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Fisica matematica. — *A mathematical model for physical theories.*
Nota I di ENZO TONTI [(*)], presentata [(**)] dal Socio B. FINZI.

RIASSUNTO. — Si presenta un modello matematico per le teorie fisiche basato sulla considerazione di coppie di spazi funzionali messi in dualità da funzionali bilineari e di corrispondenze tra questi spazi. Ognuno di tali spazi funzionali è relativo ad una variabile fisica e le corrispondenze rappresentano le equazioni che legano tra loro le diverse variabili. Dalla struttura degli operatori che descrivono tali corrispondenze si deducono, sotto forma di teoremi, le principali proprietà matematiche del modello.

1.1. INTRODUCTION

Many physical theories exhibit a common mathematical structure that is independent of the physical contents of the theory and is common to discrete and continuum theories, be they of classic, relativistic or quantum nature [(1)]. The starting point of this structure is the possibility of decomposing the *fundamental equation* [(2)] of many physical theories in three equations, known in classical fields of the macrocosm as *definition*, *balance* and *constitutive* equations, whose operators enjoy peculiar properties. The properties are as follows: the operator of balance equation is the adjoint, with respect to an opportune bilinear functional, of the operator of definition equation (if the last is linear) or of its Gateaux derivative (if it is nonlinear). Moreover, the operator of constitutive equation is symmetric (when it is linear) or has symmetric Gateaux derivative (when is nonlinear). Such a peculiar decomposition permits us to obtain a profound introspection into the mathematical structure of a theory. The fact that this decomposition can be achieved in a large number of physical theories and the fact that when it exists we can deduce easily a large number of mathematical properties, suggest constructing a mathematical model for physical theories.

1.2. THE MATHEMATICAL MODEL: THE ASSUMPTIONS

Let us suppose we have:

1) a first set of n functions of space and time coordinates $\varphi_k(t, x^1, x^2, x^3)$ (with $k = 1, 2, \cdots, n$), that will be called *configuration variables*. They

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(1) We refer the reader to the paper *On the mathematical structure of large a class of physical theories*, « Rend. Acc. Lincei », *52*, 48–56, denoted by [1].
(2) With this name we indicate the field equation in field theories, the equation of motion in mechanical theories, i.e. the equation relating the configuration of the system with the sources.

can depend only on time variable (3) or on space variables (4) can be of finite or infinite number. They can be real or complex functions, can be the components of a vector, a tensor, a spinor or may do not have special transformation properties. Every given set of these n functions, will be denoted with φ. Any linear function space of elements φ will be denoted with Φ and called *functional configuration space.*

2) a second set of n functions $\sigma_k(t, x^1, x^2, x^3)$ (with $k = 1, 2, \cdots, n$) that will be called *source variables.* They depend from space and time coordinates (5) have the same tensorial order, the same tensorial symmetry properties and the same real or complex nature of configuration variables. Every given set of these variables will be denoted with σ. Any linear function space of elements σ, denoted by Σ, will be called *functional source space.*

3) a bilinear functional defined on the elements of the two function spaces Φ and Σ that will be denoted $\langle\sigma, \varphi\rangle$. It must be such that for every $\sigma \in \Sigma$, different from the null element ϑ, there exists at least one φ such that $\langle\sigma, \varphi\rangle \neq 0$ and analogous requirement on φ. Under these conditions the two spaces are said to be *put in duality* by the bilinear functional [2, p. 88].

4) a *topology* on the spaces Φ and Σ that makes continuous every linear functional $\langle\sigma_0, \varphi\rangle$ with $\sigma_0 \in \Sigma$ and $\langle\sigma, \varphi_0\rangle$ with $\varphi_0 \in \Phi$. It can be shown that for every linear functional $l[\varphi]$ continuous with that topology a unique element $\sigma_l \in \Sigma$ can be found so that $l[\varphi] = \langle\sigma_l, \varphi\rangle$ [2, p. 91].

5) a third set of m functions $u_h(t, x^1, x^2, x^3)$ (with $h = 1, 2, \cdots, m$) of space and time coordinates, with $m \geq n$ that we shall call *first kind variables.* Every particular set of such functions will be considered as an element u. Any linear function space formed by elements u will be denoted by U.

6) a fourth set of m functions $v_h(t, x^1, x^2, x^3)$ (with $h = 1, 2, \cdots, m$) of space and time coordinates that we shall call *second kind variables* such that every v_h has the same tensor nature and the same tensorial symmetries of the first kind variables u_h. Every particular set of such functions will be denoted by u. Any linear function space formed by elements v will be denoted by V.

7) a *bilinear functional* defined on the elements of the two spaces U and V denoted by $\langle v, u\rangle$ that satisfies the same requirements of point 3).

