Electromagnetic Field Theory

2. Ampere-Maxwell' law

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Ampere-Maxwell law for magnetic field circulation

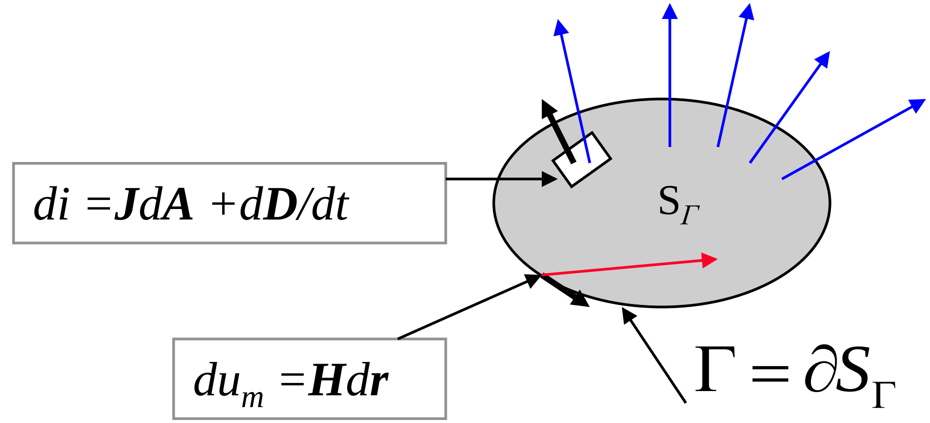
$$u_{m\Gamma} = i_{S\Gamma} + \frac{d\psi_{S\Gamma}}{dt} \Leftrightarrow$$

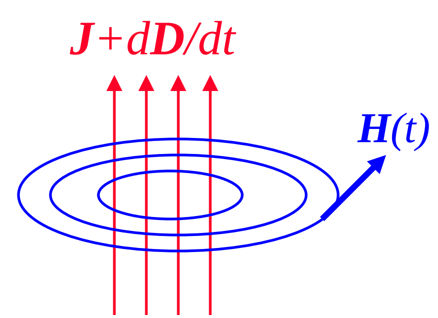
$$\oint_{\Gamma} \mathbf{H} d\mathbf{r} = \int_{S_\Gamma} \mathbf{J} d\mathbf{A} + \frac{d}{dt} \int_{S_\Gamma} \mathbf{D} d\mathbf{A}$$

1. **Global (integral) form of the law:**
2. **Physical meaning:**

   Any conductor carrying current produces a magnetic field. The time variation of electric field generates as well a magnetic field, it was called by Maxwell the "displacement current".
3. **The lines of the generated magnetic field**

- **Are closed curves**
- **They are turned around the electric or displacement current.**
- **Their direction is dependent of the current direction.**





Local forms of Ampere-Maxwell law in fixed (not mobile) bodies

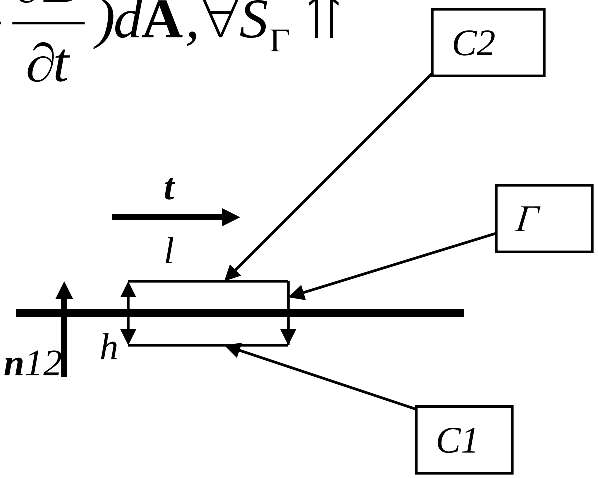
1. **Local (differential) form of the law:**

$$curl\mathbf{H} = \mathbf{J} + \underbrace{\frac{\partial \mathbf{D}}{\partial t}}_{\mathbf{J}_d} \Leftrightarrow \nabla \times \mathbf{E} = \underbrace{\mathbf{J}_t}_{\mathbf{J}+\mathbf{J}_d}$$

