Electromagnetic Field Theory

2. Gauss' law for electric and magnetic fields

Daniel Ioan

"Politehnica" Universitatea Politehnica din Bucuresti – PUB - CIEAC/LMN

http://www.lmn.pub.ro/~daniel

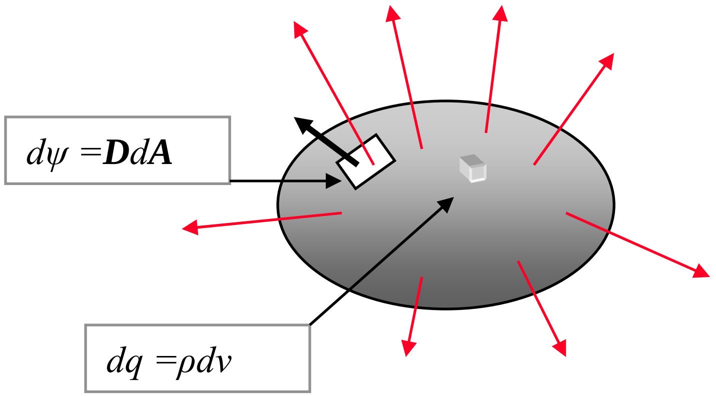
Gauss law for electric fields

1. Global (integral) form of the law:

$$\psi_\Sigma = q_{D_\Sigma} \Leftrightarrow \oint_\Sigma \mathbf{D}d\mathbf{A} = \int_{D_\Sigma} \rho dv$$

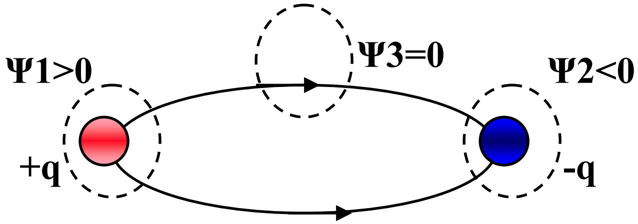
2. Physical meaning:

- Every charged body produce an electric field



3. Electric field lines:

- Open curves
- Every charged body produce an electric field
- Starts on positive charges
- Ends on negative charges
- Continuous in neutral domains



Local forms of Gauss law for electric fields

1. **Local (differential) form of the law:**

$$div\mathbf{D} = \rho$$

2. **Prove** **(based on Gauss Ostrogradski theorem):**

$$\forall D_\Sigma : \oint_{\Sigma=\partial D_\Sigma} \mathbf{D}d\mathbf{A} = \int_{D_\Sigma} div\mathbf{D}dv = \int_{D_\Sigma} \rho dv \Rightarrow div\mathbf{D} = \rho$$

3. **What is divergence?**

$$div\mathbf{D} = \lim_{V_D \to 0} \oint_{\partial D} \mathbf{D}d\mathbf{A} / V_D$$

$$div\mathbf{D} = \nabla\mathbf{D} = (\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z})(\mathbf{i}D_x + \mathbf{j}D_y + \mathbf{k}D_z) = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

–

4. **Local form on the charged surfaces**

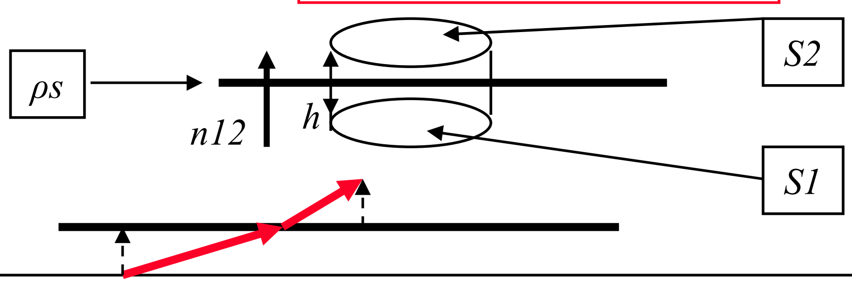
$$\oint_{\Sigma} \mathbf{D}d\mathbf{A} = \int_{S2} \mathbf{D}d\mathbf{A} + \int_{S1} \mathbf{D}d\mathbf{A} = \mathbf{n}_{12} \cdot (\mathbf{D}_2 - \mathbf{D}_1)_{ave} A =$$

$$\int_{D} \rho dv = \rho_v A h + \rho_s A \underset{h \to 0}{\Longrightarrow} \mathbf{n}_{12} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \rho_s$$

