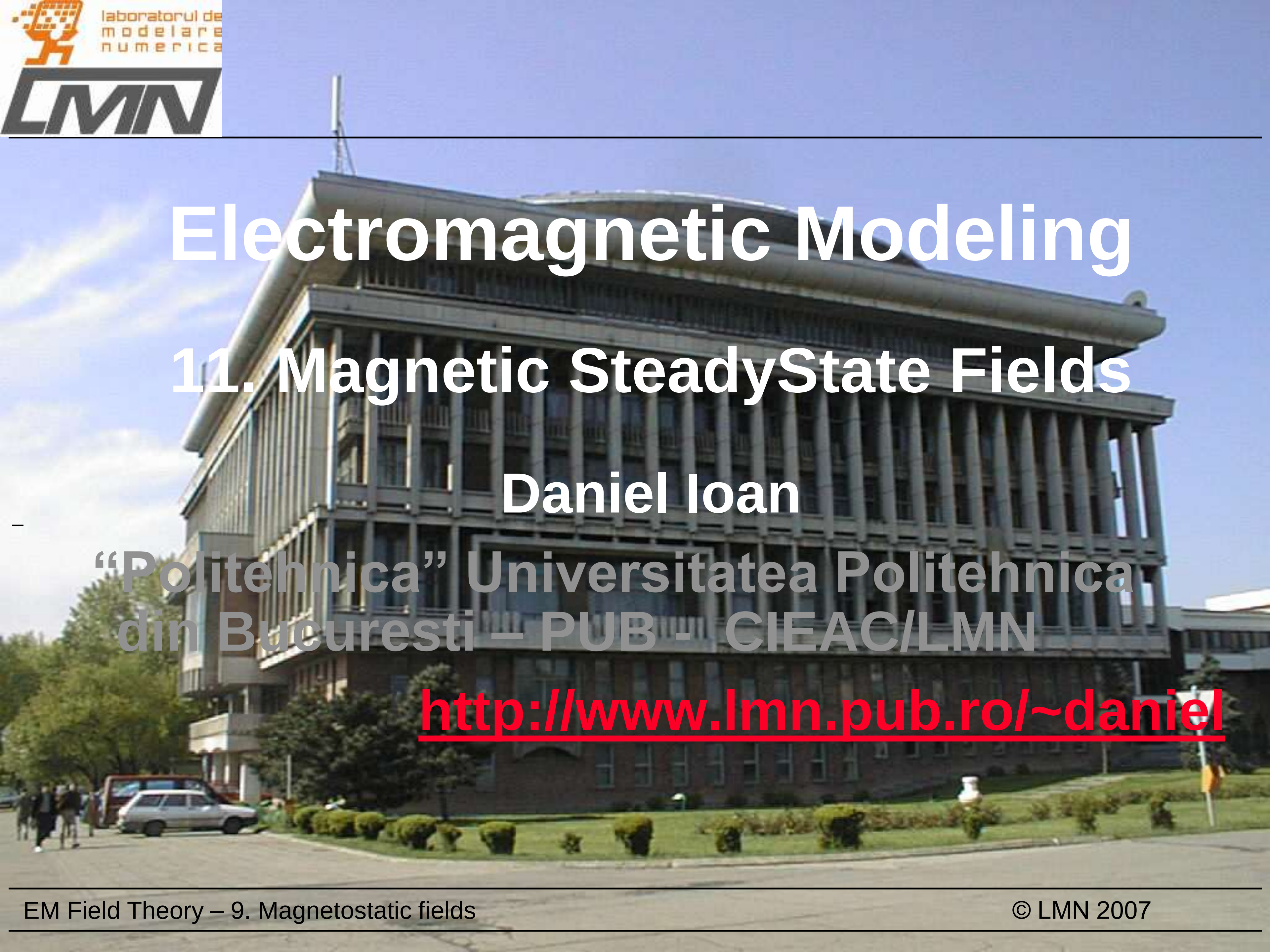

Electromagnetic Modeling

11. Magnetic SteadyState Fields

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Contents

Steady state regimes

$$1.\ \nabla\cdot\mathbf{D}=\rho$$

$$2.\ \nabla\cdot\mathbf{B}=0$$

$$3.\ \nabla\times\mathbf{E}=-\cancel{\frac{\partial\mathbf{B}}{\partial t}}$$

$$4.\ \nabla\times\mathbf{H}=\mathbf{J}+\cancel{\frac{\partial\mathbf{D}}{\partial t}}$$

$$5.\ \mathbf{D}=\varepsilon\ \mathbf{E}+\mathbf{P}_p(\mathbf{E})$$

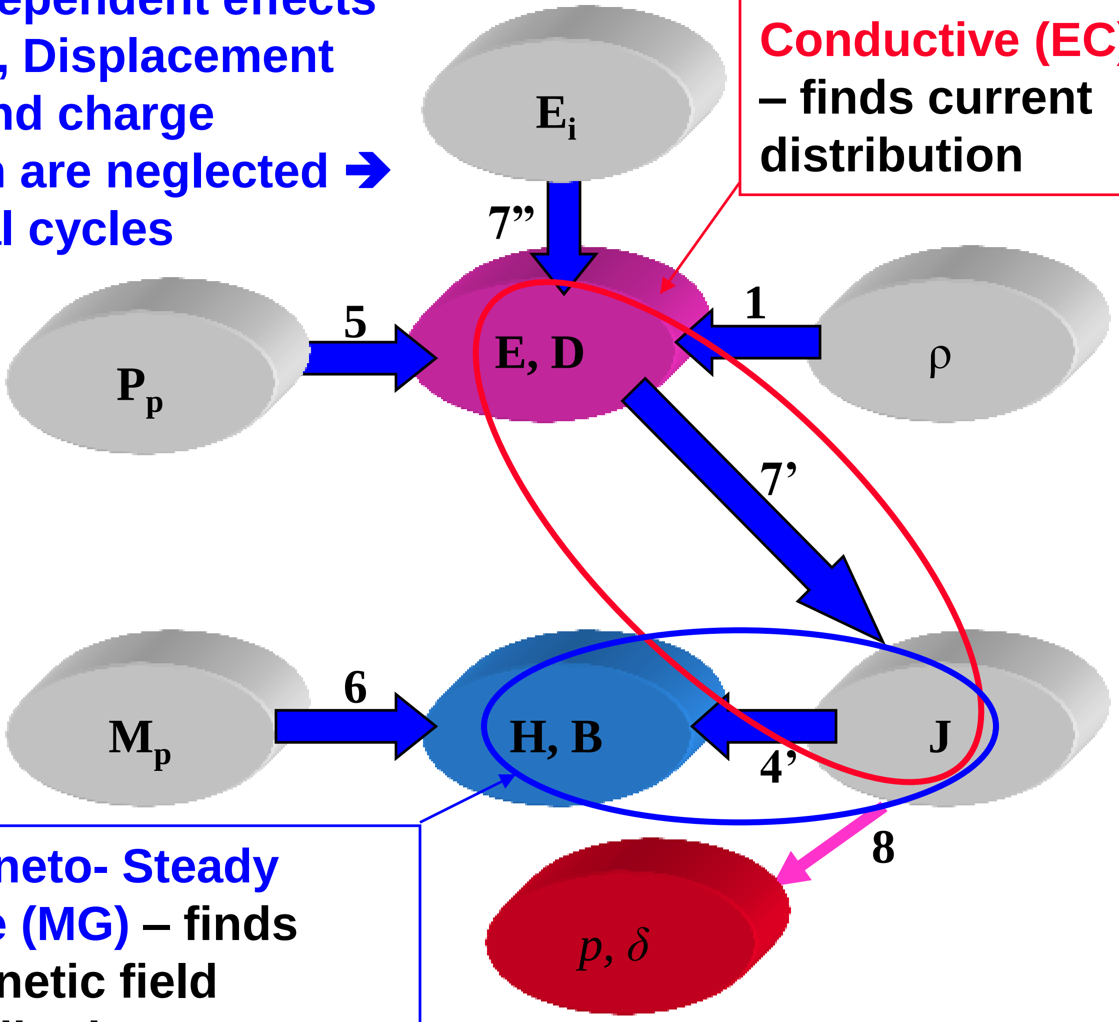
$$6.\ \mathbf{B}=\mu\ (\mathbf{H}+\mathbf{M}_p(\mathbf{H}))$$

$$7.\ \mathbf{J}=\sigma(\mathbf{E}+\mathbf{E}_i(\mathbf{E}))$$

$$8.\ p=\mathbf{E}\mathbf{J}$$

$$9.\ \nabla\cdot\mathbf{J}=-\cancel{\frac{\partial\rho}{\partial t}}$$

All time dependent effects Induction, Displacement current and charge relaxation are neglected ➔ NO causal cycles



Electro-Conductive (EC) – finds current distribution

Magneto- Steady state (MG) – finds magnetic field distribution

MG – Steady magnetic regime

- **Hypothesis:**
  - no movement
  - no time variation
  - known current distribution

$$\begin{cases} \Phi_\Sigma = 0 \Leftrightarrow \oint_\Sigma \mathbf{B} d\mathbf{A} = 0 \\ div\mathbf{B} = 0 \Rightarrow \mathbf{B} = curl\mathbf{A} \\ \mathbf{n}_{12} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0 \Leftrightarrow div_s \mathbf{B} = 0 \end{cases}$$

- **Fundamental Equations:**
- **Gauss' theorem**
- **Ampere's theorem**
- **Magnetic constitutive relation**

$$\begin{cases} u_{m\Gamma} = i_{S_\Gamma} \Leftrightarrow \oint_\Gamma \mathbf{H} d\mathbf{r} = \int_{S_\Gamma} \mathbf{J} \cdot \mathbf{n} dS \\ \mathbf{curl}\mathbf{H} = \boxed{\mathbf{J}} \Rightarrow \mathbf{H} = \mathbf{T} - \mathbf{grad} V_m \\ \mathbf{n}_{12} \times (\mathbf{H}_2 - \mathbf{H}_1) = 0, \Leftrightarrow \mathbf{H}_{t2} = \mathbf{H}_{t1} \end{cases}$$

$$\mathbf{B} = \mathbf{f}(\mathbf{H}) \Rightarrow \mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \Rightarrow \mathbf{B} = \overline{\overline{\mu}}\mathbf{H} + \mu_0 \mathbf{M}_p$$

- **Field sources:**
  - Conduction current
  - Permanent magnetization

- **MG field is similar to MS field Excepting J !!!!!**

MG:	$\mathbf{H}$	$\mathbf{B}$	$\mathbf{I}_\mathrm{p}$	μ	V_m	Φ
MS:	$\mathbf{H}$	$\mathbf{B}$	$\mathbf{I}_\mathrm{p}$	μ	V_m	Φ
EC:	$\mathbf{E}$	$\mathbf{J}$	$\mathbf{E}_\mathrm{i}$	σ	V	I
ES:	$\mathbf{E}$	$\mathbf{D}$	$\mathbf{P}_\mathrm{p}$	ε	V	Ψ

Second order equation for the scalar potential

$$\begin{cases} div\mathbf{B}=0 \Rightarrow \ div\left[\bar{\bar{\mu}}(\mathbf{T}-\mathbf{grad}V_m)+\mathbf{I}_p\right]=0 \\ \mathbf{curl}\,\mathbf{H}=\mathbf{J},\ \mathbf{J}=\mathbf{curl}\,\mathbf{T} \Rightarrow \mathbf{H}=\mathbf{T}-\mathbf{grad}V_m \\ \mathbf{B}=\bar{\bar{\mu}}\mathbf{H}+\mathbf{I}_p \Rightarrow \mathbf{B}=\bar{\bar{\mu}}(\mathbf{T}-\mathbf{grad}V_m)+\mathbf{I}_p \end{cases} \Rightarrow$$

$$\boxed{-div(\bar{\bar{\mu}}\mathbf{grad}V_m)=\rho_m}$$

$$\boxed{\rho_m=-div(\bar{\bar{\mu}}\mathbf{T}+\mathbf{I}_p)}$$

Vm=reduced potential

- **A current distribution J may be substituted by an equivalent Ip=μT, with same Vm**

