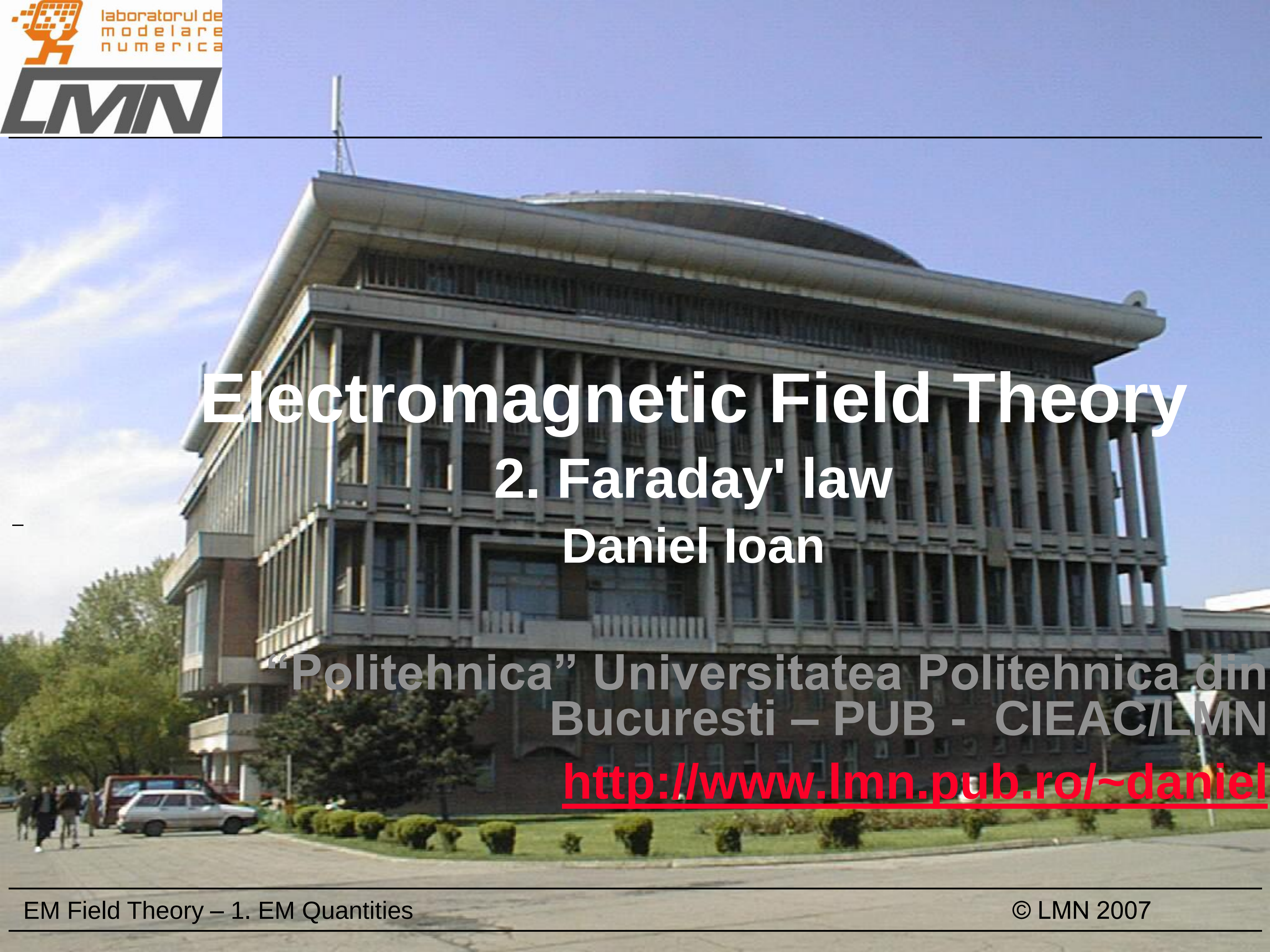


Electromagnetic Field Theory

2. Faraday' law

Daniel Ioan

"Politehnica" Universitatea Politehnica din Bucuresti – PUB - CIEAC/LMN

http://www.lmn.pub.ro/~daniel

Faraday's law of electromagnetic induction

Global (integral) form of the law:

