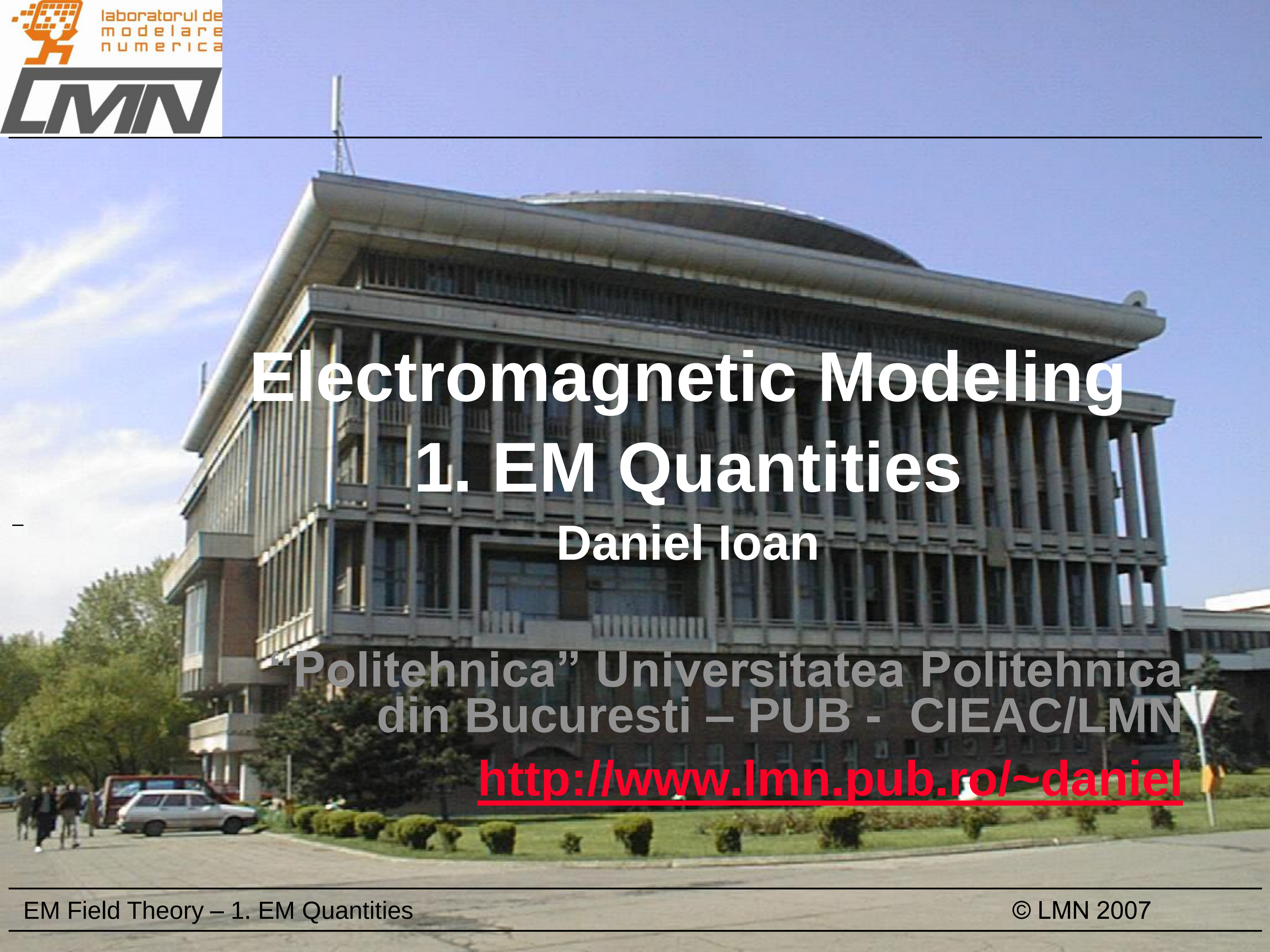

Electromagnetic Modeling
1. EM Quantities

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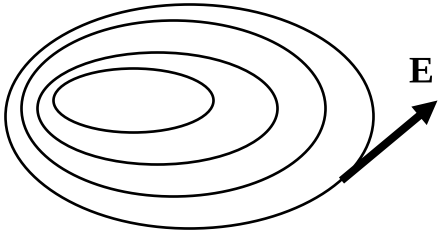
Local quantities of the EM field