- **T is a particular magnetic field of current density J (regardless boundary conditions). It is called "source field". For instance it may be a Biot-Savart-Laplace integral:**

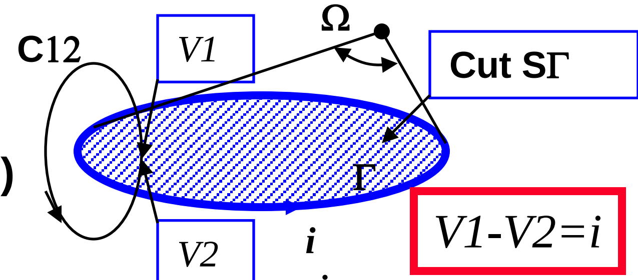
$$\mathbf{T}=\mathbf{H}_s(\mathbf{r})=\frac{1}{4\pi}\int_{R^3}\frac{\mathbf{J}\times\mathbf{R}dv}{R^3}$$

- **J=0 ➔ T=0 only in simply connected domains.**

– **Otherwise the Ampere's theorem is ot satisfied, because** \ $\oint_{\Gamma}\mathbf{H}d\mathbf{r}=-\oint_{\Gamma}gradV_m d\mathbf{r}=-\oint_{\Gamma}dV=0$

- **The multiple connected domains which surround currents should non-zero T, or they have to be transformed in simple connected domains by cuts. Ampere's theorem imposes on each cut a jump of Vm equal to current I.**

- **Each coil may be substituted by an equivalent magnetic shell having the shape of the cut and a superficial magnetization Ms=I normally oriented (potential double layer)**

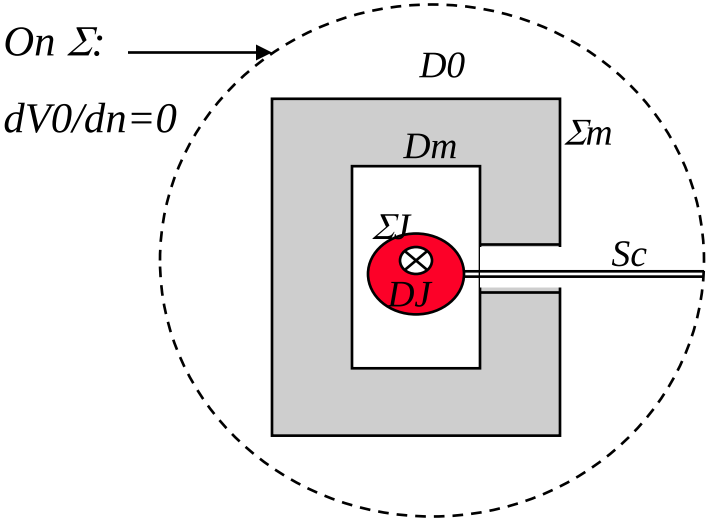


$$\mathbf{H}=\frac{i}{4\pi}\oint_{\Gamma}\frac{d\mathbf{r}\times\mathbf{R}}{R^2}\Rightarrow V_m=-\frac{i}{4\pi}\int_{S_\Gamma}\frac{\mathbf{Rn}dS}{R^3}=-\frac{i\Omega}{4\pi}\text{, or}$$

$$V_m=-\frac{M}{4\pi}\int_{S_\Gamma}(1/R^2)cos\,\alpha dS=-\frac{i}{4\pi}\int_{S_\Gamma}d\Omega=-\frac{i\Omega}{4\pi},\Rightarrow\oint_{C_{12}}\mathbf{H}d\mathbf{r}=-\oint_{C_{12}}gradV_m d\mathbf{r}=V_1-V_2=\frac{i}{4\pi}(\Omega_2-\Omega_1)=i$$

MG formulation with two scalar potentials

- **The scalar potential V is defined on sub-domains:**
  - VJ = MG scalar potential on currents domain DJ
  - V0 = MS scalar potential in air D-DJ-Dm-Sc
  - Vm = MS scalar potential in magnetic domain Dm
- **T is defined on DJ**
- **Interface conditions:**



$$\text{On } \Sigma_\text{J}: \begin{cases} \mathbf{H}_{t1} = \mathbf{H}_{t2} \Rightarrow \mathbf{T}_t - \mathbf{t}dV_J / dt = -\mathbf{t}dV_0 / dt \\ B_{n1} = B_{n2} \Rightarrow T_n - dV_J / dn = -dV_0 / dn \end{cases}$$

$$\text{On } \Sigma_\text{m}: \begin{cases} \mathbf{H}_{t1} = \mathbf{H}_{t2} \Rightarrow V_0 = V_m \\ B_{n1} = B_{n2} \Rightarrow \mu_0 dV_0 / dn = \mu_m dV_m / dn \end{cases}$$

$$\text{In } P \in S_\text{c}: \begin{cases} \mathbf{H}_{t1} = \mathbf{H}_{t2} \Rightarrow V_2(P) = V_1(P) + I \\ B_{n1} = B_{n2} \Rightarrow dV_1 / dn = dV_2 / dn \end{cases}$$

- **Advantage: avoid difference error in Dm where Hm $\simeq$ 0**

- **Approximations:**

$$\text{In } D_\text{J}: \begin{cases} T = JR/2 = IR/(2\pi a^2), \\ \mathbf{T} = \mathbf{I} \times \mathbf{R} / (2\pi a^2), \quad \mathbf{H} = \mathbf{T} - gradV_J, V_J = 0 \end{cases}$$

$$\text{On } \Sigma_\text{J}: \begin{cases} \mathbf{H}_{t1} = \mathbf{H}_{t2} \Rightarrow \mathbf{T} = -\mathbf{t}dV_0 / dt \Rightarrow \\ V_0 = -\int_0^{a\theta} T(a)dt = -I\theta/(2\pi) \\ B_{n1} = B_{n2} \Rightarrow dV_0 / dn = 0 \end{cases}$$

$$\text{On } \Sigma_\text{m}: \begin{cases} \mathbf{H}_{t1} = \mathbf{H}_{t2} \Rightarrow V_0 = V_m = ct \\ B_{n1} = B_{n2} \Rightarrow dV_0 / dn = 0 \end{cases}$$

$$\text{In } P \in S_\text{c}: \begin{cases} \mathbf{H}_{t1} = \mathbf{H}_{t2} = 0 \Rightarrow V_1(P) = 0, \quad V_2(P) = I \\ B_{n1} = B_{n2} \Rightarrow dV_1 / dn = dV_2 / dn \end{cases}$$

Second order equation for the vector potential

$$\begin{cases} div\mathbf{B}=0 \Rightarrow \mathbf{B}=\mathbf{curl}\mathbf{A} \\ \mathbf{curl}\mathbf{H}=\mathbf{J} \Rightarrow \mathbf{curl}\left[\overline{\overline{\nu}}(\mathbf{curl}\mathbf{A}-\mathbf{I}_p)\right]=\mathbf{J} \\ \mathbf{B}=\overline{\overline{\mu}}\mathbf{H}+\mathbf{I}_p \Rightarrow \mathbf{H}=\overline{\overline{\nu}}(\mathbf{B}-\mathbf{I}_p) \end{cases}$$

$$\boxed{\mathbf{curl}\left[\overline{\overline{\nu}}\mathbf{curl}\mathbf{A}\right]=\mathbf{J}_t}$$

$$\mathbf{J}_t=\mathbf{J}+\mathbf{J}_m, \mathbf{J}_m=\mathbf{curl}(\overline{\overline{\nu}}\mathbf{I}_p)$$

Total current density = conduction + magnetization

Is added to

$$\boxed{divA=0}$$

Particular cases:

- **Linear homogeneous isotropic media (Poisson vector equation):**

$$\mathbf{curl}[\mathbf{curl}\mathbf{A}]=\mu\mathbf{J}_t \Rightarrow \mathbf{grad}(div\mathbf{A})-\Delta\mathbf{A}=\mu\mathbf{J}_t \Rightarrow \boxed{\Delta\mathbf{A}=-\mu\mathbf{J}_t}$$

– • **No internal ES field sources (Laplace vector equation):**

$$\mathbf{curl}[\mathbf{curl}\mathbf{A}]=0 \Rightarrow \boxed{\Delta\mathbf{A}=0}$$

with Coulomb gauge condition: $\boxed{div\mathbf{A}=0}$

Vector boundary conditions are necessary for a unique field solution

Dirichlet b.c.

$$\boxed{\begin{array}{l}\mathbf{A}_t(P)=\mathbf{f}_{DA}(P), \text{on } S_B \neq \emptyset \\ V_m(P)=f_{DV}(P), \text{on } S_H \neq \emptyset\end{array}}$$

================or ==========

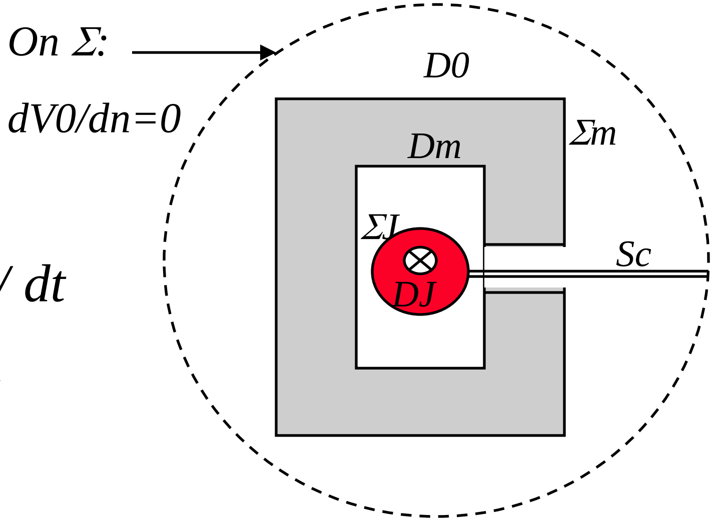
Neumann b,c.

$$\boxed{\begin{array}{l}\mathbf{n}\times(\text{curl}\mathbf{A}\times\mathbf{n})=\mathbf{f}_{NA}(P), \\ dV_m/dn=f_{NV}(P),\end{array}}\begin{array}{l}\text{on } S_H=\Sigma-S_B \\ \text{on } S_B=\Sigma-S_H\end{array}$$

$$\text{and } \boxed{\Delta V_m(P)=I_k}, \text{on } S_{cut-k}$$

AV formulation with two potentials

- **The scalar potential V is defined on sub-domains D-DJ-Sc:**
- **A is defined on DJ**
- **Interface conditions:**



$$\text{On } \Sigma_\text{J}:\begin{cases}\mathbf{H}_{t1}=\mathbf{H}_{t2}\Rightarrow \mathbf{n}\times\nu_0\mathbf{curl}\mathbf{A}\times\mathbf{n}=-\mathbf{t}dV_0/dt\\ B_{n1}=B_{n2}\Rightarrow \mathbf{n}\cdot\mathbf{curl}\mathbf{A}=-\mu dV/dn\end{cases}$$

$$\text{On } \Sigma_\text{m}:\begin{cases}\mathbf{H}_{t1}=\mathbf{H}_{t2}\Rightarrow V_0=V_m\\ B_{n1}=B_{n2}\Rightarrow \mu_0 dV_0/dn=\mu_m dV_m/dn\end{cases}$$

$$\text{In } P\in S_\text{c}:\begin{cases}\mathbf{H}_{t1}=\mathbf{H}_{t2}\Rightarrow V_2(P)=V_1(P)+I\\ B_{n1}=B_{n2}\Rightarrow dV_1/dn=dV_2/dn\end{cases}$$

- **Advantage: avoid difference error in Dm where Hm ~ 0 and it is not necessary to be computed T**

$$\boxed{\mathbf{A}(\mathbf{r})=\frac{\mu_0}{4\pi}\int_{D_J}\frac{\mathbf{J}(\mathbf{r}_0)dv}{R}}$$

