Differential Geometry, Calculus of Variations, and Their Applications

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1. INTRODUCTION

Every book on the calculus of variations starts with the typical phrase: "Let us consider a functional . . . ." Today much interest is given to the inverse problem: "Given an equation, does a functional exist that admits the given equation as its Euler-Lagrange equation?" This is known as the inverse problem of the calculus of variations.

In the literature the statement of such inverse problem varies greatly from one author to another. Let us examine these different statements.

The first distinction lies in the fact that some people limit the variational formulation to differential equations ignoring initial or boundary conditions while others take account of these.

In the first case the study is made on the mathematical form of the differential equation and the main interest is finding the lagrangian: we shall call this the formal inverse problem.

In the second case the kinds of additional conditions (initial and/or boundary conditions) are an essential part of the problem: we shall call this simply the inverse problem.

In both cases one must say whether the equation may be transformed in another by an integrating factor or not. We shall speak of extended and restricted variational formulation, respectively.

We shall deal with the inverse problem in its full meaning (not in the formal sense): a precise definition of it will be given in the next section.

We want to deal with the problem from a very general point of view, without distinguishing among ordinary or partial differential equations; among equations containing first, second, ... order derivatives; single equations or systems of them; among differential, integral, integrodifferential equations or equations with retarded arguments, etc. To make this possible we must use the operatorial notation.

A Brief Historical Survey

The history of the inverse problem has a curious feature: it is formed by two branches that developed separately for about eighty years (Fig. 1).

Both branches started in the same year (1887): one with Helmholtz, which deals with the formal inverse problem; the other with Volterra, which uses functional analysis. The singular fact is that no papers of one branch quote a paper of the other branch: the developments of the two branches are entirely separate.

Let us consider the first branch.

In 1887 Helmholtz [15] gave the necessary conditions in order that a single ordinary equation of second order may be considered as an Euler-Lagrange equation.

In 1897 Hirsch [16] gave the analogous condition for an equation of order n.

In 1928 Davis [3] gave the conditions for a partial differential equation to be of variational kind: he introduced the integrating factor.

In 1941 Douglas [6] developed Davis theory.

In 1957 an expository article of Havas [14] appeared dealing with the search for integrating factors.

