Finite Formulation of Electromagnetic Field

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***Abstract*—We show that the equations of electromagnetism can be directly obtained in a finite form, i.e., discrete, thus avoiding the traditional discretization methods of Maxwell's differential equations. The finite formulation can be used with unstructured meshes in two and three dimensions and easily permits to obtain fourth-order convergence.**

***Index Terms*—Computational electromagnetism, convergence of numerical methods, discrete electromagnetism, finite formulation.**

I. Introduction

COMPUTATIONAL electromagnetism is commonly based on a discretization of Maxwell's field equations. We show that it is possible to express the laws of electromagnetism starting directly from experimental facts by a set of algebraic equations [1]–[3].

The finite formulation is based on five items:

1) the systematic use of *global variables* instead of field functions;
2) the distinction between *source, configuration* and energy *variables*;
3) the use of *space-time elements* endowed with *inner* and *outer* orientations;
4) the use of *two cell complexes*, a primal one endowed with inner orientation and a dual one endowed with outer orientation;
5) the fact that *global variables are related to oriented space–time elements.*

A. Global Variables

We use the term *global* variable as synonym of *integral* variable. Physical measurements deal mainly with *global variables*, such as voltages, fluxes, charge contents, and charge flows, not field vectors. Global variables are continuous through the separation surface of two materials while field variables are not. This makes global variables best suited to deal with regions made of different materials.

While field variables are indispensable in a differential formulation, global variables are the natural tool for a finite formulation. Contrary to field functions, which are functions of points and instants, global variables are *domain functions* and the space and time elements to which they are related will be put inside square brackets.

The time integral of a physical variable, say, E, will be called its *impulse* and will be denoted by the corresponding calligraphic letter, say, $\mathcal{E}$.

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TABLE I
FIELD AND GLOBAL VARIABLES OF ELECTROMAGNETISM

	source				*config.*	
field	ρ	$\mathbf{J}$	$\mathbf{D}$	$\mathbf{H}$	$\mathbf{E}$	$\mathbf{B}$
space global	Q^{c}	I	Ψ	F	E	Φ
space-time global	Q^{c}	Q^{f}	Ψ	$\mathcal{F}$	$\mathcal{E}$	Φ

It is expedient to distinguish between global variables *in space* and global variables *in space–time.* So, electric current I, electromotive force E, and magnetomotive force F are global variables in space while electric charge flow Q^{f}, electromotive force impulse $\mathcal{E}$, and magnetomotive force impulse $\mathcal{F}$ are global variables in space–time.

B. Configuration, Source and Energy Variables

Source variables are, first of all, those that describe the source of the electromagnetic field, i.e., charges and currents and, in the second place, all variables linked to them by algebraic or differential operations without the intervention of physical constants. Table I collects the six main variables of this kind. *Configuration variables* are those that describe the configuration of the field, its potentials and all those variables that are linked to them by algebraic or differential operations without the intervention of physical constants. They are linked to the source ones by the constitutive equations. *Energy variables* are those obtained by the product of one source variable and one configuration variable: examples are the electric energy density $w_E = \mathbf{D} \cdot \mathbf{E}/2$, the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$.

C. Oriented Space–Time Elements

With this name we mean points ($\mathbf{P}$), lines ($\mathbf{L}$), surfaces ($\mathbf{S}$), volumes ($\mathbf{V}$), time instants ($\mathbf{I}$), and time intervals ($\mathbf{T}$). A further property of space and time elements, not commonly stressed, is that these elements can be endowed with two kinds of orientations, as shown in Fig. 1: the *inner* and the *outer* one. The four space elements endowed with inner orientation will be denoted by $\mathbf{P}$, $\mathbf{L}$, $\mathbf{S}$, $\mathbf{V}$, while those endowed with outer orientation will be denoted by $\tilde{\mathbf{P}}$, $\tilde{\mathbf{L}}$, $\tilde{\mathbf{S}}$, $\tilde{\mathbf{V}}$. In an analogous way, the primal and dual time elements will be denoted by $\mathbf{I}$, $\mathbf{T}$ and $\tilde{\mathbf{I}}$, $\tilde{\mathbf{T}}$, respectively, as shown in Fig. 3.

