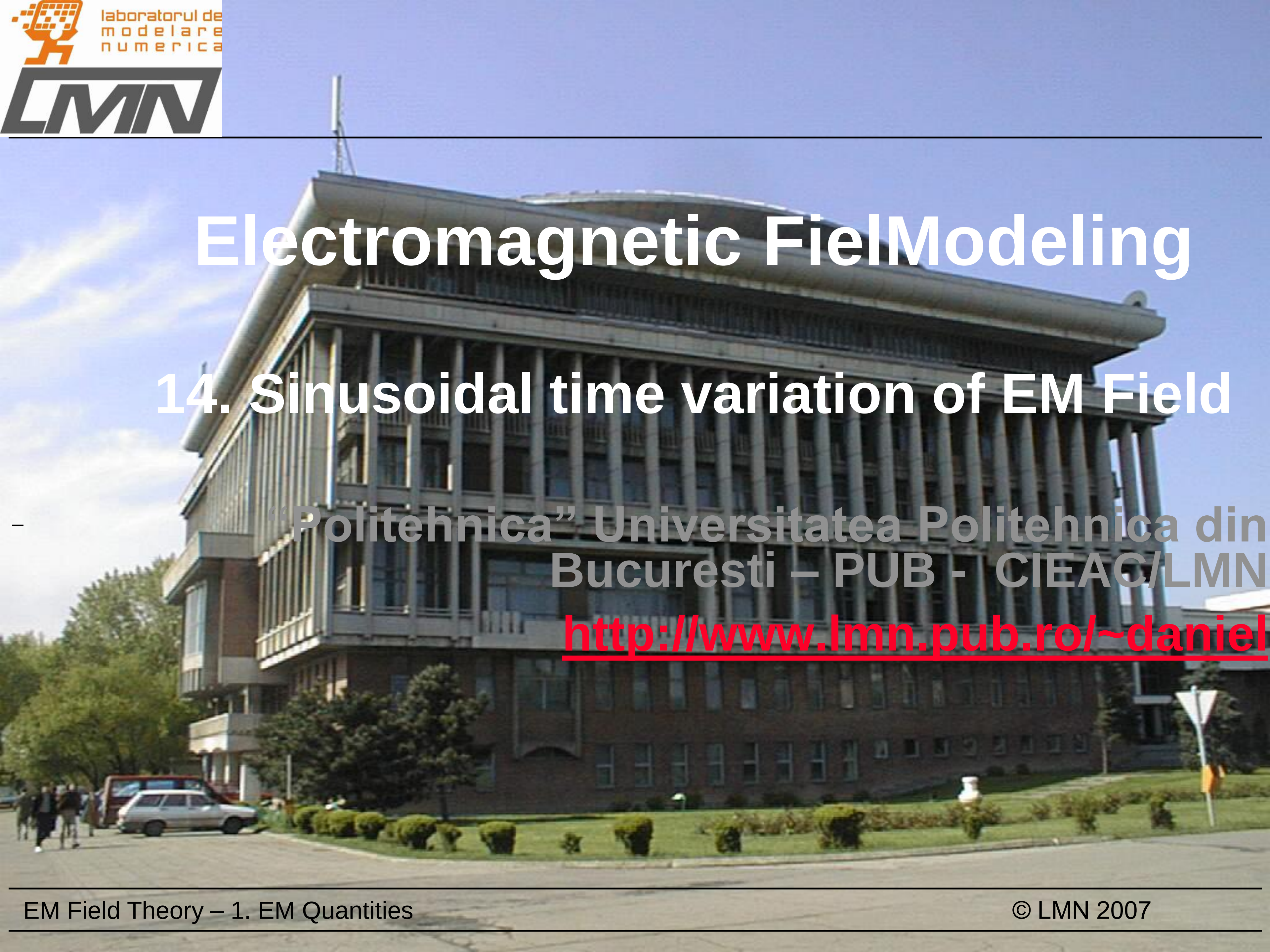

Electromagnetic FielModeling

14. Sinusoidal time variation of EM Field

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Sinusoidal (time harmonic) regime – complex representation

Sinusoidal signals, complex representation:

$$y(t) = Y\sqrt{2}\sin(\omega t + \phi) \in S \leftrightarrow \underline{Y} = \boxed{\mathscr{C}[y(t)] = Ye^{j\phi}} \in \mathbb{C} \Rightarrow$$

$$y(t) = \mathscr{C}^{-1}[\underline{Y}] = \sqrt{2}\,\mathrm{Im}\left(\underline{Y}e^{j\omega t}\right) = \sqrt{2}\,\mathrm{Im}\left(Ye^{j\phi}e^{j\omega t}\right) = \sqrt{2}\,\mathrm{Im}\left(Ye^{j(\omega t + \phi)}\right)$$

$$y'(t) = Y\sqrt{2}\omega\cos(\omega t + \phi) \leftrightarrow \underline{Y}' = j\omega\underline{Y} = Ye^{j(\phi + \pi/2)}$$

$$y(t) = \sum_{k=1,n} \lambda_k y_k(t) \leftrightarrow \underline{Y} = \sum_{k=1,n} \lambda_k \underline{Y}_k$$

–

$\mathscr{C}[.]$ complex repr. is linear and "operational"

Sinusoidal fields:

$$\boldsymbol{E}(t) = \boldsymbol{i}E_x(t) + \boldsymbol{j}E_y(t) + \boldsymbol{k}E_z(t),\ E_x, E_y, E_z \in S \Rightarrow$$

$$\underline{\boldsymbol{E}} = \boldsymbol{i}\underline{E}_x + \boldsymbol{j}\underline{E}_y + \boldsymbol{k}\underline{E}_z \in \mathbb{C}^3 \Rightarrow \boxed{\boldsymbol{G} = \frac{\partial \boldsymbol{E}}{\partial t} \in S^3 \Rightarrow \underline{\boldsymbol{G}} = j\omega\underline{\boldsymbol{E}}}$$