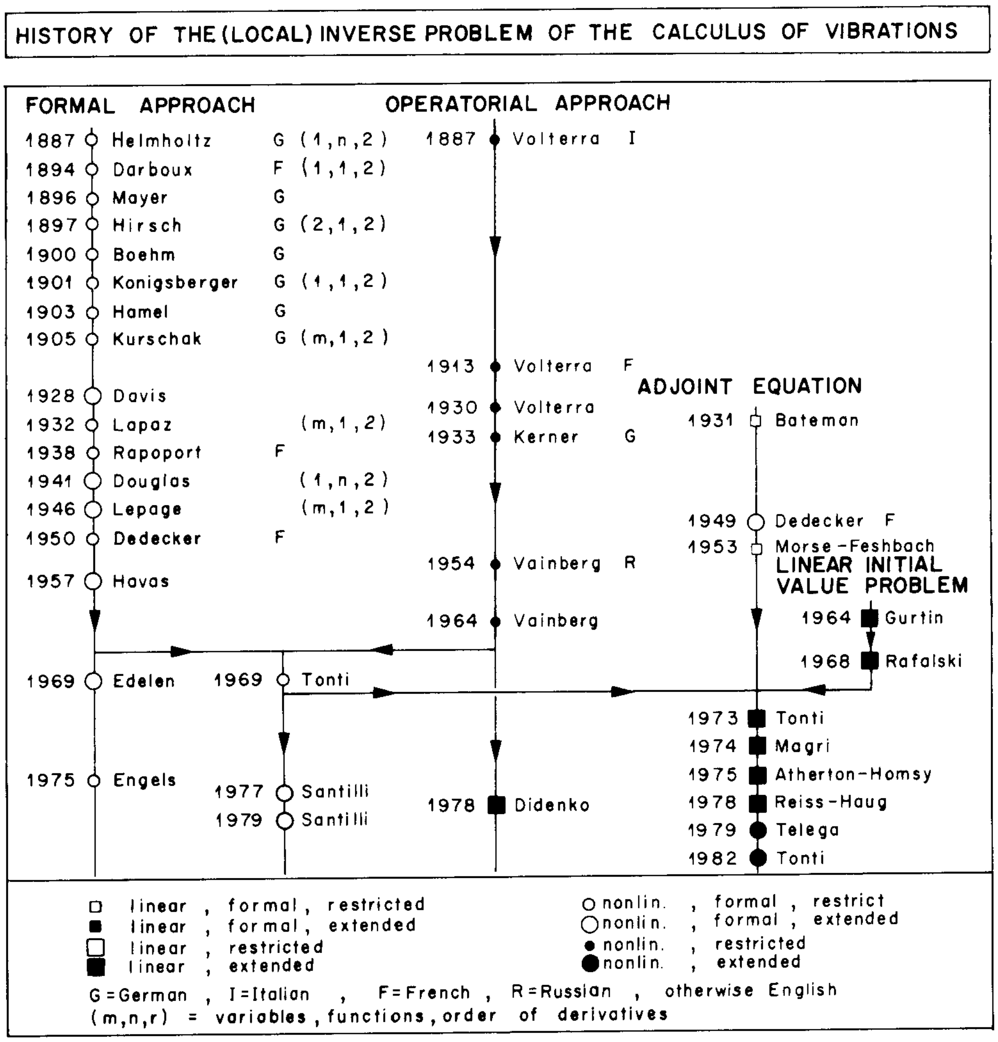

Fig. 1

The second branch starts in 1887 with three papers on the theory of functionals [42]. In paper I (p. 104) the symmetry of the second derivative of a functional appears.

In 1913 Volterra published a book in French [41] in which he gave the condition for the variational formulation and the formula to find the functional (p. 43).

In 1930 Volterra published another book in English [42] that contains the same results (Ch. V, Sect. II, Sec. 2).

In 1933 Kerner [19] published a paper giving the result of Volterra (its reference 4).

In 1954 a book of Vainberg [40] appeared (in Russian) that contained the theorem on the variational formulation. He quoted Kerner (its reference 42b).

In 1954 a book of Volterra and Pérès in French appeared [44] in which the theorem on the variational formulation was given (p. 98).

In 1959 Volterra's book [43] was reprinted by Dover.

In 1964 Vainberg's book was translated into English.

In 1969 the paper [39] appeared connecting the two branches. With the intention of giving an elementary exposition of the operatorial approach (the second branch), the author obtained the integrability conditions of Helmholtz and of other authors of the first branch. (See [28], p. 14 and p. 204.)

A third branch originated in more recent times: it is centered on the problem of giving a variational formulation to equations (including the initial/boundary conditions) that do not admit one in a classical context, as, for example, the Fourier equation of heat transfer. The inverse problem is here involved in a nonformal sense.

One of the first methods devised was that of adding the adjoint equation. This method, inaugurated by Morse and Feshbach in their book ([23], p. 298), consists of the arbitrary addition to a given linear equation of the adjoint homogeneous equation. The system of the two equations is symmetric. The adjoint function has no physical meaning. This fact and the fact that the adjoint problem introduces adjoint boundary conditions make the method artificial.

In the above-mentioned book (p. 299) the authors say: "By this arbitrary trick we are able to handle dissipative systems as though they were conservative. This is not very satisfactory if an alternate method of solution is known . . . ." Today an alternate method is known (see below) and the Morse and Feshbach technique may be abandoned.

Since 1953 many pseudovariational formulations have been devised: they have been named "quasi" or "almost" or "restricted" variational formulations. The reader may see [10] for a critical review.

The method that opened a new era is the one introduced by Gurtin in 1964 [11, 12]. He showed how to give variational formulation to linear initial value problems. This means that the initial conditions were taken into consideration from the beginning.

Gurtin's idea was the preliminary transformation of an equation into an integro-differential equation and the introduction of the convolution product of two functions.

This method opened the way to giving variational formulation to many linear initial value problems and a large number of papers appeared, mainly in engineering reviews.

The method of Gurtin was simplified in 1973 by the present author [38] who showed that the preliminary transformation of the differential equation into an integrodifferential equation is not necessary. The essential point is the introduction of a convolutive bilinear functional to give a variational formulation to a linear initial value problem whose equation has constant coefficients. In this paper Gurtin's method was included in the operatorial approach (see Fig. 1).

The idea of adapting the bilinear functional to the given operator was brought up to its apex by Magri, in 1974 [21]: he showed that every linear equation (not only those with constant coefficients) admits a variational formulation giving the explicit way to obtain the functional.

This result overthrows the common belief that equations admitting a variational formulation constitute a privileged class. At this date to every linear problem one may associate many functionals whose stationary value is attained at the solution of the problem. In general these functionals are not extremum at that point.

In 1978 Reiss and Haug [26] using Magri's result explored the possibility of finding among the many functionals those that give an extremum principle for linear initial value problems.

What about the larger class of nonlinear problems? Some attempts have been made to extend the method of adjoint equation to nonlinear problems [9]: the method suffers the same drawbacks of the linear case.

In 1979 Telega [31] first tried to extend Magri's result to nonlinear problems: the class of operators that was included was severely limited.

In 1982 the present author [32] showed that every nonlinear problem admits a variational formulation giving the explicit form of the functional. This result was further developed toward practical applications in the paper [33]: in particular nonlinear initial value problems have been solved using the Ritz method and the gradient method.

In the present paper I summarize the results of papers [32, 33] stressing the conceptual framework that makes possible the variational formulation to (practically) every nonlinear problem.