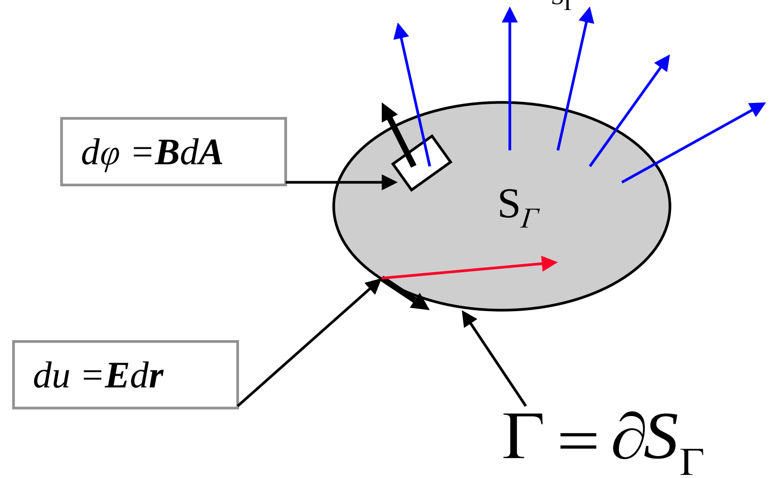
$$u_\Gamma = -\frac{d\varphi_{S\Gamma}}{dt} \Leftrightarrow \oint_\Gamma \mathbf{E}d\mathbf{r} = -\frac{d}{dt}\int_{S_\Gamma} \mathbf{B}d\mathbf{A}$$

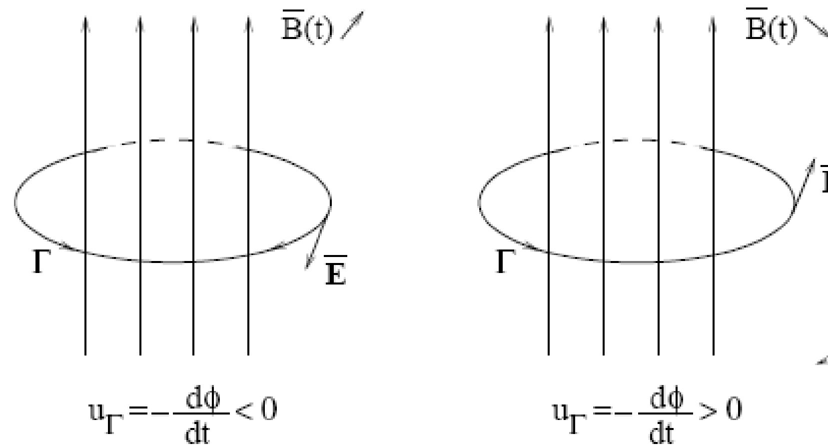
Physical meaning:

Time variation of the magnetic field induces an electric field

3. The lines of the induced electric field

- **Are closed curves**
- **They are turned around the (inductor) magnetic field lines**
- **Their direction is dependent of the magnetic field direction.**
- **If the magnetic field decreases in time, then the induced field has the direction given by the right screw rule**
- **If the magnetic field increases in time, then the induced field has an inverse direction**





Local forms of Faraday' law in fixed (not mobile) bodies

1. **Local (differential) form of the law:**

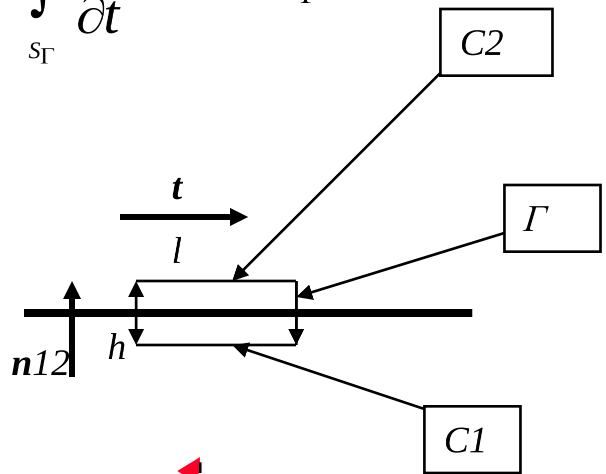
$$curl\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \Leftrightarrow \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