2. **Proof** (based on Stokes theorem):

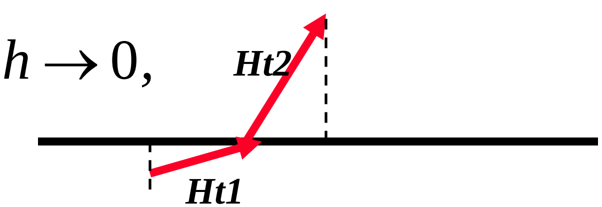
$$\oint_{\Gamma} \mathbf{H} d\mathbf{r} = \int_{S_\Gamma} curl\mathbf{H} d\mathbf{A} = \int_{S_\Gamma} \mathbf{J} d\mathbf{A} + \frac{d}{dt}\int_{S_\Gamma} \mathbf{D} d\mathbf{A} = \int_{S_\Gamma} (\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}) d\mathbf{A}, \forall S_\Gamma \Uparrow$$

3. **Conservation of the tangential component of the magnetic field strength**



$$\oint_{\Gamma} \mathbf{H} d\mathbf{r} = \oint_{C1} \mathbf{H} d\mathbf{r} + \oint_{C2} \mathbf{H} d\mathbf{r} = \mathbf{t} \cdot (\mathbf{H}_2 - \mathbf{H}_1)_{ave}\, l,$$

$$\int_{S_\Gamma} \mathbf{J} d\mathbf{A} + \frac{d}{dt}\int_{S_\Gamma} \mathbf{D} d\mathbf{A} = (J_{nave} + \frac{dD_{nave}}{dt}) lh \rightarrow J_s l, \text{when } h \rightarrow 0,$$



$$\Rightarrow \mathbf{n}_{12} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s \Rightarrow \boxed{\mathbf{H}_{t2} = \mathbf{H}_{t1}}, \text{if } J_s = 0.$$

Local and developed integral forms in mobile bodies

1. **Local (differential) form of the law:**

$$curl\,\mathbf{H} = \mathbf{J} + \frac{d_f \mathbf{D}}{dt} \Leftrightarrow$$

$$curl\,\mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} + \rho\mathbf{v} + curl(\mathbf{D}\times\mathbf{v})$$

2. **Proof, based on Stokes theorem and flux derivative:**

$$\frac{d}{dt}\int_{S_{\Gamma(t)}} \mathbf{D}(t)d\mathbf{A} = \frac{d}{dt}\int_{S_{\Gamma(t)}}\mathbf{D}d\mathbf{A} + \frac{d}{dt}\int_{S_\Gamma}\mathbf{D}(t)d\mathbf{A} \Rightarrow \frac{d}{dt}\int_{S_{\Gamma(t)}}\mathbf{D}d\mathbf{A} = \lim_{\Delta t\to 0}\left(\int_{S'_\Gamma}\mathbf{D}d\mathbf{A}' - \int_{S_\Gamma}\mathbf{D}d\mathbf{A}\right)/\Delta t =$$

$$-\int_{S_\Gamma}(\mathbf{v}\,div\mathbf{D} + curl(\mathbf{D}\times\mathbf{v}))d\mathbf{A}, \quad \frac{d}{dt}\int_{S_\Gamma}\mathbf{D}(t)d\mathbf{A} = \int_{S_\Gamma}\frac{\partial\mathbf{D}(t)}{\partial t}d\mathbf{A}, \Rightarrow \frac{d}{dt}\int_{S_{\Gamma(t)}}\mathbf{D}(t)d\mathbf{A} = \int_{S_{\Gamma(t)}}\frac{d_f\mathbf{D}(t)}{dt}d\mathbf{A},$$