Periodic non-sinusoidal fields – Fourier series

Periodic signals, Fourier series, DFT, FFT

$$y(t) = y(t+T) \in \mathcal{P} \Rightarrow y(t) = \sum_{k=0}^{\infty} a_k \cos k\omega t + b_k \sin k\omega t = \boxed{\sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}}$$

$$c_k = c_{-k}^* = \begin{cases} (a_k - jb_k)/2; k > 0 \\ a_0; k = 0 \\ (a_k + jb_k)/2; k < 0 \end{cases} \Leftrightarrow a_k = c_k + c_{-k}; b_k = c_k - c_{-k}$$

$$\leftrightarrow \underline{\mathbf{Y}} = \mathcal{F}[y(t)] = [c_k]_{k=0:\infty} \in \mathbb{C}^{\infty} \Rightarrow \mathbf{Y} = \mathcal{D}[y(t)] = [c_k]_{k=0:n} \in \mathbb{C}^{n}$$

$$y(t) = \mathcal{F}^{-1}[\underline{\mathbf{Y}}] = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} = c_0 + \sum_{k=1}^{\infty} (c_k e^{jk\omega t} + c_k^* e^{-jk\omega t})$$

$$\mathcal{F}\left[\sum_{k=1,n} \lambda_k y_k(t)\right] = \sum_{k=1,n} \lambda_k \mathcal{F}[y_k(t)]; \underline{\mathbf{Y}}' = \mathcal{F}[y'(t)] = j\omega[kc_k]_{k=0:\infty}$$

Non-periodic fields – Fourier transform

When T → ∞, Fourier series → Fourier integral

Spectrum became a complex (continous) function:

$$c_k = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\exp(-jk\omega t)dt \xrightarrow{T\to\infty} \int_{-\infty}^{\infty} y(t)\exp(-jk\omega t)dt =_{def} \mathscr{F}[y(t)] = Y(\omega) : (0,\infty) \to \mathbb{C}$$

$$y(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} \xrightarrow{T\to\infty} y(t) = \mathscr{F}^{-1}[Y(\omega)] =_{def} \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)\exp(jk\omega t)d\omega; \mathscr{F}^{-1}[\mathscr{F}[y(t)]] = y(t)$$

$$\mathscr{F}\left[\sum_{k=1,n}\lambda_k y_k(t)\right] = \sum_{k=1,n}\lambda_k \mathscr{F}[y_k(t)]; \mathscr{F}[y'(t)] = j\omega\mathscr{F}[y(t)]$$

Fourier transform is linear and “operational

If y(t) is periodical, then its Fourier transform is discrete, being a sum of Dirac impulses, weighted by the Fourier coefficients:

$$FFT : X(k) = \sum_{i=1}^{N} x(i)\omega_N^{(i-1)(k-1)}; \omega_N = \exp(-2\pi j / N) \Rightarrow Y(\omega) = \mathscr{F}[y(t)] = (T/N)FFT(y)$$

$$IFFT : x(i) = (1/N)\sum_{i=1}^{N} X(k)\omega_N^{-(i-1)(k-1)} \Rightarrow y(t) = \mathscr{F}^{-1}[Y(\omega)] = (N/T)IFFT(Y)$$

Transient fields – Laplace transform

When $t \in [0, \infty)$, for transient signals, the Laplace transform is more appropriate

$$\mathcal{L}[y(t)] =_{def} \int_{-0}^{\infty} y(t) e^{-st} dt = Y(s) : \mathbb{C} \to \mathbb{C};$$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{1}{2\pi j} \lim_{\omega \to \infty} \int_{\sigma - j\omega}^{\sigma + j\omega} e^{st} Y(s) ds$$

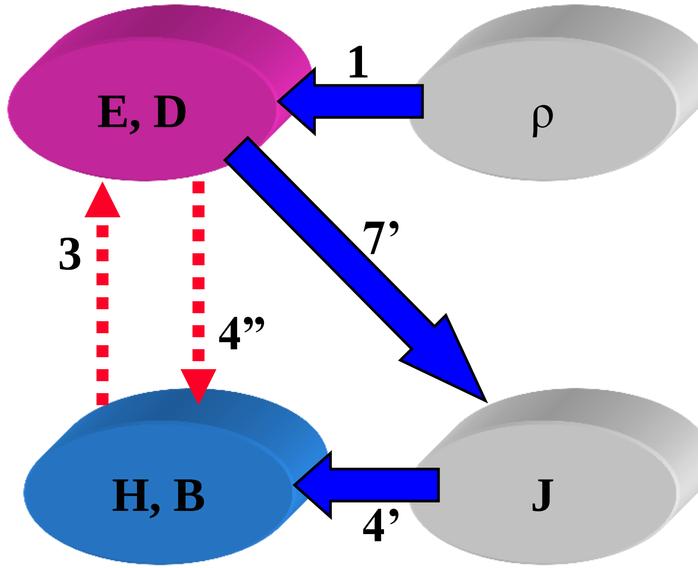
–

$$\mathcal{L}\left[\sum_{k=1,n} \lambda_k y_k(t)\right] = \sum_{k=1,n} \lambda_k \mathcal{L}[y_k(t)];$$

$$\mathcal{L}[y'(t)] = s\mathcal{L}[y(t)] - y(0_-)$$

Laplace transform is linear and “operational

Harmonic variation in time
Maxwell' equations in freq. domain



Complex representation of harmonic functions:

$$\underline{E} = \mathscr{C}[E(t)];\ \ \underline{H} = \mathscr{C}[H(t)]; \ldots$$