1. **What is a local and an instantaneous quantity ?** q(**r**,*t*), where
   - q may be : a real or a complex a scalar, a vector or a tensor
   - ***r* = *i***x + ***j***y + ***k***z is the (spatial) position vector
   - *t* is the time variable
2. **What is the Electromagnetic field?** EM = Ef + Mf able to accumulate and transfer energy even without substance, and to interact mechanically and thermal with bodies (substance)
3. **Local quantities of the electric field:**
   - ***E*** *(**r**, t)* – electric field strength [V/m]
   - ***D*** *(**r**, t)* – electric flux density [C/m^2]



4. **Local quantities of the magnetic field:**
   - ***H*** *(**r**, t)* – magnetic field strength [A/m]
   - ***B*** *(**r**, t)* – magnetic flux density [T]
5. ***E, D, B, H*** are 3D vectors: $\boldsymbol{E} = \boldsymbol{i}\, E_x + \boldsymbol{j}\, E_y + \boldsymbol{k}\, E_z$. Mathematically they are vector-fields (represented by field lines). EM field = 14 x 3 = 12 field components, scalar/real functions, w.r.t. 4 independent variables: x, y. z and t.

Global/integral quantities of the EM field

1. **What is a global quantity ?** It is an integral of a local quantity , where Ω is a 1D, 2D or a 3D spatial domain

$$Q(t) = \int_{\Omega} q(\mathbf{r},t)\,d\omega$$

2. **What is a process quantity ?** It is an integral of a instantaneous quantity along a time interval

$$W = \int_{t1}^{t2} p(t)\,dt$$

2. **Global quantities of the electric field:**
   - **Electric voltage** v [V]
   - **Electric flux** ψ [C]

$$v(t) = \int_{C} \mathbf{E}(\mathbf{r},t)\,d\mathbf{r}$$

$$\psi(t) = \int_{S} \mathbf{D}(\mathbf{r},t)\,d\mathbf{A}$$

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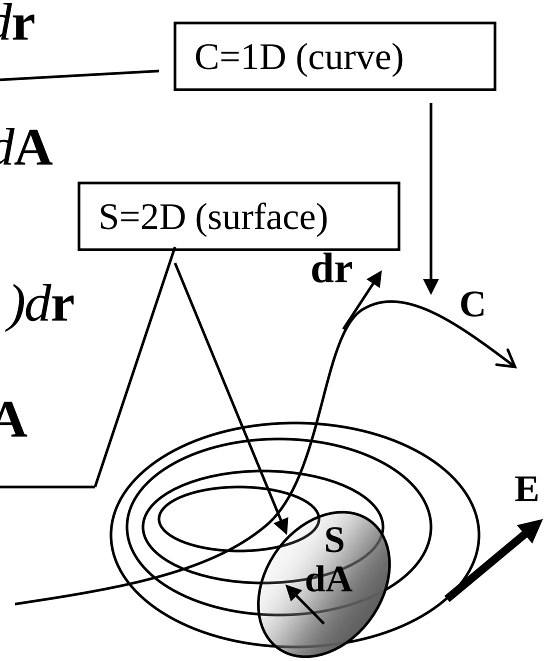
3. **Global quantities of the magnetic field:**
   - **Magnetic "voltage"** u [A]
   - **Magnetic flux** φ [Wb]