$$\text{where } \frac{d_f\mathbf{D}(t)}{dt} =_{def} \frac{\partial\mathbf{D}}{\partial t} + \mathbf{v}\,div\mathbf{D} + curl(\mathbf{D}\times\mathbf{v}), \quad curl\,\mathbf{H} = \mathbf{J} + \frac{d_f\mathbf{D}}{dt};\; div\mathbf{D} = \rho \Rightarrow$$

$$curl\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t} + \rho\mathbf{v} + curl(\mathbf{D}\times\mathbf{v}) \Rightarrow \oint_\Gamma(-\mathbf{H} + \mathbf{D}\times\mathbf{v})d\mathbf{r} + \int_{S_\Gamma}\left(\rho\mathbf{v} + \frac{\partial\mathbf{D}}{\partial t}\right)d\mathbf{A} = 0$$

3. Developed integral form

Densities of: Conduction, Displacement, Convection and Rontgen currents

Ampere theorem. The static case

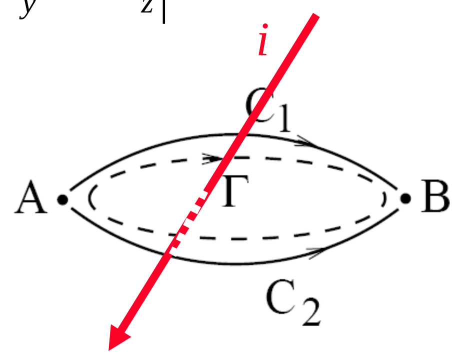
In static fields and not mobile bodies:

- **Global form**
- **Local (differential) form**
- **In Cartesian coordinates:**
- **Proof:** $curl\mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}$
- **Dependence of the magnetic voltage from the integration path**

$$u_{m\Gamma} = i_{S\Gamma} \Leftrightarrow \oint_{\Gamma} \mathbf{H}d\mathbf{r} = \int_{S_\Gamma} \mathbf{J}dA, \forall S_\Gamma$$

$$curl\mathbf{H} = \mathbf{J} \Leftrightarrow \nabla \times \mathbf{H} = \mathbf{J}$$

$$curl\mathbf{H} = \nabla \times \mathbf{H} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \mathbf{J}$$



$$u_{m\Gamma} = \oint_{\Gamma} \mathbf{H}d\mathbf{r} = \int_{C1} \mathbf{H}d\mathbf{r} + \int_{C2} \mathbf{H}d\mathbf{r} = \int_{C1} \mathbf{H}d\mathbf{r}_1 - \int_{C2} \mathbf{H}d\mathbf{r}_2 = u_1 - u_2 = i_{S\Gamma} \Rightarrow$$

$$u_1 = u_2 \; \forall C_1, C_2 \text{ with } \partial C_1 = \partial C_2 = \{A, B\} \text{ only if } \mathbf{J} = 0.$$

In this case may be defined the scalar magnetic potential: $\mathbf{H} = -gradV_m$

Application of the Ampere law

Magnetic field produced by current in a cylinder

$$u_{m\Gamma} = \oint_\Gamma \mathbf{H} d\mathbf{r} = H \int_\Gamma dr = H 2\pi r,$$