8) a *topology* for U and V spaces with the same requirements of point 4).

(3) As the lagrangian coordinates in mechanics and the extensive parameters in the irreversible thermodynamics of discrete systems.

(4) As in time-independent field theories (static and stationary fields).

(5) Source variables can depend on space and time coordinates either directly as when they are assigned (fixed or impressed sources) or indirectly as when they are linked with configuration variables of other systems (interaction) or with those of the same system (self-interaction).

Up to this point we have two pairs of function spaces in duality equipped with suitable topologies. The need to introduce a topology arises from the fact that we wish to treat subjects as stability, perturbations, convergence of iterative methods, error bounds in approximate methods and existence of solution. About mappings among these spaces we suppose to have:

9) a mapping D, generally nonlinear, between some subset $\mathfrak{D}(D) \subseteq \Phi$ (its domain) of the functional configuration space and a subset $\mathfrak{R}(D) \subseteq U$ (its range) of the function space U of first kind variables. When $m > n$ the operator D is a gradient-like operator. The equation $u = D\varphi$ will be called *definition equation*;

10) a mapping C, generally nonlinear, between a subset $\mathfrak{D}(C) \supseteq \mathfrak{R}(D)$ of the U–space and a subset $\mathfrak{R}(C)$ of the V–space. The operator C will be supposed *symmetric*, if linear, i.e. $\langle Cu', u''\rangle = \langle Cu'', u'\rangle$ or with *symmetric Gateaux derivative*, if nonlinear, i.e. $\langle C'_u u', u''\rangle = \langle C'_u u'', u'\rangle$. Moreover it is supposed that C does not contain the configuration variable φ. The equation $v = Cu$ will be called *constitutive equation*.

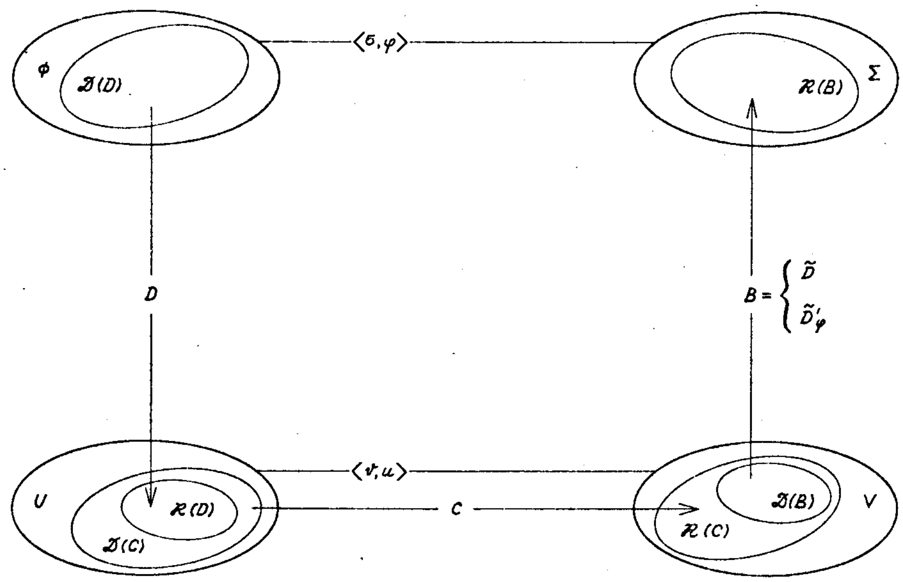

Fig. 1.

11) a *linear* mapping B between some subset $\mathfrak{D}(B) \subseteq \mathfrak{R}(C)$ of the V–space and a subset $\mathfrak{R}(B)$ of the Σ–space that be the *adjoint* of the mapping D (if D is linear) or be the adjoint of its linear Gateaux derivative (if D is nonlinear) (6), we shall use the notations $B = \tilde{D}$ and $B = \tilde{D}'_\varphi$ respectively.

(6) In physical theories the operator D is generally not continuous, being often a differential operator working on a Banach space (in particular on Hilbert and Sobolev spaces). It follows that the Gateaux derivative is not continuous in this case. Some Authors speak of Gateaux derivative only when continuity is assured [7, p. 40] [8, p. 114]. This usage is very restrictive: we adhere to the more general definition (see for ex. Tapia in [8, p. 51]).

When $m > n$ the operator B is a divergence-like operator. The equation $Bv = \sigma$ will be called *balance equation.*

We emphasize the fact that of the three mappings we shall take as primitive, only two of them are independent i.e. D and C. In the sequel will be shown that the mathematical properties of the model rest upon the properties of these two operators. The scheme of fig. 1 summarizes what we have said up to now.

