Electromagnetic Modeling

10. Electroconductive Field

Daniel Ioan

–

“Politehnica” Universitatea Politehnica din Bucuresti – PUB - CIEAC/LMN

http://www.lmn.pub.ro/~daniel

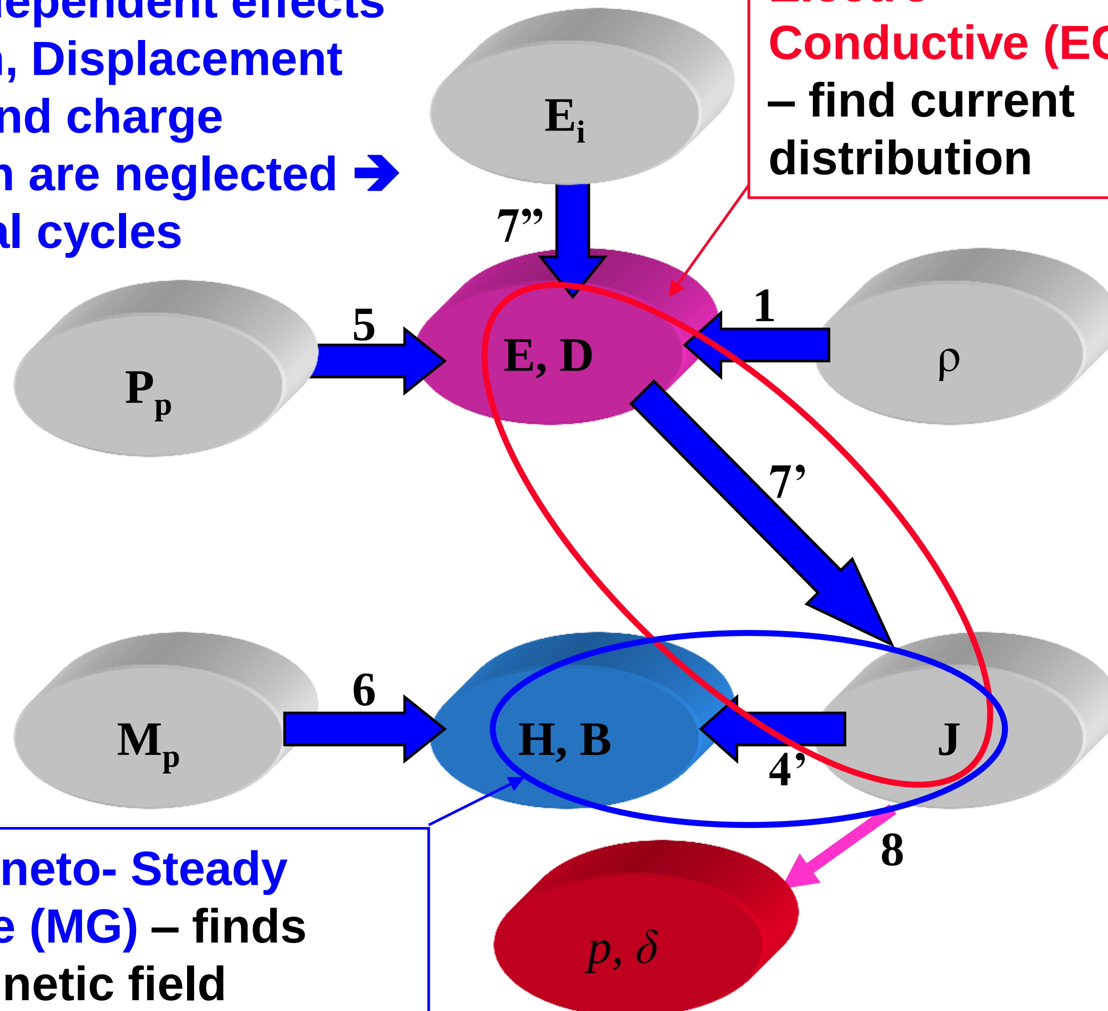
Contents

Steady state regimes

1. $\nabla \cdot \mathbf{D} = \rho$
2. $\nabla \cdot \mathbf{B} = 0$
3. $\nabla \times \mathbf{E} = \cancel{-\frac{\partial \mathbf{B}}{\partial t}}$
4. $\nabla \times \mathbf{H} = \mathbf{J} + \cancel{\frac{\partial \mathbf{D}}{\partial t}}$
5. $\mathbf{D} = \varepsilon\,\mathbf{E} + \mathbf{P}_p(\mathbf{E})$
6. $\mathbf{B} = \mu\,(\mathbf{H} + \mathbf{M}_p(\mathbf{H}))$
7. $\mathbf{J} = \sigma(\mathbf{E} + \mathbf{E}_i(\mathbf{E}))$
8. $p = \mathbf{EJ}$
9. $\nabla \cdot \mathbf{J} = \cancel{-\frac{\partial \rho}{\partial t}}$

All time dependent effects Induction, Displacement current and charge relaxation are neglected ➔ NO causal cycles

Electro-Conductive (EC) – find current distribution



Magneto- Steady state (MG) – finds magnetic field distribution

Steady-state EC regime

- **Hypothesis:**
  - no movement
  - no time variation
  - no magnetic field of interest

$$\begin{cases} i_\Sigma = 0 \Leftrightarrow \oint_\Sigma \mathbf{J}\cdot\mathbf{n}dS = 0 \\ div\mathbf{J} = 0 \Rightarrow \mathbf{J} = curl\mathbf{T} \\ \mathbf{n}_{12}\cdot(\mathbf{J}_2 - \mathbf{J}_1) = 0 \Leftrightarrow div_s\mathbf{J} = 0 \end{cases}$$

- **Fundamental Equations:**
- **Theorem of current conservation**
- **Theorem of EC potential**
- **Ohm's theorem**

$$\begin{cases} u_\Gamma = 0 \Leftrightarrow \oint_\Gamma \mathbf{E}d\mathbf{r} = 0 \\ \mathbf{curl}\mathbf{E} = 0 \Rightarrow \mathbf{E} = -\mathbf{grad}V \\ \mathbf{n}_{12}\times(\mathbf{E}_2 - \mathbf{E}_1) = 0, \Leftrightarrow \mathbf{E}_{t2} = \mathbf{E}_{t1} \end{cases}$$

$$\mathbf{J} = \mathbf{f}(\mathbf{E}) \Rightarrow \mathbf{J} = \sigma(\mathbf{E} + \mathbf{E}_i) \Leftrightarrow \mathbf{J} = \sigma\mathbf{E} + \mathbf{J}_i$$