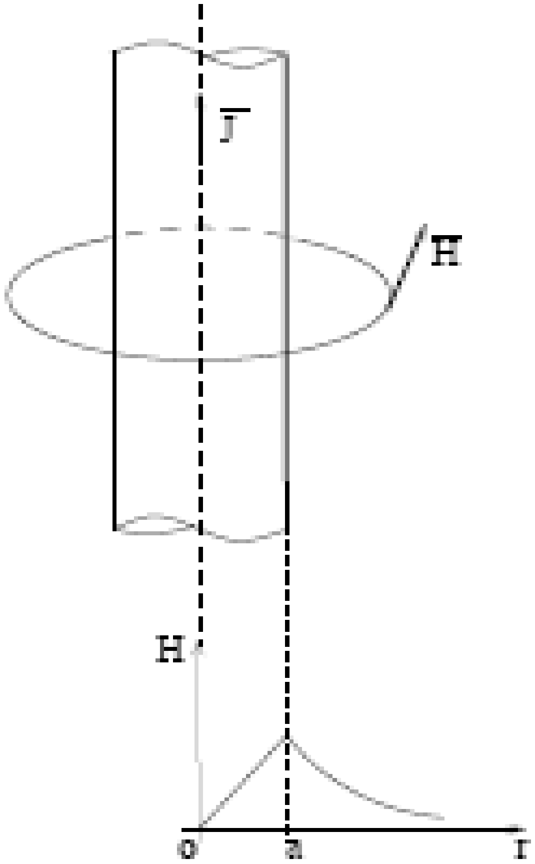
$$i_{S\Gamma} = \int_{S_\Gamma} \mathbf{J} d\mathbf{A} = J \int_{S_\Gamma} dA = J\pi r^2, \text{ for } r < a$$

$$u_{m\Gamma} = i_{S\Gamma} \Rightarrow H = Jr/2$$

$$\text{For } r > a, i_{S\Gamma} = \int_{S_\Gamma} \mathbf{J} d\mathbf{A} = \int_{S_{\Gamma a}} J dA = J\pi a^2 = I,$$

$$H = \frac{Ja^2}{2r} = \frac{I}{2\pi r}$$

$$H(r) = \begin{cases} \dfrac{Jr}{2} = \dfrac{Ir}{2\pi a^2}, \text{for } r < a \\ \dfrac{Ja^2}{2r} = \dfrac{I}{2\pi r}, \text{for } r > a \end{cases}$$



An uniform time variable electric field $D(t)$ produces same magnetic field if $J = dD/dt$.

Several forms of the Ampere's law

Movement	No	Yes
Global	$u_{m\Gamma} = i_{S\Gamma} + \frac{d\psi_{S\Gamma}}{dt}$	$u_{m\Gamma} = i_{S\Gamma} + \frac{d\psi_{S\Gamma}}{dt}$
Integral	$\oint_{\Gamma} \mathbf{H} d\mathbf{r} = \mathbf{J} + \frac{d}{dt}\int_{S_\Gamma} \mathbf{D} d\mathbf{A}$	$\oint_{\Gamma} \mathbf{H} d\mathbf{r} = \mathbf{J} + \frac{d}{dt}\int_{S_\Gamma} \mathbf{D} d\mathbf{A}$
Developed integral	$\oint_{\Gamma} \mathbf{H} d\mathbf{r} = \mathbf{J} + \int_{S_\Gamma} \frac{\partial \mathbf{D}}{\partial t} d\mathbf{A}$	$\oint_{\Gamma} \mathbf{H} d\mathbf{r} = \int_{S_\Gamma} (\mathbf{J} + \rho\mathbf{v} + \frac{\partial \mathbf{D}}{\partial t}) d\mathbf{A} + \oint_{\Gamma} (\mathbf{D} \times \mathbf{v}) d\mathbf{r}$
On surfaces	$\mathbf{n}_{12} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s$	$\mathbf{n}_{12} \times [(\mathbf{H} + \mathbf{D} \times \mathbf{v})_2 - (\mathbf{H} + \mathbf{D} \times \mathbf{v})_2] = 0$
Conserv.	$\mathbf{H}_{t1} = \mathbf{H}_{t2}$	-
Field lines	Turned around the current	-

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Not so easy questions for curious people

1. What do the surface $S_{\Gamma}(t)$ in the moving bodies ? Is it fixed or carry on by the moving body ?
2. Is still valid the H tangential component conservation in the case of a sheet carrying conduction/displacement current?
3. Why the expression of the Roengen current density was not confirmed by measurements ?

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