1.3. THE MATHEMATICAL MODEL: FIRST PROPERTIES.
a) FUNDAMENTAL EQUATION

The sequence of mappings $D : \Phi \mapsto U$, $C : U \mapsto V$, $B : V \mapsto \Sigma$ induce a mapping $F = BCD : \Phi \mapsto \Sigma$ we shall call *fundamental mapping.* The corresponding fundamental equation has the form

$$\tilde{D}CD\varphi = \sigma \qquad \tilde{D}'_\varphi CD\varphi = \sigma \tag{1.3.1}$$

in the linear and nonlinear case respectively. The fundamental mapping F enjoys many properties: we shall consider in this paper the case in which D and C are *linear* operators. In the second part we shall deal with the nonlinear case.

THEOREM 1: *If* D *and* C *are linear operators the operator* F *is symmetric.*

Proof:

$$\langle F\varphi', \varphi''\rangle = \langle\tilde{D}CD\varphi', \varphi''\rangle = \langle CD\varphi', \tilde{\tilde{D}}\varphi''\rangle = \langle C\tilde{\tilde{D}}\varphi'', D\varphi'\rangle = \langle\tilde{D}C\tilde{\tilde{D}}\varphi'', \varphi'\rangle. \tag{1.3.2}$$

But $\tilde{\tilde{D}} \supseteq D$ [9, p. 168] and then if $\varphi'' \in \mathfrak{D}(F)$

$$\langle F\varphi', \varphi''\rangle = \langle\tilde{D}CD\varphi'', \varphi'\rangle = \langle F\varphi'', \varphi'\rangle. \tag{1.3.3}$$

From the symmetry of F follow two properties: they are

THEOREM 2 (VARIATIONAL FORMULATION): *if the operator* F *is symmetric and* σ *does not depend on* φ, *the solutions of the fundamental equation, when it exist, make stationary the functional*

$$S[\varphi] \overset{\text{def}}{=} \frac{1}{2}\langle CD\varphi, D\varphi\rangle - \langle\sigma, \varphi\rangle. \tag{1.3.4}$$

Proof:

$$\delta S[\varphi] = \langle CD\varphi, D\delta\varphi\rangle - \langle\sigma, \delta\varphi\rangle = \langle\tilde{D}CD\varphi - \sigma, \delta\varphi\rangle = 0. \tag{1.3.5}$$

This theorem, stated in other words, asserts that the fundamental equation is the Euler–Lagrange equation of an action functional. We thus see that the existence of an action functional for the fundamental equation, that is assumed as postulate in field theory, is here deduced as theorem.

THEOREM 3 (RECIPROCITY THEOREM): *if the operator* F *is symmetric let us be* σ' *and* σ'' *two different sources and* φ', φ'' *two corresponding solutions then*

$$\langle \sigma', \varphi'' \rangle = \langle \sigma'', \varphi' \rangle . \tag{1.3.6}$$

Proof:

$$\langle \sigma', \varphi'' \rangle = \langle F\varphi', \varphi'' \rangle = \langle F\varphi'', \varphi' \rangle = \langle \sigma'', \varphi' \rangle . \qquad \text{(q.e.d.)} \tag{1.3.7}$$

A frequent case is that the operator C be definite positive. When this happens we have the following properties:

THEOREM 4 (MINIMUM OF THE FUNCTIONAL): *if* C *is a positive definite operator, i.e.* $\langle Cu, u \rangle > 0$ *for* $u \neq 0$ *then the solution of the fundamental equation, when exists, makes minimum the action functional* S *of Theorem 2.*

Proof: being $\delta S[\varphi] = \langle \tilde{D}CD\varphi - \sigma, \delta\varphi \rangle$ will be

$$\delta^2 S[\varphi] = \langle \tilde{D}CD\delta\varphi, \delta\varphi \rangle = \langle CD\delta\varphi, D\delta\varphi \rangle = \langle C\delta u, \delta u \rangle > 0 . \tag{1.3.8}$$

THEOREM 5: *if* C *is a positive definite operator the fundamental operator has the same null manifold of the definition operator*:

$$\mathfrak{N}(F) = \mathfrak{N}(D)$$

Proof:

$$\varphi_0 \in \mathfrak{N}(F) \Rightarrow F\varphi_0 = 0 \Rightarrow \langle F\varphi_0, \varphi_0 \rangle = \langle CD\varphi_0, D\varphi_0 \rangle = 0 \Rightarrow \Rightarrow D\varphi_0 = 0 \Rightarrow \varphi_0 = \mathfrak{N}(D) \tag{1.3.9}$$

$$\varphi_0 \in \mathfrak{N}(D) \Rightarrow D\varphi_0 = 0 \Rightarrow CD\varphi_0 = 0 \Rightarrow \tilde{D}CD\varphi_0 = 0 \Rightarrow \Rightarrow F\varphi_0 = 0 \Rightarrow \varphi_0 = \mathfrak{N}(F) . \tag{1.3.10}$$

From this theorem follows as a lemma the

THEOREM 6: (UNIQUENESS). *If the operator* C *is positive definite and the operator* D *has no null manifold, the solution of the fundamental equation, when exists, is unique.*

The existence of a null manifold of the definition operator D implies the existence of a compatibility condition on the source term σ irrespectively of the positive definite character of the operator C.

