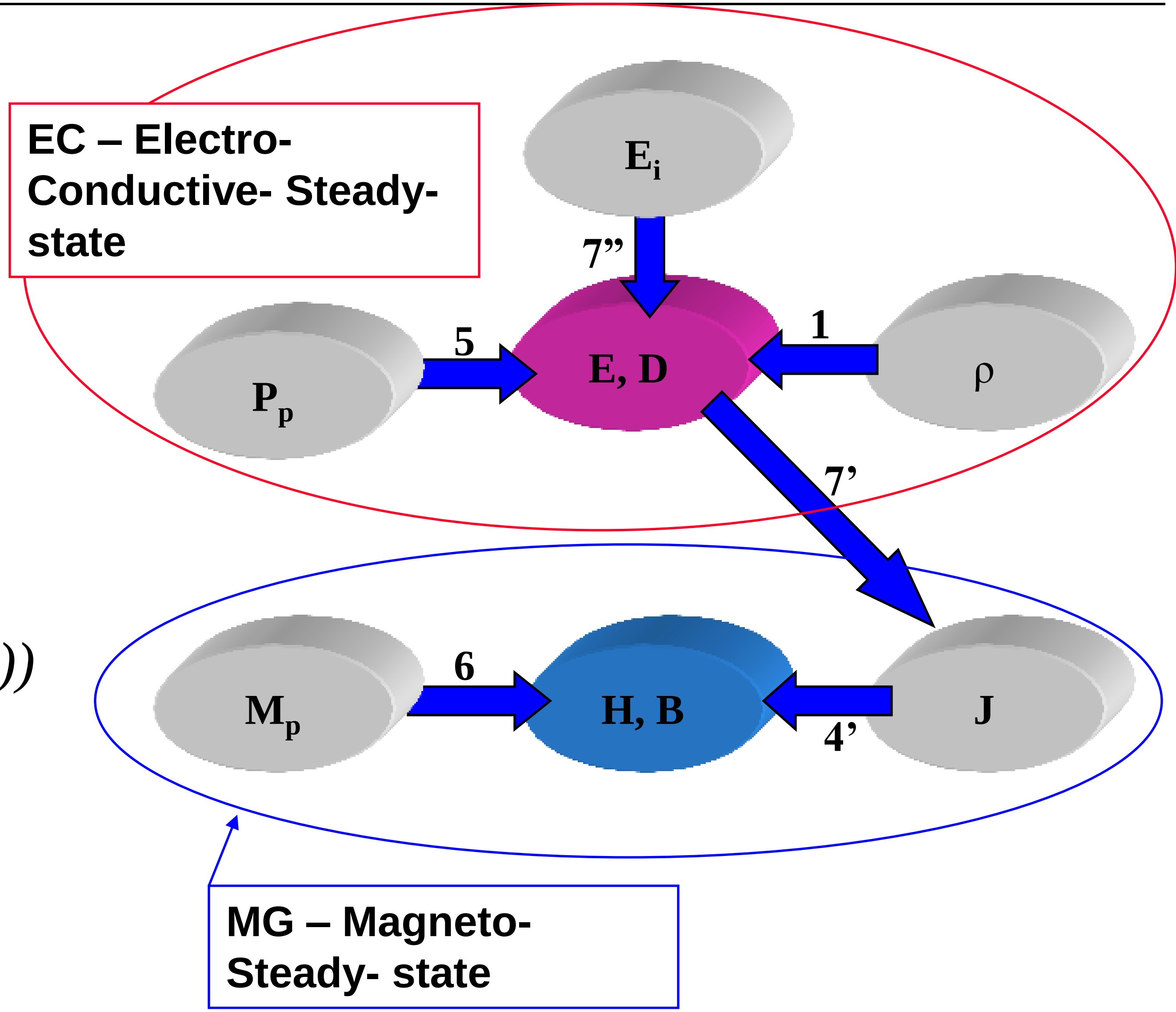
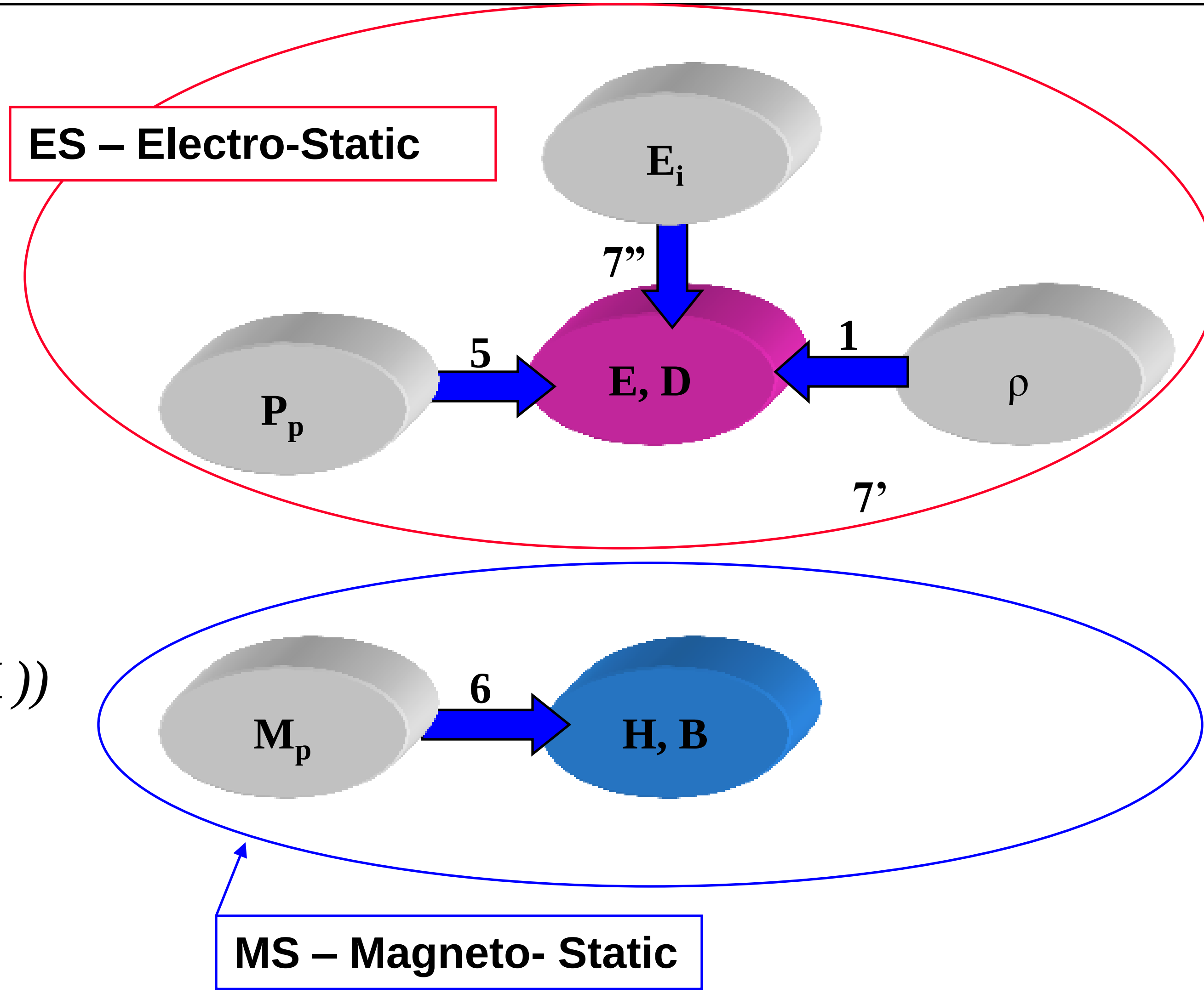