- **Field sources:**
  - Intrinsic electric field

- **EC field is similar to MS field and it is similar to ES field in uncharged domains:**

EC:	$\mathbf{E}$	$\mathbf{J}$	$\mathbf{E}_\mathrm{i}$	σ	V	I
MS:	$\mathbf{H}$	$\mathbf{B}$	$\mathbf{I}_\mathrm{p}$	μ	V_m	Φ
ES:	$\mathbf{E}$	$\mathbf{D}$	$\mathbf{P}_\mathrm{p}$	ε	V	ψ

Second order equations. Fundamental EC problem

$$\begin{cases} div\mathbf{J}=0 \Rightarrow \mathbf{J}=curl\mathbf{T} \\ \mathbf{curl}\,\mathbf{E}=0 \Rightarrow \mathbf{E}=-\mathbf{grad}V \\ \mathbf{J}=\mathbf{f}(\mathbf{E}) \Rightarrow \mathbf{J}=\sigma(\mathbf{E}+\mathbf{E}_i) \Leftrightarrow \mathbf{J}=\sigma\mathbf{E}+\mathbf{J}_i,\ \mathbf{J}_i=\sigma\mathbf{E}_i, \\ \mathbf{E}=\rho\mathbf{J}-\mathbf{E}_i,\ \rho=1/\sigma \end{cases}$$

T in EC → **A** in MS

for the scalar potential:

$$\boxed{-div(\overline{\overline{\sigma}}\,\mathbf{grad}V)=\rho_c}$$

$$\boxed{\rho_c=-div\mathbf{J}_i}$$

Conduction "source"

for the vector potential:

$$\boxed{\mathbf{curl}\left[\overline{\overline{\rho}}\,\mathbf{curl}\,\mathbf{T}\right]=\mathbf{S}}$$

$$\boxed{\mathbf{S}=\mathbf{curl}\,\mathbf{E}_i}$$

Particular cases:

- **Linear homogeneous isotropic media (Poisson equation):**

$$-div(\mathbf{grad}V)=\rho_c/\sigma \Rightarrow \boxed{\Delta V=-\rho\rho_c,}\ \mathbf{curl}(\mathbf{curl}\,\mathbf{T})=\overline{\overline{\sigma}}\mathbf{S} \Rightarrow grad\,\cancel{div\mathbf{T}}-\Delta\mathbf{T}=\overline{\overline{\sigma}}\mathbf{S}, \Rightarrow \boxed{-\Delta\mathbf{T}=\overline{\overline{\sigma}}\mathbf{S}}$$

Coulomb gauge cond.

- **No internal ES field sources (Laplace equation):**

$$div(\sigma\,\mathbf{grad}V)=0 \Rightarrow div(\mathbf{grad}V)=0 \Leftrightarrow \boxed{\Delta V=0}\ \mathbf{curl}(\mathbf{curl}\,\mathbf{T})=0,\ \boxed{div\mathbf{T}=0} \Rightarrow \boxed{-\Delta\mathbf{T}=0}$$

Boundary conditions: Et on SE and Jn on SJ

•Dirichlet b.c.

$$\boxed{\begin{gathered} V(P)=f_{DV}(P),\quad \text{on } S_E\neq\emptyset \\ \mathbf{n}\times\mathbf{T}_t(P)\times\mathbf{n}=\mathbf{f}_{DT}(P), \text{on } S_J \end{gathered}}$$

Neumann b,c.

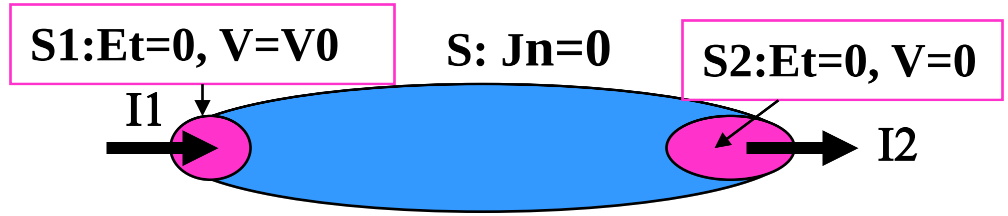
$$\boxed{\begin{gathered} \frac{\partial V}{\partial n}=f_{NV}(P)\quad \text{on } S_J=\Sigma-S_E \\ \mathbf{n}\times(\rho\,\mathrm{curl}\mathbf{T}\times\mathbf{n})=\mathbf{f}_{NT}(P),\ \text{on } S_E \end{gathered}}$$

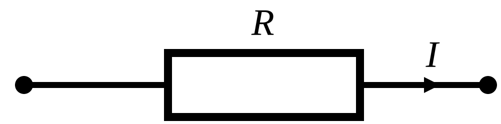
Partial (nodal) resistances/conductances

- **Perfect conductors (superconductors):** $\rho=1/\sigma \rightarrow 0$, **E=0, V=ct.**
- **Perfect insulators:** $\sigma \rightarrow 0$, **J=0, T=ct**
- **Dipolar resistor: two superconducting terminals S1, S2, surrounding by a perfect insulator:**

  I1 = I2 = I;   U = V1-V2 = V0;