THEOREM 7: *if the definition operator* D *has a null manifold, denoted with* $\varphi_0 = L\chi$ *the general solution of the homogeneous equation* $D\varphi = 0$ *then in order that the fundamental problem admits a solution must be* $\tilde{L}\sigma = 0$ (7).

Proof:

$$DL\chi \equiv 0 \Rightarrow \tilde{L}\tilde{D} = 0 \Rightarrow \tilde{L}\tilde{D}CD\varphi = \tilde{L}\sigma = 0 . \tag{1.3.11}$$

(7) Because the symmetry of C does not enter in this theorem while it is essential for the variational formulation (see Theorem 2) we see that the link between gauge invariance and conservation laws is essentially of non-variational nature. Noether theorem requiring a variational principle is then very demanding.

The property $DL\chi = 0$ is commonly known in physics as gauge invariance. The theorem establishes a link between the gauge invariance of first kind variables and the existence of compatibility conditions that usually mean conservation laws [3].

1.4 THE MATHEMATICAL MODEL: *b*) CANONICAL FORM

If the constitutive mapping is one to one we can consider the inverse mapping C^{-1}. In this case we can reduce the three basic equations to the following two equations

$$D\varphi = C^{-1} v \quad , \quad \tilde{D} v = \sigma . \tag{1.4.1}$$

These two sets will be called the *canonical system*.

THEOREM 8 (VARIATIONAL FORMULATION): *the solutions of the canonical system, with* σ *assigned, make stationary the functional*

$$\bar{S}[\varphi , v] \overset{\text{def}}{=} \langle v , D\varphi\rangle - \frac{1}{2}\langle v , C^{-1} v\rangle - \langle\sigma , \varphi\rangle \tag{1.4.2}$$

Proof:

$$\delta\bar{S}[\varphi , v] = \langle\delta v , D\varphi - C^{-1} v\rangle + \langle\tilde{D}v - \sigma , \delta\varphi\rangle = 0 . \tag{1.4.3}$$

The functional $\bar{S}[\varphi , v]$ will be called the *canonical action functional*.

The canonical equations can be written in a matrix-differential form as follows

$$\left[\begin{array}{c|c} 0 & \tilde{D} \\ \hline D & -C^{-1}\end{array}\right]\left[\begin{array}{c}\varphi \\ \hline v\end{array}\right] = \left[\begin{array}{c}\sigma \\ \hline 0\end{array}\right] . \tag{1.4.4}$$

If we introduce two vectors $\psi = (\varphi_1 , \cdots , \varphi_n ; v_1 , \cdots , v_m)$ and $\chi = (\sigma_1 , \cdots , \sigma_n ; 0 , \cdots , 0)$ putting

$$L = \left[\begin{array}{c|c} 0 & \tilde{D} \\ \hline D & 0\end{array}\right] \qquad K = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & -C^{-1}\end{array}\right] \tag{1.4.5}$$

the canonical system can be written as

$$L\psi + K\psi = \chi . \tag{1.4.6}$$

Often D is a first order linear operator: in this case the matrix-differential operator L can be decomposed in the form

$$L = \sum_{0}^{3}{}_{\alpha} L_{\alpha} \frac{\partial}{\partial x^{\alpha}} \tag{1.4.7}$$

where L_α denotes some square matrices of $(m + n)^2$ elements. The canonical system then assumes the typical form

$$\sum_{0}^{3}{}_{\alpha} L_{\alpha} \frac{\partial}{\partial x^{\alpha}} \psi + K\psi = \chi \tag{1.4.8}$$

used in the matrix-algebraic approach to the relativistic theory of particles of arbitrary spin [4, p. 378] [5, p. 270] [6, p. 143].