2. **Proof (based on Stokes theorem):**

$$\oint_{\Gamma} \mathbf{E} d\mathbf{r} = \int_{S_\Gamma} curl\mathbf{E} d\mathbf{A} = -\frac{d}{dt}\int_{S_\Gamma} \mathbf{B} d\mathbf{A} = -\int_{S_\Gamma} \frac{\partial \mathbf{B}}{\partial t} d\mathbf{A}, \forall S_\Gamma \Uparrow$$

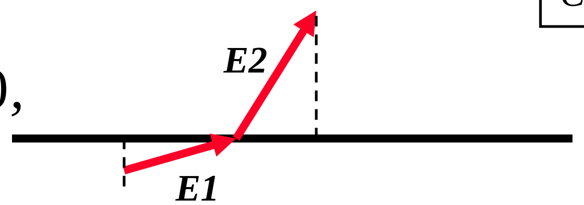
3. **Conservation of the tangential component of the electric field strength**

–



$$\oint_{\Gamma} \mathbf{E} d\mathbf{r} = \oint_{C1} \mathbf{E} d\mathbf{r} + \oint_{C2} \mathbf{E} d\mathbf{r} = \mathbf{t}\cdot(\mathbf{E}_2 - \mathbf{E}_1)_{ave} l,$$

$$\frac{d}{dt}\int_{S_\Gamma} \mathbf{B} d\mathbf{A} = \frac{dB_{nave}}{dt} lh \to 0, \text{when } h \to 0,$$

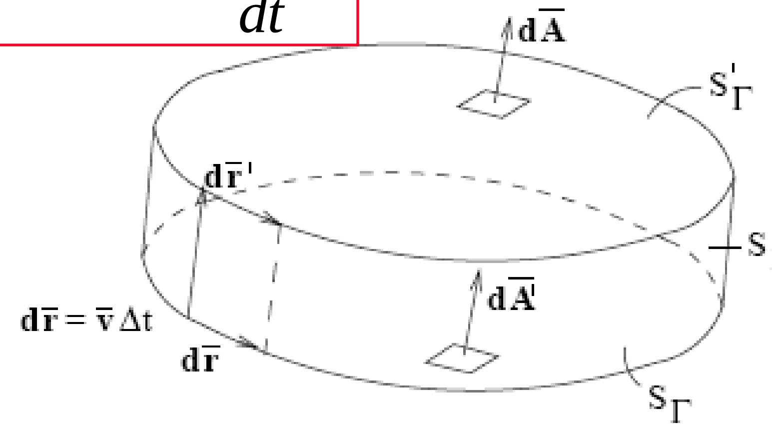


$$\Rightarrow \mathbf{n}_{12} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0, \Leftrightarrow \boxed{\mathbf{E}_{t2} = \mathbf{E}_{t1}}$$

Local and developed integral forms in mobile bodies

1. **Local (differential) form of the law:**

$$curl\,\mathbf{E} = -\frac{d_f\mathbf{B}}{dt}$$



2. **Proof, based on Stokes theorem and flux derivative:**

$$\frac{d}{dt}\int_{S_{\Gamma(t)}}\mathbf{B}(t)d\mathbf{A} = \frac{d}{dt}\int_{S_{\Gamma(t)}}\mathbf{B}d\mathbf{A} + \frac{d}{dt}\int_{S_\Gamma}\mathbf{B}(t)d\mathbf{A} \Rightarrow$$

$$\frac{d}{dt}\int_{S_{\Gamma(t)}}\mathbf{B}d\mathbf{A} = \lim_{\Delta t\to 0}\Big(\int_{S'_\Gamma}\mathbf{B}d\mathbf{A}' - \int_{S_\Gamma}\mathbf{B}d\mathbf{A}\Big)/\Delta t = -\lim_{\Delta t\to 0}\int_{Sl}\mathbf{B}d\mathbf{A}'/\Delta t = -\oint_{\Gamma}\mathbf{B}(d\mathbf{r}\times\mathbf{v}) = \int_{S_\Gamma} curl(\mathbf{B}\times\mathbf{v})d\mathbf{A}$$

because $0 = \oint_\Sigma \mathbf{B}d\mathbf{A} = \int_{S'_\Gamma}\mathbf{B}d\mathbf{A} - \int_{S_\Gamma}\mathbf{B}d\mathbf{A}' + \int_{Sl}\mathbf{B}d\mathbf{A}$ and on Sl $d\mathbf{A} = d\mathbf{r}\times\mathbf{s} = \Delta t(d\mathbf{r}\times\mathbf{v})$