  U = R I; G=1/R

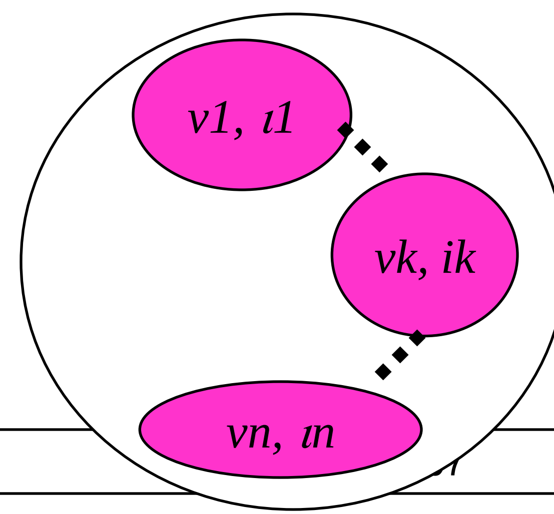




$$V(\mathbf{r}'')=-\int_{S_D}\sigma\frac{dG}{dn'}\cdot f_D(\mathbf{r}')dS'-\int_{S_N}\sigma G f_N(\mathbf{r}')dS'=-V_0\int_{S1}\sigma\frac{dG}{dn'}dS'$$

$$G=I/U=\int_{S1}\int_{S1}\sigma'\sigma''\frac{d^2G}{dn'\,dn''}dS'\,dS'';\quad g_{jk}=g_{kj}=\int_{Sj}\int_{Sk}\sigma'\sigma''\frac{d^2G}{dn'\,dn''}dS'\,dS''$$

$$\mathbf{i}=\mathbf{G}\mathbf{v}\Leftrightarrow\begin{bmatrix}i_1\\ i_2\\ \\ i_n\end{bmatrix}=\begin{bmatrix}g_{11} & g_{12} & \cdots & g_{1n}\\ g_{21} & g_{22} & \cdots & g_{2n}\\ \cdots & & & \\ g_{n1} & g_{n2} & \cdots & g_{nn}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ \\ v_n\end{bmatrix}$$



$$\Rightarrow \mathbf{v}=\mathbf{R}\mathbf{i}\Leftrightarrow\mathbf{R}=\mathbf{G}^{-1},\ \mathbf{R}=\mathbf{R}^{\mathrm{T}}>0,\quad \mathbf{G}=\mathbf{G}^{\mathrm{T}}>0$$

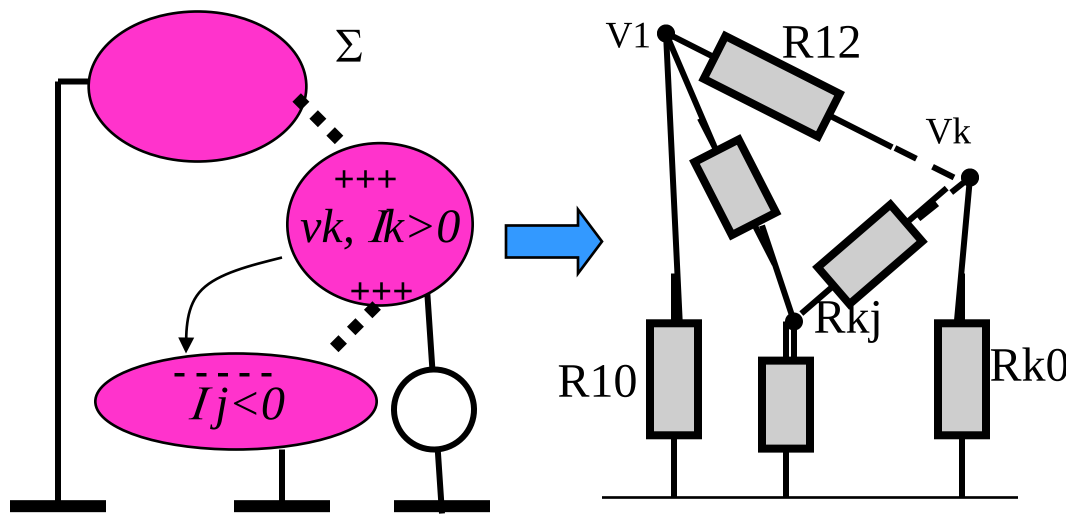
Equivalent (branch) resistances/conductances. Power

$$i_1 = g_{11}\cdot v_1 + g_{12}\cdot v_2 + \ldots + g_{1n}\cdot v_n = G_{10}\cdot v_1 + G_{12}\cdot(v_1 - v_2) + \ldots + G_{1n}\cdot(v_1 - v_n)$$

....

$$G_{kj} = -g_{kj} > 0, \qquad G_{k0} = g_{k1} + g_{k2} + \ldots + g_{kn} > 0$$

$$R_{kj} = 1/G_{kj} > 0,\ R_{k0} = 1/R_{k0} > 0$$



Electric power

$$P = \int_D p dv = \int_D \mathbf{J}\cdot\mathbf{E} dv = -\oint_\Sigma V\mathbf{J}\cdot\mathbf{n} dS$$

$$div(V\mathbf{J}) = \mathbf{J}\cdot gradV + V div\mathbf{J} = -\mathbf{J}\cdot\mathbf{E}$$

In conductors with perfect terminals and insulating boundary:

$$P = -\oint_\Sigma V\mathbf{J}\cdot\mathbf{n} dS = -\sum_{k=1}^{n} V_k \int_{Sk}\mathbf{J}\cdot\mathbf{n} dS = \sum_{k=1}^{n} V_k I_k \Rightarrow \quad P = <\mathbf{E},\mathbf{J}> = \mathbf{v}^{\mathrm{T}}\cdot\mathbf{i} = \mathbf{i}^{\mathrm{T}}\cdot\mathbf{v} = \mathbf{i}^{\mathrm{T}}\mathbf{R}\mathbf{i} = \mathbf{v}^{\mathrm{T}}\mathbf{G}\mathbf{v} > 0$$

Tellegen's theorem: **regardless material relations, the total pseudo-energy is zero in zero boundary conditions.** $< \mathbf{J}', \mathbf{E}'' > - \mathbf{i}'^{T}\cdot\mathbf{v}'' = 0 \Rightarrow \mathbf{J}\perp\mathbf{E}$