Complex form of the Maxwell's equations:

$$\nabla \cdot \underline{\mathbf{D}} = \rho$$

$$\nabla \cdot \underline{\mathbf{B}} = 0$$

$$\nabla \times \underline{\mathbf{E}} = -j\omega \underline{\mathbf{B}}$$

$$\nabla \times \underline{\mathbf{H}} = \underline{\mathbf{J}} + j\omega \underline{\mathbf{D}}$$

$$\underline{\mathbf{D}} = \varepsilon\ \underline{\mathbf{E}}$$

$$\underline{\mathbf{B}} = \mu\ \underline{\mathbf{H}}$$

$$\underline{\mathbf{J}} = \sigma \underline{\mathbf{E}}$$

$$\nabla \cdot (\varepsilon\ \underline{\mathbf{E}}) = \rho$$

$$\nabla \times \underline{\mathbf{E}} = -j\omega\mu\ \underline{\mathbf{H}}$$

$$\nabla \times \underline{\mathbf{H}} = (\sigma + j\omega\varepsilon\ )\underline{\mathbf{E}}$$

$$\nabla \times \underline{\mathbf{E}} = -s\mu\ \underline{\mathbf{H}}$$

$$\nabla \times \underline{\mathbf{H}} = (\sigma + s\varepsilon\ )\underline{\mathbf{E}}$$

To apply the complex transform, media should have linear constitutive relations
Sources: boundary conditions
Fields: E, H
Potentials: A, *V*
Material constants: ε, μ, σ
PDE of complex elliptic type.
After Laplace transform

Operational form of Maxwell' equations

Laplace transform of transient fields:

$$\boldsymbol{E}(s) = \mathcal{L}[\boldsymbol{E}(t)] : \mathbb{C} \to \mathbb{C}^3;\ \boldsymbol{H}(s) = \mathcal{L}[\boldsymbol{H}(t)]; ....$$

Laplace transform of Maxwell's equations:

$$\nabla \cdot \boldsymbol{D}(s) = \rho(s)$$

$$\nabla \cdot \boldsymbol{B}(s) = 0$$

$$\nabla \times \boldsymbol{E}(s) = -s\boldsymbol{B}(s) - \boldsymbol{B}_0$$

$$\nabla \times \boldsymbol{H}(s) = \boldsymbol{J}(s) + s\boldsymbol{D}(s) - \boldsymbol{D}_0$$

$$\boldsymbol{D}(s) = \varepsilon \boldsymbol{E}(s)$$

$$\boldsymbol{B}(s) = \mu \boldsymbol{H}(s)$$

$$\boldsymbol{J}(s) = \sigma \boldsymbol{E}(s)$$

Fourier transform:

$$\nabla \cdot \boldsymbol{D}(\omega) = \rho(s)$$

$$\nabla \cdot \boldsymbol{B}(\omega) = 0$$

$$\nabla \times \boldsymbol{E}(\omega) = -j\omega \boldsymbol{B}(\omega)$$

$$\nabla \times \boldsymbol{H}(\omega) = \boldsymbol{J}(s) + j\omega \boldsymbol{D}(\omega)$$

$$\boldsymbol{D}(\omega) = \varepsilon \boldsymbol{E}(\omega)$$

$$\boldsymbol{B}(\omega) = \mu \boldsymbol{H}(\omega)$$

$$\boldsymbol{J}(\omega) = \sigma \boldsymbol{E}(\omega)$$

$$s \to j\omega$$

Helmholtz equations

Laplace transform of Maxwell (FW) equations:

$$\nabla \cdot \boldsymbol{D}(s) = \rho(s)$$

$$\nabla \cdot \boldsymbol{B}(s) = 0$$

$$\nabla \times \boldsymbol{E}(s) = -s\boldsymbol{B}(s) + \boldsymbol{B}_0 = -s\mu \boldsymbol{H}(s) + \boldsymbol{B}_0 \Rightarrow$$

$$\nabla \times \left(\nu \nabla \times \boldsymbol{E}(s)\right) = -s\nabla \times \boldsymbol{H}(s) + \nu \boldsymbol{B}_0$$

$$\nabla \times \boldsymbol{H}(s) = \boldsymbol{J}(s) + s\boldsymbol{D}(s) - \boldsymbol{D}_0 = \sigma \boldsymbol{E}(s) + s\varepsilon \boldsymbol{E}(s) - \boldsymbol{D}_0$$

$$\boldsymbol{D}(s) = \varepsilon \boldsymbol{E}(s)$$

$$\boldsymbol{B}(s) = \mu \boldsymbol{H}(s) \Leftrightarrow \boldsymbol{H}(s) = \nu \boldsymbol{B}(s)$$

$$\boldsymbol{J}(s) = \sigma \boldsymbol{E}(s)$$

Helmholtz equation in the (complex) frequency domain:

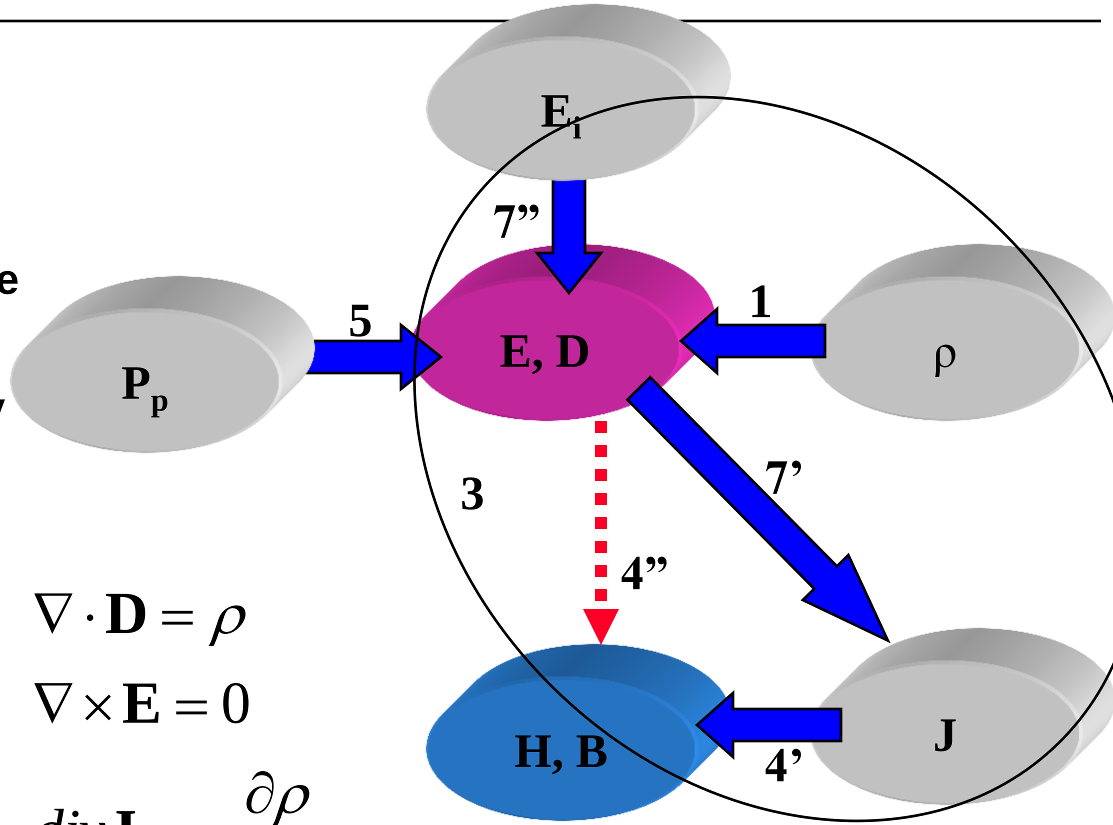
$$\Rightarrow \boxed{\nabla \times \left(\nu \nabla \times \boldsymbol{E}(s)\right) + \left(s^2 \varepsilon + s\sigma\right)\boldsymbol{E}(s) = s\boldsymbol{D}_0} + \nu \boldsymbol{B}_0 \xrightarrow{s \to j\omega}$$

$$\nabla \times \left(\nu \nabla \times \boldsymbol{E}\right) + \left(-\omega^2 \varepsilon + j\omega\sigma\right)\boldsymbol{E} = 0$$

Electro Quasi-Static regime

EQS hypothesis:

1. **No movement**
2. **Slow time variation so that Electromagnetic induction may be neglected**
3. **No interest in Magnetic field, only in charge relaxation due to the parasitic conduction**



EQS fundamental equations:

- Gauss theorem: $\nabla \cdot \mathbf{D} = \rho$
- Potential theorem: $\nabla \times \mathbf{E} = 0$
- Current-charge conservation: $div\mathbf{J} = -\frac{\partial \rho}{\partial t}$
- Polarization and conduction constitutive relations: $\mathbf{D} = \varepsilon\ \mathbf{E} + \mathbf{P}_p(\mathbf{E})$, $\mathbf{J} = \sigma(\mathbf{E} + \mathbf{E}_i(\mathbf{E}))$

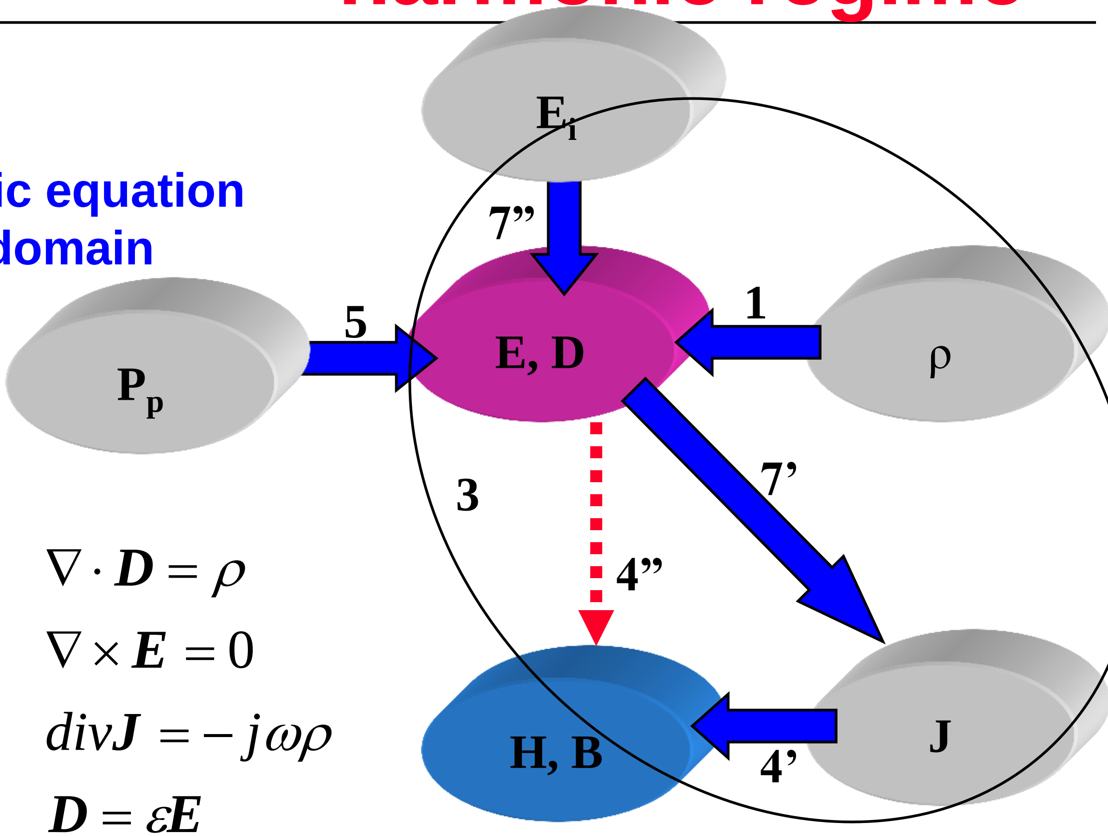
PDE of parabolic type for potential.
Field diffuses in space