THEOREM 9 (SYMMETRY OF THE OPERATORS L AND K). *The matrix-differential operator* L *and the operator* K *are symmetric with respect to the bilinear functional*

$$\langle \chi , \psi \rangle \overset{\text{def}}{=} \langle \sigma , \varphi \rangle + \langle v , u \rangle . \tag{1.4.9}$$

Proof:

$$\langle L\psi' , \psi'' \rangle = \langle \tilde{D} v' , \varphi'' \rangle + \langle v'' , D\varphi' \rangle = \langle v' , D\varphi'' \rangle + \langle \tilde{D} v'' , \varphi' \rangle = \langle L\psi'' , \psi' \rangle \tag{1.4.10}$$

$$\langle K\psi' , \psi'' \rangle = \langle - C^{-1} v' , v'' \rangle = \langle - C^{-1} v'' , v' \rangle = \langle K\psi'' , \psi' \rangle \tag{1.4.11}$$

Using the bilinear functional (1.4.9) the canonical action functional (1.4.2) can be written

$$\bar{S}[\psi] = \frac{1}{2} \langle L\psi , \psi \rangle + \frac{1}{2} \langle K\psi , \psi \rangle - \langle \chi , \psi \rangle \tag{1.4.12}$$

(the proof is straightforward).

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Fisica matematica. — *A mathematical model for physical theories* (*). Nota II di ENZO TONTI, presentata (**) dal Socio B. FINZI.

RIASSUNTO. — In questa Nota si continua l'esame delle proprietà di un modello matematico di una teoria fisica, presentato in una Nota precedente. Tali proprietà riguardano in particolare la formulazione variazionale, l'invertibilità del legame costitutivo, la decomposizione dell'equazione fondamentale in una parte spaziale ed una temporale, nonché la costruzione dello schema duale.

1.1. INTRODUCTION

This is the second part of a paper which deals with a mathematical model for physical theories [3]. In this paper we prove a number of mathematical properties that follow from the assumptions given in [3]. In this paper we take away the limitation concerning the linearity of definition and constitutive operators used in the properties shown in the preceding paper.

1.2. INVERTIBLE CONSTITUTIVE MAPPINGS

Many mathematical properties of the model are based on the possibility to invert the constitutive mapping C. The necessary and sufficient condition is that C be one-to-one. This leads to investigate sufficient conditions in order that C be one-to-one. When C is linear a sufficient condition is that it be positive definite i.e. $\langle Cu, u\rangle > 0$ for $u \neq \vartheta$ (ϑ is the null element of the U-space). This property is frequently met in physical theories.

When C is nonlinear we have the

THEOREM 10 (INVERTIBILITY THEOREM): *a sufficient condition in order that a mapping* C *be one-to-one (and then be invertible) is that* C *be strictly monotone, i.e.*

$$\langle C(u') - C(u''), u' - u''\rangle > 0 \qquad \text{for} \quad u' \neq u'' \tag{1.2.1}$$ (1).

Proof: if C is strictly monotone and $u' \neq u''$ must be $C(u') \neq C(u'')$. This assures that two different elements u' and u'' cannot correspond to the same element v and then the mapping is one-to-one. Because the condition of being strictly monotone reduces to that of being positive definite in the linear

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(1) When $>$ is replaced by $\geq$ we have the definition of *monotone* operator.

case, we shall consider in the sequel only strictly monotone operators. What can be said about the inverse of a strictly monotone operator? We have

THEOREM 11: *The inverse of a strictly monotone operator is also a strictly monotone operator.*

Proof: with the position $u = C^{-1}(v)$ relation (1.2.1) becomes

$$\langle v' - v'', C^{-1}(v') - C^{-1}(v'')\rangle > 0 \quad \text{for} \quad v' \neq v''. \tag{1.2.2}$$

1.3. SPACE AND TIME PART OF THE FUNDAMENTAL MAPPING [2]

When configuration variables depend on space and time coordinates it can happen that the definition operator D be the sum of two operators, generally nonlinear, formed with space and time derivatives respectively. In this case we can decompose the operator D and the set of first kind variables according to the scheme

$$\left[\frac{u_t}{u_s}\right] = \left[\frac{D_t}{D_s}\right] \varphi . \tag{1.3.1}$$

This amounts to considering the U-space as the sum of two subspaces U_t and U_s i.e. $U = U_t \oplus U_s$. When this happens the balance equation can be written in the form ($\tilde{D}$ is the adjoint of D)

$$[\tilde{D}_t \mid \tilde{D}_s] \left[\frac{v_t}{v_s}\right] = \sigma \qquad \tilde{D}_t \to \tilde{D}'_{t\varphi} \text{ and } \tilde{D}_s \to \tilde{D}'_{s\varphi} \text{ in the non linear case} \tag{1.3.2}$$

and the V-space can be conceived as the sum of two subspaces $V = V_t \oplus V_s$.

Moreover the constitutive operator C can often be decomposed according to the scheme

$$\left[\frac{v_t}{v_s}\right] = \left[\begin{array}{c|c} C_t & 0 \\ \hline 0 & C_s \end{array}\right] \left[\frac{u_t}{u_s}\right] \tag{1.2.3.}$$

where C_t and C_s can be nonlinear operators.