2. OPERATORIAL NOTATION

For better adherence to physical applications we shall consider operators with domain and range in two different spaces U and V.

Let us remember that an operator A is said to be equal to an operator B if D(A) = D(B) and if A(u) = B(u) for every $u \in D(A)$. An operator A is called a restriction of an operator B if $D(A) \subset D(B)$ and if A(u) = B(u) for every $u \in D(A)$. The operator B is called extension of the operator A. Every element u_0 of D(A) such that $A(u) = O_V$ is called a null element of A ([2], p. 91), the set of null elements is called the null manifold of A.

A differential equation is formed by a differential expression equated to zero. The differential expression is called formal differential operator ([41], p. 146; [38]). To form a full operator we must select a set of functions, for example those of an assigned functional class. The operator so defined has a very large domain: any supplementary boundary or initial condition gives rise to a new operator that is a restriction of the initial operator.

To give a variational formulation to a problem, say $N(u) = O_V$, we need a real-valued, bilinear functional $V \times U \to R$, denoted $\langle v, u \rangle$, that is nondegenerate: we shall call it the scalar product of $v \in V$ and $u \in U$. The two spaces U and V are said to be put in duality by the bilinear functional $\langle v, u \rangle$ and V is called the dual of U and is denoted by U*.

It is at this stage that we introduce norms on U and V such that the bilinear functional be continuous in both arguments.

We call the adjoint of a linear operator L with respect to a given bilinear functional $\langle v, u \rangle$ the linear operator L* that satisfies the relation

$$\langle Lp, q \rangle = \langle L^* q, p \rangle \tag{2.1}$$

for every $p \in D(L)$ and every possible q which will form the domain D(L*).

An operator L is said to be symmetric if

$$\langle Lp, q \rangle = \langle Lq, p \rangle \tag{2.2}$$

for every $p, q \in D(L)$. Comparing (2.2) with (2.1) we see that in general it will be $D(L^*) \supset D(L)$, i.e., the operator L* is an extension of the operator L. If the two domains coincide we have $L = L^*$ and the operator is called self-adjoint.

Given a formal differential operator $\mathscr{L}$, the formal differential operator $\mathscr{L}^*$ that satisfies the relation

$$\int u \mathscr{L} \bar{u}\, dx = \int \bar{u} \mathscr{L}^* u\, dx + \{\text{boundary terms}\} \tag{2.3}$$

is called formal adjoint. If $\mathscr{L} = \mathscr{L}^*$ the formal operator is called formally symmetric or formally self-adjoint, the two names are equivalent ([18], p. 274).

If F is a real functional, i.e., $F : D(F) \subset U \to R$ and if N is an operator $N : D(N) \subset U \to V = U^*$ and if

$$\delta F\{u\} = F'_u\{u; \delta u\} = \langle N(u), \delta u \rangle \tag{2.4}$$

is true with the condition D(F) = D(N), then the operator N is called the gradient of the functional F and F is called the potential of N; N is said to be a potential operator. The symbol δ is the usual one of the calculus of variations: δF coincides with the Gâteaux differential of the functional.

To say that the functional F is stationary at u_0 means that $\delta F\{u\} = 0$ at $u_0 \in D(F)$. The elements u_0 for which the functional is stationary are called critical points and the set formed by them is called the critical manifold.

Let $N'_u(u, \delta u)$ denote the Gâteaux differential of N and $N'_u(u; \cdot)$ the (linear) Gâteaux derivative calculated in u.

At this point we are able to enunciate two forms of the inverse problem.

Inverse Problem in the Restricted Sense. Given a problem $N(u) = O_V$ with $D(N) \subset U$ and $R(N) \subset V = U^*$ find a functional F, if any, whose gradient is the operator N.

Inverse Problem in the Extended Sense. Given a problem $N(u) = O_V$ with $D(N) \subset U$ and $R(N) \subset V = U^*$ find a functional F, if any, whose critical points are the solutions of the problem and vice versa.

The inverse problem in the "extended" sense requires only the coincidence of the critical manifold of the functional F with the null manifold of the operator N. The inverse problem in the "restricted" sense requires a stricter link between N and F: N must be the gradient of F with respect to a given bilinear functional. In this case it follows that the domain of N coincides with the domain of F and moreover the null manifold of N coincides with the critical manifold of F.

Stated in another way: in the extended sense the gradient of the functional F will be an operator $\bar{N}$ linked in some way with N and with the same null manifold while in the restricted sense the gradient of F must coincide with N.

The inverse problem in the restricted sense was solved for the first time in 1913 by Volterra [41,43] with the following theorem.