Diagram of fundamental static EM fields

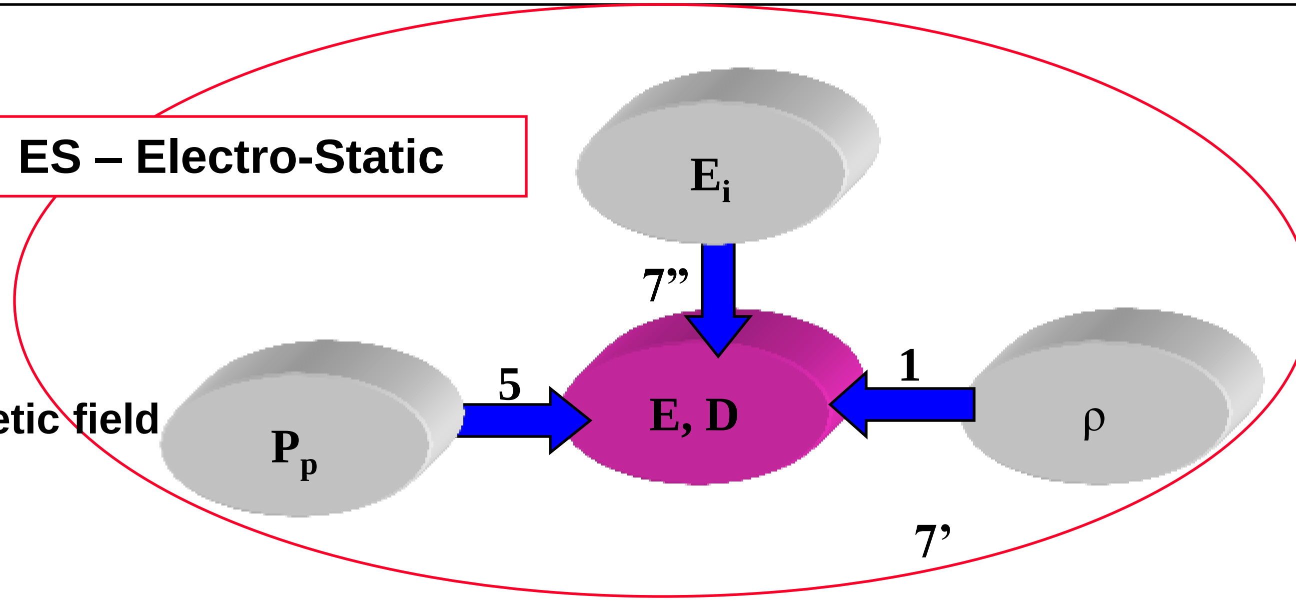
1. $\nabla \cdot \mathbf{D} = \rho$
2. $\nabla \cdot \mathbf{B} = 0$
3. $\nabla \times \mathbf{E} = \cancel{-\frac{\partial \mathbf{B}}{\partial t}}$
4. $\nabla \times \mathbf{H} = \cancel{\mathbf{J}} + \cancel{\frac{\partial \mathbf{D}}{\partial t}}$
5. $\mathbf{D} = \varepsilon \, \mathbf{E} + \mathbf{P}_p(\mathbf{E})$
6. $\mathbf{B} = \mu \, (\mathbf{H} + \mathbf{M}_p(\mathbf{H}))$
7. ~~$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{E}_i(\mathbf{E}))$~~
8. ~~$p = \mathbf{E} \, \mathbf{J}$~~



Electro-Static field

ES hypothesis:

1. **No movement**
2. **No time variation**
3. **No energy transfer**
4. **No interest for magnetic field**



ES fundamental equations:

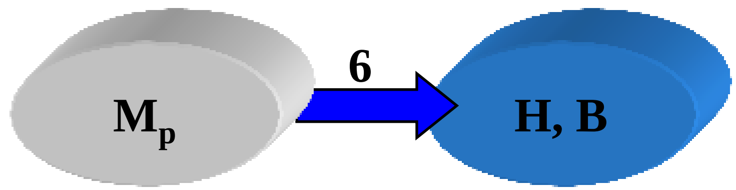
- Gauss theorem: $\nabla \cdot \mathbf{D} = \rho$
- Potential theorem: $\nabla \times \mathbf{E} = 0$
- Polarization theorem: $\mathbf{D} = \varepsilon\, \mathbf{E} + \mathbf{P}_p(\mathbf{E})$
- Electrostatic equilibrium condition in conductors: $\sigma(\mathbf{E} + \mathbf{E}_i(\mathbf{E})) = 0$

Sources: ρ, **P_p, E_i**
Fields: D, (curl-free) E
Potential: *V*
Material constants - only one: ε
PDE of elliptic type for potential. ES fields are instantaneous distributed in space

Magneto-Static field

MS hypothesis:

1. **No movement**
2. **No time variation**
3. **No energy transfer**
4. **No interest for electric field, only for magnetic field produced by permanent magnets**



Sources: M_p
Fields: (curl-free) H, (div-free) B
Potential: V
Material constants - only one: μ
PDE of elliptic type for potential.
Field is instantaneous distributed in space

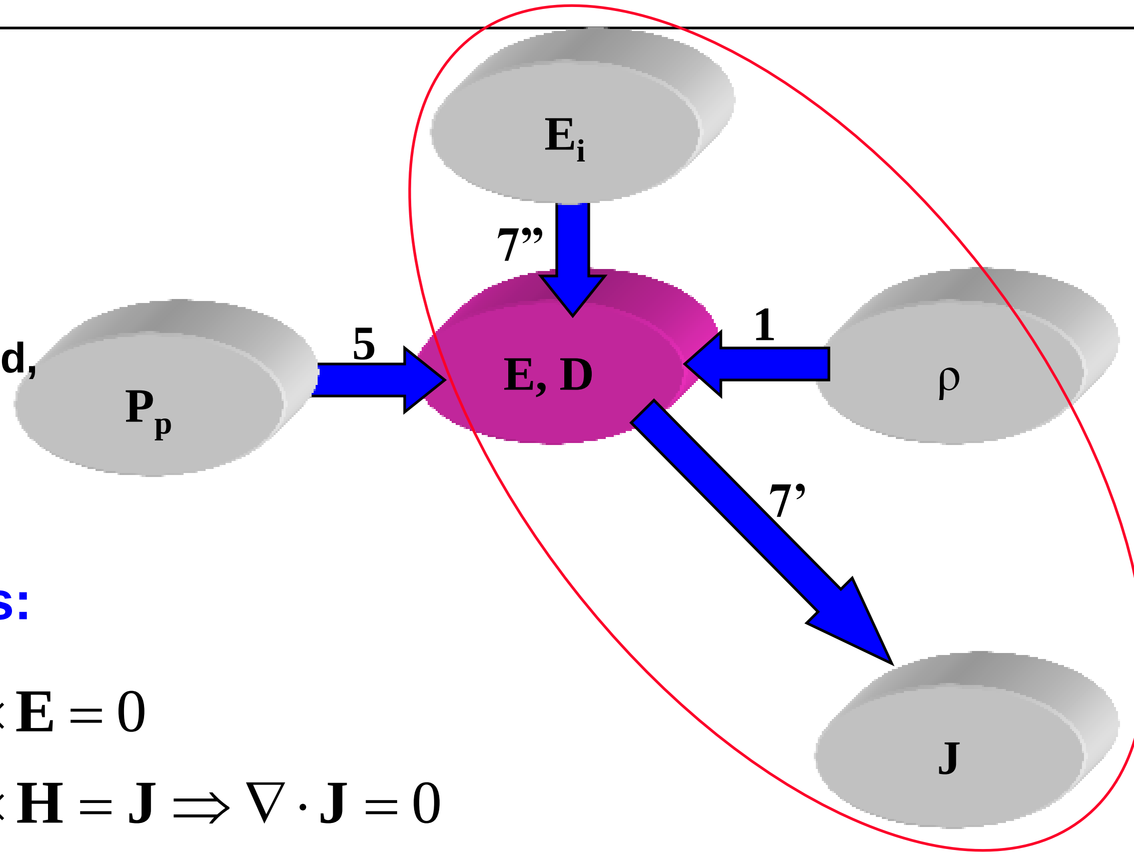
MS fundamental equations:

- Gauss theorem: $\nabla \cdot \mathbf{B} = 0$
- Potential theorem: $\nabla \times \mathbf{H} = 0$
- Magnetization theorem: $\mathbf{B} = \mu \left(\mathbf{H} + \mathbf{M}_p(\mathbf{H}) \right)$

Electro-Conductive regime

EC hypothesis:

1. **No movement**
2. **No time variation**
3. **No interest for magnetic field, only for d.c. current distribution in massive conductors**



EC fundamental equations:

- Theorem of the current conservation:
- Potential theorem:
- Ohm’s law:

$$\nabla \times \mathbf{E} = 0$$

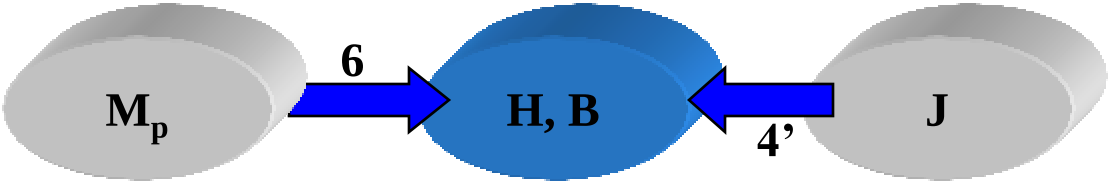
$$\nabla \times \mathbf{H} = \mathbf{J} \Rightarrow \nabla \cdot \mathbf{J} = 0$$

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{E}_i(\mathbf{E}))$$

Sources: $\mathbf{E_i}$
Fields: **(curl-free) E, (div-free) J**
Potential: *V*

Material constants - only one: σ
PDE of elliptic type for potential.
Field is instantaneous distributed in space

Magneto-Steady-State regime



MG hypothesis:

1. **No movement**
2. **No time variation**
3. **Current distribution is known from a previous EC analysis**

MG fundamental equations:

- Gauss theorem: $\nabla \cdot \mathbf{B} = 0$
- Ampere theorem: $\nabla \times \mathbf{H} = \mathbf{J}$
- Magnetization theorem $\mathbf{B} = \mu\,(\mathbf{H} + \mathbf{M}_p(\mathbf{H}))$

Sources: J
Fields: H, (div-free) B
Potential: *A*

Material constants - only one: μ
PDE of elliptic type for potential.
Field is instantaneous distributed in space

Electro Quasi-Static regime

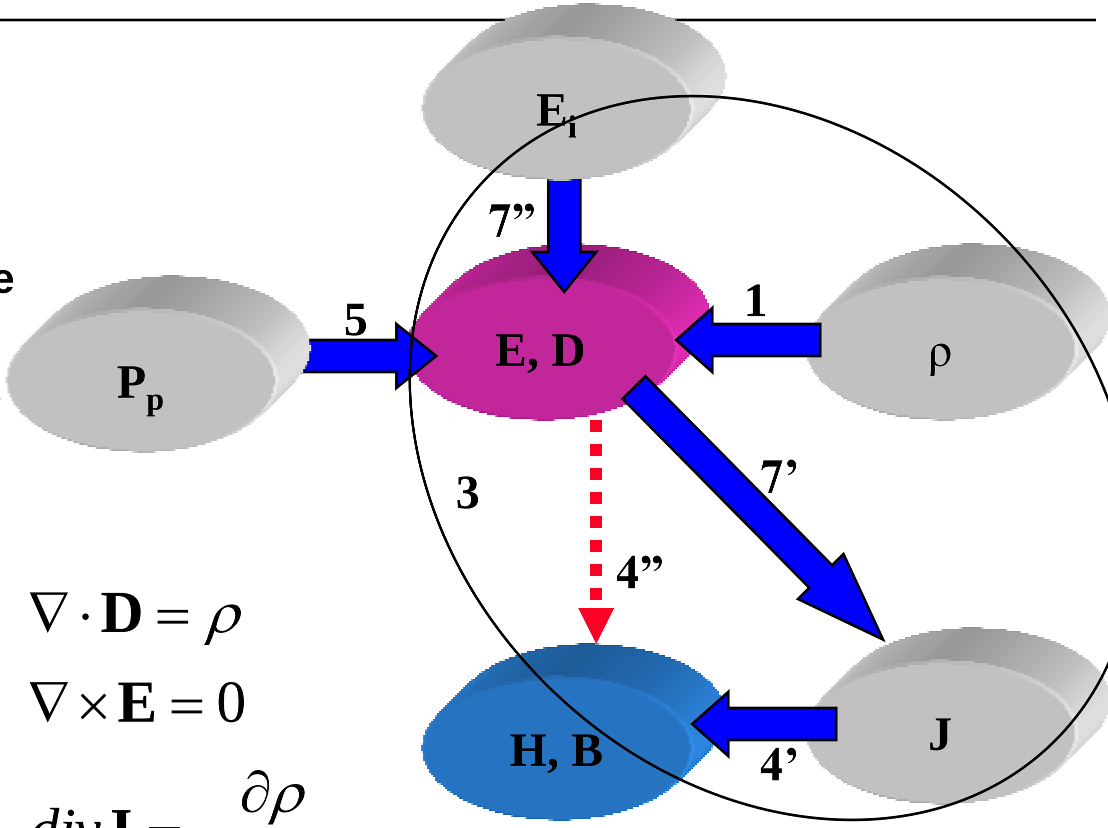
EQS hypothesis:

1. **No movement**
2. **Slow time variation so that Electromagnetic induction may be neglected**
3. **No interest in Magnetic field, only in charge relaxation due to the parasitic conduction**

EQS fundamental equations:

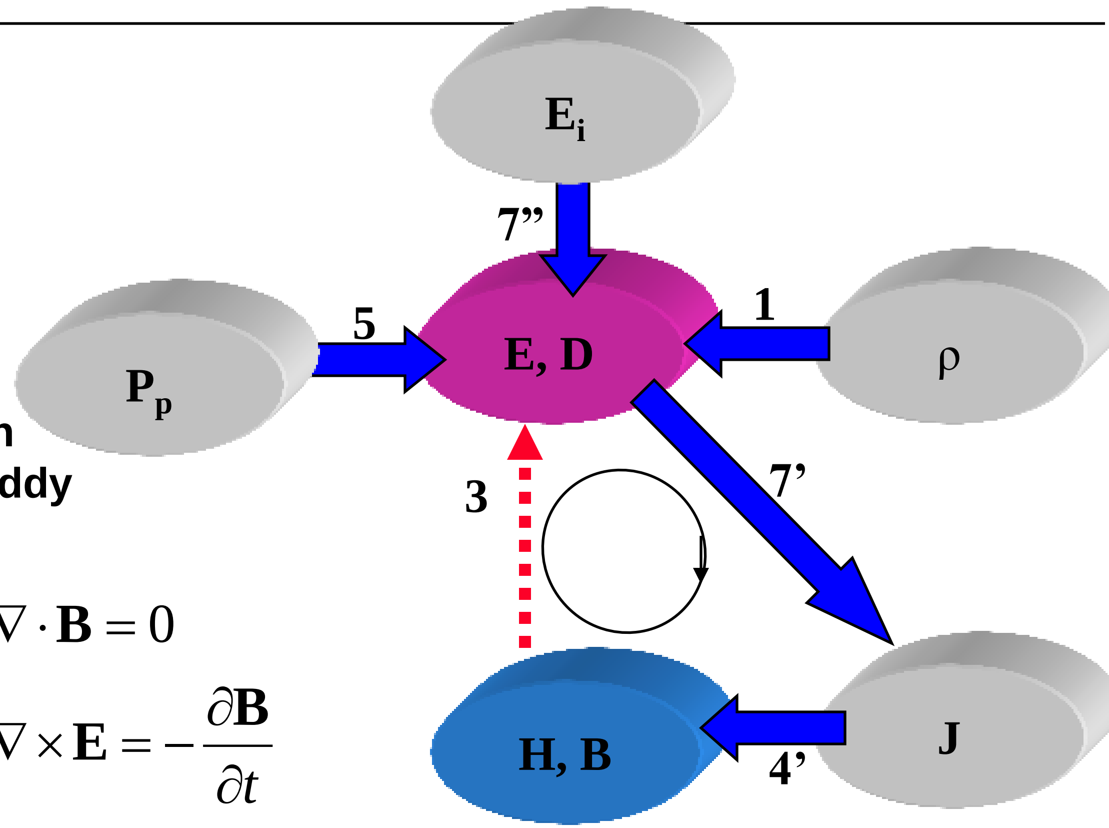
- Gauss theorem: $\nabla \cdot \mathbf{D} = \rho$
- Potential theorem: $\nabla \times \mathbf{E} = 0$
- Current-charge conservation: $div\mathbf{J} = -\dfrac{\partial \rho}{\partial t}$
- Polarization and conduction constitutive relations: $\mathbf{D} = \varepsilon\,\mathbf{E} + \mathbf{P}_p(\mathbf{E})$, $\mathbf{J} = \sigma(\mathbf{E} + \mathbf{E}_i(\mathbf{E}))$

PDE of parabolic type for potential.
Field diffuses in space



Sources: $\mathbf{E}_i$
Fields: ρ, J, D,(curl-free) E
Potential: *V*
Material constants: ε, σ

Magneto-Quasi-Static regime



MQS hypothesis:

1. **No movement**
2. **Slow time variation so that Displacement current may be neglected**
3. **No interest in charge distribution (it is supposed relaxed), but in eddy currents, skin effects, etc**

MQS fundamental equations:

- Gauss theorem: $\nabla \cdot \mathbf{B} = 0$
- Potential theorem: $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
- Current-charge conservation: $\nabla \times \mathbf{H} = \mathbf{J}$
- Polarization and conduction constitutive relations:

$$\mathbf{B} = \mu\,(\mathbf{H} + \mathbf{M}_p(\mathbf{H}))$$

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{E}_i(\mathbf{E}))$$

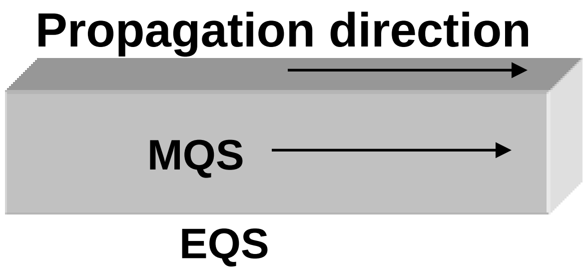
$$p = \mathbf{E} \cdot \mathbf{J}$$

Sources: $\mathbf{E}_i$
Fields: **E, H, (div-free) J, B**
Potentials: **A**, V
Material constants: **μ, σ**
PDE of parabolic type
Field diffuses in space

Electro-Magneto-Quasi-Static regime

EMQS hypothesis:

1. Computational domain is decomposed in conductive and dielectric parts. Time variation is slow, so that in conductors may be neglected displacement currents and in dielectrics may be neglected EM induction
2. Charge distribution in conductors are relaxed, no interest for magnetic field in dielectrics, no semiconductors in the computational domain
3. Are considered: eddy currents and skin effects in conductors, charge relaxation in lossy dielectrics and propagation along the interface



EMQS fundamental equations:

- EQS equations in dielectrics
- MQS equations in conductors

Fields in dielectrics: (curl-free) E, D, J, ρ
Fields in conductors: E, H, (div-free) J, B
Potentials: A, *V*
Material constants: σ, in conductors μ, in dielectrics ε
PDE of **parabolic(3D)/hyperbolic(2D)** type
Field **diffuses** in space and it is **propagating** along the conductor boundary surface

Full Wave Loss-Less regime

$$\nabla \cdot \mathbf{D} = \rho$$

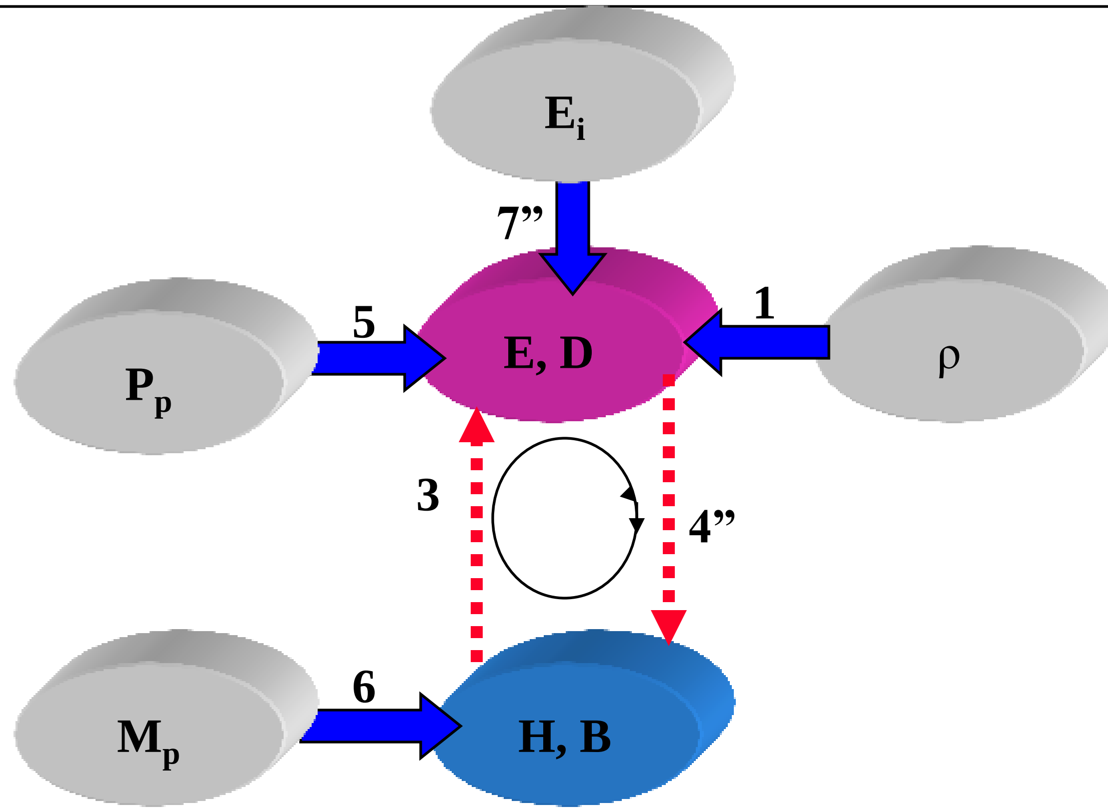
$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$