Sources: $\mathbf{E}_i$
Fields: **ρ, J, D,(curl-free) E**
Potential: *V*
Material constants: ε, σ

Electro Quasi-Static time harmonic regime

$$\begin{cases} \nabla \cdot \boldsymbol{D} = \rho \\ \nabla \times \boldsymbol{E} = 0 \\ div \boldsymbol{J} = -\dfrac{\partial \rho}{\partial t} \\ \boldsymbol{D} = \varepsilon \boldsymbol{E} + \boldsymbol{P}_p(\boldsymbol{E}) \\ \boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{E}_i(\boldsymbol{E})) \end{cases}$$

→ **Parabolic equation in time domain**



$$\begin{cases} \nabla \boldsymbol{D} = \rho \\ \nabla \times \boldsymbol{E} = 0 \Rightarrow \boldsymbol{E} = -gradV \\ div \boldsymbol{J} = -s\rho + \rho_0 \\ \boldsymbol{D} = \varepsilon \boldsymbol{E} \\ \boldsymbol{J} = \sigma \boldsymbol{E} + \boldsymbol{J}_i(s) \end{cases}$$

→

$$\nabla \cdot \boldsymbol{D} = \rho$$
$$\nabla \times \boldsymbol{E} = 0$$
$$div \boldsymbol{J} = -j\omega\rho$$
$$\boldsymbol{D} = \varepsilon \boldsymbol{E}$$
$$\boldsymbol{J} = \sigma \boldsymbol{E} + \boldsymbol{J}_i(\omega)$$

$$\Rightarrow div\left((s\varepsilon - \sigma) gradV + \boldsymbol{J}_i(s)\right) = \rho_0$$

Elliptic (Poisson) equation in frequency domain

Sources: ρ0, Ji
Fields: ρ, J, D,(curl-free) E
Potential: *V*
Material constants: ε, σ

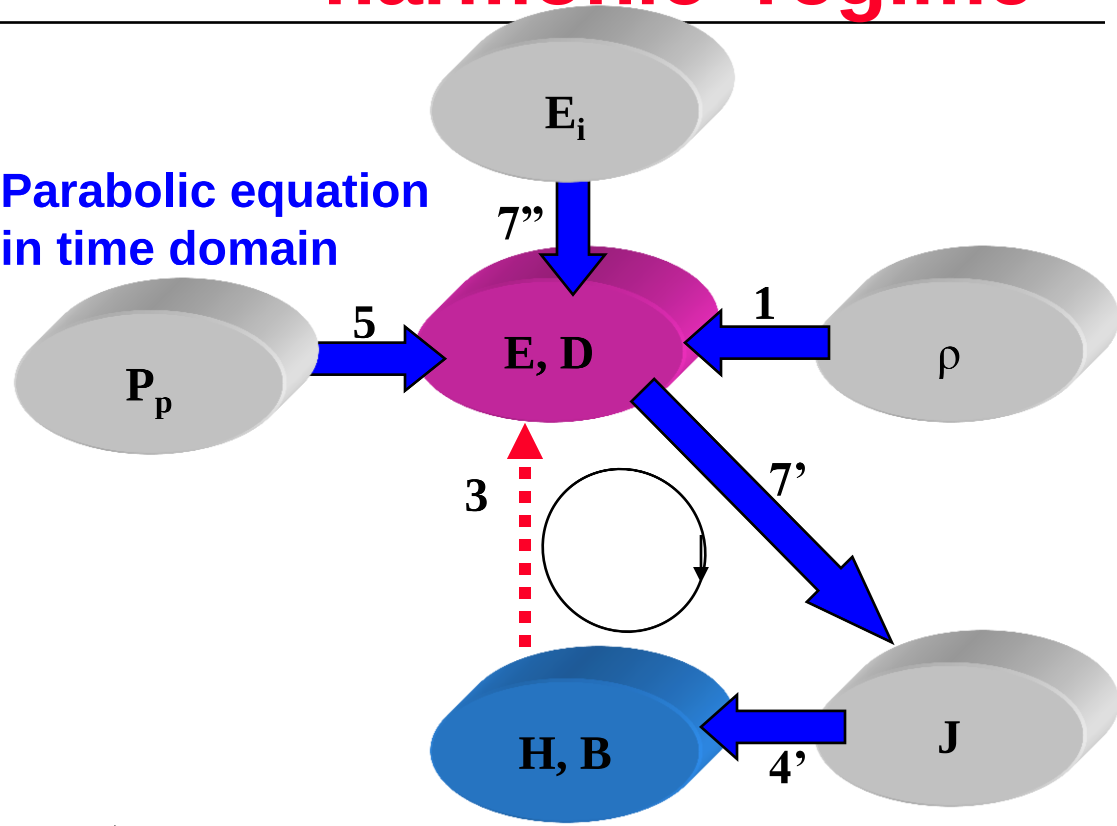
Magneto-Quasi-Static, time harmonic regime

$$\begin{cases} \nabla\cdot\boldsymbol{B}=0; \quad \nabla\times\boldsymbol{E}=-\dfrac{\partial\boldsymbol{B}}{\partial t} \\ \nabla\times\boldsymbol{H}=\boldsymbol{J}; \quad \boldsymbol{B}=\mu\boldsymbol{H} \\ \boldsymbol{J}=\sigma\boldsymbol{E}+\boldsymbol{J}_i(t) \end{cases}$$

→ Parabolic equation in time domain

$$\begin{cases} \nabla\cdot\boldsymbol{B}=0; \quad \nabla\times\boldsymbol{E}=-s\boldsymbol{B}+\boldsymbol{B}_0 \\ \nabla\times\boldsymbol{H}=\boldsymbol{J}; \quad \boldsymbol{B}=\mu\boldsymbol{H} \\ \boldsymbol{J}=\sigma\boldsymbol{E}+\boldsymbol{J}_i(s) \end{cases}$$