Theorem. The necessary and sufficient condition in order that an operator $N: D(N) \subset U \to R(N) \subset V = U^*$, whose domain is simply connected, be the gradient of a functional is that

$$\langle N'_u(u;p), q \rangle = \langle N'_u(u;q), p \rangle \tag{2.5}$$

Putting $w(s) = su + (1 - s)u_0$ the functional is

$$F\{u\} = F\{u_0\} + \int_0^1 \langle N(w(s)), \partial w(s)/\partial s \rangle \, ds \tag{2.6}$$

The condition (2.5) expresses the symmetry of the Gâteaux derivative of N and we shall call it the Volterra symmetry condition.

The condition (2.5) is necessary: the hypothesis that D(N) be simply connected (an implicit hypothesis in the original Volterra formulation) is almost always satisfied in practice. In fact, usually the domain is either a linear or a convex set. In the particular case in which the operator is linear, condition (2.5) becomes

$$\langle Lp, q \rangle = \langle Lq, p \rangle \tag{2.7}$$

i.e., the linear operator must be symmetric (not necessarily self-adjoint).

In the linear case the functional (2.6) may be cast in the closed form

$$F\{u\} = F\{u_0\} + 1/2 \langle Lu, u \rangle - \langle f, u \rangle \tag{2.8}$$

Historical Remark

Volterra's theorem is usually called Vainberg's theorem or Kerner's theorem. But Vainberg [40] quoted Kerner (its reference 42.b) and Kerner [19] quoted Volterra (its reference 4). The theorem was contained in the book published by Volterra in 1913 [41] (in French) and in the book (1930) [43] (in English); the latter was reprinted by Dover in 1959.

3. RELATIVITY OF SYMMETRY

The keystone for giving variational formulation of problems lies in the observation that the symmetry of an operator is relative to the bilinear functional considered. Then, contrary to a common belief, the adjoint of an operator is not necessarily unique, and there can be many different possible bilinear functionals. Moreover, if a given operator does not satisfy the Volterra symmetry condition with respect to a given bilinear functional we may look for other bilinear functionals with respect to which this condition is satisfied. The following example will clarify this statement. The linear operator

$$D = \{d/dt,\ u(0) = 0, u \in C^1(0,T)\} \tag{3.1}$$

is not symmetric with respect to the cartesian bilinear functional

$$\langle v,u\rangle_0 = \int_0^T v(t)\, u(t)\, dt \tag{3.2}$$

because its adjoint is

$$D^* = \{-d/dt,\ v(T) = 0,\ v \in AC(0,T)\} \tag{3.3}$$

But if we consider the convolutive bilinear functional

$$\langle v,u\rangle_C = \int_0^T v(T-t)\, u(t)\, dt \tag{3.4}$$

the adjoint becomes

$$D^* = \{d/dt,\ v(0) = 0,\ v \in AC(0,T)\} \tag{3.5}$$

Since $D \subset D^*$ the operator is symmetric with respect to the convolutive bilinear functional.

This observation opens the way to the variational formulation for problems that do not admit one in the "classical" sense.

At this point a question arises: given an operator, does a bilinear functional exist such that it lets the given operator satisfy the Volterra symmetry condition?

The answer was given for linear operators in 1975 by Magri [21] and for nonlinear operators in 1982 by the present author [32].

Magri has shown how to find a bilinear functional that makes symmetric a given linear operator: the operative rule is the following.

Let L be a linear invertible operator with domain in a vector space U and range in a vector space V. Let

$$Lu = f \tag{3.6}$$

be the given problem.

Let us consider the cartesian bilinear functional on $V \times U$

$$\langle v,u\rangle_0 = \int_0^T v(t)\, u(t)\, dt \tag{3.7}$$

and let us suppose that the spaces U and V be such that the functional (3.7) is nondegenerate.

Let us introduce a real bilinear functional (v,v) on $V \times V$

$$(v,v) = \langle v, Kv\rangle = \int_0^T v(t) \int_0^T k(t,s)\, v(s)\, ds\, dt \tag{3.8}$$

that is symmetric and nondegenerate. These two conditions are satisfied if $k(t,s) = k(s,t)$ and if the integral transform

$$w(t) = \int_0^T k(t,s)\, v(s)\, ds \tag{3.9}$$

is invertible.

Let us define a new bilinear functional $\langle v,u \rangle$ on $V \times U$ by

$$\langle v,u \rangle = (v, Lu) = \langle v, KLu \rangle_0 = \int_0^T v(t) \int_0^T k(t,s)Lu(s)\, ds\, dt \tag{3.10}$$

The bilinear functional $\langle v,u \rangle$ is nondegenerate because L is invertible. The operator L is symmetric with respect to this bilinear functional and the solution of problem (3.6) is the critical point of the functional

$$F\{u\} - F\{u_0\} = 1/2 \langle Lu, u \rangle - \langle f,u \rangle = 1/2\,(Lu, Lu) - (f, Lu)$$
$$= 1/2 \langle Lu, KLu \rangle_0 - \langle f, KLu \rangle_0 \tag{3.11}$$

To extend this result to nonlinear operators we need to introduce the notion of integrating operator.