- **Approximations:**

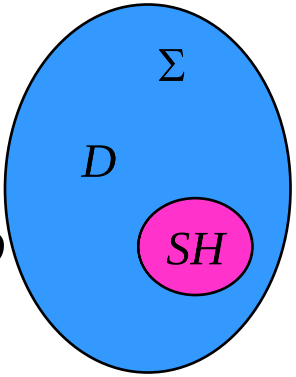
$$\begin{cases}\text{In } D_\text{J}:\begin{cases}H=JR/2=IR/(2\pi a^2),\\ A(R)=\mu_0\int_0^R H(r)dr=\mu_0 JR^2/4\\ \mathbf{A}=\mu_0\mathbf{J}R^2/4,\quad \mathbf{B}=\mathbf{curl}\mathbf{A}\end{cases}\\ \text{On } \Sigma_\text{J}:\begin{cases}\mathbf{H}_{t1}=\mathbf{H}_{t2}\Rightarrow I/(2\pi a)=-dV_0/dt\Rightarrow\\ V_0=-\int_0^{a\theta}Hdt=-I\theta/(2\pi)\\ B_{n1}=B_{n2}\Rightarrow dV_0/dn=0\end{cases}\\ \text{On } \Sigma_\text{m}:\begin{cases}\mathbf{H}_{t1}=\mathbf{H}_{t2}\Rightarrow V_0=V_m=ct\\ B_{n1}=B_{n2}\Rightarrow dV_0/dn=0\end{cases}\\ \text{In } P\in S_\text{c}:\begin{cases}\mathbf{H}_{t1}=\mathbf{H}_{t2}=0\Rightarrow V_1(P)=0,\ V_2(P)=I\\ B_{n1}=B_{n2}\Rightarrow dV_1/dn=dV_2/dn\end{cases}\end{cases}$$

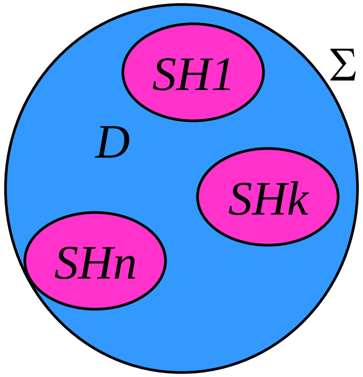
The fundamental MG problem in terms of fields

Input (known) data:

- Computational domain *D* bounded by Σ
- (CM) Material characteristics $\mu(\mathbf{r})>0$ in *D*
- (CD) Internal field sources J(r), **M**p(r) in *D*
- (CΣ’) Boundary conditions (external sources), the invariant field components:

Ht(r) on *SH* connected and **Bn(r)** on *SB=Σ-SH*





Output data (solution): **H(r), B(r)** in D

Equations:

$$\begin{cases} div\mathbf{B}=0 \\ \mathbf{curl}\mathbf{H}=\mathbf{J} \\ \mathbf{B}=\bar{\bar{\mu}}\mathbf{H}+\mathbf{I}_p \end{cases}$$

For non-connected Dirichlet surfaces $S_H=\bigcup_{k=1}^{n}S_{Hk}, S_{Hk}\cap S_{Hj}=\emptyset$

according to MS-MG similitude in addition to (CΣ’) solution uniqueness requires :

(CΣ”) $\boxed{U_k=\int_{PkP_0}\mathbf{H}_t d\mathbf{r} \text{ or } \Phi_k=\int_{S_{Fk}}\mathbf{B}_n dS,}$ for $k=1,2,..,n\text{-}1,$ and $U_n=0.$

Examples: perfect ferromagnetic bodies (with Ht=0), excited in “magnetic voltage” or in flux

MG boundary conditions in terms of potentials

- (CΣ) **for scalar potential**:

  Vm(r) = fDV(r) on SH and dVm/dn=fNV(r) on SB=Σ- SH

  V”(r) = V’(r)+Ik and dV”/dn=dV’/dn on Sck

- **(CΣ”’) for vector potential:**

  At(r) = fDA(r) on SB and **nx(curlAxn)=fNA(r)** on SH=Σ–SB

$$\mathbf{B}_n = \mathbf{n}\cdot\mathbf{curl}\mathbf{A} = \mathbf{curl}\mathbf{A}_t = \mathbf{curl}(\mathbf{f}_{DA}), \quad \mathbf{H}_\mathrm{t} = \mathbf{n}\times\overline{\overline{\nu}}(\mathbf{B}-\mathbf{I}_p)\times\mathbf{n} \Rightarrow \mathbf{H}_\mathrm{t} = \nu\,\mathbf{f}_{NA}$$

$$\Phi_k = \int_{S_{Hk}} \mathbf{B}_n dS = \int_{S_{Hk}} (curl\mathbf{A})\mathbf{n} dS = \oint_{\partial S_{Hk}} \mathbf{A} d\mathbf{r} = \oint_{\partial S_{Hk}} \mathbf{f}_{DA} d\mathbf{r}$$

In these conditions B, H are unique but A it is not. For uniqueness of A, additional boundary conditions are necessary and gauge conditions have to be added.

$$\boxed{div\mathbf{A} = 0} \Rightarrow \mathbf{curl}(\nu\mathbf{curl}\mathbf{A}) + \boxed{\mathbf{grad}(\nu div\mathbf{A})} = \mathbf{J},$$

nx(Axn) = $\mathbf{f}_{DA}$(r) on $S_B=S_{DA}$ and

nx(curlAxn) = $\mathbf{f}_{NA}$(r), $\boxed{\mathbf{nA=0}}$ on $S_H=\Sigma-S_B=S_{NA}$

MG+MS

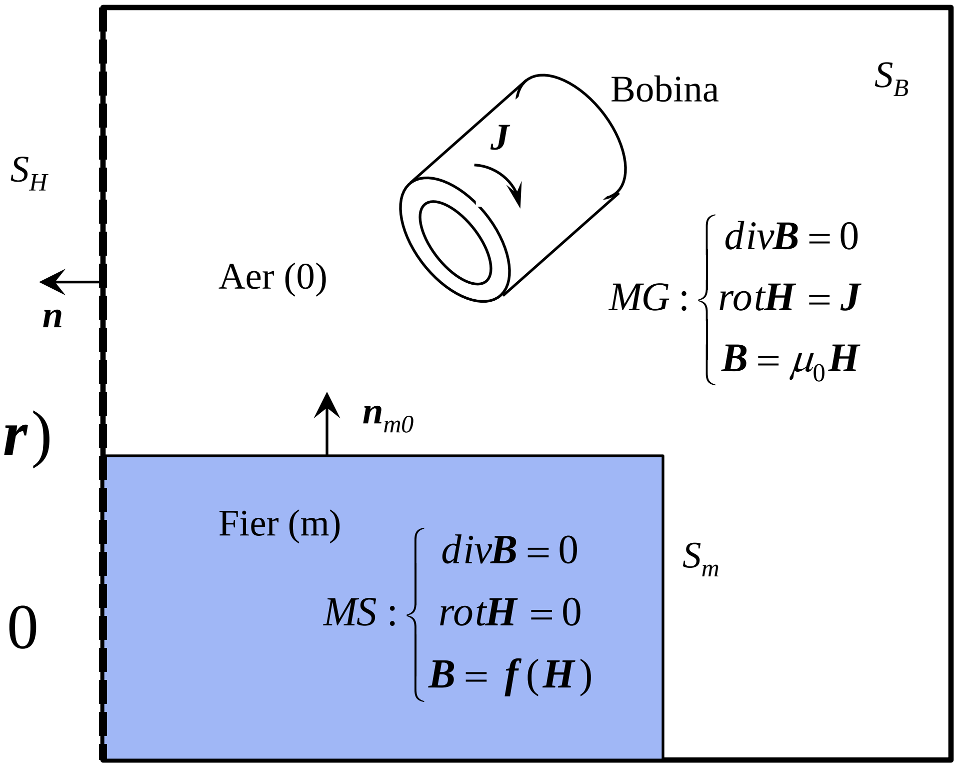
- **In most practical cases:**

boundary conditions

$$S_H: \qquad \boldsymbol{H}_t = \boldsymbol{n}\times(\boldsymbol{H}\times\boldsymbol{n}) = -\boldsymbol{J}_s = \boldsymbol{h}(\boldsymbol{r})$$

$$\Rightarrow \boldsymbol{n}\times\boldsymbol{H} = \boldsymbol{0}, \quad \text{if } \boldsymbol{h} = \boldsymbol{0}$$

$$S_B: \qquad B_n = \boldsymbol{n}\cdot\boldsymbol{B} = -b(\boldsymbol{r}) \Rightarrow \boldsymbol{n}\cdot\boldsymbol{B} = 0$$



interface conditions

$$S_m: \qquad \nabla_s\times\boldsymbol{H} = \boldsymbol{n}_{12}\times(\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{0} \Rightarrow \boldsymbol{n}_{m0}\times\boldsymbol{H}_0 = \boldsymbol{n}_{m0}\times\boldsymbol{H}_m \Rightarrow \boldsymbol{H}_{t0} = \boldsymbol{H}_{tm}$$

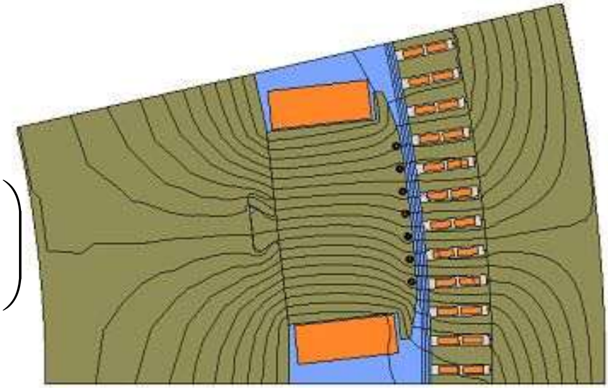
$$\nabla_s\cdot\boldsymbol{B} = \boldsymbol{n}_{12}\cdot(\boldsymbol{B}_2 - \boldsymbol{B}_1) = 0 \Rightarrow \boldsymbol{n}_{m0}\cdot\boldsymbol{B}_0 = \boldsymbol{n}_{m0}\cdot\boldsymbol{B}_m \Rightarrow B_{n0} = B_{nm}$$

The fundamental MG problem in 2D

- J and A are along Oz, B,H in plane xOy:

$$\mathbf{J}=\mathbf{k}J(x,y),\mathbf{A}=\mathbf{k}A(x,y),\mathbf{B}(x,y)=\mathbf{i}B_x+\mathbf{j}B_y,\mathbf{H}=\mathbf{i}H_x+\mathbf{j}H_y$$

$$\nabla\times\mathbf{H}=\mathbf{J}\Rightarrow\mathbf{k}J(x,y)=\nabla\times\mathbf{H}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \partial/\partial x&\partial/\partial y&0\\ H_x&H_y&0\end{vmatrix}=\mathbf{k}\left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\right)$$



$$\mathbf{B}=\nabla\times\mathbf{A}\Rightarrow\mathbf{i}B_x+\mathbf{j}B_y=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \partial/\partial x&\partial/\partial y&0\\ 0&0&A\end{vmatrix}=\mathbf{i}\frac{\partial A}{\partial y}-\mathbf{j}\frac{\partial A}{\partial x}\Rightarrow$$