5. **Conservation of the normal component of flux density**

If there are no singular surfaces, ρ_s=0 and

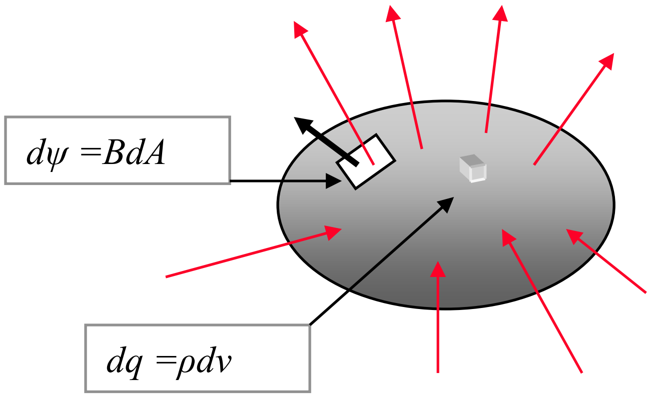
$$\mathbf{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = 0 \Leftrightarrow D_{n1} = D_{n2}$$



Gauss law for magnetic fields

1. **Global (integral) form of the law:**

$$\varphi_\Sigma = 0 \Leftrightarrow \oint_\Sigma \mathbf{B} d\mathbf{A} = 0$$



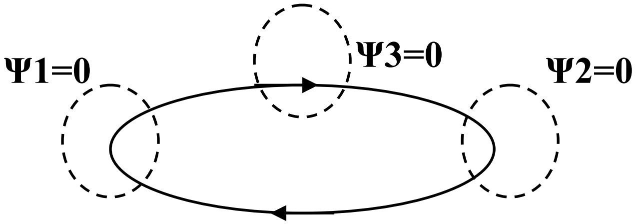
2. **Physical meaning:**

- There are no "magnetic charges"

–

3. **Magnetic field lines:**

- Continuous curves without boundary (no start, no end point), e.g. closed curves.



Local forms of Gauss law for magnetic fields

1. **Local (differential) form of the law:** $$div\mathbf{B} = 0$$

2. **Prove** **(based on Gauss Ostrogradski theorem):** $$\forall D_{\Sigma} : \oint_{\Sigma=\partial D_{\Sigma}} \mathbf{B} d\mathbf{A} = \int_{D_{\Sigma}} div\mathbf{B} dv = 0 \Rightarrow div\mathbf{B} = 0$$

1. **PDE form in Cartesian coordinates:**

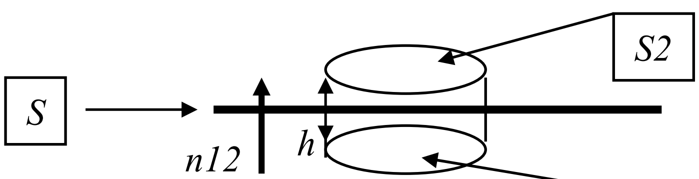
$$div\mathbf{B} = \nabla\mathbf{B} = (\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z})(\mathbf{i}B_x + \mathbf{j}B_y + \mathbf{k}B_z) = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$

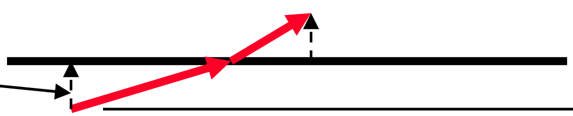
4. **Conservation of the normal component of the flux density**

$$\oint_{\Sigma} \mathbf{B} d\mathbf{A} = \int_{S2} \mathbf{B} d\mathbf{A} + \int_{S1} \mathbf{B} d\mathbf{A} = \mathbf{n}_{12} \cdot (\mathbf{B}_2 - \mathbf{B}_1)_{ave} A = 0$$

$$\underset{h \to 0}{\Longrightarrow} \mathbf{n}_{12} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0$$

$$\mathbf{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0 \Rightarrow B_{n1} = B_{n2}$$



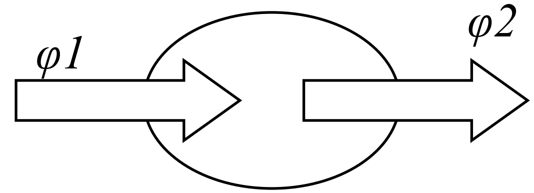


Integral consequences

1. **Magnetic flux conservation**

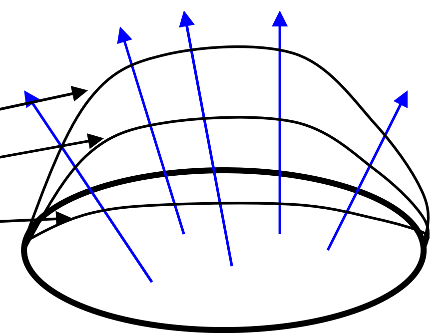
$$\varphi = \oint_{\Sigma = S1 \cup S2} \mathbf{B}d\mathbf{A} = \int_{S1} \mathbf{B}d\mathbf{A} + \int_{S2} \mathbf{B}d\mathbf{A} =$$

$$\int_{S1} \mathbf{B}d\mathbf{A}_1 - \int_{S2} \mathbf{B}d\mathbf{A}_2 = \varphi_1 - \varphi_2 = 0 \Longrightarrow \boxed{\varphi_1 = \varphi_2}$$