Under these hypotheses on the decomposition of D and C the fundamental mapping becomes

$$\tilde{D}_t C_t D_t \varphi + \tilde{D}_s C_s D_s \varphi = \sigma . \tag{1.3.4}$$

(2) In order to have an example to support the mind, the reader can think of the elastodynamic field whose fundamental equation is

$$\left[+\frac{\partial}{\partial t}\right] \rho\, a_{hk} \left[\frac{\partial}{\partial t}\right] u^k + \left[-\nabla^k\right] C_{hkrs} \left[\frac{1}{2}(\nabla^r u^s + \nabla^s u^r)\right] = f_h$$

(Navier equation) where the operator C_s is Hooke tensor C_{hkrs}, C_t is $\rho\, a_{hk}$, φ is the displacement vector u^k, σ is the body force f_h, a_{hk} the metric tensor, D_s is the symmetrical part of gradient of the displacement vector u^k, D_t the time derivative.

The subspaces U_t, U_s (and V_t, V_s) can be disjointed and conceived as two distinct spaces.

The corresponding scheme is shown in fig. 1

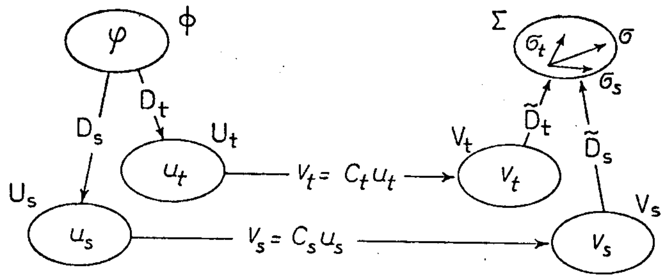

Fig. 1.

The decomposition into a time and space part of the operator C has several mathematical advantages. For example in many physical theories the operator C is not monotone, while C_s is.

Another property is expressed by the following

THEOREM 12: *if* D_s *is a linear operator with dense domain in the* Φ*–space then if* C_s *is monotone the operator* $F_s = \tilde{D}_s C_s D_s$ *is also monotone.*

Proof:

$$(1.3.5)\quad \langle C_s(u') - C_s(u''), u' - u''\rangle = \langle C_s(D_s\varphi') - C_s(D_s\varphi''), D_s\varphi' - D_s\varphi''\rangle = \\ = \langle \tilde{D}_s C_s(D_s\varphi') - \tilde{D}_s C_s(D_s\varphi''), \varphi' - \varphi''\rangle \geq 0 .$$

From this property it follows that the fundamental mapping F written in the form

$$(1.3.6)\qquad F_t\varphi = -F_s\varphi + \sigma$$

has the typical structure of monotonic evolution equations to which many Theorems about existence, uniqueness and continuous dependence on initial data can be applied [1] [4].

1.4. THE POTENTIALS

One of the assumptions of the mathematical model (n. 10) is that the constitutive mapping $C : U \mapsto V$ be symmetric (if linear) or have a symmetric Gateaux derivative (if nonlinear). Such operators enjoy the property that the circulation of the vector $v = C(u)$ along a line in the U–space connecting two fixed points does not depend on the line chosen [5]. In other words the mapping $v = C(u)$ can be regarded as describing a conservative vector field in the U–space. This fact leads us to consider a potential that is a functional defined by

$$(1.4.1)\qquad E[u] = E[u_0] + \int_{\lambda=0}^{\lambda=1} \langle C(\eta), \delta\eta\rangle \quad \text{and then} \quad \delta E[u] = \langle C(u), \delta u\rangle$$

being $\eta = \eta(\lambda)$ so that $\eta(0) = u_0$, $\eta(1) = u$. For this reason the operator C is said to be a *potential operator*. It is also called the *gradient* of the functional $E[u]$. When C is a linear operator we can choose $\eta(\lambda) = \lambda u$ and eq. (1.4.1) reduces to

$$E[u] = \int_0^1 \langle C\lambda u \, , u d\lambda \rangle = \frac{1}{2} \langle u \, , Cu \rangle \tag{1.4.2}$$

a well known result. The link between C and E is reinforced by

THEOREM 13. *If* C *is a monotone (resp. strictly monotone) operator, the potential* $E[u]$ *is a convex (resp. strictly convex) functional and viceversa*:

$$E[\lambda u' + (1-\lambda) u''] \leq \lambda E[u'] + (1-\lambda) E[u''] \qquad (\text{resp.} <). \tag{1.4.3}$$

For the proof see [1; Theorem 1.2].

THEOREM 14: (VARIATIONAL FORMULATION IN THE NONLINEAR CASE). *The solution of the fundamental equation (with* $\sigma = 0$*) makes stationary the functional* $S[\varphi] = E[D(\varphi)]$ *being* $E[u]$ *given by eq.* (1.4.1.).