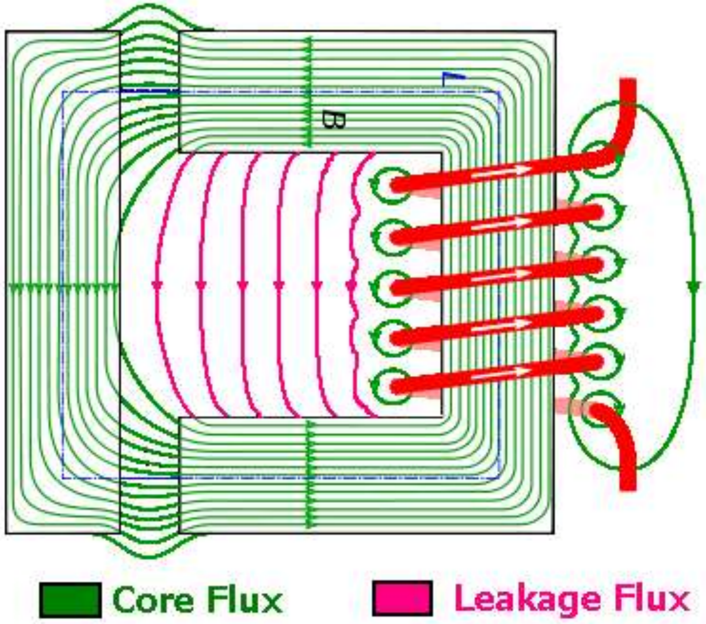
$$B_x=\frac{\partial A}{\partial y};B_y=-\frac{\partial A}{\partial x};$$

$$\mathbf{B}=\mu\mathbf{H}\Leftrightarrow\mathbf{H}=\nu\mathbf{B},\Rightarrow H_x=\nu\frac{\partial A}{\partial y};H_y=-\nu\frac{\partial A}{\partial x}\Rightarrow$$

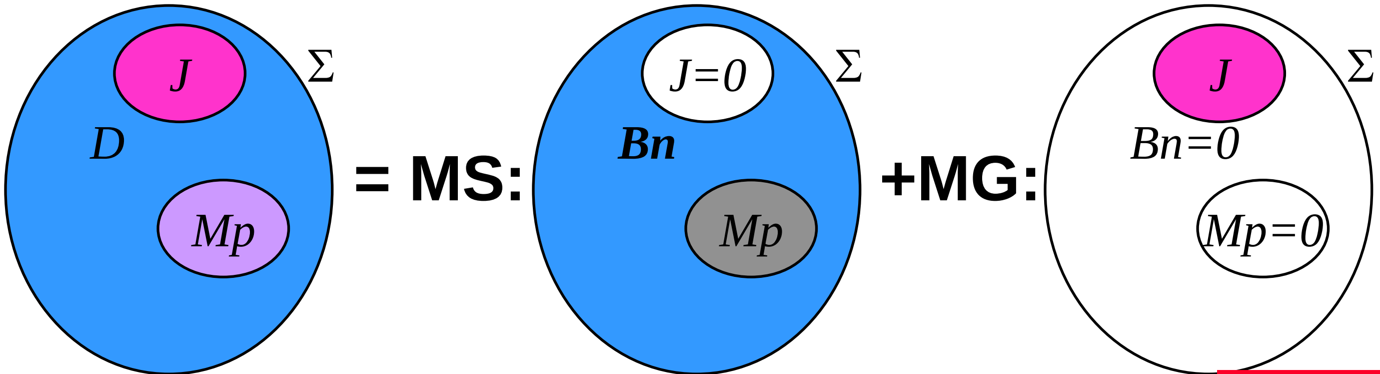
$$\left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\right)=J\Leftrightarrow\left(\frac{\partial}{\partial x}\left(\nu\frac{\partial A}{\partial y}\right)+\frac{\partial}{\partial y}\left(\nu\frac{\partial A}{\partial x}\right)\right)=J$$

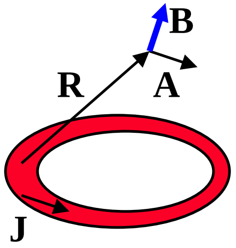
$$\Rightarrow\boxed{\nabla(\nu\nabla A)=J\Leftrightarrow div(\nu gradA)=J}$$

Poisson scalar eq.



Superposition.
Integral MG solutions in R3





$$\boxed{S\left(\sum_{k=1}^{n}\lambda_k \mathbf{C}_k\right)=\sum_{k=1}^{n}\lambda_k S(\mathbf{C}_k)}$$

In linear media, between field sources C = [CD, CS] and solutions F = [B, H] is a linear relationship: S: C→ F

Coulomb integrals: solutions in vacuum extended to R3: $\Delta V_m = -\rho_m / \mu_0 \Rightarrow$

$$V_m(\mathbf{r}) = \frac{1}{4\pi\mu_0}\int_{R^3}\frac{\rho_m(\mathbf{r}_0)dv}{R} = -\frac{1}{4\pi}\int_{R^3}\frac{div(\mu_r\mathbf{T}+\mathbf{M}_p)dv}{R}, \Rightarrow$$

$$\mathbf{H}(\mathbf{r}) = -gradV_m = -\frac{1}{4\pi}\int_{R^3}\frac{\mathbf{R}div(\mu_r\mathbf{T}+\mathbf{M}_p)dv}{R^3},$$

Biot-Savart-Laplace integrals $\Delta\mathbf{A} = -\mu_0\mathbf{J} \Rightarrow$

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{R^3}\frac{\mathbf{J}(\mathbf{r}_0)dv}{R}, \quad \mathbf{B}(\mathbf{r}) = curl\mathbf{A} = \frac{\mu_0}{4\pi}\int_{R^3}\frac{\mathbf{J}\times\mathbf{R}dv}{R^3},}$$

When $\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M}),$ **it is actually an integral equation in H:**

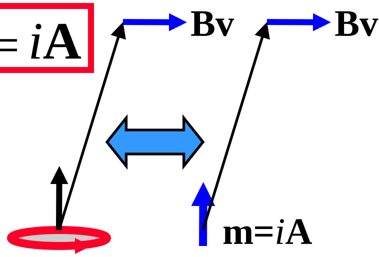
$\mathbf{M} = \chi_m\mathbf{H}+\mathbf{M}_p$

$$4\pi\left(\left(\chi_m+1\right)\mathbf{H}+\mathbf{M}_p\right)-\int_{R^3}\left(\mathbf{J}+curl(\chi_m\mathbf{H}+\mathbf{M}_p)\right)\times\mathbf{R}/R^3dv=0$$

MG field of a set of small coils

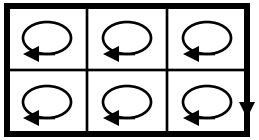
Any small coil in vacuum is equivalent from both pov field and mechanical interactions (forces and torques) with a small magnetized particle m= *i*A

The moment of the magnetic shell is $\mathbf{m} = \mathbf{M}_s A = \mathbf{n}\, M_s A = M_s \mathbf{A} \Rightarrow \boxed{\mathbf{m} = i\mathbf{A}}$



According to MS-MG similitude : $\mathbf{H}_{ext} = \frac{1}{4\pi}\left[\frac{3(\mathbf{m}\cdot\mathbf{R})\mathbf{R}}{R^5} - \frac{\mathbf{m}}{R^3}\right]$

A large coil is equivalent to a set of small coils. Permanent and temporal magnets produce a similar fields as coils.



A set of n small particles having several shapes, magnetized or carrying currents produce the field:

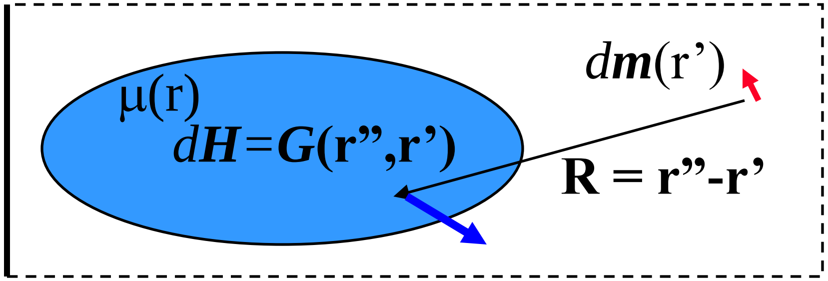
$$\mathbf{H}_j = \frac{1}{4\pi}\sum_{\substack{k=1\\k\neq j}}^{n}\left[\frac{3(\mathbf{m}_k\cdot\mathbf{R})_k\,\mathbf{R}_k}{R_k^{\;5}} - \frac{\mathbf{m}_k}{R_k^{\;3}}\right]$$

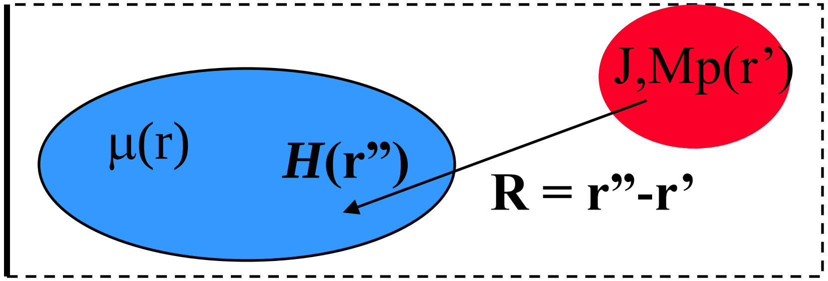
According to MS-MG similitude, their moments are obtained by solving :

$$\boxed{\mathbf{m}_j - \frac{V_j\chi_{mj}(1+D_j\chi_{mj})^{-1}}{4\pi}\sum_{\substack{k=1\\k\neq j}}^{n}\left(\frac{3(\mathbf{m}_k\cdot\mathbf{R}_k)\mathbf{R}_k}{R_k^{\;5}} - \frac{\mathbf{m}_k}{R_k^{\;3}}\right) = \mathbf{m}_{pj},\, j = 1,\ldots,n}$$

where $\mathbf{m}_{pj} = \frac{1}{4\pi V_j}\int_{D_j}\int_{D_j}\frac{\mathbf{J}\times\mathbf{R}}{R^3}dv + \int_{D_j}\mathbf{M}_p dv \cong i_j\mathbf{A}_j + \mathbf{M}_{pj}V_j$

Green function of a non-homogeneous domain





Green function is defined as in MS as the field of a punctual unitary magnetic moment of a small coil or magnetized particle m(r)=δ(r-r’)u:

$$\mathbf{H}(r'')=\overline{\overline{G}}(r'',r')\mathbf{m}(r')\Rightarrow\boxed{-div\left(\mu grad\overline{\overline{G}}(r'',r')\mathbf{u}\right)=-div(\delta(r-r')\mathbf{u})}$$

The components of G are obtained by successively orienting of u=i,j,k

By superposition is obtained the magnetic field for an arbitrary distribution of currents J=curlT:

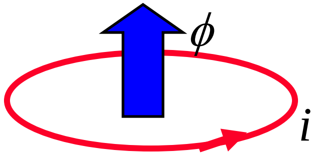
$$\boxed{\mathbf{H}(\mathbf{r}'')=\int_D\overline{\overline{G}}(\mathbf{r}'',\mathbf{r}')\mu_r(\mathbf{r}')\mathbf{T}(\mathbf{r}')dv}$$

The Green function G of a bounded domain is the field of a punctual unitary momentum in a domain with zero b.c.: Bn=0 on SB, Ht=0 on SH and Uk=0

By superposition is obtained the magnetic field of an arbitrary current distribution and permanent magnetization in the same zero boundary conditions. Then may be superposed the contribution of non-zero b.c.