$$\frac{d}{dt}\int_{S_\Gamma}\mathbf{B}(t)d\mathbf{A} = \int_{S_\Gamma}\frac{\partial\mathbf{B}(t)}{\partial t}d\mathbf{A}, \Rightarrow \frac{d}{dt}\int_{S_{\Gamma(t)}}\mathbf{B}(t)d\mathbf{A} = \int_{S_{\Gamma(t)}}\frac{d_f\mathbf{B}(t)}{dt}d\mathbf{A}, \text{where}$$

$$\frac{d_f\mathbf{B}(t)}{dt} =_{def} \frac{\partial\mathbf{B}}{\partial t} + curl(\mathbf{B}\times\mathbf{v})$$

$$curl\,\mathbf{E} = -\frac{d_f\mathbf{B}}{dt} \Leftrightarrow curl\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} - curl(\mathbf{B}\times\mathbf{v}) \Rightarrow$$

$$\oint_\Gamma(\mathbf{E}+\mathbf{B}\times\mathbf{v})d\mathbf{r} + \int_{S_\Gamma}\frac{\partial\mathbf{B}}{\partial t}d\mathbf{A} = 0$$

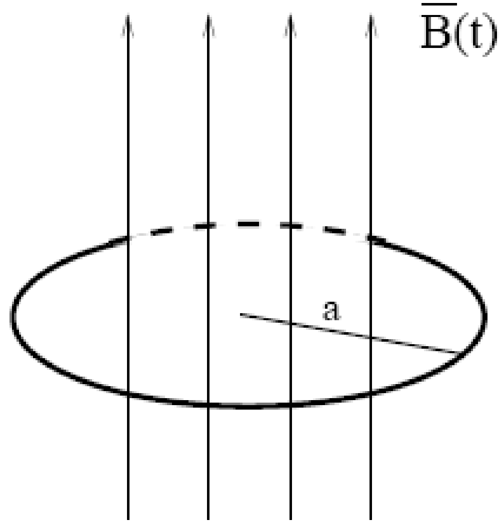
3. Developed integral form

Applications of Faraday law

- **1. EM induction in fixed bodies- Transformer principle**

$$\mathbf{B}(t) = \mathbf{k}B_0 \sin(\omega t),$$

$$u_\Gamma = -\frac{d\varphi_{S\Gamma}}{dt} = -AB_0\omega\cos(\omega t)$$

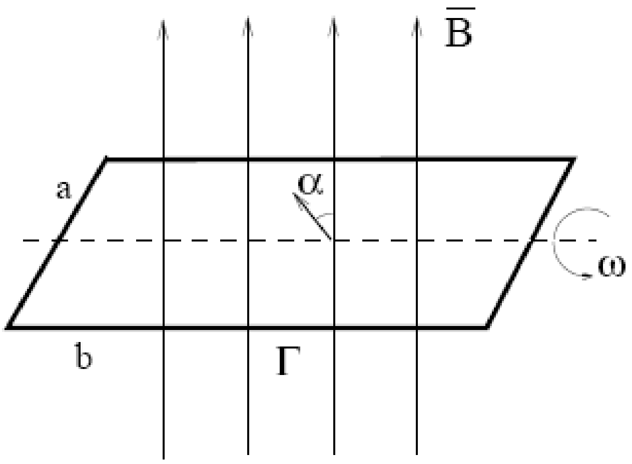


- **2. EM induction in constant magnetic field- Rotating generator principle**

$$\varphi_{S\Gamma} = \int_{S_\Gamma} \mathbf{B}d\mathbf{A} = BA\cos(\omega t),$$

$$u_\Gamma = -\frac{d\varphi_{S\Gamma}}{dt} = BA\omega\sin(\omega t)\text{ or}$$

$$u_\Gamma = -\int_\Gamma (\mathbf{B}\times\mathbf{v})d\mathbf{r} = B\omega ab\sin(\omega t)$$



Static potential theorem

In static fields and not mobile bodies:

$$u_\Gamma = 0 \Leftrightarrow \oint \mathbf{E}d\mathbf{r} = 0, \forall S_\Gamma$$

- **Global form**
- **Local (differential) form**

$$curl\mathbf{E} = 0 \Rightarrow \mathbf{E} = -gradV \Leftrightarrow \mathbf{E} = -\nabla V$$