$$u(t) = \int_{C} \mathbf{H}(\mathbf{r},t)\,d\mathbf{r}$$

$$\varphi(t) = \int_{S} \mathbf{B}(\mathbf{r},t)\,d\mathbf{A}$$

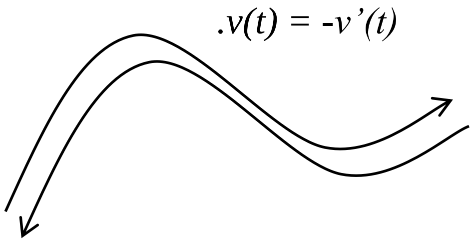


4. *v, ψ, u, φ* are scalar quantities . $d\mathbf{A}=\mathbf{n}dA$, $d\mathbf{r}=\mathbf{t}dr$

Mathematically they are real functions w.r.t. a real variable t.

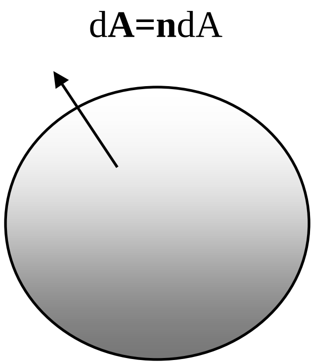
Orientation rules

- Open curves may have an arbitrary orientation ➔ quantity sign is associated to the orientation

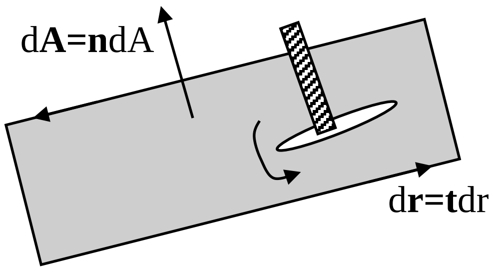


$$v(t) = \int_C \mathbf{E}(\mathbf{r},t)d\mathbf{r} = -\int_C \mathbf{E}(\mathbf{r},t)d\mathbf{r}' = -v'(t)$$

- Orientation of the close curves is associated to the orientation of the surface boarded by it, according to the “right screw rule”



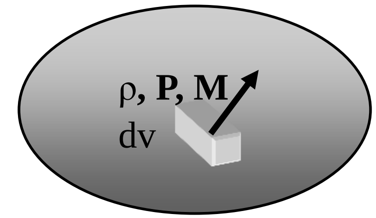
- Closed surfaces are “externally” orientated
- Orientation of the open surfaces is associated to the orientation of the board (boundary curve)



$$\psi(t) = \int_S \mathbf{D}(\mathbf{r},t)d\mathbf{A} = -\int_S \mathbf{D}(\mathbf{r},t)d\mathbf{A}' = -\psi'(t)$$

EM quantities of the substance

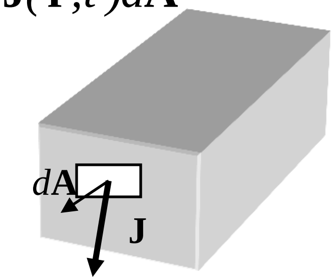
1. **Local EM fundamental quantities:**
   - $\rho\,(\mathbf{r}, t)$ – charge density [C/m^3]
   - $\mathbf{J}\,(\mathbf{r}, t)$ – current density [A/m^2]



2. **Global EM fundamental quantities:**
   - $q(\mathbf{r}, t)$ – (electric) charge [C]
   - $i(\mathbf{r}, t)$ – (electric) current [A]

$$q(t) = \int_D \rho(\mathbf{r},t)\,dv$$

$$i(t) = \int_S \mathbf{J}(\mathbf{r},t)\,d\mathbf{A}$$



3. **Other EM local quantities of the substance**
   - $\mathbf{P}\,(\mathbf{r}, t)$ – Polarization [C/m^2]
   - $\mathbf{M}\,(\mathbf{r}, t)$ – Magnetization [A/m]