Maxwell equations for inductances

- **IF μ → infinity, then H→0 and the body is similar to a conductor in ES.**
- *Vm* = ct , **H_t=0, on the boundary, hence ext. field lines are perpendicular on it**
- **By ES→ MS similitude the Maxwell relations for capacitances are transformed in the linear relations for n perfect ferromagnetic bodies :**



$$\boxed{\varphi = \mathbf{Li}} \Leftrightarrow \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \\ \varphi_n \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{2n} \\ \cdots & & & \\ L_{n1} & L_{n2} & \cdots & L_{nn} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ \\ i_n \end{bmatrix}$$

–

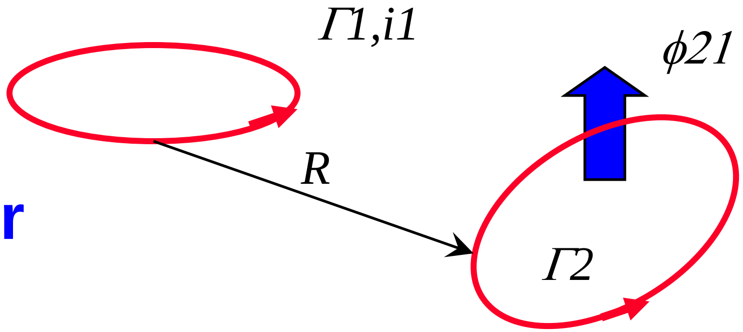
$$\boxed{\Rightarrow \mathbf{i} = \mathbf{\Gamma}\varphi \Leftrightarrow \mathbf{\Gamma} = \mathbf{L}^{-1}}$$

- **Coil fluxes:** $\varphi=[\varphi_1; \varphi_2;...; \varphi_\nu]$
- **Currents:** $\mathbf{i}=[i_1;i_2;\ldots;i_n]$
- **Matrix of coil inductances L**
- **Matrix of reverse inductances Γ**

Laplace formula for mutula inductances

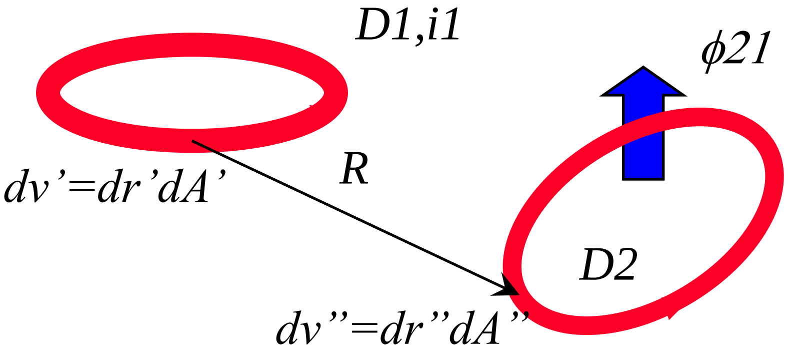
$$\boldsymbol{A}(\boldsymbol{r})=\frac{\mu_0}{4\pi}\int_{D_1}\frac{\boldsymbol{J}(\boldsymbol{r}_0)dv}{R}=\frac{\mu_0 i_1}{4\pi}\oint_{\Gamma_1}\frac{dr}{R}\Rightarrow\varphi_{21}=\frac{\mu_0 i_1}{4\pi}\oint_{\Gamma_2}\oint_{\Gamma_1}\frac{dr'dr''}{R}\Rightarrow$$

$$\boxed{L_{21}=L_{12}=\frac{\mu_0}{4\pi}\oint_{\Gamma_2}\oint_{\Gamma_1}\frac{dr'dr''}{R}}$$



- **For self inductance (j=k), conductor thickness should be considered:**

$$\boxed{L_{jk}=L_{jk}=\frac{\mu_0}{4\pi}\int_{D_j}\int_{D_k}\frac{dv'dv''}{A'A''R}}$$



Flux and current are averaged in conductors cross-section

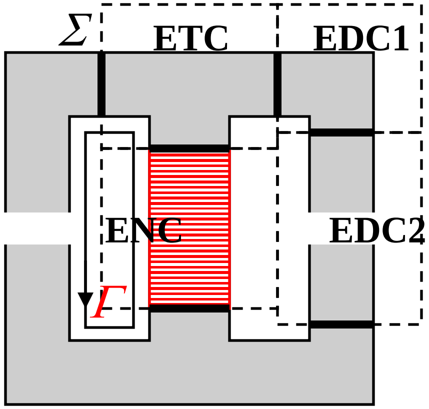
Magnetic circuits

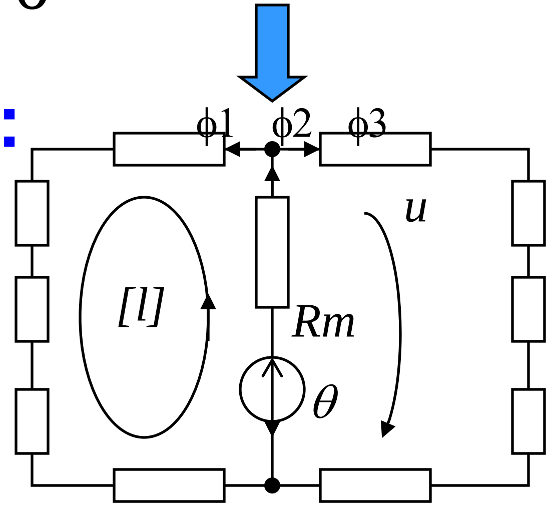
- **Flux law ➔ KFL:**

$$\oint_{\Sigma} \mathbf{B}\cdot\mathbf{n}dS = 0 \Rightarrow \boxed{\sum_{k\in(n)} \varphi_k = 0} \Rightarrow \varphi_1 - \varphi_2 + \varphi_3 = 0$$

- **Voltage theorem ➔ KVL:**

$$\oint_{\Gamma} \mathbf{H}d\mathbf{r} = 0 \Rightarrow \boxed{\sum_{k\in[l]} u_k = 0} \Rightarrow u_1 + u_2 + u_3 + \ldots = 0$$





- **Constitutive relations (MS problem):**
  - ETC – tripolar element (linear or not!)
  - EDC1 - dipolar element (linear or not!)
  - ENC – coil: field source
  - EDC2 - airgap (linear)

$$\boldsymbol{B} = \mu\boldsymbol{H} \Rightarrow \boxed{u_k = R_{mk}\varphi_k,}\ u_k = \int_{C_k}\boldsymbol{H}d\boldsymbol{r}, \varphi_k = \int_{S_k}\boldsymbol{B}d\boldsymbol{S}$$

$$\oint_{\Gamma}\boldsymbol{H}d\boldsymbol{r} = \int_{S_\Gamma}\boldsymbol{J}d\boldsymbol{A} \Rightarrow \boxed{R_{mk}\varphi_k + u_k = \theta_k}\ \text{where}\ \theta_k = n_k i_k \ \text{ is m.m.f.}$$

Energy of MS field, Tellegen's and reciprocity theorems

$$W_m = \int_D w_m dv = \frac{1}{2}\int_D \mu \mathbf{H}^2 dv = -\frac{1}{2}\int_D \mathbf{I}_p \cdot \mathbf{H} dv - \frac{1}{2}\oint_\Sigma V\mathbf{B}\cdot\mathbf{n} dS > 0$$

In domains bounded by perfect ferromagnetic bodies or with zero boundary conditions:

$$\oint_\Sigma V\mathbf{B}\cdot\mathbf{n} dS = -\mathbf{v}^{\mathrm{T}}\cdot\varphi$$

Tellegen's theorem: **regardless material relations, the total pseudo-energy is zero in zero boundary conditions.**

$$\text{If } div\mathbf{B}' = 0,\ \ curl\mathbf{H}'' = 0 \Rightarrow \boxed{<\mathbf{B}',\mathbf{H}''> - \varphi'^{T}\cdot\mathbf{v}'' = 0} \Rightarrow \mathbf{B}\perp\mathbf{H}$$

- **Reciprocity theorem:** **in linear reciprocal materials ($\mu = \mu^T$) the relation between sources and responses is symmetric. Consequently, the Green function is symmetric:**

$$<\mathbf{M}_1,\mathbf{H}_2> - <\mathbf{M}_2,\mathbf{H}_1> = \int_D\int_D (\mathbf{M}_1^{T}\cdot\overline{\overline{G}}\mathbf{M}_2 - \mathbf{M}_2^{T}\cdot\overline{\overline{G}}\mathbf{M}_1) dv' dv'' = 0$$

$$\text{If } \ \mathbf{M}_1 = \mathbf{i}\delta(\mathbf{r}-\mathbf{r}'), \mathbf{M}_2 = \mathbf{j}\delta(\mathbf{r}-\mathbf{r}') \Rightarrow G_{xy}(\mathbf{r}',\mathbf{r}'') = G_{yx}(\mathbf{r}',\mathbf{r}'')$$

$$\text{If } \ \mathbf{M}_1 = \mathbf{i}\delta(\mathbf{r}-\mathbf{r}'),\ \ \mathbf{M}_2 = \mathbf{j}\delta(\mathbf{r}-\mathbf{r}'') \Rightarrow G_{xy}(\mathbf{r}',\mathbf{r}'') = G_{yx}(\mathbf{r}'',\mathbf{r}')$$

$$\Rightarrow \ \boxed{\overline{\overline{G}}(\mathbf{r}',\mathbf{r}'') = \overline{\overline{G}}(\mathbf{r}'',\mathbf{r}') = \overline{\overline{G}}^{T}(\mathbf{r}',\mathbf{r}'')}$$

Variational MG formulations

- **The MS "energy" functional in terms of scalar potential is similar to the ES one**

$$\boxed{F(V_m)=\frac{1}{2}\int_D\left[\mu(gradV_m)^2+div(\mathbf{I}_p)V_m\right]dv+\int_{S_N}V_mB_ndS<F(V_m+\delta V)}$$

Neumann are natural boundary conditions while Dirichlet are essential boundary conditions. Weak (integral-differential) formulations:

$$\boxed{\int_D(\mu\, gradV_m\cdot grad\delta V+\delta V div\mathbf{I}_p)dv+\int_{S_{NV}=S_B}\delta VD_ndS=0,}\ \mathrm{f}_{\mathrm{N}}=D_n=-\mu dV_m/dn$$

- **The MS weak formulation in terms of vector potential:**

$$\mathbf{curl}\left[\overline{\overline{\nu}}\mathbf{curl}\mathbf{A}\right]=\mathbf{J}_m,\mathbf{J}_m=\mathbf{curl}(\overline{\overline{\nu}}\mathbf{I}_p)\Rightarrow\int_D\delta\mathbf{A}\cdot\left[\mathbf{curl}\left(\overline{\overline{\nu}}\mathbf{curl}\mathbf{A}\right)-\mathbf{J}_m\right]dv=0$$

$$\nabla\cdot(\delta\mathbf{A}\times\nu\nabla\times\mathbf{A})=\nu\nabla\times\mathbf{A}\cdot\nabla\times\delta\mathbf{A}-\delta\mathbf{A}\cdot\nabla\times(\nu\nabla\times\mathbf{A}),\qquad \mathbf{n}\times\delta\mathbf{A}=0\text{ on }S_{DA}\Rightarrow$$

$$\boxed{\int_D\left[\overline{\overline{\nu}}\mathbf{curl}\,\delta\mathbf{A}\cdot\mathbf{curl}\mathbf{A}-\delta\mathbf{A}\cdot\mathbf{J}_m\right]dv+\int_{S_{NA}=S_H}\delta\mathbf{A}\cdot(\mathbf{n}\times\overline{\overline{\nu}}\mathbf{curl}\mathbf{A})dS=0,}\ \mathbf{f}_{\mathrm{NA}}=\mathbf{n}\times\mathbf{H}$$

Neumann are again natural b. c. and Dirichlet are essential b. c. also for A.