$$\mathbf{D} = \varepsilon\ \mathbf{E} + \mathbf{P}_p(\mathbf{E})$$

$$\mathbf{B} = \mu\ (\mathbf{H} + \mathbf{M}_p(\mathbf{H}))$$



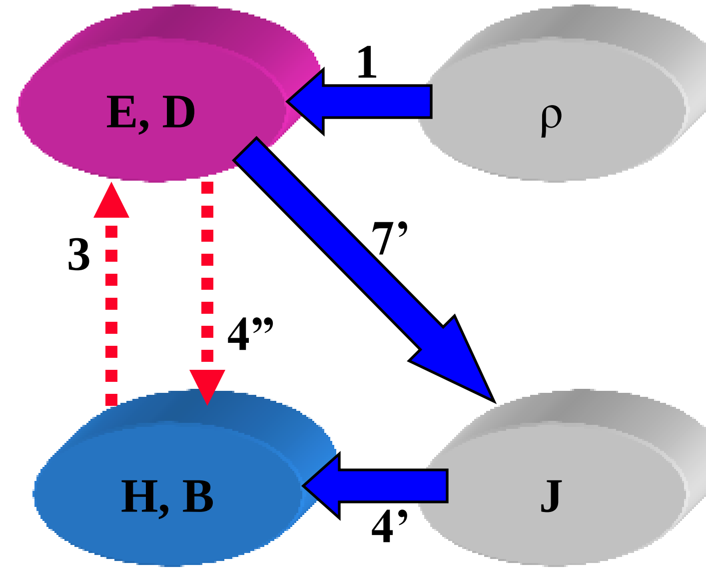
Sources: $E_{i,}$, M_p, P_p,
Fields: E, D, H, (div-free) B
Potentials: A, *V*
Material constants: ε, μ
PDE of hyperbolic type
Field is propagating in space

LL hypothesis:

1. **No movement**
2. **No conductive losses (σ=0, J=0)**
3. **No hysteretic losses of dielectric or magnetic nature**

laboratorul de modelare numerica LMN

Harmonic variation in time
Maxwell' equations in freq. domain



Complex representation of harmonic functions:

$$y(t) = Y\sqrt{2}\sin(\omega t + \phi) \leftrightarrow \underline{Y} = Ye^{j\phi} \Rightarrow$$

$$y'(t) = Y\sqrt{2}\omega\cos(\omega t + \phi) \leftrightarrow \underline{Y}' = j\omega\underline{Y} = Ye^{j(\phi+\pi/2)}$$

$$y(t) = \sum_{k=1,n}\lambda_k y_k(t) \leftrightarrow \underline{Y} = \sum_{k=1,n}\lambda_k \underline{Y}_k$$

Complex form of the Maxwell's equations:

$$\nabla\cdot\underline{\mathbf{D}} = \rho$$

$$\nabla\cdot\underline{\mathbf{B}} = 0$$

$$\nabla\times\underline{\mathbf{E}} = -j\omega\underline{\mathbf{B}}$$

$$\nabla\times\underline{\mathbf{H}} = \underline{\mathbf{J}} + j\omega\underline{\mathbf{D}}$$

$$\underline{\mathbf{D}} = \varepsilon\,\underline{\mathbf{E}}$$

$$\underline{\mathbf{B}} = \mu\,\underline{\mathbf{H}}$$

$$\underline{\mathbf{J}} = \sigma\underline{\mathbf{E}}$$

$$\nabla\cdot(\varepsilon\,\underline{\mathbf{E}}) = \rho$$

$$\nabla\times\underline{\mathbf{E}} = -j\omega\mu\,\underline{\mathbf{H}}$$

$$\nabla\times\underline{\mathbf{H}} = (\sigma + j\omega\varepsilon)\underline{\mathbf{E}}$$

$$\nabla\times\underline{\mathbf{E}} = -s\mu\,\underline{\mathbf{H}}$$

$$\nabla\times\underline{\mathbf{H}} = (\sigma + s\varepsilon)\underline{\mathbf{E}}$$

To apply the complex transform, media should have linear constitutive relations

Sources: boundary conditions

Fields: $\underline{\mathbf{E}}$, $\underline{\mathbf{H}}$

Potentials: $\underline{\mathbf{A}}$, $\underline{V}$

Material constants: ε, μ, σ

PDE of complex elliptic type.

After Laplace transform

Summary of the EM field regimes

Regime	Eq. type	Fields	Material const.	Phenomenon
ES	Elliptic	D, curl-free E	ε	Distribution
MS	Elliptic	Curl-free H, div-free B	μ	Distribution
EC	Elliptic	Curl-free E, div-free J	σ	Distribution
MG	Elliptic	H, div-free B	μ	Distribution
EQS	Parabolic	D, J, curl-free E	ε, σ	Diffusion
MQS	Parabolic	E, H, div-free B, div-free J	μ, σ	Diffusion
LL	Hyperbolic	E, H, div-free D, div-free B,	ε, μ	Propagation
FW	Hyperbolic	E,D, H, div-free B	ε, μ, σ	Propagation
Freq.	Elliptic	E, H	ε, μ, σ	Distribution

Summary of the EM field laws

Law	Global/ Integral	Local	On sufaces	Field lines
Electric flux	$\psi_\Sigma = q_{D_\Sigma} \Leftrightarrow$ $\oint_\Sigma \boldsymbol{D}\cdot d\boldsymbol{A} = \int_{D_\Sigma} \rho dv$	$div\boldsymbol{D} = \rho \Leftrightarrow$ $\nabla\cdot\boldsymbol{D} = \rho$	$div_s\boldsymbol{D} = \rho_s \Rightarrow$ $D_{n1} = D_{n2}$	D - open from + to -
Magnetic flux	$\varphi_\Sigma = 0 \Leftrightarrow$ $\oint_\Sigma \boldsymbol{B}\cdot d\boldsymbol{A} = 0$	$div\boldsymbol{B} = 0 \Leftrightarrow$ $\nabla\cdot\boldsymbol{B} = 0$	$div_s\boldsymbol{B} = 0 \Rightarrow$ $B_{n1} = B_{n2}$	B – closed
EM induction	$u_\Gamma = -\frac{d\varphi_{S_\Gamma}}{dt} \Rightarrow$ $\oint_\Gamma \boldsymbol{E}\cdot d\boldsymbol{r} = -\frac{d}{dt}\int_{S_\Gamma} \boldsymbol{B}\cdot d\boldsymbol{A}$	$rot\boldsymbol{E} = -\frac{d_f\boldsymbol{B}}{dt} \Rightarrow$ $\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}$	$rot_s\boldsymbol{E} = 0 \Rightarrow$ $\boldsymbol{E}_{t1} = \boldsymbol{E}_{t2}$	E – closed arround B
Magnetic circulation	$u_{m\Gamma} = i_{S_\Gamma} + \frac{d\psi_{S_\Gamma}}{dt} \Rightarrow$ $\oint_\Gamma \boldsymbol{H}\cdot d\boldsymbol{r} = \int_{S_\Gamma} \boldsymbol{J}\cdot d\boldsymbol{A} + \frac{d}{dt}\int_{S_\Gamma} \boldsymbol{D}\cdot d\boldsymbol{A}$	$rot\boldsymbol{H} = \boldsymbol{J} + \frac{d_f\boldsymbol{D}}{dt} \Rightarrow$ $\nabla\times\boldsymbol{H} = \boldsymbol{J} + \frac{\partial\boldsymbol{D}}{\partial t}$	$rot_s\boldsymbol{H} = \boldsymbol{J}_s \Rightarrow$ $\boldsymbol{H}_{t1} = \boldsymbol{H}_{t2}$	H – closed arround J+Jd
E-D	$\Psi_S = f(U_C) \Leftarrow$	$\boldsymbol{D} = \mathrm{f}(\boldsymbol{E}) = \overline{\overline{\varepsilon}}\,\boldsymbol{E} + \boldsymbol{P}_p$	$tg\alpha_1 / tg\alpha_2 = \varepsilon_1 / \varepsilon_2$	E open,D closed
B-H	$\Phi_S = f(U_{mC}) \Leftarrow$	$\boldsymbol{B} = \mathrm{f}(\boldsymbol{H}) = \overline{\overline{\mu}}\boldsymbol{H} + \mu_0\boldsymbol{M}_p$	$tg\alpha_1 / tg\alpha_2 = \mu_1 / \mu_2$	H open,J closed
Conduction	$I_S = f(U_C) \Leftarrow$	$\boldsymbol{J} = \mathrm{f}(\boldsymbol{E}) = \overline{\overline{\sigma}}\boldsymbol{E} + \boldsymbol{J}_i$	$tg\alpha_1 / tg\alpha_2 = \sigma_1 / \sigma_2$	E open,J closed
Energy transfer	$p = ui$	$p = \boldsymbol{J}\cdot\boldsymbol{E}$	-	Energy: sgn(p)
Mass transfer	$m = kit$	$\boldsymbol{\delta} = k\boldsymbol{J}$	-	Mass - δ