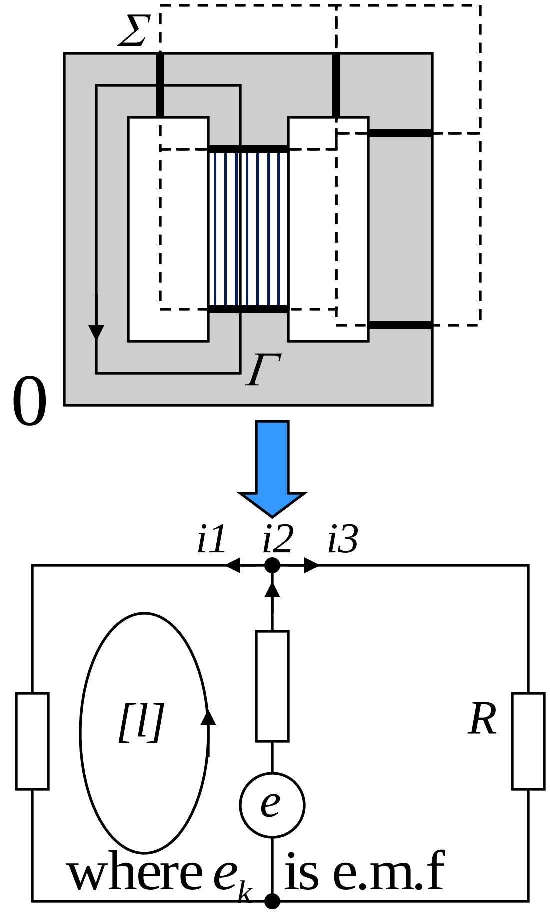
D.c. electric circuits

- **KCL:**

$$\oint_{\Sigma}\mathbf{J}\cdot\mathbf{n}dS=0\Rightarrow\boxed{\sum_{k\in(n)}i_k=0}\Rightarrow i_1-i_2+i_3=0$$

- **KVL:**

$$\oint_{\Gamma}\mathbf{E}d\mathbf{r}=0\Rightarrow\boxed{\sum_{k\in[l]}u_k=0}\Rightarrow u_1+u_2+u_3+\ldots=0$$



- **Constitutive relations:**

$$\mathbf{E}=\rho\mathbf{J}\Rightarrow\boxed{u_k=R_k i_k,}\quad \text{where } u_k=\int_{C_k}\mathbf{E}d\mathbf{r},\, i_k=\int_{S_k}\mathbf{J}\cdot\mathbf{n}dS$$

$$\text{Active elements}: \mathbf{E}=\rho\mathbf{J}-\mathbf{E}_i\Rightarrow\boxed{u_k=R_k i_k+e_k}\ .$$

$$\text{Nonlinear elements}: \mathbf{J}=\mathbf{f}(\mathbf{E})\Rightarrow\boxed{u_k=f_k(i_k)}$$

$$\text{Linear multipoles}: \boxed{\mathbf{v}=\mathbf{R}\mathbf{i},\quad \mathbf{i}=\mathbf{G}\mathbf{u}}$$

where e_k is e.m.f

- **Power:**

$$\boxed{P=\sum_{k=1}^{L}u_k i_k}\Rightarrow\sum_{k=1}^{L}R_k i_k^{\,2}=\sum_{k=1}^{L}e_k i_k$$

$$\text{Active multipoles}: \boxed{P=\mathbf{v}^{\mathrm{T}}\cdot\mathbf{i}=\mathbf{i}^{\mathrm{T}}\cdot\mathbf{v}}$$

$$\text{Passive multipoles}\ \boxed{P=\mathbf{i}^{\mathrm{T}}\mathbf{R}\mathbf{i}=\mathbf{v}^{\mathrm{T}}\mathbf{G}\mathbf{v}>0}$$

EC – ES - MS similitude

EC:	**E**	**J**	$\mathbf{E}_i$	σ	V	I
MS:	**H**	**B**	$\mathbf{I}_p$	μ	V_m	Φ
ES:	**E**	**D**	$\mathbf{P}_p$	ε	V	ψ

➔

EC:	**R**	**G**	P/2
MS:	$\mathbf{R}_m$	**P**	W_m
ES:	**S**	**C**	W_e

$$\boxed{G = C\big|_{\varepsilon \to \sigma}} \qquad C = \frac{\varepsilon A}{d} \;\Rightarrow\; G = \frac{\sigma A}{d} \;\Rightarrow\; R = \frac{1}{G} = \rho\frac{d}{A}$$

Reciprocity ➔

Symetric conductance matrix: $\boldsymbol{C} = \boldsymbol{C}^T \;\Rightarrow\; \boxed{\boldsymbol{G} = \boldsymbol{G}^T, \;\; \boldsymbol{R} = \boldsymbol{R}^T}$

Cohn-Vratsanos: Increase of **σ** or metallization ➔ ΔG>0:

$$\frac{\delta C}{\delta \varepsilon} = -\frac{<V',\Delta V')>}{V^2} > 0 \Rightarrow \boxed{\frac{\delta G}{\delta \sigma} = -\frac{<V',\Delta V')>}{V^2} > 0}$$

All other ES theorems can be translated in EC terms!

EC applications

- Distribution of currents in massive conductors
- Extraction of interconnect resistances
- Electric lighting and heating
- Cables and power/signal transmission lines
- Pcb
- Electrochemical plating/coating
- Cathodic protection
- Grounding protection (Lightning rod)