Acc. Preis91-MAG-5 A is unique if to the Galerkin variation formulation are added

$$\int_D\left[\nu\mathbf{curl}\,\delta\mathbf{A}\cdot\mathbf{curl}\mathbf{A}-\delta\mathbf{A}\cdot\mathbf{J}_m\right]dv+\int_{S_{NA}}\delta\mathbf{A}\cdot(\mathbf{n}\times\nu\mathbf{curl}\mathbf{A})dS+$$

$$-\int_D\nu div\,\delta\mathbf{A}div\mathbf{A}dv-\int_{S_{DA}}\delta\mathbf{A}\cdot\nu\mathbf{n}div\mathbf{A}dS=0\Leftrightarrow\quad\boxed{\mathbf{curl}\left[\nu\mathbf{curl}\mathbf{A}\right]+\mathbf{grad}\left[\nu div\mathbf{A}\right]=\mathbf{J}_m}$$

Weak form of MG field equations

- **Strong (differential form):**

$$\begin{cases} div\boldsymbol{B} = 0 \Rightarrow \boldsymbol{B} = \nabla \times \boldsymbol{A} \\ \nabla \times \boldsymbol{H} = \boldsymbol{J} \Rightarrow \nabla \times \left[\overline{\overline{\nu}}(\nabla \times \boldsymbol{A} - \boldsymbol{I}_p)\right] = \boldsymbol{J} \\ \boldsymbol{B} = \overline{\overline{\mu}}\boldsymbol{H} + \boldsymbol{I}_p \Rightarrow \boldsymbol{H} = \overline{\overline{\nu}}(\boldsymbol{B} - \boldsymbol{I}_p) \end{cases}$$

➡

$$\boxed{\begin{aligned} &\nabla \times \left[\overline{\overline{\nu}}\nabla \times \boldsymbol{A}\right] = \boldsymbol{J}_t \\ &\boldsymbol{J}_t = \boldsymbol{J} + \boldsymbol{J}_m, \boldsymbol{J}_m = \nabla \times (\overline{\overline{\nu}}\boldsymbol{I}_p) \end{aligned}}$$

- **Boundary conditions:**

$$S_H: \quad \boldsymbol{H}_t = \boldsymbol{n} \times (\boldsymbol{H} \times \boldsymbol{n}) = -\boldsymbol{J}_s = \boldsymbol{h}(\boldsymbol{r}) \Rightarrow \boldsymbol{n} \times \boldsymbol{H} = \boldsymbol{0} \Rightarrow \boldsymbol{n} \times (\nabla \times \boldsymbol{A}) = 0$$

$$S_B: \quad B_n = \boldsymbol{n} \cdot \boldsymbol{B} = -b(\boldsymbol{r}) \Rightarrow \boldsymbol{n} \cdot \boldsymbol{B} = 0 \Rightarrow \boldsymbol{n} \times \boldsymbol{A} = 0$$

- **Weak gauged form and energy functional:**

$$\int_\Omega \nu\left((\nabla \times \boldsymbol{W})(\nabla \times \boldsymbol{A}) + (\nabla \boldsymbol{W})(\nabla \boldsymbol{A})\right)dv = \int_\Omega \boldsymbol{W} \cdot \boldsymbol{J}dv - \int_\Omega \nabla \times \boldsymbol{W} \cdot \boldsymbol{I}_p dv + \int_{S_H} \boldsymbol{W} \cdot \boldsymbol{J}_s dA$$

$$F(\boldsymbol{A}) = \frac{1}{2}\int_\Omega \nu\left([\nabla \times \boldsymbol{A}]^2 + [\nabla \boldsymbol{A}]^2\right)dv - \int_\Omega \boldsymbol{A} \cdot \boldsymbol{J}dv + \int_\Omega \nabla \times \boldsymbol{A} \cdot \boldsymbol{I}_p dv - \int_{S_H} \boldsymbol{A} \cdot \boldsymbol{J}_s dA$$

- **Weak un-gauged form and energy functional:**

$$\int_\Omega \nabla \times \boldsymbol{W}[\nu \nabla \times \boldsymbol{A}]dv = \int_\Omega \nabla \times \boldsymbol{W} \cdot \boldsymbol{T}dv - \int_\Omega \nabla \times \boldsymbol{W} \cdot \boldsymbol{I}_p dv; \quad \boldsymbol{J} = \nabla \times \boldsymbol{T}, \boldsymbol{J}_s = 0$$

$$F(\boldsymbol{A}) = \frac{1}{2}\int_\Omega \left(\nu[\nabla \times \boldsymbol{A}]^2 + \nabla \times \boldsymbol{A} \cdot (\boldsymbol{I}_p - T)\right)dv$$

Nonlinear magnetic media: isotropic/anisotropic

- **Variable permeability**

$$\begin{cases} div\boldsymbol{B}=0\Rightarrow \boldsymbol{B}=\nabla\times\boldsymbol{A}\\ \nabla\times\boldsymbol{H}=\boldsymbol{J}\Rightarrow\nabla\times\left[\nu(\nabla\times\boldsymbol{A})\right]=\boldsymbol{J}\\ \boldsymbol{B}=\mu(|\boldsymbol{H}|)\boldsymbol{H}\Rightarrow\boldsymbol{H}=\nu(|\boldsymbol{B}|)\boldsymbol{B}=f(B)\boldsymbol{B}/B\end{cases}$$

- **Weak form:**

$$\int_\Omega \nu(|\nabla\times\boldsymbol{A}|)\left((\nabla\times\boldsymbol{W})(\nabla\times\boldsymbol{A})+(\nabla\boldsymbol{W})(\nabla\boldsymbol{A})\right)dv=$$

$$-\int_\Omega \boldsymbol{W}\cdot\boldsymbol{J}dv+\int_{S_H}\boldsymbol{W}\cdot\boldsymbol{J}_s dA$$

- **Energy functional:**

$$\boxed{F(\boldsymbol{A})=\int_\Omega\left(\int_0^{|\nabla\times\boldsymbol{A}|}f(b)db\right)dv-\int_\Omega\boldsymbol{A}\cdot\boldsymbol{J}dv-\int_{S_H}\boldsymbol{A}\cdot\boldsymbol{J}_s dA}$$

Variable magnetization

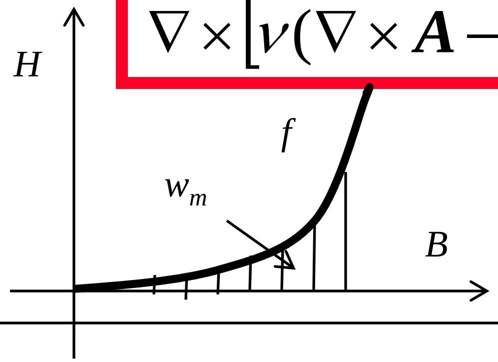
$$\begin{cases} div\boldsymbol{B}=0\Rightarrow \boldsymbol{B}=\nabla\times\boldsymbol{A}\\ \nabla\times\boldsymbol{H}=\boldsymbol{J}\Rightarrow\nabla\times\left[\nu(\nabla\times\boldsymbol{A}-\boldsymbol{I}(\boldsymbol{B}))\right]=\boldsymbol{J}\\ \boldsymbol{B}=\boldsymbol{f}(\boldsymbol{H})=\mu\boldsymbol{H}+\boldsymbol{I}(\boldsymbol{H})\Rightarrow\\ \boldsymbol{H}=\boldsymbol{g}(\boldsymbol{B})=\nu(\boldsymbol{B}-\boldsymbol{I}(\boldsymbol{B}))=\nu\boldsymbol{B}-\boldsymbol{M}(\boldsymbol{B})\end{cases}$$

$$\boldsymbol{B}=\boldsymbol{f}(\boldsymbol{H})=\mu\boldsymbol{H}+\boldsymbol{I}(\boldsymbol{H})\Rightarrow$$

$$\boldsymbol{H}=\boldsymbol{g}(\boldsymbol{B})=\nu(\boldsymbol{B}-\boldsymbol{I}(\boldsymbol{B}))=\nu\boldsymbol{B}-\boldsymbol{M}(\boldsymbol{B})$$

$$\boldsymbol{I}(\boldsymbol{B})=\boldsymbol{F}_B(\boldsymbol{B})=\boldsymbol{f}(\boldsymbol{H})-\mu\boldsymbol{H}\Rightarrow$$

$$\boxed{\nabla\times\left[\nu(\nabla\times\boldsymbol{A}-\boldsymbol{F}_B(\nabla\times\boldsymbol{A}))\right]=\boldsymbol{J}}$$



MG applications

Based on the force of the electromagnets

- Electromagnets
- Relays
- Sensors
- Electromagnetic latches

Conversion of mechanical to electrical energy

- D.C. Generators
- A.C. generators

Conversion of electrical to mechanical energy

- Motors
- Meters
- Actuators, linear, and rotational

Direct, shape and control electron or ion beams

- CRT - cathode-ray tubes
- Electromagnets for particle accelerators
- Computer tomograph coils

Others

Correct mathematical formulation of MG (curl-curl) fundamental problems

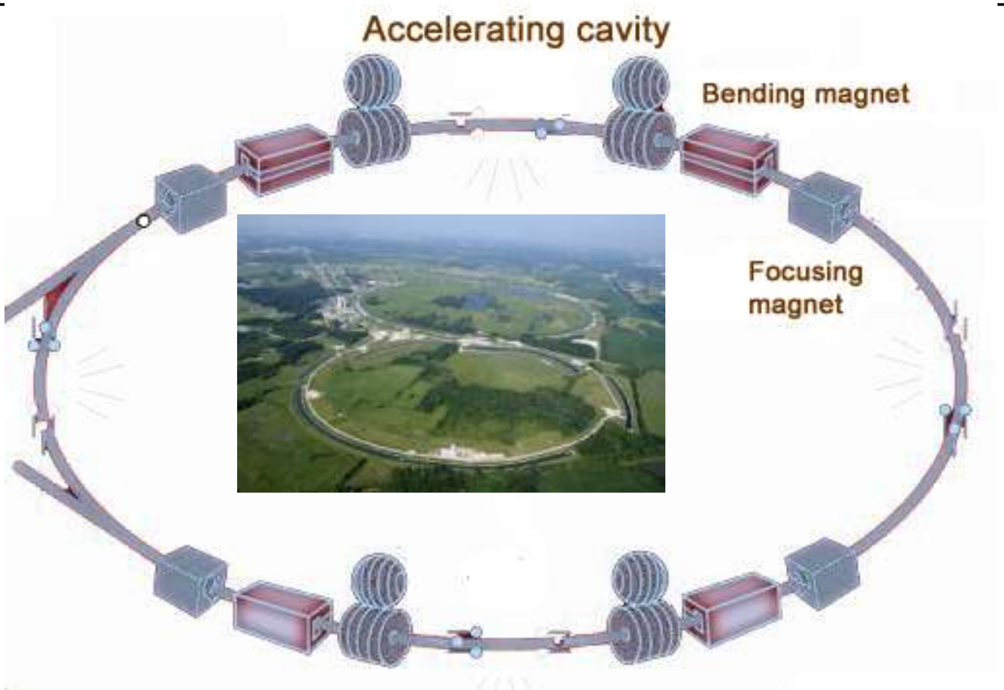
- **Known data:**
  - Computational domain: Ω - Lipchitz type
  - Material characteristics ($\boldsymbol{B}= \mu \boldsymbol{H}$), $\mu=1/\nu=f(\mathbf{r}): \Omega\rightarrow IR, \mu >0$
  - Internal sources of field (current density): $\boldsymbol{J}=\boldsymbol{g}(\mathbf{r}): \Omega\rightarrow \mathrm{IR}^3$,
  - Boundary cond. (ext. sources): $\begin{cases} \boldsymbol{n}\cdot\boldsymbol{B}(P)=0, P\in S_B\subset\partial\Omega \\ \boldsymbol{n}\times\boldsymbol{H}=\boldsymbol{J}_s(P), P\in S_H=\partial\Omega-S_B \end{cases}$
- **Solution (vector potential):** A: $\Omega\rightarrow \mathrm{IR}^3$,with $\mathbf{B} = curl\mathbf{A}$
- **Equation (weak formulation):** Find $\boldsymbol{A}\in H_B(curl,\Omega)$, s.t.

$$\int_\Omega \left(\nu(\nabla\times\boldsymbol{A})\cdot(\nabla\times\boldsymbol{W})+\nu(\nabla\boldsymbol{A})\cdot(\nabla\boldsymbol{W})-\boldsymbol{W}\cdot\boldsymbol{J}\right)dv-\int_{S_H}\boldsymbol{W}\cdot\boldsymbol{J}_s dS=0,$$