- **In Cartesian coordinates:**

$$\mathbf{E} = -(\mathbf{i}\frac{\partial V}{\partial x} + \mathbf{j}\frac{\partial V}{\partial y} + \mathbf{k}\frac{\partial V}{\partial k})$$

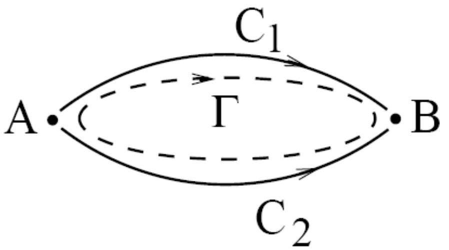
- **Proof:**

$$curl\mathbf{E} = -curl(gradV) = -\nabla \times (\nabla V) = 0$$

- **Gradient definition**

$$gradV = \lim_{W \to 0} \frac{1}{W} \int_{\partial\Omega} Vd\mathbf{A}, \text{ where } W = Vol(\Omega) = \int_{\Omega} dv$$

- **Independence from the shape of the static voltage**



$$u_\Gamma = \oint_\Gamma \mathbf{E}d\mathbf{r} = \int_{C1} \mathbf{E}d\mathbf{r} + \int_{C2} \mathbf{E}d\mathbf{r} = \int_{C1} \mathbf{E}d\mathbf{r}_1 - \int_{C2} \mathbf{E}d\mathbf{r}_2 = u_1 - u_2 = 0 \Rightarrow$$

$$u_1 = u_2 \; \forall C_1, C_2 \text{ with } \partial C_1 = \partial C_2 = \{A, B\}$$

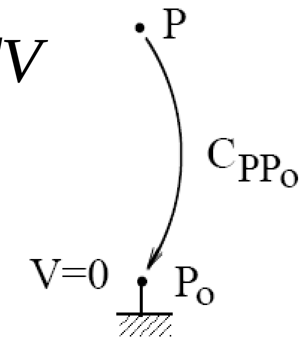
Integral definition of the potential

1. **Potential uniqueness**

$$\mathbf{E} = grad(V + C) = gradV$$

2. **Integral definition of the scalar potential**

$$V(P) = \int_{C_{PO}} \mathbf{E} d\mathbf{r}$$



3. **Poof:**

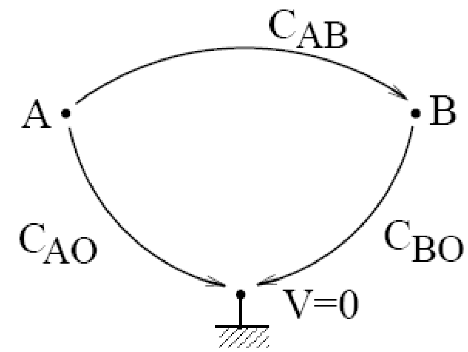
$$\int_{C_{PO}} \mathbf{E} d\mathbf{r} = -\int_{C_{PO}} gradV d\mathbf{r} =$$

$$-\int_{C_{PO}} (\mathbf{i}\frac{\partial V}{\partial x} + \mathbf{j}\frac{\partial V}{\partial y} + \mathbf{k}\frac{\partial V}{\partial k}) d\mathbf{r} = -\int_{C_{PO}} dV = V(O) - V(P) = V(P)$$

–

4. **Voltage difference:**

$$u_{AB} = \int_{C_{AB}} \mathbf{E} d\mathbf{r} = \int_{C_{AO \cup OB}} \mathbf{E} d\mathbf{r} = \int_{C_{AO}} \mathbf{E} d\mathbf{r} - \int_{C_{OB}} \mathbf{E} d\mathbf{r} = V_A - V_B$$



$$u_{AB} = V_A - V_B$$

Applications of the static potential theorem

- **The potential of a charged sphere in vacuum**

For $R > a, D = \dfrac{q}{4\pi R^2}$

$$D = \varepsilon_0 E \Rightarrow E = D/\varepsilon_0 = \frac{q}{4\pi\varepsilon_0 R^2}$$

$$V(R) = \int_{CPP_0} \mathbf{E} d\mathbf{r} = \frac{q}{4\pi\varepsilon_0}\int_{CPP_0}\frac{dr}{r^2} = -\frac{q}{4\pi\varepsilon_0 r}\Bigg|_R^{R_0} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R} - \frac{1}{R_0}\right)$$