Spatial differential operators

1. **Divergence,** from vector field to scalar field: $div\circ \equiv \nabla\cdot\circ$
2. **Curl (rotor),** from vector field to vector field: $rot\circ \equiv curl\circ \equiv \nabla\times\circ$
3. **Gradient,** from scalar field to vector field: $grad\circ \equiv \nabla\circ$

Nabla (del) operator [1/m] = vector + derivative: $\nabla\circ = \boldsymbol{i}\frac{\partial\circ}{\partial x} + \boldsymbol{j}\frac{\partial\circ}{\partial y} + \boldsymbol{k}\frac{\partial\circ}{\partial z}$

Meaning: div = field productivity, curl = rotational tendency, grad = maxim variation (value, direction)

$$\nabla\cdot\boldsymbol{D} = \boldsymbol{i}\frac{\partial\boldsymbol{D}}{\partial x} + \boldsymbol{j}\frac{\partial\boldsymbol{D}}{\partial y} + \boldsymbol{k}\frac{\partial\boldsymbol{D}}{\partial z} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z};$$

$$\nabla\times\boldsymbol{E} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \boldsymbol{i}\left(\frac{\partial E_z}{\partial y} - \ldots\right)\ldots$$

$$\nabla V = \boldsymbol{i}\frac{\partial V}{\partial x} + \boldsymbol{j}\frac{\partial V}{\partial y} + \boldsymbol{k}\frac{\partial V}{\partial z}$$

Differential identities:

$$\nabla\cdot(\nabla\times\boldsymbol{G}) = 0 \Leftrightarrow \quad div(rot\circ) = 0\ ;\nabla\cdot(\nabla V) = div(grad V) = \Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2};$$

$$\nabla\times\nabla V = 0 \Leftrightarrow \quad rot(grad\circ) = 0; \quad grad(div\boldsymbol{G}) = \nabla(\nabla\cdot\boldsymbol{G}) + \Delta\boldsymbol{G} = \nabla\times(\nabla\times\boldsymbol{G}) = rotrot\boldsymbol{G} + \Delta\boldsymbol{G}$$

De Rahm sequence

1. **Divergence:** $div : H(div;\Omega) \rightarrow L^2(\Omega)$
2. **Curl**: $curl : H(curl;\Omega) \rightarrow \left[L^2(\Omega)\right]^3$
3. **Gradient:** $grad : H^1(\Omega) \rightarrow \left[L^2(\Omega)\right]^3$

Sobolev spaces of functions with differentials in L2 (it is a Hibert space)

If Ω is simply connected, the sequence (de Rham) is an exact one:

$$IR \xrightarrow{id} H^1(\Omega) \xrightarrow{\nabla} H(curl;\Omega) \xrightarrow{\nabla\times} H(div;\Omega) \xrightarrow{\nabla\cdot} L^2(\Omega) \xrightarrow{0} 0$$

– That means (kernel of each operator is exactly the range of previous one):

$$\ker(grad) = IR \Rightarrow grad(ct) = 0;\ \text{zero gradient} \Rightarrow \text{constant scalar field}$$

$$\ker(curl) = \nabla H^1(\Omega) \Rightarrow curl(grad V) = 0;\ \text{\& any curl-free field has a scalar potential}$$

$$\ker(div) = \nabla \times H(curl;\Omega) \Rightarrow div(curl\mathbf{G}) = 0;\ \text{\& any div-free field has a vector potential}$$

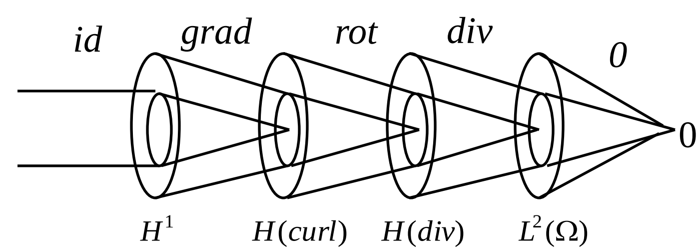
Helmholtz decomposition:

$$\forall \mathbf{G} \in L^2(\Omega)^3 \Rightarrow \exists V \in H^1(\Omega), \mathbf{A} \in H(curl;\Omega), s.t.$$

$$\mathbf{G} = grad V + curl\mathbf{A};\ grad V \perp curl\mathbf{A}$$

1(Ω)

Any vector field has two sources: *div* and *curl*



General chain spaces for d=3

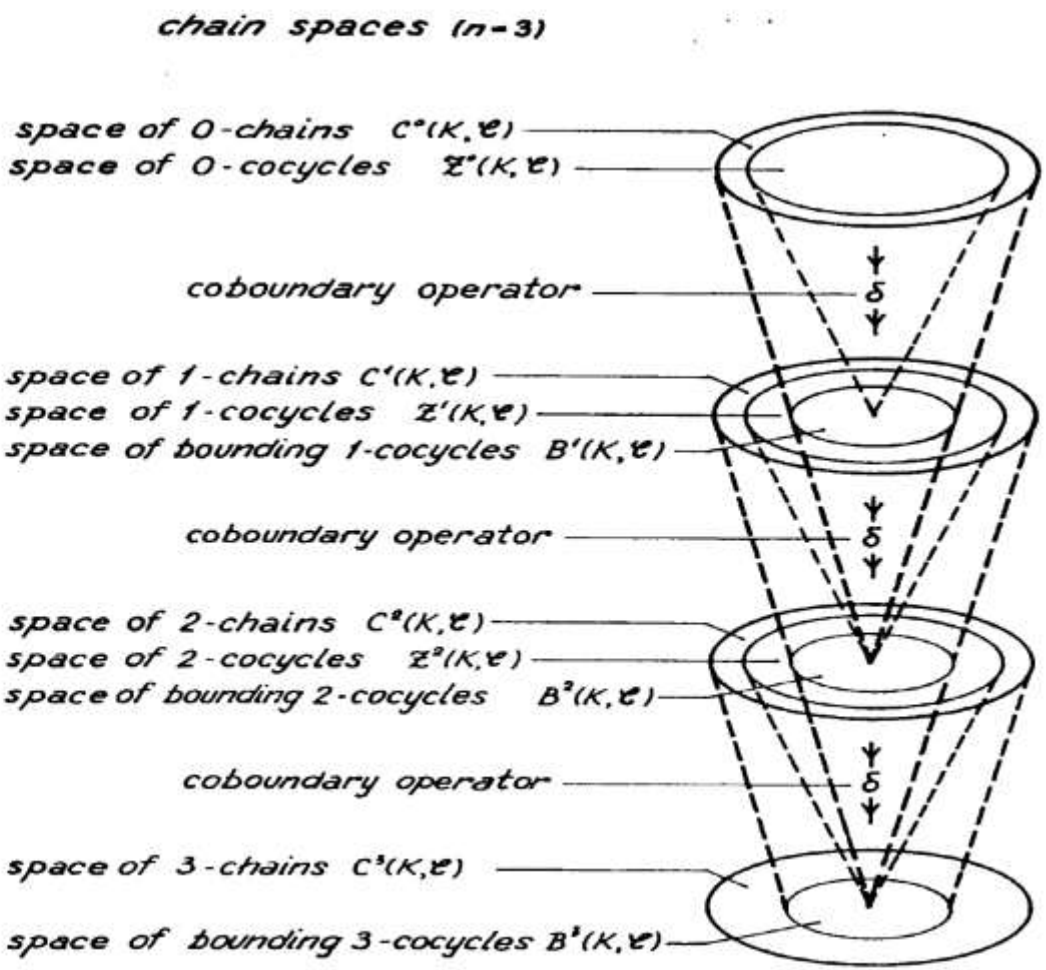


Fig. 3.12.1

If Ω is not contractible:
The scalar potential is not uniquely defined

$$\ker(grad) \subset IR$$

$$\ker(curl) \subset \nabla H^1(\Omega)$$

$$\ker(div) \subset \nabla \times H(curl;\Omega)$$

The Rham sequence is also exact for fields with essential zero boundary conditions (potentials are well defined).
In the case of hybrid boundary conditions, the sequence is not exact.