4. INTEGRATING OPERATOR

An observation of capital importance in our problem is the following: the change of the bilinear functional is equivalent to the application (on the left) to the given operator L of a suitable linear operator R.

Let us show this equivalence in a particular case referring to the operator D defined by Eq. (3.1). Let us define the convolution operator

$$Cv(t) = v(T - t) \tag{4.1}$$

and the convolutive bilinear functional

$$\langle v,u \rangle_C = \langle Cv, u \rangle_0 \tag{4.2}$$

We have the symmetry of D with respect to $\langle v,u \rangle_C$, and thus

$$\langle CDu, \bar{u} \rangle_0 = \langle Du, \bar{u} \rangle_C = \langle D\bar{u}, u \rangle_C = \langle CD\bar{u}, u \rangle_0 \tag{4.3}$$

that proves the symmetry of CD with respect to the bilinear functional $\langle v,u \rangle_0$.

More in general if $L: U \to V$ is a linear operator symmetric with respect to the bilinear functional

$$\langle v,u \rangle_R = \langle Rv, u \rangle_0 \tag{4.4}$$

where $R: V \to V$ is a linear, invertible operator, then $RL: U \to V$ is symmetric with respect to the cartesian bilinear functional $\langle v,u \rangle_0$.

The operator R has the same role of the integrating factor used with differential equations: we shall call it an integrating operator. The requirement that R be invertible (i.e., kernel-free) assures that the bilinear functional be nondegenerate.

5. HOW TO FIND THE INTEGRATING OPERATOR

Let us start with a linear system of algebraic equations

$$Ax = b \tag{5.1}$$

If we apply to both members the adjoint matrix A^* we obtain the system

$$A^*Ax = A^*b \tag{5.2}$$

whose matrix is now symmetric with respect to the cartesian bilinear form

$$\langle x,x \rangle = \sum_k x_k x^k \tag{5.3}$$

and then the solutions of problem (5.1) make stationary the function

$$f(x) = 1/2 \langle A^*Ax, x\rangle - \langle A^*f, x\rangle \qquad (5.4)$$

Then the matrix A^* is an integrating operator, provided that it is invertible; if it were not there would be critical values of f(x) that are not solutions of problem (5.1).

Is it possible to extend this procedure to general linear operators, say to differential operators? Let us consider the operator

$$Du = f \quad D = \{d/dt,\ u(0) = 0,\ u \in C^1(0,T),\ f \in C(0,T)\} \qquad (5.5)$$

If we consider the adjoint with respect to the cartesian bilinear functional (3.2) given by (3.3), we see that it is applicable to both members of problem (5.5) only if the domain of D^* contains the given function f. This implies that $f \in AC(0,T)$ and that $f(T) = 0$.

While the derivability requirement is satisfied, because $C(0,T) \in AC(0,T)$, the second condition is not, a priori, satisfied.

One may be tempted to add to f(t) the additional condition $f(T) = 0$. But we take as a fundamental principle that of imposing no supplementary conditions on the functions entering a problem different from the ones that are assigned to the problem. In fact, if the given problem expresses a physical law or a geometric condition or a technical process, every additional condition imposed would exclude possible source distributions or possible configurations. Even the simple condition that the unknown functions be of class $C_0^\infty(0,T)$ would result in an inadmissible restriction of the domain and consequently a restriction of the range.

Observing that the main hindrance to the application of the adjoint operator D^* is the final condition $f(T) = 0$, the idea arises of performing a preliminary transformation, like the following:

$$\bar{f}(t) = \int_0^T k(t,s)\, f(s)\, ds \qquad (5.6)$$

in which the kernel k(t, s) must satisfy the following conditions:

$$k(T,s) = 0 \quad k(t,s) = k(s,t)$$

and moreover be such that the integral operator K defined by (5.6) be invertible. In such a way the final condition $\bar{f}(T) = 0$ is satisfied.

One then obtains the integrodifferential equation

$$\int_0^T k(t,s)\, d/ds\, u(s)\, ds = \int_0^T k(t,s)\, f(s)\, ds \qquad (5.7)$$

that has the same solution of the given problem. Now we may apply the operator D^* (adjoint of D with respect to the ordinary cartesian bilinear functional):

$$-d/dt \int_0^T k(t,s)\, d/ds\, u(s)\, ds = -d/dt \int_0^T k(t,s)\, f(s)\, ds \qquad (5.8)$$

Let K denote the integral operator (5.6). We may write problem (5.8) as follows:

$$D^*KDu = D^*Kf \qquad (5.9)$$

We see that the operator D^*K is the integrating operator we were searching for. In fact,

$$\langle (D^*KD)u, \bar{u}\rangle_0 = \langle (D\bar{u}), K(Du)\rangle_0 = \langle (Du), K(D\bar{u})\rangle_0 = \langle (D^*KD)\bar{u}, u\rangle_0 \qquad (5.10)$$

The corresponding functional is

$$\bar{F}\{u\} = \bar{F}\{u_0\} + 1/2 \langle Du, KDu\rangle_0 - \langle f, KDu\rangle_0 \qquad (5.11)$$

If the operator K is also positive definite then the critical points of the functional are points of minimum.