For $R_0 \to \infty$

$$V(R) = \frac{q}{4\pi\varepsilon_0 R}, \text{ where } R > a.$$

For $R < a, E = \dfrac{qR}{4\pi\varepsilon_0 a^3}$

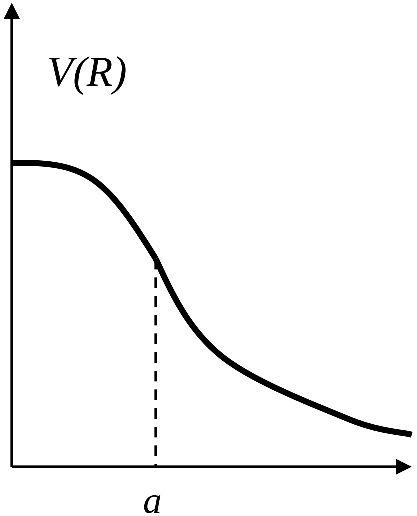
$$V(R) - V(a) = \int_{CPP_a} \mathbf{E} d\mathbf{r} = \frac{q}{4\pi\varepsilon_0 a^3}\int_{CPP_a} r dr = \frac{qr^2}{8\pi\varepsilon_0 a^3}\Bigg|_R^a = \frac{q(a^2 - R^2)}{8\pi\varepsilon_0 a^3}$$

$$V(R) = V(a) + \frac{q(a^2 - R^2)}{8\pi\varepsilon_0 a^3} = \frac{q}{4\pi\varepsilon_0 a} + \frac{q(a^2 - R^2)}{8\pi\varepsilon_0 a^3}$$

$$V(0) = \frac{3q}{8\pi\varepsilon_0 a}$$

$$V(R) = \begin{cases} \dfrac{q}{4\pi\varepsilon_0 a} + \dfrac{q(a^2 - R^2)}{8\pi\varepsilon_0 a^3} + C \text{ for } R < a; \\ \dfrac{q}{4\pi\varepsilon_0 R} + C, \text{ for } R > a. \end{cases}$$

–



Several forms of the Faraday's EM induction law

Movement	No	Yes
Global	$u_\Gamma = -\frac{d\varphi_{S\Gamma}}{dt}$	$u_\Gamma = -\frac{d\varphi_{S\Gamma}}{dt}$
Integral	$\oint_\Gamma \mathbf{E}d\mathbf{r} = -\frac{d}{dt}\int_{S_\Gamma} \mathbf{B}d\mathbf{A}$	$\oint_\Gamma \mathbf{E}d\mathbf{r} = -\frac{d}{dt}\int_{S_\Gamma} \mathbf{B}d\mathbf{A}$
Developed integral	$\oint_\Gamma \mathbf{E}d\mathbf{r} = -\int_{S_\Gamma} \frac{\partial \mathbf{B}}{\partial t}d\mathbf{A}$	$\oint_\Gamma \mathbf{E}d\mathbf{r} = -\int_{S_\Gamma} \frac{\partial \mathbf{B}}{\partial t}d\mathbf{A} - \oint_\Gamma (\mathbf{B}\times\mathbf{v})d\mathbf{r}$
On surfaces	$\mathbf{n}_{12}\times(\mathbf{E}_2 - \mathbf{E}_1) = 0$	$\mathbf{n}_{12}\times[(\mathbf{E}+\mathbf{B}\times\mathbf{v})_2 - (\mathbf{E}+\mathbf{B}\times\mathbf{v})_2] = 0$
Conserv.	$\mathbf{E}_{t1} = \mathbf{E}_{t2}$	-
Field lines	Tourned around the inductor field	-

–

Not so easy questions for curious people

1. What do the surface $S_\Gamma(t)$ in the moving bodies ? Is it fixed or carry on by the moving body ? What is the Faraday paradox and how it is solved ? Does the Earth's magnetic field rotate with the planet?
2. How looks like the local form of the Faraday law in orthogonal coordinates?
3. What kind of function (continuous, derivable, integral, etc) should be the scalar potential and the electric/magnetic field?
4. May be the static electric field lines closed ?
5. May be defined a scalar potential in the variable field ?
6. How the distribution theory (e.g. Dirac generalized function) may be used to write Farday law.
7. Is still valid the E tangential component conservation in the case of a sheet carrying magnetic flux.