→



$$\boldsymbol{J}=\sigma\boldsymbol{E}+\boldsymbol{J}_i(s)$$

$$\nabla\cdot\boldsymbol{B}=0;\Rightarrow\boldsymbol{B}=\nabla\times\boldsymbol{A}$$

$$\nabla\times\boldsymbol{E}+s\boldsymbol{B}=\boldsymbol{B}_0\Rightarrow\nabla\times(\boldsymbol{E}+s\boldsymbol{A}-\boldsymbol{A}_0)=0\Rightarrow$$

$$\boldsymbol{E}+s\boldsymbol{A}-\boldsymbol{A}_0=-grad V$$

$$\nabla\times\boldsymbol{H}=\sigma\boldsymbol{E}+\boldsymbol{J}_i(s); \quad \boldsymbol{B}=\mu\boldsymbol{H}\Rightarrow$$

$$\boxed{\nabla\times\left(\nu\nabla\times\boldsymbol{A}\right)+\sigma\left(s\boldsymbol{A}+\boldsymbol{A}_0+grad V\right)=\boldsymbol{J}_i(s)}$$

Sources: J_i

Fields: E, H, (div-free) J, B

Potentials: A, *V*

Material constants: μ, σ

Helmholtz equation in fr. domain

laboratorul de modelare numerica

Full Wave Loss-Less time harmonic regime

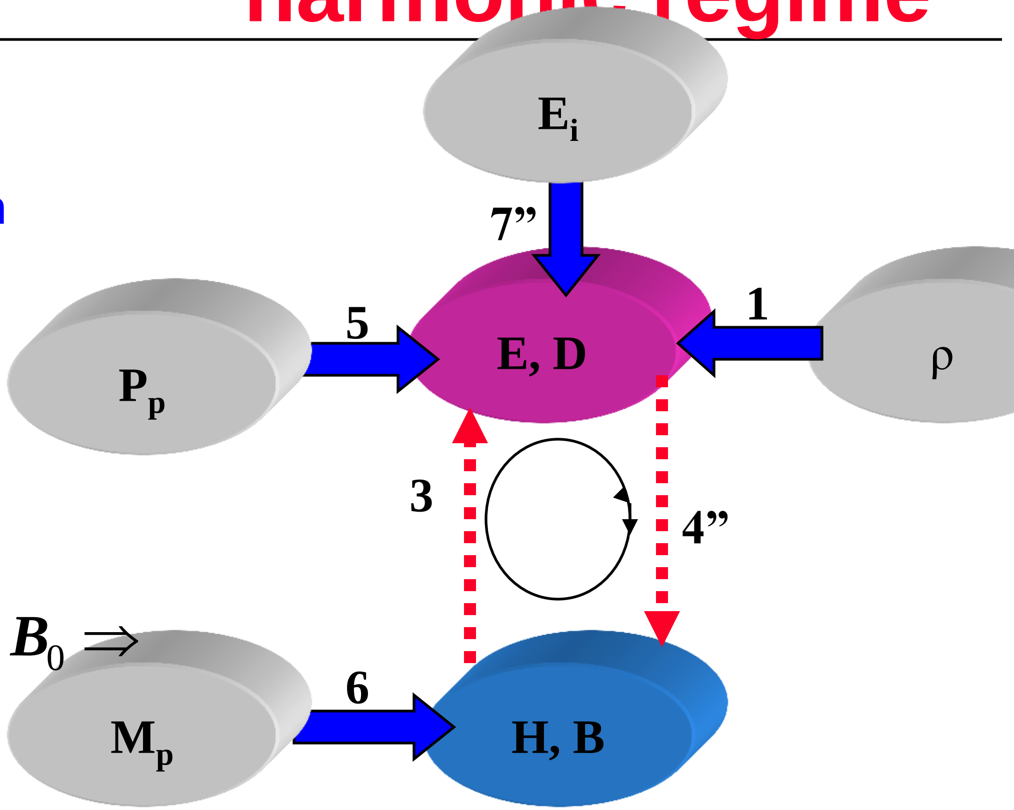
$$\nabla \cdot \boldsymbol{D} = \rho ; \nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

Hyperbolic equation in time domain

$$\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t}$$

$$\boldsymbol{D} = \varepsilon \ \boldsymbol{E} ; \boldsymbol{B} = \mu \ \boldsymbol{H}$$



$$^{-}\nabla \times \boldsymbol{E}(s) = -s\boldsymbol{B}(s) + \boldsymbol{B}_0 = -s\mu \boldsymbol{H}(s) + \boldsymbol{B}_0 \Rightarrow$$

$$\nabla \times \left(\nu \nabla \times \boldsymbol{E}(s)\right) = -s\nabla \times \boldsymbol{H}(s) + \nu \boldsymbol{B}_0$$

$$\nabla \times \boldsymbol{H}(s) = s\boldsymbol{D}(s) - \boldsymbol{D}_0 = s\varepsilon \boldsymbol{E}(s) - \boldsymbol{D}_0$$

$$\boldsymbol{D}(s) = \varepsilon \boldsymbol{E}(s); \boldsymbol{B}(s) = \mu \boldsymbol{H}(s) \Leftrightarrow \boldsymbol{H}(s) = \nu \boldsymbol{B}(s)$$

$$\Rightarrow \nabla \times \left(\nu \nabla \times \boldsymbol{E}(s)\right) + s^2 \varepsilon \boldsymbol{E}(s) = s\boldsymbol{D}_0 + \nu \boldsymbol{B}_0$$

Helmholtz equation in frequency domain

Sources: J_i, D_0, B_0,
Fields: E, D, H, (div-free) B
Potentials: A, *V*
Material constants: ε, μ
Field **is propagating** in space

Weak forms of operational (Helmhotz) equation

$$\nabla\times\left(\nu\nabla\times\boldsymbol{E}(s)\right)+\left(s^2\varepsilon+s\sigma\right)\boldsymbol{E}(s)=s\boldsymbol{D}_0+\nu\boldsymbol{B}_0$$