The integrating operator is

$$Rv(t) = D^*Kv(t) = -d/dt \int_0^T K(t,s)\, v(s)\, ds \tag{5.12}$$

that is, an integrodifferential operator.

It is not difficult to find integral operators K meeting these requirements: all Green functions of linear positive definite operators may be used. For example, the inverse of the operator

$$S = \{-d^2/dt^2,\ u(0) = 0,\ u(T) = 0;\ u \in C^2(0,1)\} \tag{5.13}$$

is

$$w(t) = Kv(t) = \int_0^T \{-(t-s)\, H(t-s) - (T-s)\}\, v(s)\, ds \tag{5.14}$$

being $H(t)$, the Heaviside function. It is $w(T) = 0$.

Then, at least for linear operators, we have succeeded in finding a variational formulation under the hypothesis that L is invertible.

In particular, if L is symmetric with respect to the cartesian bilinear functional putting $K = L^{-1}$ the functional

$$\bar{F}\{u\} = \bar{F}\{u_0\} + 1/2\, \langle Lu, KLu\rangle - \langle f, KLu\rangle \tag{5.15}$$

reduces itself to

$$F\{u\} = F\{u_0\} + 1/2\, \langle Lu, u\rangle - \langle f, u\rangle \tag{5.16}$$

that is, the classical one.

Then, in the linear case, the extended variational formulation contains the restricted one when this exists.

6. NONLINEAR PROBLEMS

It is possible to extend this procedure to nonlinear problems. We have the following [32]

Theorem. Let us consider two linear spaces U and V such that a nondegenerate, real valued, bilinear functional $\langle v, u\rangle$ may be defined; let the two spaces be endowed with a norm that makes $\langle v, u\rangle$ continuous in both arguments. Let

$$N(u) = O_V \tag{6.1}$$

be a problem, whose operator $N: D(N) \in U \rightarrow R(N) \in V$ is such that its domain is simply connected and it admits (linear) Gâteaux derivative $N'_u(u;\cdot)$ for every $u \in D(N)$. Let $N'^*_u(u;\cdot)$ be its adjoint with respect to the bilinear functional $\langle v, u\rangle$: if $N'^*_u(u;\cdot)$ is invertible for every $u \in D(N)$ and $D(N'_u)$ is dense in U, then for every operator $K: D(K) \subset V \rightarrow U$ such that

(1) $D(K) \supset R(N)$,

(2) $R(K) \subset D(N'^*_u)$,

(3) is linear,

(4) is invertible, i.e., kernel-free,

(5) is symmetric, i.e., $\langle v, Kv\rangle = \langle v, Kv\rangle$,

(6) is positive definite, i.e., $\langle v, Kv\rangle > 0 \quad (v \neq O_V)$,

the operator $\bar{N}$ defined by

$$\bar{N}(u) = N_u'^*(u; KN(u)) \tag{6.2}$$

has the following properties:

(a) it has the same domain of N;
(b) it has the same null manifold of N;
(c) it is potential (i.e., it satisfies the Volterra condition);
(d) it is a gradient of the functional

$$\bar{F}\{u\} = \bar{F}\{u_0\} + 1/2 \langle N(u), KN(u) \rangle \tag{6.3}$$

(e) the functional is minimum at the critical points.

Proof. If $u \in D(N)$, for the properties (1) and (2) also $u \in D(\bar{N})$. Contrarily, if $u \in D(\bar{N})$ it follows from (6.2) that $u \in D(N)$: this proves property (a).

If u_0 is a solution of $N(u) = O_V$, on account of the linearity of $N_u'^*(u; \cdot)$ and of K we have

$$\bar{N}(u) = N_u'^*(u_0; KN(u_0)) = N_u'^*(u_0; KO_V) = N_u'^*(u_0; O_u) = O_V \tag{6.4}$$

i.e., it is a solution of $\bar{N}(u) = O_V$, too. Contrarily, if u_0 is a solution of $\bar{N}(u) = O_V$, since $N_u'^*$ and K are invertible, it is

$$K^{-1}(N_u'^*)^{-1}(u_0; \bar{N}(u_0)) = O_V \tag{6.5}$$

i.e., u_0 is also a solution on $N(u) = O_V$. This proves property (b).