laboratorul de modelare numerica

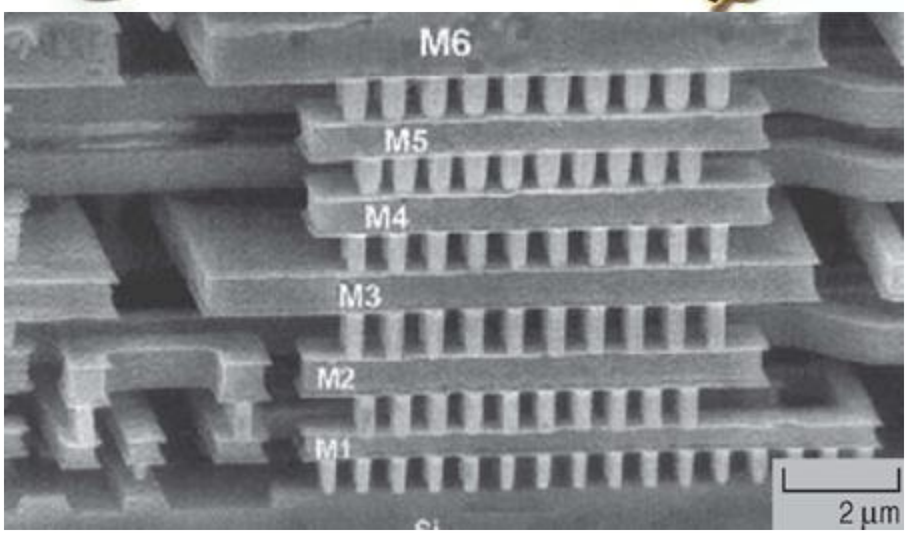
NASA – City lights

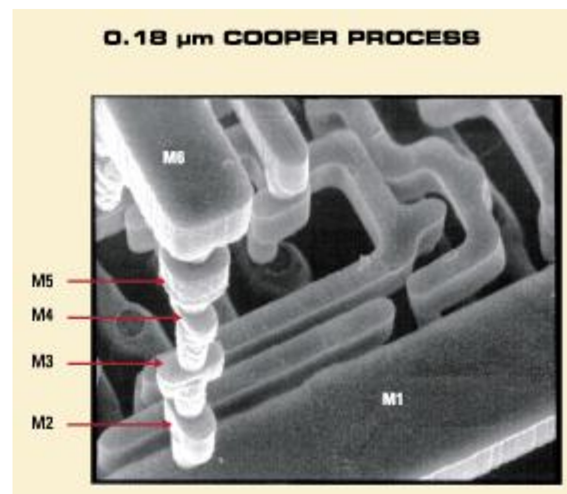
Paris – the city of lights

New York

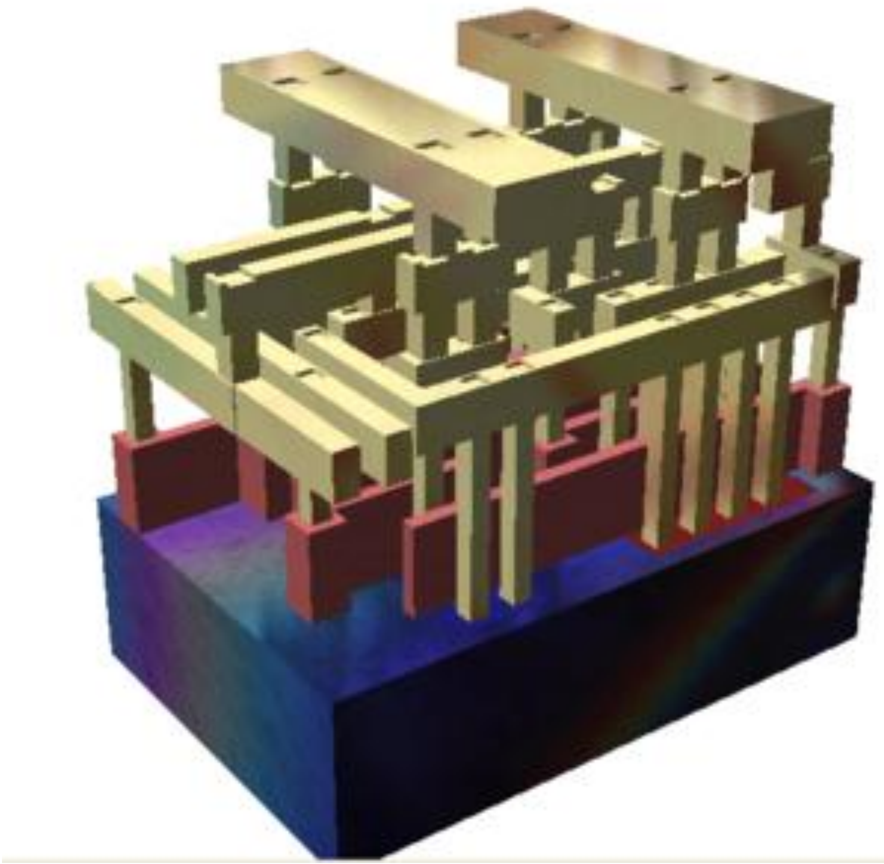
Tokyo

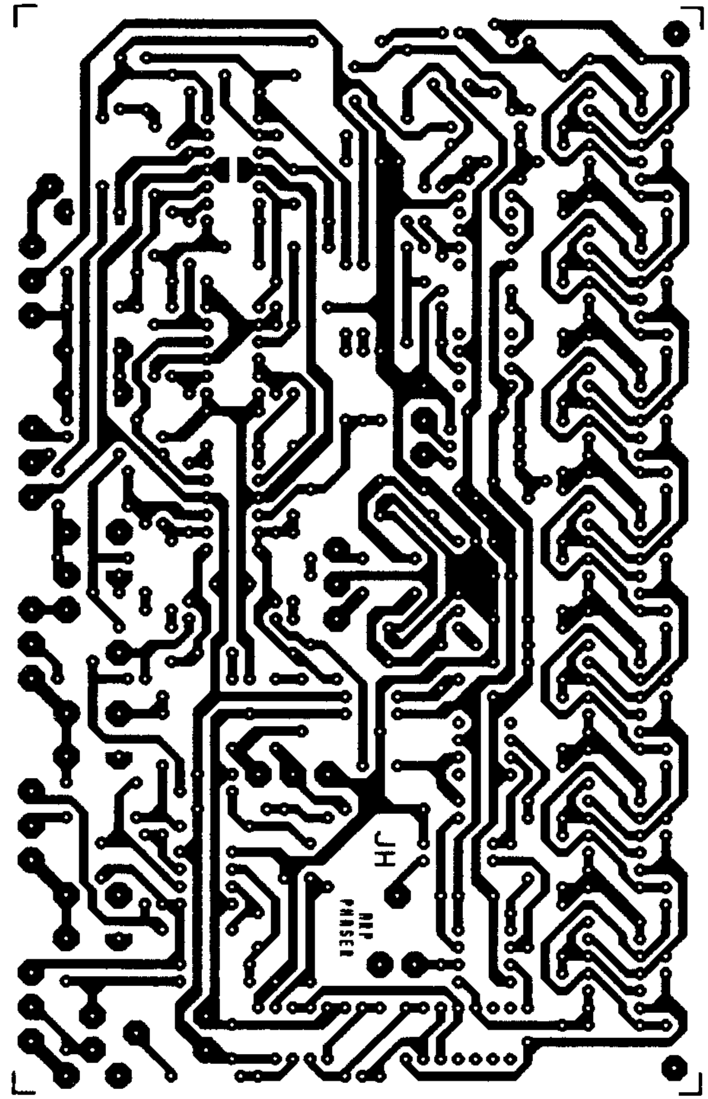
Cables and interconnects



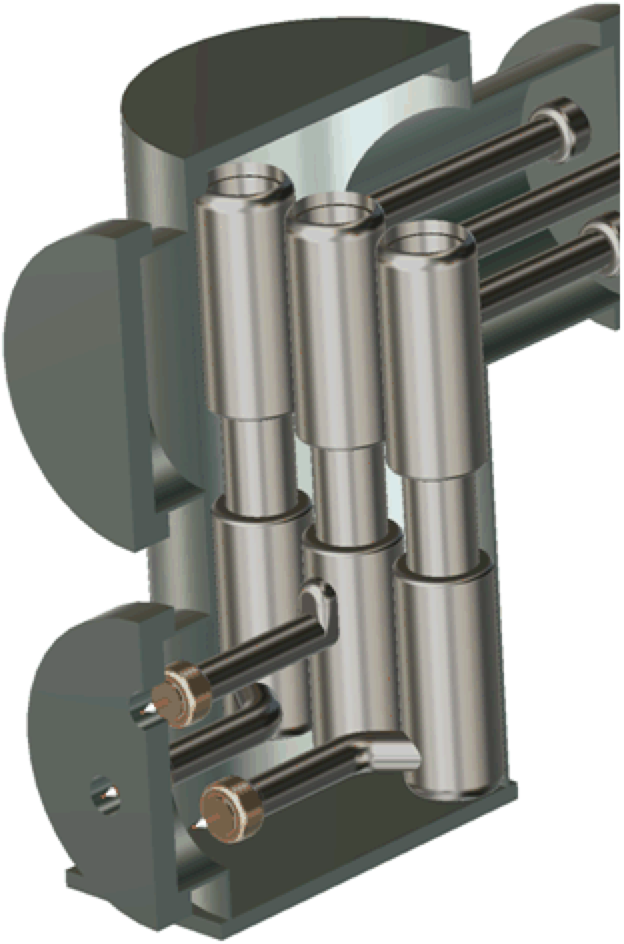


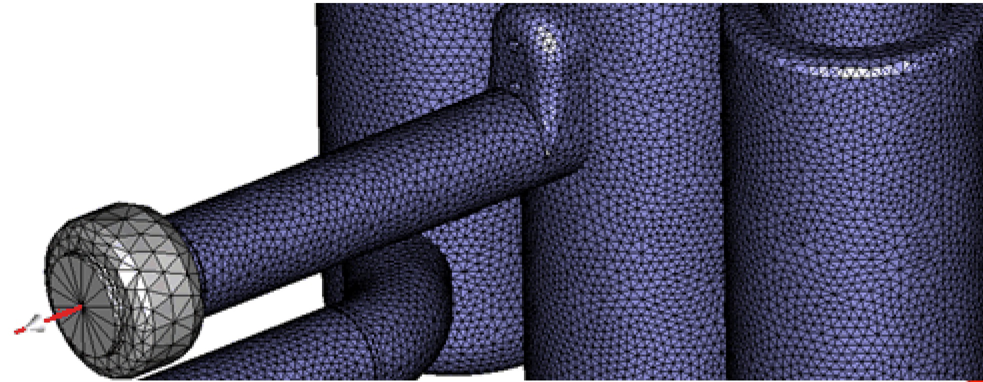
VLSI interconnects and PCB

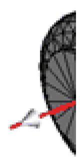


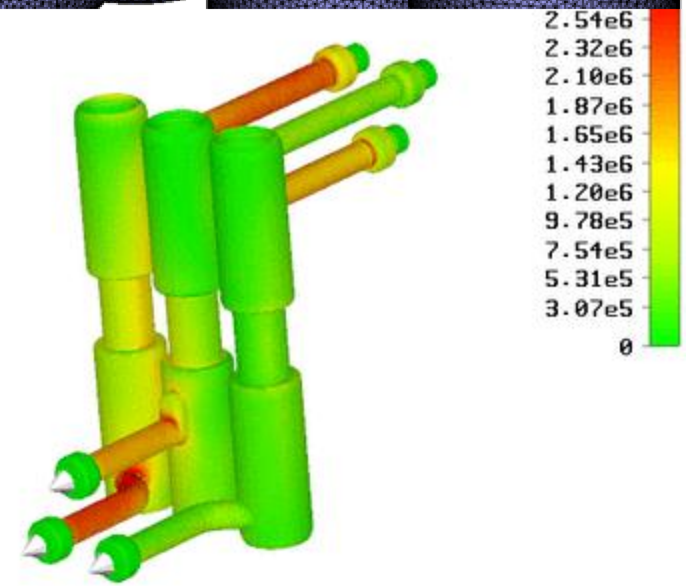


Models of power switches





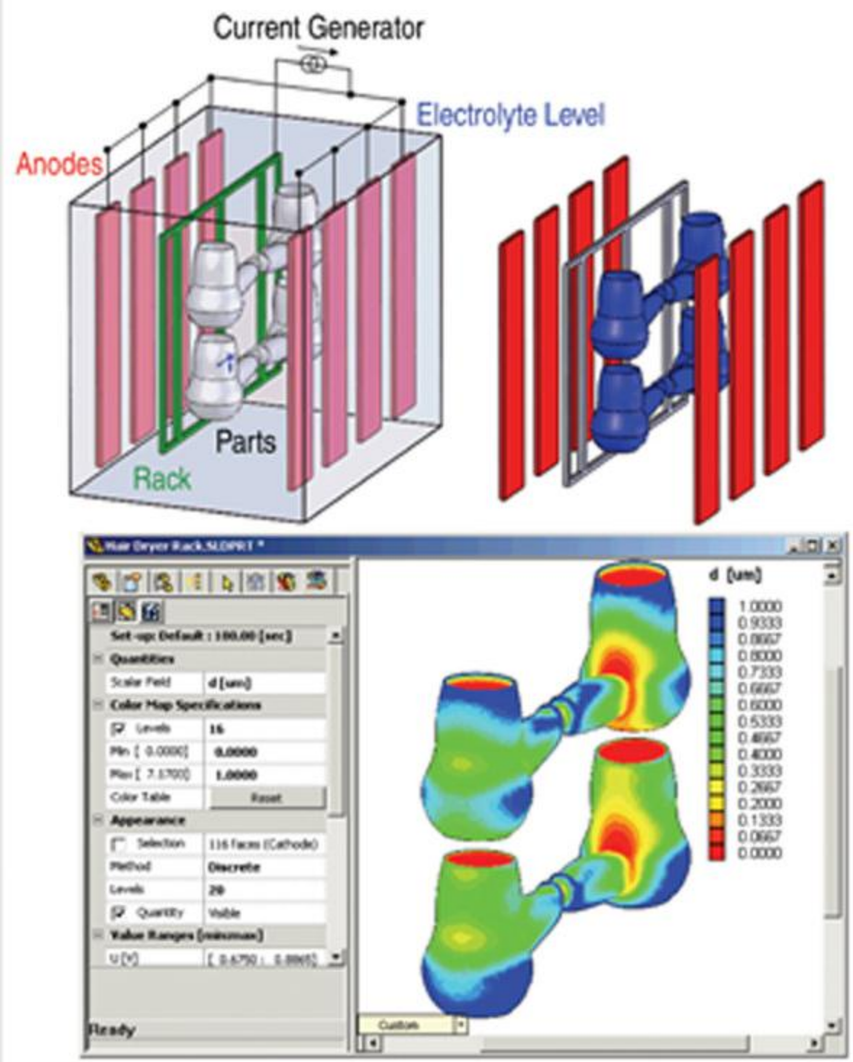




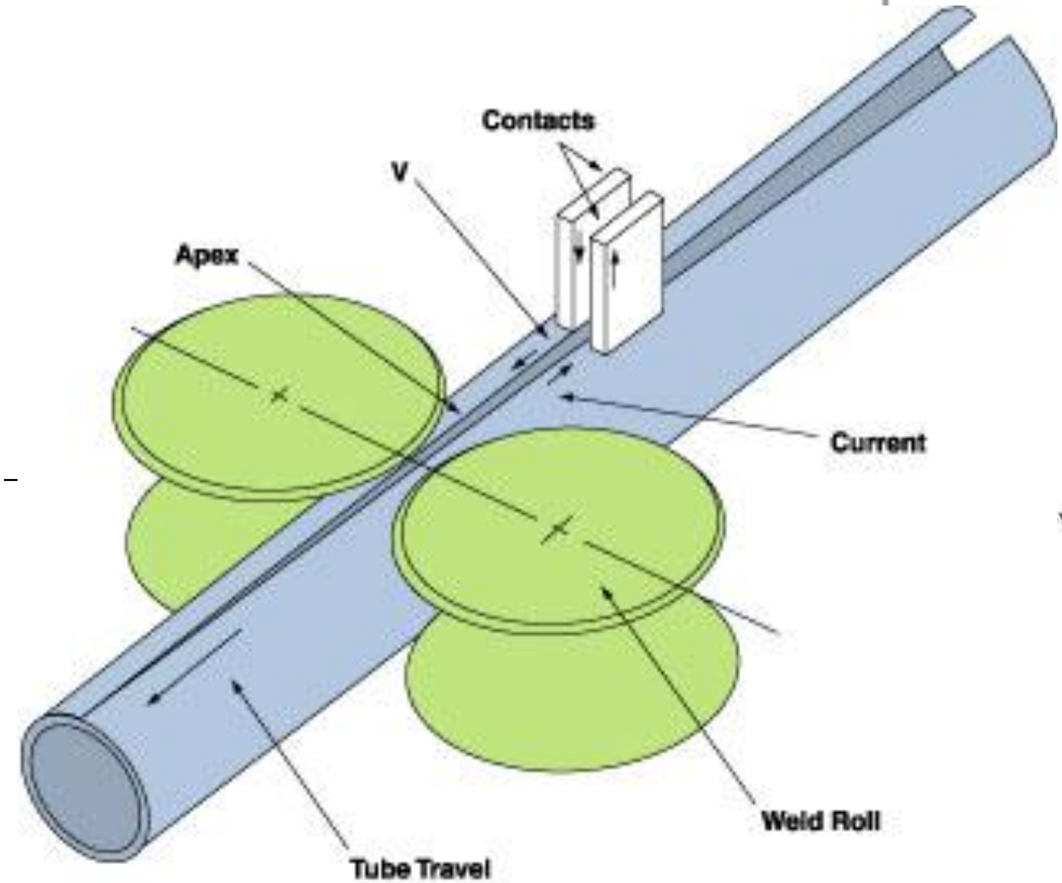
Power transport

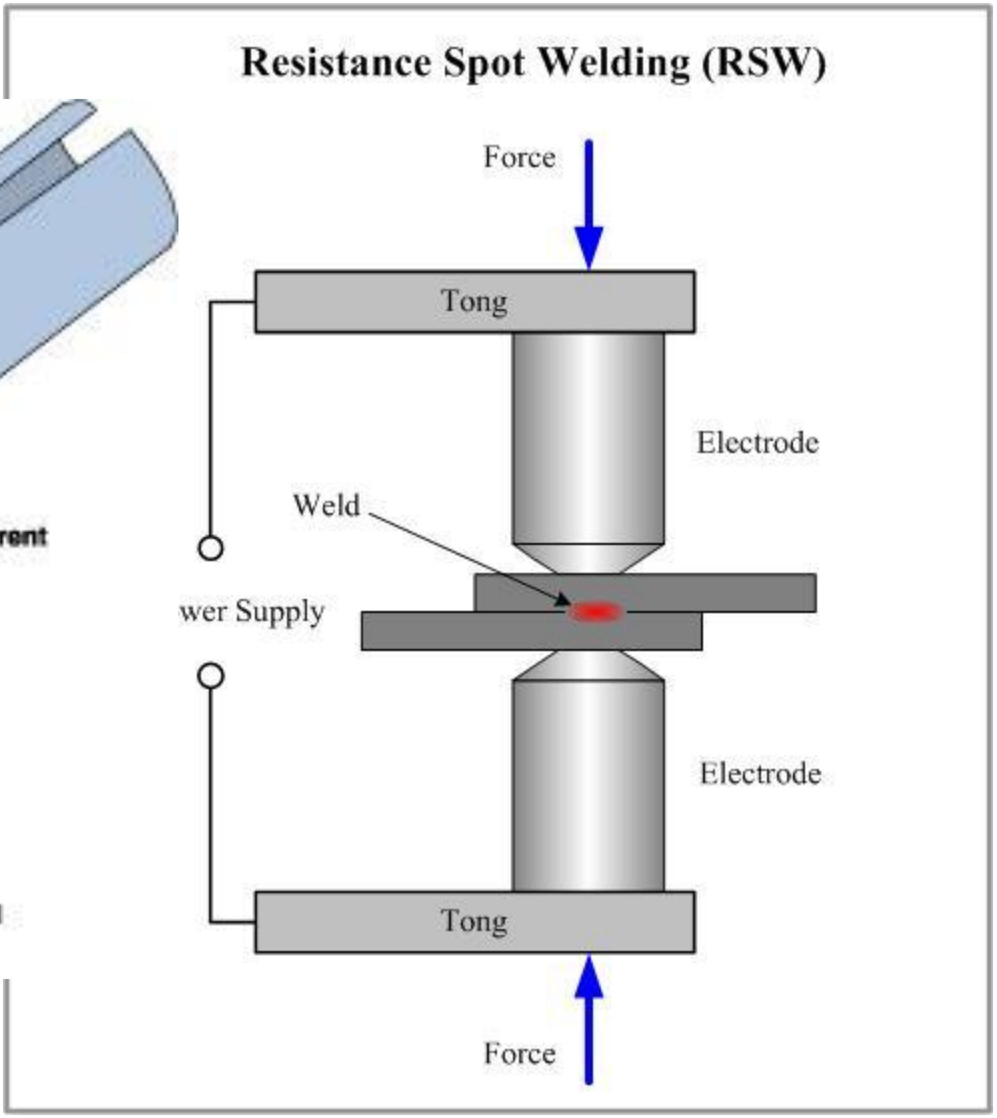
Electrochemical application

- Electroplating simulation.
- Current density and thickness distribution
- Anodising.
- Electroforming.
- Electrochemical machining and etching.
- Cathodic and anodic protection.



Electric welding





Electric welding robots

EC summary. Equations, interface and boundary conditions