Proof.

$$\begin{aligned} \delta_\varphi S[\varphi] = \delta_\varphi E[D\varphi] &= \langle CD(\varphi) \, , \delta D(\varphi) \rangle = \langle CD\varphi \, , D'_\varphi \delta\varphi \rangle = \\ &= \langle \widetilde{D}'_\varphi CD(\varphi) \, , \delta\varphi \rangle = 0 \, . \end{aligned} \tag{1.4.4}$$

$S[\varphi]$ will be called the *action functional.*

THEOREM 15: *If* C *is invertible mapping the inverse operator* C^{-1} *is also of potential kind.*

Proof. It suffices to show that the elementary circulation

$$\langle \delta v \, , C^{-1}(v) \rangle \tag{1.4.5}$$

is the variation of a functional (automatically the circulation does not depend the line connecting two points).

From the identity

$$\langle \delta v \, , u \rangle \equiv \delta \langle v \, , u \rangle - \langle v \, , \delta u \rangle \tag{1.4.6}$$

it follows

$$\begin{aligned} \langle \delta v \, , C^{-1}(v) \rangle &\equiv \delta \langle v \, , C^{-1}(v) \rangle - \delta E[C^{-1}(v)] = \\ &= \delta \{ \langle v, C^{-1}(v) \rangle - E[C^{-1}(v)] \} = \delta \bar{E}[v] \, . \end{aligned} \tag{1.4.7}$$

The new functional

$$\bar{E}[v] \stackrel{\text{def}}{=} [v \, , C^{-1}(v)] - E[C^{-1}(v)] \tag{1.4.8}$$

will be called the *dual potential.* The transform (1.4.8) is known as Legendre transform.

Combining Theorem 13 with this result we can state the

THEOREM 16: *If* C *is strictly monotone then the dual potential* $\bar{E}[v]$ *is convex.*

1.5. Dual balance equation

If we look at definition equation $u = D(\varphi)$ as an equation in which u is assigned and φ must be found we are faced with *compatibility conditions* on u (that are existence conditions for φ). If these conditions are found, be they $R(u) = 0$ we shall call the operator R an *annichilator* of D because $RD(\varphi) \equiv 0$.

This means that null manifold of R contains the range of D i.e. $\mathfrak{N}(R) \supseteq \mathfrak{R}(D)$.

If *all* elements u_0 for which $Ru_0 = 0$ can be cast into the form $u_0 = D\varphi$ then we call R a *minimal annichilator* because its null manifold coincides with the range of $D : \mathfrak{N}(R) = \mathfrak{R}(D)$.

In this case the compatibility condition $Ru = 0$ is not only necessary but also sufficient to assure that the equation $u = D\varphi$ admits a solution.

While the domain of R lies in the U–space, its range lies in another function space we choose *linear* and that we shall denote with T and call *dual source space*.

If definition equation is of the form $u = u_0 + D(\varphi)$ then the compatibility condition is $R(u - u_0) = 0$. If D and R are linear, this equation can be written $Ru = \tau$. The "incompatibility" term τ that can be viewed as a dual source variable. The equation $Ru = \tau$ is then called dual *balance equation*. Alongside the linear T–space we are lead to introduce another linear function space whose elements are of the same tensorial order as those of T. This space will be denoted with Ψ and called *dual configuration space*. These two spaces are put in duality introducing the bilinear functional denoted with $\langle \psi , \tau \rangle$. The space Ψ and the bilinear map $\langle \psi , \tau \rangle$ will be chosen so that the duality be separating and both spaces will be equipped with topologies that make the bilinear functional $\langle \psi , \tau \rangle$ continuous.

1.6. Relation between the dual balance and the dual definition operator (linear case)

With the bilinear form we can define the adjoint of the operator when the last is *linear* and when its domain is *dense* in the U–space

$$\langle \psi , Ru \rangle = \langle \tilde{R}\psi , u \rangle . \tag{1.6.1}$$

Now we can easily see that *the equation* $v = R\psi$ *gives a solution of the homogeneous balance equation* $\tilde{D}v = 0$.

We have in fact the following

THEOREM 17: *If* R *is a linear operator with domain dense in the* U–*space and range in* T, *that be an annichilator of* D, *then* $\tilde{D}$ *is an annichilator of* $\tilde{R}$.

Proof.

$$\langle \psi , RD\varphi \rangle = \langle \tilde{R}\psi , D\varphi \rangle = \langle \tilde{D}\tilde{R}\psi , \varphi \rangle . \tag{1.6.2}$$

Now if $\varphi \in \mathfrak{D}(D)$ $RD\varphi = 0$ because R is an annichilator of D: then $\langle \psi , RD\varphi \rangle = 0$ for every $\psi \in \Psi$. In particular this is true if $\psi \in \mathfrak{D}(\tilde{R})$ then from $\langle \tilde{D}\tilde{R}\psi , \varphi \rangle = 0$ being $\varphi \in \mathfrak{D}(D)$ and $\bar{\mathfrak{D}}(D) = U$ follows $\tilde{D}\tilde{R}\psi = 0$. Thus $\tilde{D}$ is annichilator of $\tilde{R}$.