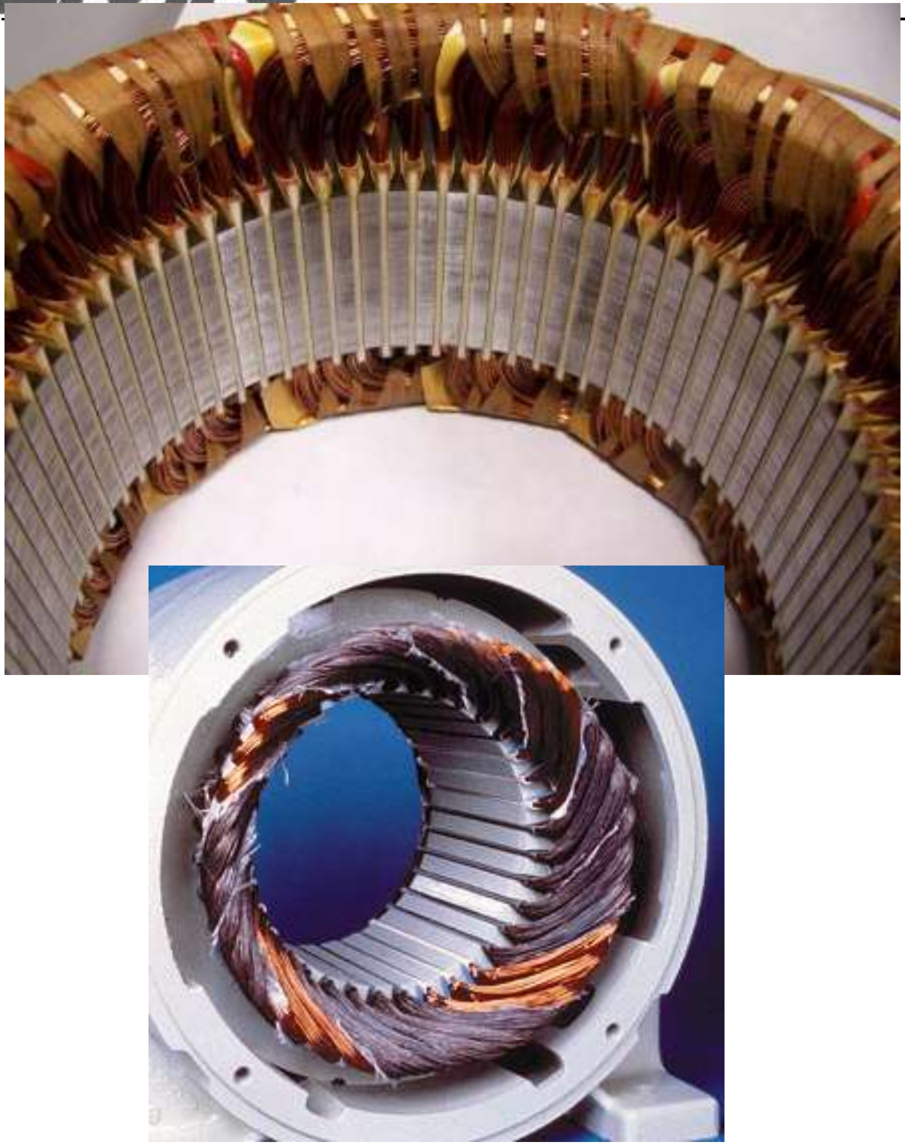
$$\forall\boldsymbol{W}\in H_B(curl,\Omega),\text{with } H_B(curl,\Omega)=\left\{\boldsymbol{A}\in L^2(\Omega)\,|\,\nabla\times\boldsymbol{A}\in L^2(\Omega),\boldsymbol{A}\big|_{S_B}=0\right\}$$

- **Existence, uniqueness and stability of solutions granted by the Lax-Milgram theorem**

Accelerators electromagnets

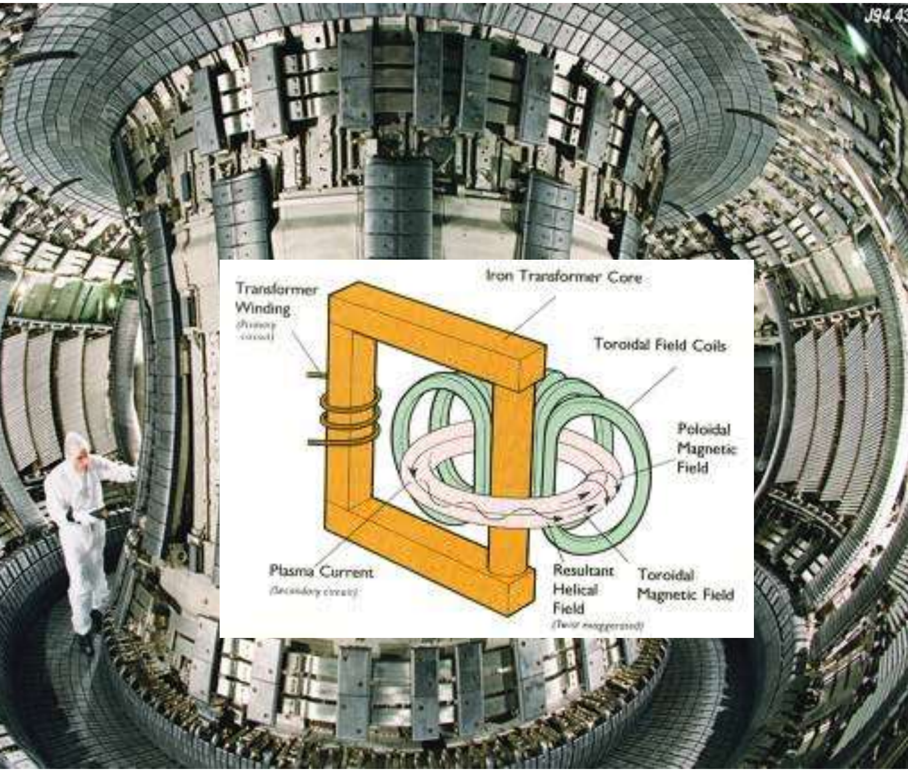


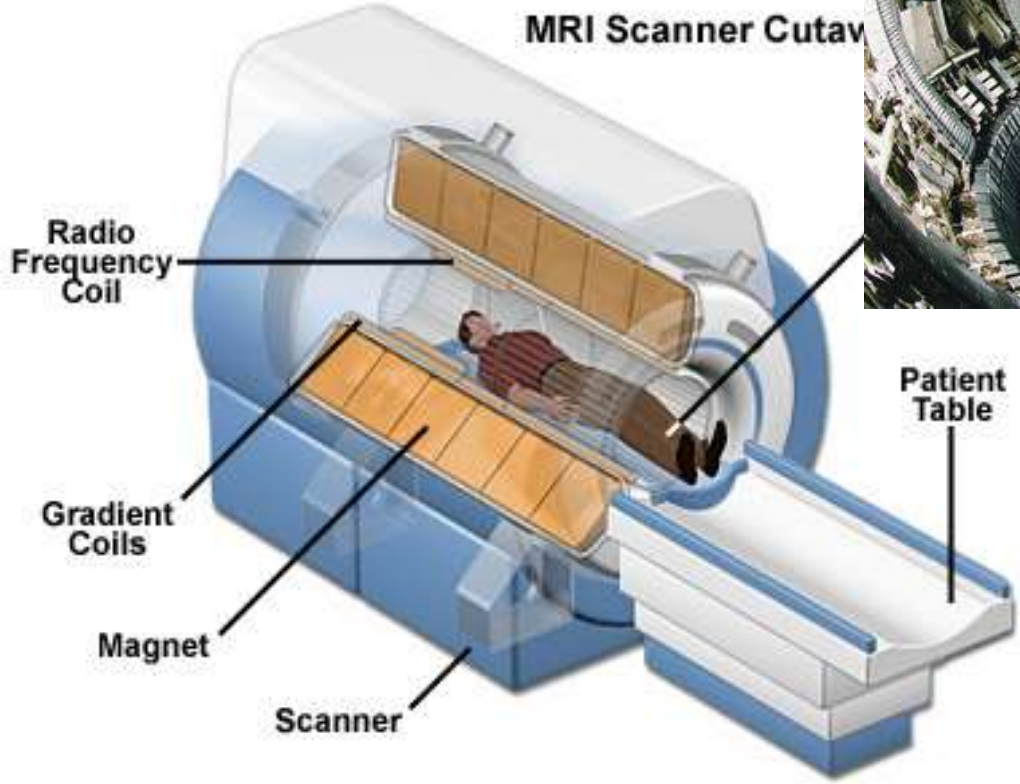
Motors and generators:



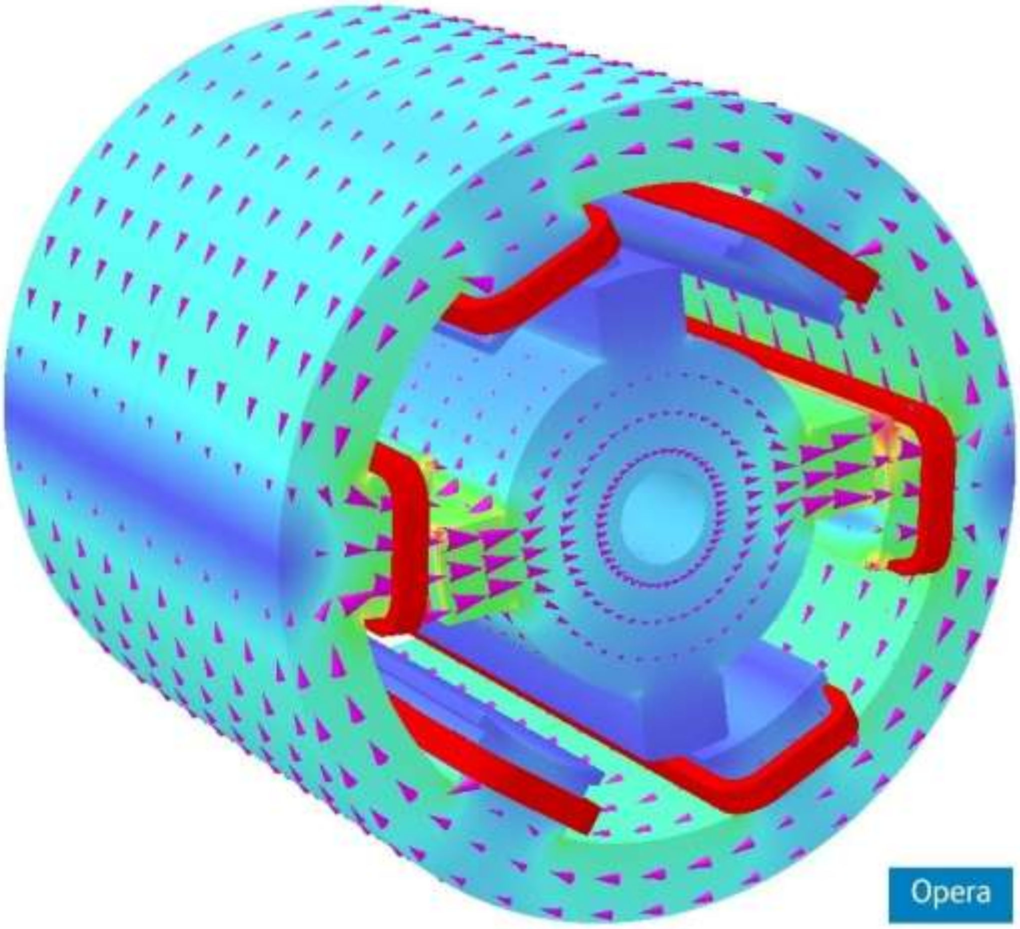
MG Applications:

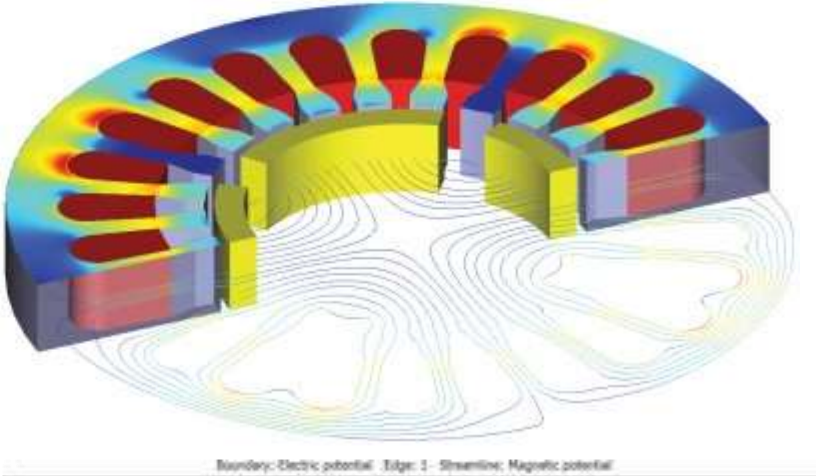
MRI, Tokamak:

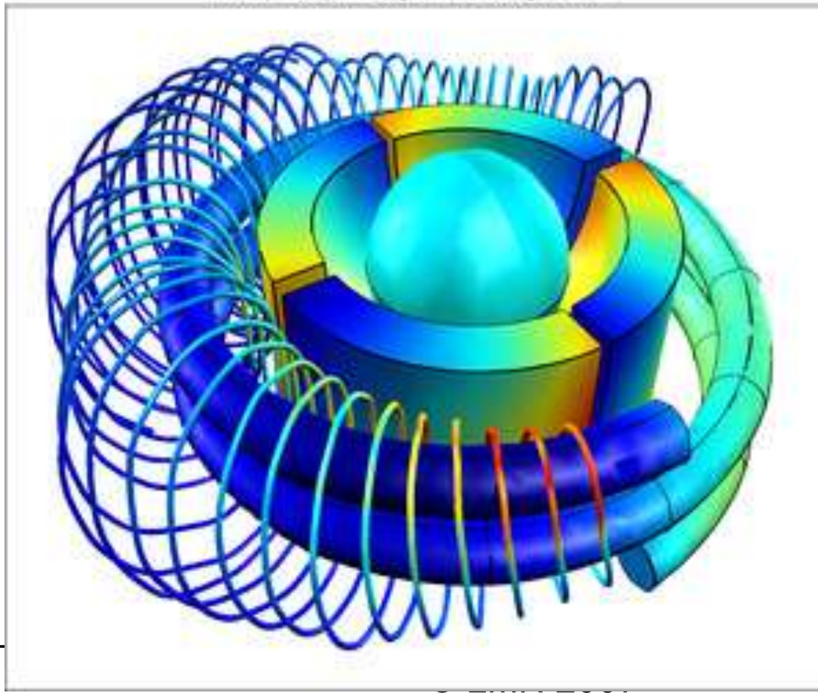




Magnetic CAD:







MG benchmarks

- http://www.compumag.org/jsite/team.html

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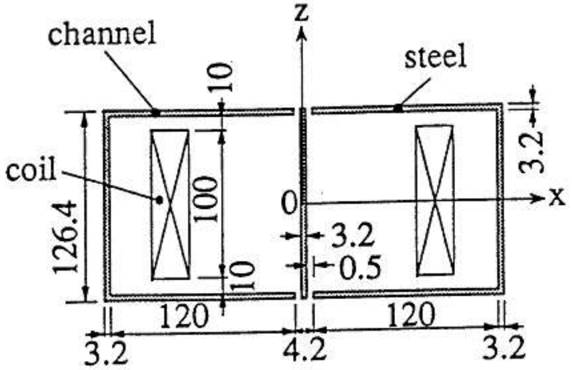
Home ▸ TEAM

Testing Electromagnetic Analysis Methods (T.E.A.M.)

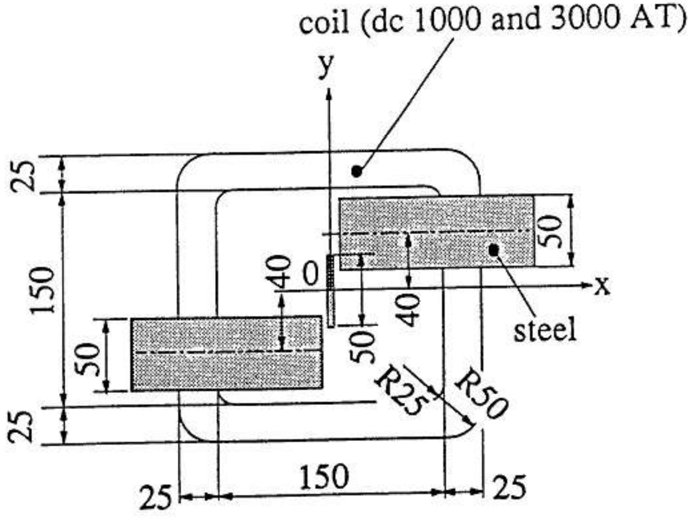
Nonlinear 3D - MG TEAM problems

- Problem 13

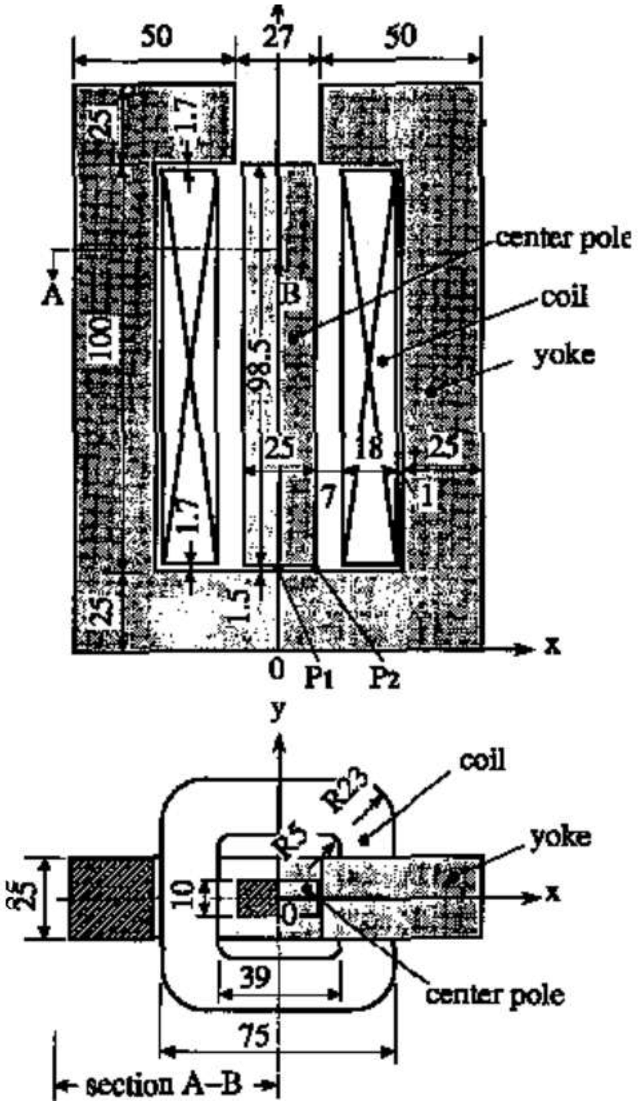
Problem 20



(a) front view



(b) plan view



MG summary. Equations, interface and boundary conditions

$$\begin{cases} div\mathbf{B}=0 \\ curl\boldsymbol{H}=\boldsymbol{J} \\ \mathbf{B}=\bar{\bar{\mu}}\mathbf{H}+(\mathbf{I}_p) \end{cases} \Longleftrightarrow \begin{cases} -div(\bar{\bar{\mu}}\,\mathbf{grad}V)=\rho_m \\ \rho_m=-div\mathbf{I}_p \\ \mathbf{H}=-\mathbf{grad}V \end{cases} \Longleftrightarrow \begin{cases} \mathbf{curl}\left[\bar{\bar{\nu}}\,\mathbf{curl}\mathbf{A}\right]=\mathbf{J}_m \\ \mathbf{J}_m=\mathbf{curl}(\bar{\bar{\nu}}\mathbf{I}_p) \\ \mathbf{B}=curl\mathbf{A},\ div\mathbf{A}=0 \end{cases}$$

$$\begin{cases} B_{n1}=B_{n2} \\ \mathbf{H}_{t1}=\mathbf{H}_{t2} \end{cases} \Longleftrightarrow \begin{cases} \mu_1\frac{\partial V_1}{\partial n}=\mu_2\frac{\partial V_2}{\partial n} \\ V_1=V_2 \end{cases} \Longleftrightarrow \begin{cases} \mathbf{A}_1=\mathbf{A}_2 \\ \nu_1\mathbf{n}\times curl\mathbf{A}_1\times\mathbf{n}=\nu_2\mathbf{n}\times curl\mathbf{A}_2\times\mathbf{n} \end{cases}$$

$$\begin{cases} \mathbf{H}_t=\mathbf{f}_H(P)\ \text{ on }\ S_H \\ B_n=f_B(P)\ \text{ on }\ S_B \\ \int_{PkP_0}\mathbf{H}_t d\mathbf{r}=U_k\ \text{ or }\ \int_{S_{Ek}}B_n dS=\Phi_k, \\ \text{for each }\ S_{Hk}, k=1,2,..,n\text{-}1,\text{and}\ U_n=0 \end{cases} \Longleftrightarrow \begin{cases} V=f_D(P) \\ \frac{dV}{dn}=f_N(P) \end{cases} \Longleftrightarrow \begin{cases} \mathbf{n}\times\mathbf{A}=\mathbf{f}_D(P)\ \text{ on }\ S_{DA} \\ \mathbf{n}\times curl\mathbf{A}=\mathbf{f}_N(P)\ \text{ on }\ S_{NA} \end{cases}$$

- **Circuit parameters:** $\varphi=\mathbf{P}\mathbf{v},\quad \mathbf{v}=\mathbf{R}\varphi,\quad \mathbf{R}=\mathbf{P}^{-1}$

$$\mathbf{R}=\mathbf{R}^T>0,\quad \mathbf{P}=\mathbf{P}^T>0$$

MG forces

- **Magnetized particle** $\mathbf{F}_m = \mathbf{grad}(\mathbf{m} \cdot \underset{\uparrow}{\mathbf{B}_\mathbf{v}})$ $\mathbf{T}_m = \mathbf{r} \times \mathbf{F}_m + \mathbf{m} \times \mathbf{B}_v$
- **Linear magnetic particle** $\mathbf{m} = V\chi_m(1 + \overline{\overline{D}}\chi_m)^{-1}\mathbf{H}_v$
- **Perfect ferromagnetic bodies:**

$$\mathbf{F} = \oint_\Sigma w_m \mathbf{n} dS, \qquad \mathbf{T} = \oint_\Sigma w_m (\mathbf{r} \times \mathbf{n}) dS$$

$$X_k = -\frac{1}{2}\varphi^\mathbf{T} \frac{\partial \mathbf{R}}{\partial x_k}\varphi; \quad X_k = \frac{1}{2}\mathbf{v}^\mathbf{T} \frac{\partial \mathbf{P}}{\partial x_k}\mathbf{v}$$

- **In general**

$$X_{k\,mg} = -\left.\frac{\partial W_m}{\partial x_k}\right|_{\varphi = const.} \qquad X_{k\,mg} = -\left.\frac{\partial W_m^{\ *}}{\partial x_k}\right|_{v = const.}$$

- **Maxwell's tensor**

$$\mathbf{f} = -\frac{H^2}{2}(\mathbf{grad}\,\mu) + \mathbf{grad}\left(\frac{H^2}{2}\tau\frac{\partial \mu}{\partial \tau}\right) = div\left[\mathbf{H} \wedge \mathbf{B}^T + \overline{\overline{\mathbf{I}}}\left(\frac{H^2}{2}\tau\frac{\partial \mu}{\partial \tau} - w_m\right)\right]$$

Not so easy questions for curious people

1. How are first order MG equations?
2. What type of potentials may be defined in MG regime?
3. How are the second order equations for these potentials?
4. How are the boundary conditions for each potential to be unique?
5. What are MG boundary conditions in semi-bounded domains ?
6. Are Biot-Savart-Laplace integrals convergent ?
7. How are the equations of MG-2D field?
8. How is defined Green function for MG field?
9. How may be computed inductances, using magnetic circuits?
10. How may be used similitude with ES field to compute inductances?
11. What space may be used for trial and test functions in weak MG formulation?
12. How are the integral equations of MG field?
13. What about nonlinear magnetic materials ? Uniqueness, energy, forces.
14. What are the main novelties and difficulties of MG regime, compared with other static and steady state regimes?