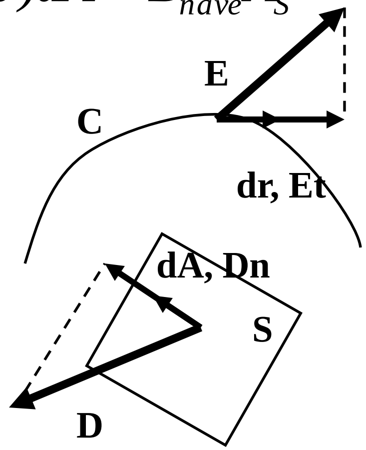
4. **Other EM global quantities of the substance**
   - $\mathbf{p}\,(\mathbf{r}, t)$ – Electric momentum [Cm]
   - $\mathbf{m}\,(\mathbf{r}, t)$ – Magnetic momentum [Am^2]

$$\mathbf{p}(t) = \int_D \mathbf{P}(\mathbf{r},t)\,dv$$

$$\mathbf{m}(t) = \int_D \mathbf{M}(\mathbf{r},t)\,dv$$

Global quantities as products

1. **The integrals as products (average value definition):**
   - $q_{ave} = Q / M_\Omega$ $\quad Q(t) = \int_\Omega q(\mathbf{r},t)d\omega = q_{ave}M_\Omega \quad M_\Omega = \int_\Omega d\omega = \begin{cases} l, \Omega = C(1D) \\ A, \Omega = S(2D) \\ V, \Omega = D(3D) \end{cases}$
2. **Voltages as products:**
   - ***Electric voltage*** $\quad v(t) = \int_C \mathbf{E}(\mathbf{r},t)d\mathbf{r} = \int_C E_t(\mathbf{r},t)dr = E_{tave}l_C$
   - ***Magnetic voltage*** $\quad u(t) = \int_C \mathbf{H}(\mathbf{r},t)d\mathbf{r} = \int_C H_t(\mathbf{r},t)dr = H_{tave}l_C$
3. **Fluxes as products:**
   - **Magnetic flux:** $\quad \psi(t) = \int_S \mathbf{D}(\mathbf{r},t)d\mathbf{A} = \int_S D_n(\mathbf{r},t)dA = D_{nave}A_S$
   - **Electric flux:** $\quad \varphi(t) = \int_S \mathbf{B}(\mathbf{r},t)d\mathbf{A} = \int_S B_n(\mathbf{r},t)dA = B_{nave}A_S$
4. **The charge as a product:**
   - **Electric charge:** $\quad q(t) = \int_D \rho dv = \rho_{ave}V_D$
5. **Moments as products:**
   - Electric momentum $\quad p(t) = \int_D \mathbf{P}(\mathbf{r},t)dv = \mathbf{P}_{ave}V_D$
   - Magnetic momentum $\quad m(t) = \int_D \mathbf{M}(\mathbf{r},t)dv = \mathbf{M}_{ave}V_D$



Singular distributions of charge and current

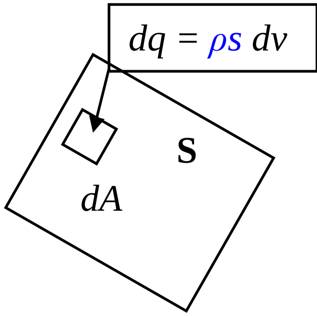
1. **Local quantities as "derivatives" of the global quantities:**

- $\rho_{ave} = q_D / V_D$ ➔ $\rho = \lim \Delta q / \Delta V$ for $\Delta V \to 0$