Helmholtz equation for ED-FW field in frequency domain

In LL: σ= 0; in MQS: ε =0, D0=0

With following boundary conditions:

$$\boldsymbol{E}_t:\quad \boldsymbol{n}\times\boldsymbol{E}=\boldsymbol{p}\quad on\ S_E$$ **essential**

–

$$\boldsymbol{H}_t(ABC):\quad \boldsymbol{n}\times\left(\nu\nabla\times\boldsymbol{E}\right)+\gamma\boldsymbol{n}\times(\boldsymbol{E}\times\boldsymbol{n})=\boldsymbol{q}\quad on\ S_H$$ **natural**

the weak form of this equation is

$$\int_\Omega\left(\nu(\nabla\times\boldsymbol{W})\cdot(\nabla\times\boldsymbol{E})+\left(s^2\varepsilon+s\sigma\right)\boldsymbol{W}\cdot\boldsymbol{E}\right)dv+$$

$$\int_{S_H}\boldsymbol{W}(\boldsymbol{q}+\gamma\boldsymbol{n}\times(\boldsymbol{E}\times\boldsymbol{n})dS=s\int_\Omega\left(\boldsymbol{W}\cdot(\boldsymbol{J}_i+s\boldsymbol{D}_0+\nu\boldsymbol{B}_0\right)dv$$

Green functions, solutions of Helmholtz equations

- **Helmholtz equations for electro-dynamic potentials in LL-FW**

$$\nabla \cdot \boldsymbol{B} = 0 \Rightarrow \boldsymbol{B} = \nabla \times \boldsymbol{A}; \boldsymbol{D} = \varepsilon_0 \boldsymbol{E}; \boldsymbol{B} = \mu_0 \boldsymbol{H}; \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \Rightarrow \boldsymbol{E} = -\nabla V - j\omega \boldsymbol{A}$$

$$\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J}_i \Rightarrow \nabla \times \boldsymbol{H} = j\omega\varepsilon_0 \boldsymbol{E} + \boldsymbol{J}_i \Rightarrow \nabla \times \nabla \times \boldsymbol{A} + j\omega\varepsilon_0\mu_0(\nabla V + j\omega \boldsymbol{A}) = \mu_0 \boldsymbol{J}_i$$

$$\nabla(\nabla \cdot \boldsymbol{A}) - \Delta \boldsymbol{A} - \varepsilon_0\mu_0\omega^2 \boldsymbol{A} + j\omega\varepsilon_0\mu_0 \nabla V = \mu_0 \boldsymbol{J}_i; \Rightarrow \boxed{\Delta \boldsymbol{A} + \varepsilon_0\mu_0\omega^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}_i}$$

$$\nabla \cdot \boldsymbol{A} = -j\omega\varepsilon_0\mu_0 V \Leftrightarrow \boxed{div\boldsymbol{A} = -\varepsilon_0\mu_0 \frac{\partial V}{\partial V}}$$

Lorenz gauge conditions

–

$$\nabla \cdot \boldsymbol{D} = \rho \Rightarrow \nabla \cdot \boldsymbol{E} = \rho / \varepsilon_0 \Rightarrow \Delta V + j\omega \nabla \cdot \boldsymbol{A} = \rho / \varepsilon \Rightarrow \boxed{\Delta V + \varepsilon_0\mu_0\omega^2 V = \rho / \varepsilon}$$

Both A and V satisfy Helmholtz equations:

$$\Delta Y + k^2 Y = f; k^2 = \varepsilon_0\mu_0\omega^2 = (\omega / c_0)^2 \Rightarrow Y(\boldsymbol{r}) = \int_\Omega G(\boldsymbol{r}, \boldsymbol{r}') f(\boldsymbol{r}') dv;$$

$$\Delta G + k^2 G = \delta(\boldsymbol{r}, \boldsymbol{r}') \Rightarrow G(\boldsymbol{r}, \boldsymbol{r}') = \frac{1}{4\pi} \frac{\exp(-jkR)}{R}; R = |\boldsymbol{r} - \boldsymbol{r}'| \Rightarrow$$

$$\boxed{V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \int_{\partial\Omega_c} \frac{\exp(-jkR)\rho_s(\boldsymbol{r}')}{R} dS;} \quad \boxed{\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int_{\partial\Omega_c} \frac{\exp(-jkR)\boldsymbol{J}_s(\boldsymbol{r}')}{R} dS;}$$

Integral equations (EFIE, MFIE) on PEC scatter objects

- **Scattered field:**

$$\boldsymbol{E}^s = -\nabla V - j\omega \boldsymbol{A} = -\frac{\nabla}{4\pi\varepsilon_0}\int_{\partial\Omega_c}\frac{\exp(-jkR)\rho_s(\mathbf{r}')}{R}dS' - \frac{j\omega\mu_0}{4\pi}\int_{\partial\Omega_c}\frac{\exp(-jkR)\boldsymbol{J}_s(\mathbf{r}')}{R}dS'$$

$$\nabla \boldsymbol{J}_s = -j\omega\rho_s \Rightarrow \rho_s = (j/\omega)\nabla \boldsymbol{J}_s \Rightarrow$$

$$\boldsymbol{E}^s = -\frac{j\nabla}{4\pi\omega\varepsilon_0}\int_{\partial\Omega_c}\frac{\exp(-jkR)\nabla'\boldsymbol{J}_s(\mathbf{r}')}{R}dS' - \frac{j\omega\mu_0}{4\pi}\int_{\partial\Omega_c}\frac{\exp(-jkR)\boldsymbol{J}_s(\mathbf{r}')}{R}dS'$$

$$\boldsymbol{E}_t^i + \boldsymbol{E}_t^s = 0 \Rightarrow$$

Electric Field Integral Equation (EFIE):

$$\frac{-j\nabla}{4\pi\omega\varepsilon_0}\int_{\partial\Omega_c}\frac{\exp(-jkR)}{R}\nabla'\boldsymbol{J}_s(\mathbf{r}')dS'\Big|_t + \frac{j\omega\mu_0}{4\pi}\int_{\partial\Omega_c}\frac{\exp(-jkR)}{R}\boldsymbol{J}_s(\mathbf{r}')dS'\Big|_t = \boldsymbol{E}_t^i$$