An obvious question can be raised: is the solution $v = \tilde{R}\psi$ *general*, i.e. such that all elements v_0 such that $Dv_0 = 0$ are of the form $v_0 = \tilde{R}\psi$? This implies that $\mathfrak{R}(\tilde{R}) = \mathfrak{N}(\tilde{D})$. As we shall now see the answer is linked with the question: is the condition $Ru = 0$ *sufficient* to assure that $u = D\varphi$? We have in fact the following

THEOREM 18: *Let* U *and* T *be two complete topological vector spaces. If* R *is a closed linear operator with domain dense in* U *and closed range in* T *that is a minimal annichilator of* D, *and if* D *is a closed linear operator with closed range then the operator* $\tilde{D}$ *is a minimal annichilator of* $\tilde{R}$.

Proof. The hypothesis that R be a minimal annichilator means $\mathfrak{N}(R) = \mathfrak{R}(D)$. Then $\mathfrak{N}^{\perp}(R) = \mathfrak{R}^{\perp}(D)$. But, on account of the general property

$$\mathfrak{N}^{\perp}(R) = \bar{\mathfrak{R}}(\tilde{R}) \quad \text{and} \quad \mathfrak{R}^{\perp}(D) = \bar{\mathfrak{N}}(\tilde{D}) . \tag{1.6.3}$$

Then $\bar{\mathfrak{R}}(\tilde{R}) = \bar{\mathfrak{N}}(\tilde{D})$. Because $\tilde{D}$ is a closed operator its null space is also closed [6] i.e. $\bar{\mathfrak{N}}(\tilde{D}) = \mathfrak{N}(\tilde{D})$. Because R is closed with closed range then also $\tilde{R}$ has closed range [6] then $\bar{\mathfrak{R}}(\tilde{R}) = \mathfrak{R}(\tilde{R})$. It follows

$$\mathfrak{R}(\tilde{R}) = \mathfrak{N}(\tilde{D}) \tag{1.6.4}$$

then $\tilde{D}$ is a minimal annichilator for $\tilde{R}$.

From this Theorem it follows that under the conditions given in the Theorem the equation $v = \tilde{R}\psi$ gives the *general solution* of the balance equation $\tilde{D}v = 0$.

1.7. THE GENERALIZED THEOREM OF VIRTUAL WORKS

The principle of virtual works of mechanics is a formulation of equilibrium expressed as a link between source variables (the forces) and configuration variables (the position vectors). The actual dependence of sources from configuration, i.e. constitutive equations, does not enter into the principle. We now show that the principle can be restated as a Theorem valid in the mathematical model on account of the relation between definition and balance operators.

THEOREM 19 (GENERALIZED THEOREM OF VIRTUAL WORKS): *the balance equation is equivalent to the equation*

$$\langle v , \delta u \rangle = \langle \sigma , \delta\varphi \rangle \tag{1.7.1}$$

Proof.

$$(1.7.2) \qquad \langle v , \delta u\rangle = \langle v , \delta D(\varphi)\rangle = \langle v , D'_\varphi \delta\varphi\rangle = \langle \tilde{D}'_\varphi v , \delta\varphi\rangle = \langle \sigma , \delta\varphi\rangle .$$

COROLLARY 19-bis: *if* D *is a linear operator, balance equation is equivalent to the equation*

$$(1.7.3) \qquad \langle v , u\rangle = \langle \sigma , \varphi\rangle$$

THEOREM 20 (DUAL GENERALIZED THEOREM OF VIRTUAL WORKS). *If the annichilator* R *is linear then the dual balance equation is equivalent to the equation*

$$(1.7.4) \qquad \langle \delta v , u\rangle = \langle \delta\psi , \tau\rangle .$$

Proof.

$$(1.7.5) \qquad \langle \delta\psi , \tau\rangle = \langle \delta\psi , Ru\rangle = \langle \tilde{R}\delta\psi , u\rangle = \langle \delta\tilde{R}\psi , u\rangle = \langle \delta v , u\rangle .$$

1.8. THE DUAL SCHEME

In order to relate dual source variables with dual configuration variables we need a mapping $V \mapsto U$. When the constitutive mapping C can be inverted then its inverse C^{-1} realizes the mapping $V \mapsto U$. When this happens we can consider the dual scheme $\psi \to v \to u \to \tau$. The mapping $RC^{-1}\tilde{R}\psi = \tau$ will be called the dual fundamental mapping.

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