2. **Charge distribution over a surface:**

- **Superficial charge density on an charged sheet** $\rho_s = \lim \Delta q / \Delta A$ for $\Delta A \to 0$ [C/m[2]]

$$q(t) = \int_S \rho_s dA = \rho_{save} A_S$$



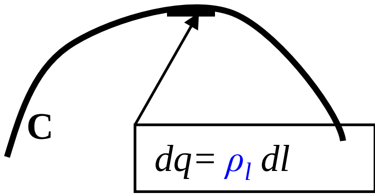
- **Line charge density on an charged wire** $\rho_l = \lim \Delta q / \Delta l$ for $\Delta l \to 0$ [C/m]

$$q(t) = \int_S \rho_s dA = \rho_{save} A_S$$

- **Punctual charge distribution q [C]**

- **Total charge in the domain Ω with several charge distributions**

$$q(t) = \int_{D \subset \Omega} \rho_v dv + \int_{S \subset \Omega} \rho_s dA + \int_{C \subset \Omega} \rho_l dl + \sum_{k=1,n} q_k$$

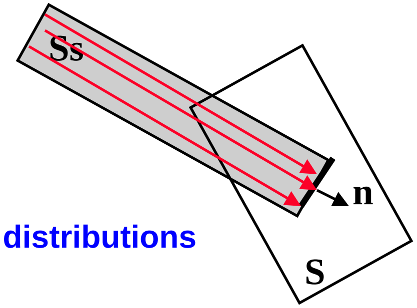


3. **Current distribution on a surface:**

- **Superficial current density in a sheet J$_s$** $= \lim \Delta q / \Delta l$ for $\Delta l \to 0$ [A/m]

- **Current I in a wire**

$$i(t) = \int_{C = S \cap Ss} \mathbf{J}_s(\mathbf{r}, t)\mathbf{n} dl$$



- **Total current which cross the section S of a domain Ω with several current distributions**

$$i(t) = \int_{S \cap \Omega \cap D} \mathbf{J}(\mathbf{r}, t) d\mathbf{A} + \int_{S \cap \Omega \cap Ss} \mathbf{J}_s(\mathbf{r}, t) dl + \sum_{k=1,m} i_k$$

Mathematic aspects

1. Classic approach:

$$\boldsymbol{E},\boldsymbol{D},\boldsymbol{B},\boldsymbol{H},\boldsymbol{J} = \mathrm{f}(\boldsymbol{r},t);\mathrm{f}:\Omega\times(t_{\min},t_{\max})\to IR^3$$

$$\rho = f(\boldsymbol{r},t); f:\Omega\times(t_{\min},t_{\max})\to IR$$

f, f classic functions(e.g piecewise - continous,bounded), Riemann integrals:

$$u=\int_C \boldsymbol{E}\cdot d\boldsymbol{r} = E_{t,med} l_C; \quad \oint_{\Gamma=\partial S} \boldsymbol{E}\cdot d\boldsymbol{r} = \int_S (\nabla\times\boldsymbol{E})\cdot\boldsymbol{n}dS \quad \text{Stokes} \Leftarrow \mathrm{f}\in C^1(\Omega)$$

$$\psi=\int_S \boldsymbol{D}\cdot d\boldsymbol{A} = D_{n,med} A_S; \quad \oint_{\Sigma=\partial\Omega} \boldsymbol{D}\cdot d\boldsymbol{A} = \int_\Omega (\nabla\cdot\boldsymbol{D})dv \quad \text{Gauss} \Leftarrow \mathrm{f}\in C^1(\Omega)$$

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2. Geometric differential approach:

$$\text{Manifolds}: \quad \Omega = \begin{cases} C\text{ - 1D(curve)} \\ S\text{ - 2D(surface)} \\ \mathscr{D}\text{ – 3D(volume)} \end{cases}$$

$$u=\int_C \boldsymbol{E}\cdot d\boldsymbol{r} = \int_C e; \quad e\text{ - differntial form of degree 1(as } \boldsymbol{H})$$

$$d\Omega = \begin{cases} ds,\ \Omega = C\text{ - 1D(curve)} \\ dS,\ \Omega = S\text{ - 2D(surface)} \\ dv,\ \Omega = \mathscr{D}\text{ - 3D(volume)} \end{cases}$$

$$\psi=\int_S \boldsymbol{D}\cdot d\boldsymbol{A} = \int_S d; \quad d\text{ - differntial form of degree 2 (as } \boldsymbol{B},\boldsymbol{J})$$

$$q=\int_\Omega \rho\cdot dv = \int_S \rho; \quad \rho\text{ - differntial form of degree 3.}$$

$$\text{Stokes}: \int_\Omega \mathrm{d}\omega = \int_{\partial\Omega}\omega,$$

$$\text{Geen}: \int_\Omega \varphi\, div b + \int_\Omega b\cdot grad\varphi = \int_{\partial\Omega} n\cdot b\varphi$$

Mathematic aspects (cont.)