Magnetic Field Integral Equation (MFIE):

$$\frac{1}{2}n\times\boldsymbol{J}_s(\mathbf{r}) + \frac{1}{4\pi}\int_{\partial\Omega_c}\nabla\left(\frac{\exp(-jkR)}{R}\right)\times\boldsymbol{J}_s(\mathbf{r}')dS'\Big|_t = -\boldsymbol{H}_t^i$$

Multipolar electric circuit element with distributed parameters

- **Boundary conditions:**

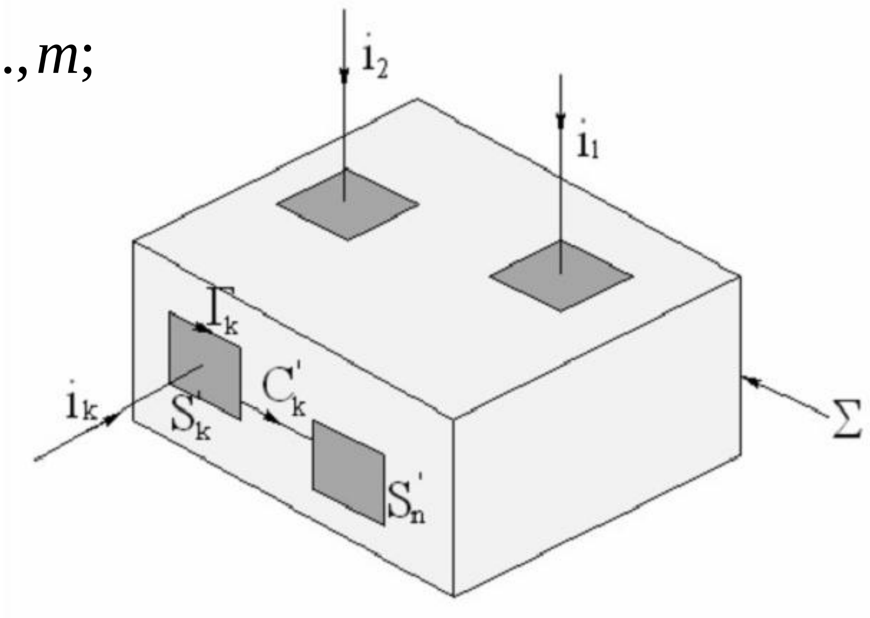
$$\boldsymbol{E}_t = \boldsymbol{n}\times(\boldsymbol{E}\times\boldsymbol{n}) = 0 \quad \text{on} \quad S_k, k = 1,2,\ldots,m;$$

$$\boldsymbol{n}\cdot(\nabla\times\boldsymbol{E}) = 0 \quad \text{on} \quad \partial\Omega;$$

$$\boldsymbol{n}\cdot(\nabla\times\boldsymbol{H}) = 0 \quad \text{on} \quad \partial\Omega - \bigcup_{k=1}^{m} S_k;$$

$$\int_{C_{km}\in\partial\Omega} E dr = V_k \in \mathbb{C}, k = 1,2,..,n;$$

$$\oint_{\partial S_k} H dr = I_k \in \mathbb{C}, k = n+1,2,..,m-1;$$



- **Solution uniqueness:**

$$P_s = P + jQ = \int_{\partial\Omega}\left(\boldsymbol{E}\times\boldsymbol{H}^*\right)d\boldsymbol{S} = \int_{\Omega}\left(\sigma\boldsymbol{E}\boldsymbol{E}^* + j\omega(\boldsymbol{B}\boldsymbol{H}^* - \boldsymbol{E}\boldsymbol{D}^*)\right)dv = \sum_{k=1}^{m} V_k I_k^*$$

- **Circuit functions:**

$$\begin{bmatrix} \mathbf{V}_a \\ \mathbf{I}_b \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_{aa} & \mathbf{A}_{ab} \\ \mathbf{B}_{ba} & \mathbf{Y}_{bb} \end{bmatrix} \begin{bmatrix} \mathbf{I}_a \\ \mathbf{V}_b \end{bmatrix},$$

Eigen-frequencies

A dielectric cavity with PEC boundary. Find resonant frequencies:

$$\nabla \times \mathbf{E} + j\omega\mu\mathbf{H} = 0,$$
$$\nabla \times \mathbf{H} - j\omega\varepsilon\mathbf{E} = 0,$$
$$\nabla(\mu\mathbf{H}) = 0,$$
$$\nabla(\varepsilon\mathbf{E}) = 0,$$

$$\nabla \times \mu^{-1}\nabla \times \mathbf{E} = \omega^2\varepsilon\mathbf{E}, \text{ in } \Omega$$
$$\nabla\varepsilon\mathbf{E} = 0, \text{ in } \Omega$$
$$\mathbf{E} \times \mathbf{n} = 0, \text{ pe } \partial\Omega.$$

Weak (variational) form:

$$Find \quad \lambda > 0, \quad \boldsymbol{u} \in H_0(curl, \Omega), s.t.$$
$$\int_\Omega \mu^{-1}(\nabla \times \boldsymbol{u}) \cdot (\nabla \times \boldsymbol{v}) d\boldsymbol{x} = \lambda \int_\Omega \varepsilon \boldsymbol{u} \cdot \boldsymbol{v} d\boldsymbol{x}, \forall \boldsymbol{v} \in H_0(curl, \Omega);$$

$H_0(curl, \Omega)$ **Sobolev space of curl-conform functions with zero boundary conditions**

Resonant cavity at dasy.de