Since $N_u'^*(u; w)$ is linear on w from Eq. (6.2) we have

$$\bar{N}_u'(u; \delta u) = (N_u'^*)_u'(u; KN(u), \delta u) + N_u'^*(u; KN_u'(u; \delta u)) \tag{6.6}$$

Now

$$\langle \bar{N}_u'(u; \delta u), w \rangle = \langle (N_u'^*)_u'(u; KN(u), \delta u), w \rangle + \langle N_u'^*(u; KN_u'(u; \delta u)), w \rangle \tag{6.7}$$

From the relation that defines the adjoint

$$\langle N_u'(u; p), q \rangle = \langle N_u'^*(u; q), p \rangle \tag{6.8}$$

by differentiation with respect to u we obtain

$$\langle N_{uu}''(u; p, \delta u), q \rangle = \langle (N_u'^*)_u'(u; q, \delta u), p \rangle \tag{6.9}$$

Relation (6.7) becomes

$$\langle N_u'(u; \delta u), w \rangle = \langle N_{uu}''(u; w, \delta u), KN(u) \rangle + \langle N_u'(u; w), KN_u'(u; \delta u) \rangle \tag{6.10}$$

The second Gâteaux derivative is symmetric:

$$N_{uu}''(u; p, q) = d^2/dadb[N(u + ap + bq)] = d^2/dbda[N(u + ap + bq)] = N_{uu}''(u; q, p) \tag{6.11}$$

From this property and from the symmetry of K it follows that

$$\langle \bar{N}_u'(u; \delta u), w \rangle = \langle \bar{N}_u'(u; w), \delta u \rangle \tag{6.12}$$

which proves the symmetry of $\bar{N}_u'$. Then property (c) is proved.

Putting $w(s) = su + (1 - s)u_0$ the functional is given by the general formula

$$F\{u\} = \int_0^T \langle \bar{N}(w(s)), \partial w(s)/\partial s \rangle \, ds = \int_0^T \langle N_u'^*(w; KN(w)), \partial w/\partial s \rangle \, ds$$

$$= \int_0^T \langle N'_u(w\,;\,\partial w/\partial s),\ KN(w)\rangle\, ds = \int_0^T \langle \delta N(w),\ KN(w)\rangle$$

$$= \int_0^T \delta[(1/2)\langle N(w),\ KN(w)\rangle] = (1/2)[\langle N(w),\ KN(w)\rangle]_0^1$$

$$= F\{u_0\} + (1/2)\langle N(u),\ KN(u)\rangle \tag{6.13}$$

Taking account of the symmetry of K we obtain

$$\delta F\{u\} = \langle \delta N(u),\ KN(u)\rangle = \langle N'_u(u\,;\delta u),\ KN(u)\rangle = \langle N'^*_u(u\,;KN(u)),\ \delta u\rangle \tag{6.14}$$

and then if

$$\delta F\{u\} = \langle N'^*_u(u\,;\,KN(u)),\ \delta u\rangle = 0 \tag{6.15}$$

since $\delta u \in D(N'_u)$ and this domain is dense in U, and $\langle v, u\rangle$ is nondegenerate and continuous, it follows that

$$N'^*_u(u\,;KN(u)) = O_V \tag{6.16}$$

which proves property (d).

Since

$$F\{u\} - F\{u_0\} = (1/2)\langle N(u),\ KN(u)\rangle > 0 \tag{6.17}$$

for every $N(u) \neq O_V$ it follows that $F\{u\}$ is minimum at the solution: this proves property (e).

We remark that the continuity of the operator N is not required and that the Gâteaux derivative (not the Fréchet one) is involved. Q.E.D.

The theorem shows how, under mild conditions on the operator, one may give variational formulation in the extended sense to nonlinear problems without changing the initial or boundary conditions or even the functional class of the functions entering the problem. Moreover, it is always possible to find a functional that is minimum at the critical points.

In the particular case that N is a potential operator the functional F does not reduce to the potential of N. To have this inclusion a further generalization of the theorem is necessary; this has been done in [33].

7. CRITICAL REMARKS

The preceding theorem enables us to give many variational formulations to a problem whose operator satisfies few requirements. This means that one may actually characterize the solutions of the problem $N(u) = O_V$ as those elements of the domain of N that make minimum some functional.

From this it follows that the common belief that the existence of a variational principle for a given problem may be used as a criterion to accept or to refuse possible physical laws is meaningless.

But the usual belief, often expressed, sometimes written and never proved, that dissipative phenomena cannot be described by a variational principle is invalid even in a classical context.

For example, if we throw a book on a table its motion is uniformly retarded according to the law

$$m\,\ddot{q}(t) = -k\,m\,g \tag{7.1}$$

where k is the dynamic frictional coefficient. Nothing is more dissipative than this motion! Yet the Lagrangian exists: it is

$$L(q, \dot{q}) = 1/2\ m\ \dot{q}^2(t) - k\ m\ g\ q(t) \tag{7.2}$$

Another well-known example is that of the equation of the harmonic oscillator with damping term, i.e.,

$$m\ \ddot{q}(t) + h\ \dot{q}(t) + k\ q(t) = 0 \tag{7.3}$$

It admits the integrating factor exp (h/mt). The corresponding lagrangian is

$$L(q, \dot{q}) = 1/2\ \exp\ (h/mt)\{m\dot{q}^2(t) + kq^2(t)\} \tag{7.4}$$

Yet the motion is dissipative.