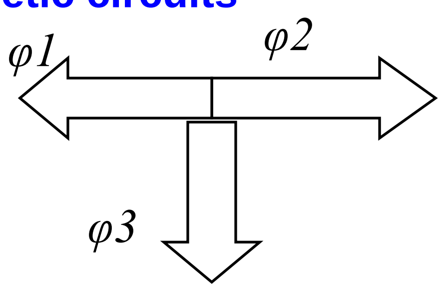
2. **Invariance of the magnetic flux w.r.t. surface shape (for a fixed border)**

$$\varphi_1 = \varphi_2 = \varphi_2 \ldots$$



2. **Kirchhoff flux law for magnetic circuits**

$$\boxed{\sum_{k \in (n)} \varphi_k = 0}$$



Magnetic vector potential

1. **Definition of the magnetic vector potential A:**

$$div\mathbf{B}=0 \Rightarrow \boxed{\mathbf{B}=curl\mathbf{A}}$$

$$div\mathbf{B}=divcurl\mathbf{A}=\nabla\cdot(\nabla\times\mathbf{A})=0$$

$$\varphi=\int_S \mathbf{B}d\mathbf{S}=\int_S curl\mathbf{A}d\mathbf{S}=\int_{\partial S}\mathbf{A}d\mathbf{r}\Rightarrow \boxed{\varphi=\int_{\partial S}\mathbf{A}d\mathbf{r}}$$

2. **Curl definition:**

$$curl\mathbf{A}=\mathbf{n}\lim_{A_s\to 0}\oint_{\partial S}\mathbf{A}d\mathbf{r}/A_s$$

3. **PDE form in Cartesian coordinates:**

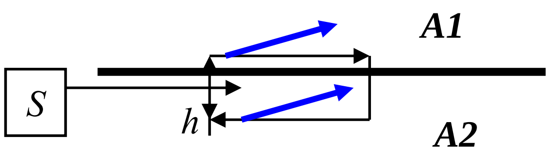
$$curl\mathbf{A}=\nabla\times\mathbf{A}=(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z})\times(\mathbf{i}A_x+\mathbf{j}A_y+\mathbf{k}A_z)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}$$

4. **Gauge condition:**

$$\mathbf{B}=curl\mathbf{A}=curl(\mathbf{A}+\mathbf{A}_0)\Rightarrow curl\mathbf{A}_0=0\Rightarrow \mathbf{A}_0=grad\lambda$$

$$div\mathbf{A}=?, div\mathbf{A}=\beta(=0).\Rightarrow div\mathbf{A}_0=divgrad\lambda=\Delta\lambda=0$$

5. **Continuity of the (tangential component of the) vector potential**



$$\varphi=B_{ave}lh=\int_{\partial S}\mathbf{A}d\mathbf{r}=(A_{t1}-A_{t2})l\to 0\Rightarrow(A_{t1}-A_{t2})=0,\forall\mathbf{n}_S\Rightarrow \mathbf{A}_{t1}=\mathbf{A}_{t2}$$

$$div\mathbf{A}=0\Rightarrow \mathbf{A}_{n1}=\mathbf{A}_{n2}\Rightarrow \boxed{\mathbf{A}_1=\mathbf{A}_2}$$

Several forms of the Gauss law for electric and magnetic fields

Field:	Electric	Magnetic
Global	$\psi_\Sigma = q_{D_\Sigma}$	$\varphi_\Sigma = 0$
Integral	$\oint_\Sigma \mathbf{D} d\mathbf{A} = \int_{D_\Sigma} \rho dv$	$\oint_\Sigma \mathbf{B} d\mathbf{A} = 0$
Local differential	$div\mathbf{D} = \rho$	$div\mathbf{B} = 0$
On surfaces	$\mathbf{n}_{12} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \rho_s$	$\mathbf{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0$
Conserv.	$D_{n1} = D_{n2}$	$B_{n1} = B_{n2}$
Field lines	Open, from positive to negative charges	Continuous, closed in most cases

Not so easy questions for curious people

1. Prove the gauss and Stokes theorems
2. Find a magnetic field having no closed curves as field lines
3. Fid the conditions to define a unique magnetic vector potential of a known magnetic field.
4. If you know two components of the field density (e.g. Bx and By) could you find the third one using Gauss law?
5. How the distribution theory (e.g. Dirac generalized function) may be used to write the Gauss laws.
6. Is still valid the B normal component conservation in the case of a sheet carrying magnetic flux ?
7. What form has the Gauss law in the case of a charged wire? What happens in the case of the punctual body?