$$\begin{cases} div\mathbf{J} = 0 \\ \mathbf{curl}\mathbf{E} = 0 \\ \mathbf{J} = \overline{\overline{\sigma}}\mathbf{E} + (\mathbf{J}_i) \end{cases} \Leftrightarrow \begin{cases} -div(\overline{\overline{\sigma}}\mathbf{grad}V) = (\rho_c) \\ \rho_c = -div\mathbf{J}_i \\ \mathbf{E} = -\mathbf{grad}V \end{cases} \Leftrightarrow \begin{cases} \mathbf{curl}\left[\overline{\overline{\rho}}\mathbf{curl}\mathbf{T}\right] = \mathbf{S} \\ \mathbf{S} = \mathbf{curl}(\mathbf{E}_i) \\ \mathbf{J} = curl\mathbf{T},\ div\mathbf{T} = 0 \end{cases}$$

$$\begin{cases} J_{n1} = J_{n2} \\ \mathbf{E}_{t1} = \mathbf{E}_{t2} \end{cases} \Leftrightarrow \begin{cases} \sigma_1 \dfrac{\partial V_1}{\partial n} = \sigma_2 \dfrac{\partial V_2}{\partial n} \\ V_1 = V_2 \end{cases} \Leftrightarrow \begin{cases} \mathbf{T}_1 = \mathbf{T}_2 \\ \mathbf{n} \times \overline{\overline{\rho}}_1 curl\mathbf{T}_1 \times \mathbf{n} = \mathbf{n} \times \overline{\overline{\rho}}_2 curl\mathbf{T}_2 \times \mathbf{n} \end{cases}$$

$$-\begin{cases} \begin{rcases} \mathbf{E}_t = \mathbf{f}_E(P) \ \text{ on } \ S_E \\ J_n = f_J(P) \ \text{ on } \ S_J \end{rcases} \Leftrightarrow \begin{cases} V = f_D(P) \text{ on } S_E \\ \dfrac{dV}{dn} = f_N(P) \text{ on } S_J \end{cases} \Leftrightarrow \begin{cases} \mathbf{n} \times \mathbf{T} = \mathbf{f}_D(P) \text{ on } S_J \\ \mathbf{n} \times curl\mathbf{T} = \mathbf{f}_N(P) \text{ on } S_E \end{cases} \\ \int_{PkP_0} \mathbf{E}_t d\mathbf{r} = V_k \quad \text{or} \quad \int_{S_{Ek}} J_n dS = I_k, \\ \text{for each } \ S_{Ek}, k = 1,2,.., n\text{-}1, \text{and } V_n = 0 \end{cases}$$

For T uniqueness:

$$\mathbf{curl}\left[\overline{\overline{\rho}}\mathbf{curl}\mathbf{T}\right] + \mathbf{grad}\left[\overline{\overline{\rho}}div\mathbf{T}\right] = \mathbf{S}$$

$$\begin{cases} \mathbf{n} \times \mathbf{T} = \mathbf{f}_D(P) \text{ on } S_J \\ \mathbf{n} \cdot \mathbf{T} = 0, \mathbf{n} \times curl\mathbf{T} = \mathbf{f}_N(P) \text{ on } S_E \end{cases}$$

- **Circuit parameters:** $\mathbf{i} = \mathbf{G}\mathbf{v}, \quad \mathbf{v} = \mathbf{R}\mathbf{i}, \quad \mathbf{R} = \mathbf{G}^{-1}$
- **Electric power:** $< \mathbf{J}', \mathbf{E}'' > - \mathbf{i}'^T \cdot \mathbf{v}'' = 0 \Rightarrow \mathbf{J} \perp \mathbf{E} \Rightarrow \quad \mathbf{R} = \mathbf{R}^T > 0, \quad \mathbf{G} = \mathbf{G}^T > 0$

Not so easy questions for curious people

1. Give examples of wrong and well EC field formulations.
2. What about nonlinear conductors? Equations, uniqueness, power.
3. How may be defined the dipolar resistance using linear relation and by using power approach?
4. How many field problems should be solved to extract the resistance matrix of a m-terminal linear multipole?
5. Describe the similitude between ES, MS and EC. How can it used to compute R?
6. How can be computed the electroplated thickness?
7. How can be computed the temperature generated by a current distribution in a conductor body?
8. How are projection/minimization weak complementary EC formulations?
9. What space may be used for trial and test functions in the weak EC formulation ?
10. What are the scissor relations for resistance computation?
11. How are Tellegen, reciprocity and conductivity variation theorems for EC field?
12. How can be applied Adjoint Field Technique (AFT) to compute the resistance senzitivity?