Integral-differential consequences in Sobolev spaces

Lipschitz-domain: Ω and ∂Ω are piecewise defined by Lipschitz functions w.r.t. local coordinates, s.t. Ω lies locally on one side of ∂Ω. It enables the definition of an outer unit vector **n** almost everywhere on ∂Ω and

$C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega), C^\infty(\overline{\Omega})^3$ in $H(curl,\Omega), H(div,\Omega) \Rightarrow tr_{\partial\Omega}(u) = u|_{\partial\Omega}$ extended in H

Here, general Stokes theorem: for any differential form ω is valid: $\int_\Omega d\omega = \int_{\partial\Omega} \omega \Rightarrow$

Gauss-Ostrogradski theorem: $\int_\Omega div\mathbf{v}dx = \int_{\partial\Omega} \mathbf{v}\cdot\mathbf{n}dx; \mathbf{v} \in \forall H(div,\Omega)$

Green identities:

$$\int_\Omega grad u\cdot\mathbf{v}dx = -\int_\Omega u div\mathbf{v}dx + \int_{\partial\Omega} tr_{\partial\Omega}(u)\mathbf{v}\cdot\mathbf{n}dx; \forall u \in H^1(\Omega)\forall \mathbf{v} \in H(div,\Omega)$$

$$\int_\Omega curl\mathbf{u}\cdot\mathbf{v}dx = \int_\Omega \mathbf{u}\cdot curl\mathbf{v}dx - \int_{\partial\Omega} tr_\tau(\mathbf{u})\cdot\mathbf{v}dx; \forall \mathbf{u} \in H(curl,\Omega); \forall \mathbf{v} \in H^1(\Omega)^3$$

$$div\mathbf{G} = \nabla(\nabla V + \nabla\times\mathbf{A}) = f \Rightarrow \Delta V = f \Rightarrow V(\mathbf{x}) = \int_\Omega G(\mathbf{x},\mathbf{y}) f(\mathbf{y})dx + \int_{\partial\Omega} \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n} V(\mathbf{y})dS,$$

$$curl\mathbf{G} = \nabla\times(\nabla V + \nabla\times\mathbf{A}) = \mathbf{g} \Rightarrow \Delta\mathbf{A} = \mathbf{g} \Rightarrow \mathbf{A}(\mathbf{x}) = \int_\Omega G(\mathbf{x},\mathbf{y})\mathbf{g}(\mathbf{y})dx + \int_{\partial\Omega} \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n} \mathbf{A}(\mathbf{y})dS$$

$$\Delta G = \delta(\mathbf{x}-\mathbf{y}) \to V(\mathbf{x}) = \int_{IR^3} G(\mathbf{x},\mathbf{y}) f(\mathbf{y})dx; \quad \mathbf{A}(\mathbf{x}) = \int_{IR^3} G(\mathbf{x},\mathbf{y})\mathbf{g}(\mathbf{y})dx; \quad G(\mathbf{x},\mathbf{y}) = \frac{-1}{4\pi\|\mathbf{x}-\mathbf{y}\|},$$

Tonti's diagram (Maxwell house)

In several regimes, it is reduced:

ES: front wall (V,E,D,ρ)

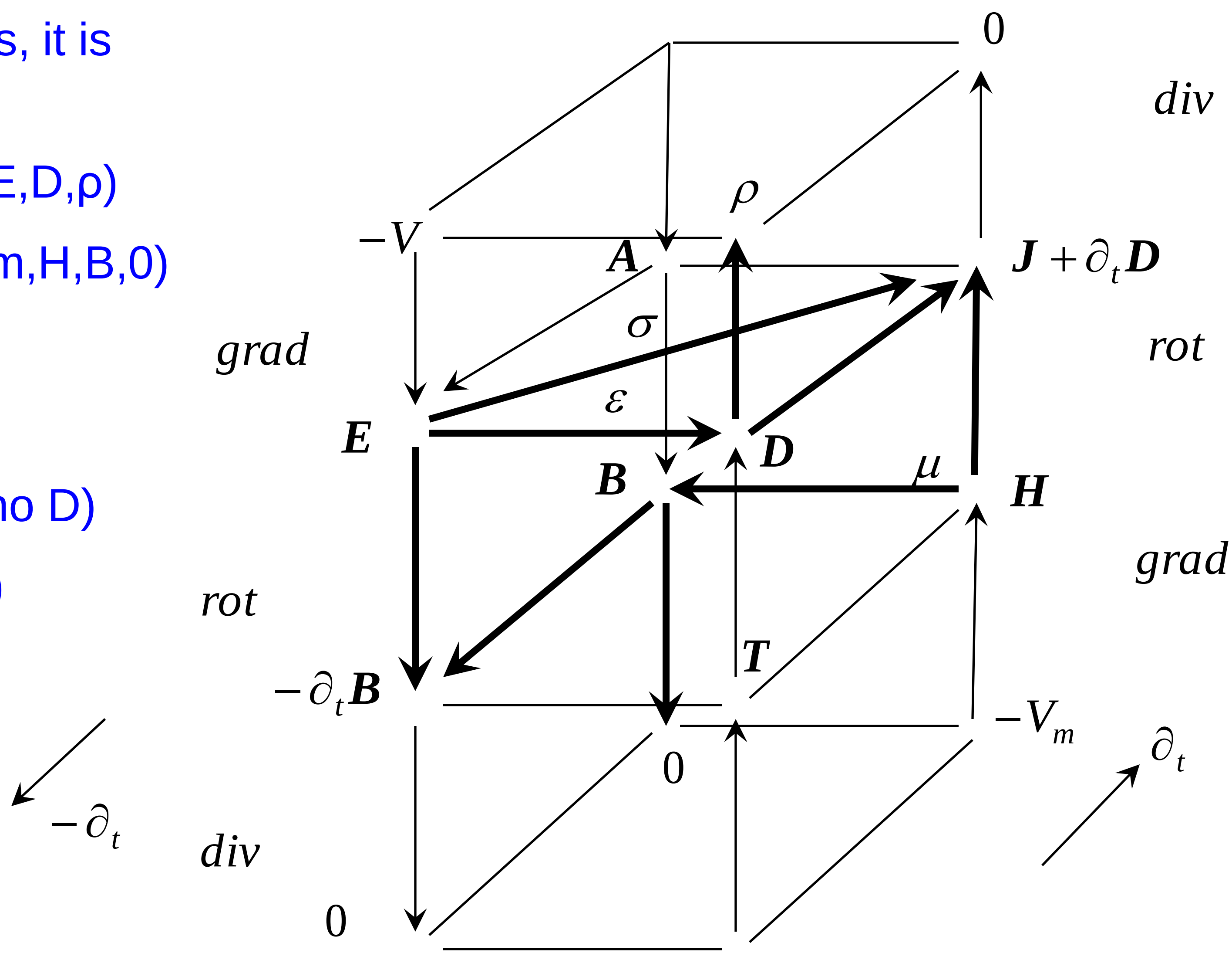
MS: back wall (Vm,H,B,0)