3. Functional analysis approach:

$\rho \in L^2(\Omega) = \{ f : \Omega \rightarrow IR \,|\, \| f \| =_{def} \int_\Omega f^2 d\Omega < \infty \}$; Lebesgue integral

$\mathrm{L}^2(\Omega) = (\Omega)/N; N = \ker(\|.\|) = \{ f{:}f = 0, \text{a.e.(almost evrywere)} \}$ (p.p.t,a.p..t)

is a Banch/Hilbert space (it is complete),but $\mathrm{f} \in \mathrm{L}^2(\Omega)$ (Lebesgue space of square integrable functions) is not a classic function,it is a equivalence class.

$$\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{B}, \boldsymbol{H}, \boldsymbol{J} \in \left[L^2(\Omega)\right]^3; <\boldsymbol{E}, \boldsymbol{D}> = \int_\Omega \boldsymbol{E}\boldsymbol{D} d\Omega \propto \mathrm{W}; \| \boldsymbol{E} \| = \sqrt{<\boldsymbol{E}, E>} = \sqrt{\int_\Omega \boldsymbol{E}^2 d\Omega} \propto \sqrt{\mathrm{W}}$$

$\|.\|$ - norm = a function measure. Distance between functions(fields) : $d(f,g) = \| f - g \|$

$<.,.>$ scalar product. $<f,g> = \| f \| \cdot \| g \| \cdot \cos\alpha; f \perp g \Leftrightarrow <f,g> = 0 \Leftrightarrow \alpha = \pi/2$

Sobolev spaces (when Ω is enough smooth,e.g. it satisfies Lieschitz conditions) :

$$\oint_{\Gamma=\partial S} \boldsymbol{E} \cdot d\boldsymbol{r} = \int_S (\nabla \times \boldsymbol{E}) \cdot d\boldsymbol{S} \quad \text{Stokes} \Leftarrow \boldsymbol{E} \in H(curl, \Omega) = \{ \boldsymbol{f} \in \left[L^2(\Omega)\right]^3 \,|\, \nabla \times \boldsymbol{f} \in \left[L^2(\Omega)\right]^3 \}$$

$$\oint_{\Sigma=\partial\Omega} \boldsymbol{D} \cdot d\boldsymbol{S} = \int_\Omega (\nabla \cdot \boldsymbol{D}) dv \quad \text{Gauss} \Leftarrow \boldsymbol{D} \in H(div, \Omega) = \{ \boldsymbol{f} \in \left[L^2(\Omega)\right]^3 \,|\, \nabla \cdot \boldsymbol{f} \in L^2(\Omega) \}.$$

Mathematic aspects (cont.)

4. Distributions approach

$$\mathcal{D}(\Omega) = \{\varphi : \Omega \to IR \,|\, d^k\varphi / dx^k \in C(\Omega), k \in IN\} = C^\infty(\Omega) \quad \text{test functions}$$

$$\mathcal{D}'(\Omega) = \{T : \mathcal{D}(\Omega) \to IR \,|\, T(\varphi) = < f, \varphi > = \int_\Omega f\varphi d\Omega, \ \text{linear functional}\}$$

$$\rho \in \mathcal{D}'(\Omega) \text{ - } \ \text{Distribution space} \Rightarrow \rho \text{ is derivable, even if it is not continous!}$$