These two examples show that even on the classical ground a selection criterion does not exist.

The fact is that the variational formulation is based on the form of the equation. We know that every linear second order ordinary differential equation can be cast in self-adjoint form. The transformation to a self-adjoint form changes the form of the equation, not the solution set, i.e., the substance. Then a mathematical requirement (variational formulation) that is based on the form (self-adjointness) when we let it be altered by an integrating factor cannot be reasonably used as a discriminating criterion.

8. CRITIQUE OF THE HAMILTON PRINCIPLE

Let us consider the Hamilton principle in mechanics. This principle ignores the initial condition on the velocity and adds a fictitious final condition. So the initial value problem

$$m\ddot{q} + kq = 0 \qquad q(0) = a;\ \dot{q}(0) = b \tag{8.1}$$

has an operator that is formally symmetric (with respect to the cartesian bilinear form). The term arising from the integration by parts is

$$m\{q(T)\ \delta q(T) - q(0)\ \delta q(0)\} \tag{8.2}$$

The given initial conditions imply $\delta q(0) = 0$ and $\delta\dot{q}(0) = 0$ and then only the second term vanishes. Of course we do not know $q(T)$. What do we do? We add a fictitious final condition $q(T) = c$ so that $\delta q(T) = 0$. The whole boundary term vanishes and the operator is now symmetric. The functional is

$$A\{q\} = 1/2 \int_0^T \{m\dot{q}^2 - kq^2\}\, dt \tag{8.3}$$

We arrive in this way at the Hamilton principle. In its statement the natural motion is compared with those motions that respect the same initial and final conditions.

It is evident that in order to obtain a variational formulation we have had to alter the problem: we have forgotten the initial condition on the velocities and added a fictitious final condition. In operatorial language this means that the domain of the functional is not the domain of the operator.

This expedient has invaded all physics: it is used in all evolution phenomena, in field theories of classical, relativistic, and quantum physics.

There is no longer a reason to use the mathematical trick necessary for the Hamilton principle when we know that the solution of the problem of motion makes minimum a functional like

$$F\{q\} = 1/2 \int_0^T \{\ddot{q}(t) - f(t;q,\dot{q})\} \int_0^T k(t,s)\{\ddot{q}(s) - f(s;q,\dot{q})\}\, ds\, dt \tag{8.4}$$

The lagrangian of this functional is no longer a function of q, $\dot{q}$ as in the classical case, but is an operator. Many formalisms used dealing with the calculus of variations, like the theory of exterior forms, cohomology theory, the notions of jets, spray, etc., cannot be applied.

Faced with such a loss one may be tempted to reject the extended variational formulation considering in some way unacceptable an integrating operator.

But where is the line of demarcation between integrating factor and integrating operator?

May we accept the integrating factor exp (h/mt) to make formally symmetric the operator of the damped harmonic oscillator and refuse an integral operator that makes the operator symmetric only because it destroys the differential nature of the equation?

What is more important: to change the form of the problem keeping intact the solution manifold or to change the solution manifold to preserve the form of the problem?

A mathematician is free to change the additional conditions or the functional class if the equation is for him a pretext to utilize a given algorithm. But a mathematician, a physicist, an engineer cannot do the same if the equation describes a phenomenon or a process that he must study or solve.

When we form an equation and the additional conditions to describe a process, what we have in mind is to characterize the solutions of the problem among all functions of a certain set. The form of the equation is immaterial. Two mathematical formulations that lead to the same set of solutions are equally acceptable. So when we know the Green function of an operator we may transform a differential equation into an integral one without changing the solution. For example, the problem of the harmonic oscillator (8.1) is equivalent to the integral equation

$$q(t) = -k/m \int_0^T (t - s)\, H(t - s)\, q(s)\, ds + a + bt \tag{8.5}$$

The form is changed but the content is the same. The passage from (7.1) to (8.5) is equivalent to the application of the integral operator

$$R(\cdot) = \int_0^T (t - s)\, H(t - s) \cdots ds + a + bt \tag{8.6}$$

in which the kernel is the propagator for the problem

$$\ddot{q}(t) = f(t) \qquad q(0) = 0 \qquad \dot{q}(0) = 0 \tag{8.7}$$

This is an example of application of an operator to a given problem that changes the form but not the content of the problem.

There is one last thing to be said on the Hamilton principle: we cannot apply to it the direct methods of the calculus of variations, say the Ritz method, because we don't know the final value q(T). On the contrary, a direct method has been applied with success to the functional (8.4): see [33].

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