EC: V,E,J,0

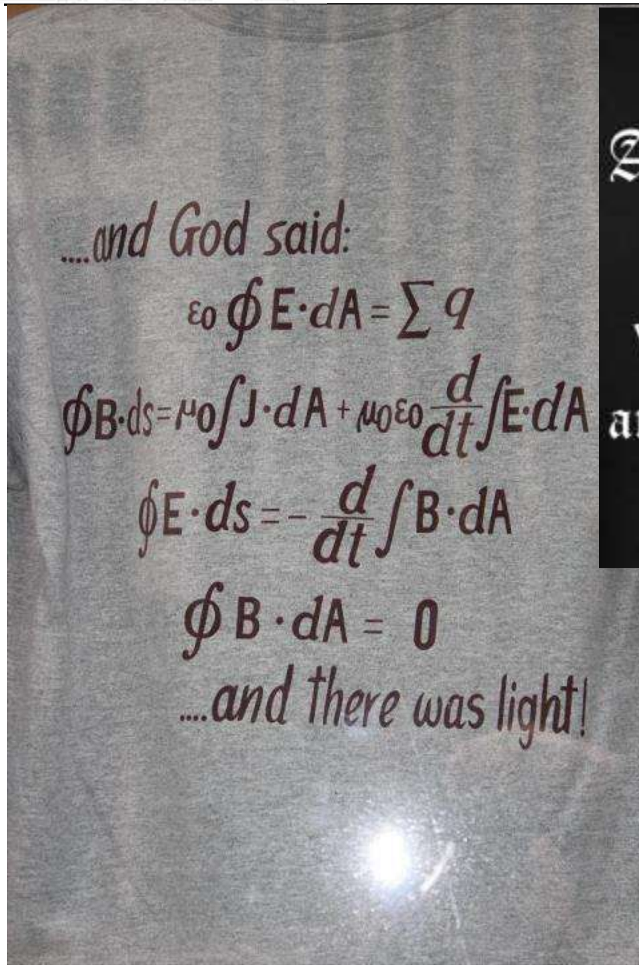
MG: back wall

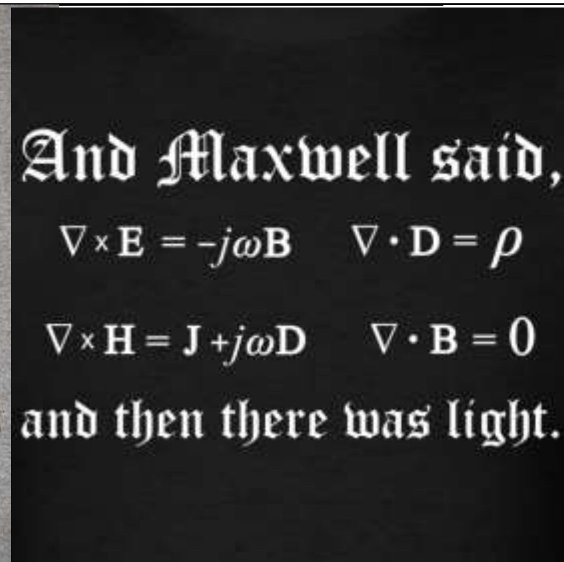
MQS: E,J, H, B (no D)

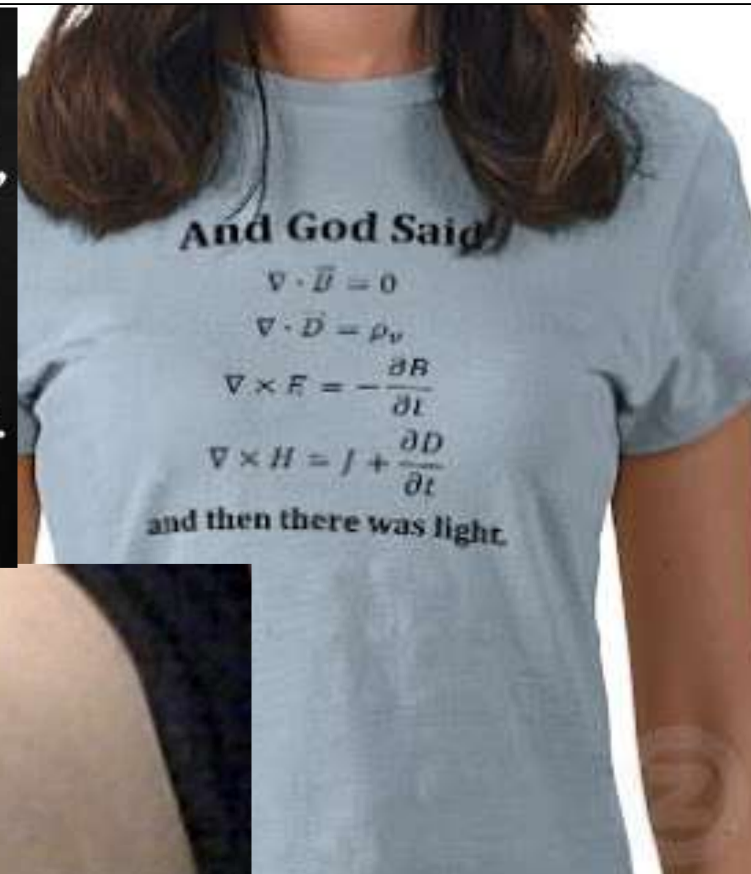
EQS: E,J,D (noB)

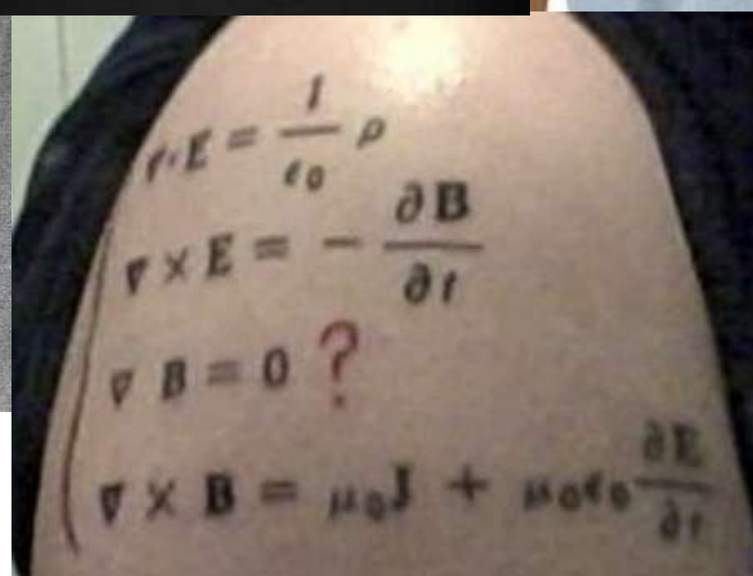


Pop art (find mitakes, pls.!)









Not so easy questions for curious people

1. Are all four Maxwell equations independent?
2. What is Lorentz Ether Theory? Is it the ether immovable or not ?
3. How the EM quantities are transformed when the reference system is changed with a moving one?
4. How looks like the Maxwell-Hertz equations in global form ? Are the surfaces S in these equations dragged by matter ?
5. How Maxwell discovered the displacement current ?
6. May be explained Hertz's experiment using solely the EM induction ?
7. What speed is the maxim speed of the EM waves ?
8. Why the EC steady state current can not have permanent polarization as a source?