Examples. Dirac distribution describes concentrated charge in $\boldsymbol{r}_0$

$$h(x) = \text{Heaviside, unit step function} \Rightarrow$$

$$\int_{-\infty}^{\infty} \delta(x)\varphi(x)dx = \varphi(0); \quad \delta(x) = h'(x) = \frac{dh}{dx} \ \text{Dirac impulse} \Rightarrow$$

$$\rho = q \cdot \delta(\mathbf{r} - \mathbf{r}_0) = q \cdot \delta(x - x_0) \cdot \delta(y - y_0)\delta(z - z_0)$$

$$\int_{IR^3} \rho dv = q \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x - x_0) \cdot \delta(y - y_0)\delta(z - z_0) dxdydz = q$$

Mathematic aspects (cont.)

Example. Surface charge distribution:

$$S : \boldsymbol{r} = \boldsymbol{f}_S(u,v) : (0,1)\times(0,1) \to IR^3 \text{ - parametric equation of surface}$$

$$\rho_s(u,v) \text{ - surface charge density } [\mathrm{C/m^2}].$$

Charge density as distribution:

$$\rho = \rho_s(u,v)\cdot\delta\big(\boldsymbol{r} - \boldsymbol{f}_S(u,v)\big) \Rightarrow \int_{IR^3} \rho\, dv = \int_S \rho_s\, dA = \int_0^1\int_0^1 \rho_s h_u h_v\, du\, dv = q_S$$

Classic approach:

$$q(t) = \int_{D\subset\Omega} \rho_v\, dv + \int_{S\subset\Omega} \rho_s\, dA + \int_{C\subset\Omega} \rho_l\, dl + \sum_{k=1,n} q_k$$

Distribution approach:

$$q(t) = \int_\Omega \rho\, dv; \quad \rho = \rho_v(\boldsymbol{r}) + \rho_s\cdot\delta(\boldsymbol{r} - \boldsymbol{f}_S) + \rho_l\cdot\delta(\boldsymbol{r} - \boldsymbol{f}_C) + \sum_{k=1}^{n} q_k\cdot\delta(\boldsymbol{r} - \boldsymbol{r}_k)$$

EM fundamental quantities Summary

Field	Local	Global
Electric	Field strength: **E** [V/m]/1Form	Voltage [V] $v(t)=\int_C \mathbf{E}(\mathbf{r},t)d\mathbf{r}$
	Flux density: **D** [C/m²]/2Form	Electric flux $\psi(t)=\int_S \mathbf{D}(\mathbf{r},t)d\mathbf{A}$
Magnetic	Field strength: **H** [V/m] /1Form	Magnetic Voltage [A] $u(t)=\int_C \mathbf{H}(\mathbf{r},t)d\mathbf{r}$
	Flux density: **B** [T]/2Form	Magnetic flux [Wb] $\varphi(t)=\int_S \mathbf{B}(\mathbf{r},t)d\mathbf{A}$
Substance	Charge density ρ [C/m²]/3Form	(Electric) charge [C] $q(t)=\int_D \rho(\mathbf{r},t)dv$
	Current density **J** [A/m²]/2Form	(Electric) current [A] $i(t)=\int_S \mathbf{J}(\mathbf{r},t)d\mathbf{A}$

Not so easy questions for curious people

1. What is the electromagnetic (electric, magnetic) field ?
2. What is a (open/close) curve (or surface) ?
3. What is an (orthogonal) system of coordinates?
4. What happen when the Cartesian system of coordinates is changed?
5. How are expressions of field integrals/derivatives in several coordinates?
6. What is the main difference between field strength and flux density
7. How the distribution theory (e.g. Dirac generalized function) may be used to describe charge/current singular distributions?
8. How the fundamental EM quantities can be measured?
9. What is SI, and how the measurement units of the fundamental EM quantities may be defined?

Readings:

Alain Bossavit, *Differential Geometry for the student of numerical methods in Electromagnetism, 1991,* http://butler.cc.tut.fi/~bossavit/Books/DGSNME/DGSNME.pdf