D. Cell Complexes

Since a finite formulation requires space and time elements, not only points and instants, it appear natural to introduce *cell complexes* instead of coordinate systems. Cell complexes exhibit vertices, edges, faces, and volumes. In a finite formulation, a pivotal role is played by the *dual* complex. If we make use of a simplicial complex as a primal complex, then the commonest

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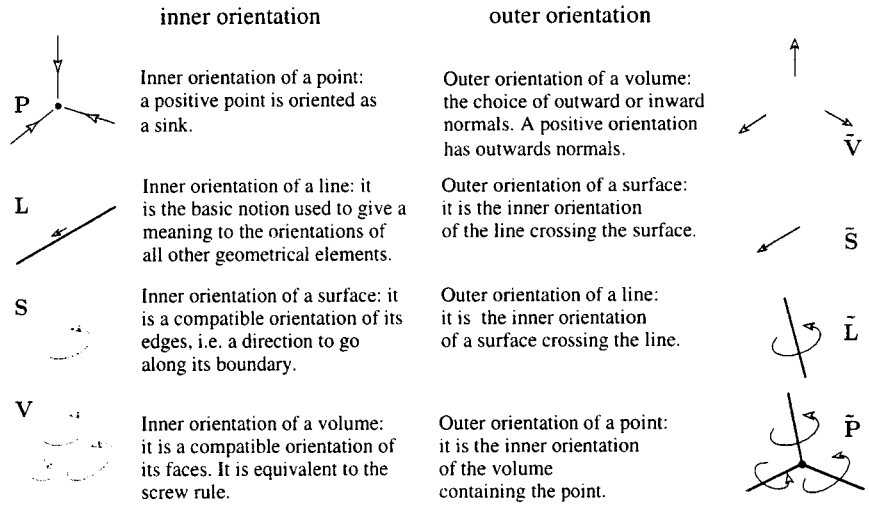


Fig. 1. Two notions of inner and outer orientations in 3-D space.

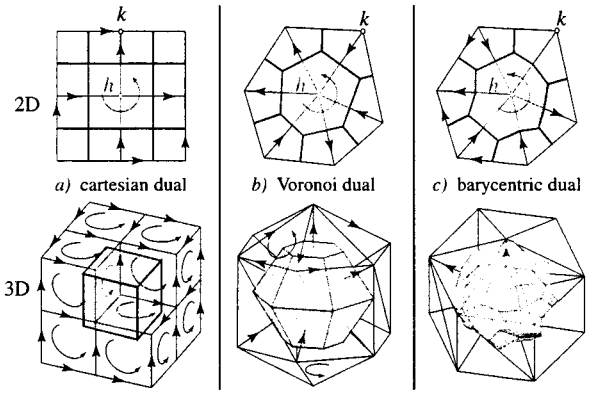


Fig. 2. Primal and dual cell complexes.

dual complexes are either the barycentric or the Voronoi dual, as shown in Fig. 2.

A cell complex and its dual enjoy a peculiar property: once the vertices, edges, faces, and cells of the *primal* complex have been endowed with *inner* orientation, then on the cells, faces, edges, and vertices of its dual is induced an *outer* orientation.

It follows that a pair formed by a cell complex and its dual is the natural frame to exhibit all space elements and their orientations.

E. Physical Variables and Space–Time Elements

With these premises, it is possible to build up a finite formulation of electromagnetism. We need two principles that have been inferred from a detailed analysis of global variables of classical field theories [3].

First Principle: In every physical theory, global configuration variables are associated with space and time elements that are endowed with inner orientation, while global source variables are associated with space and time elements that are endowed with outer orientation.

Second Principle: In every physical theory, there are physical laws that link global variables related to an oriented space–time element, say, Ω, with others related to its oriented boundary, say, $\partial\Omega$.

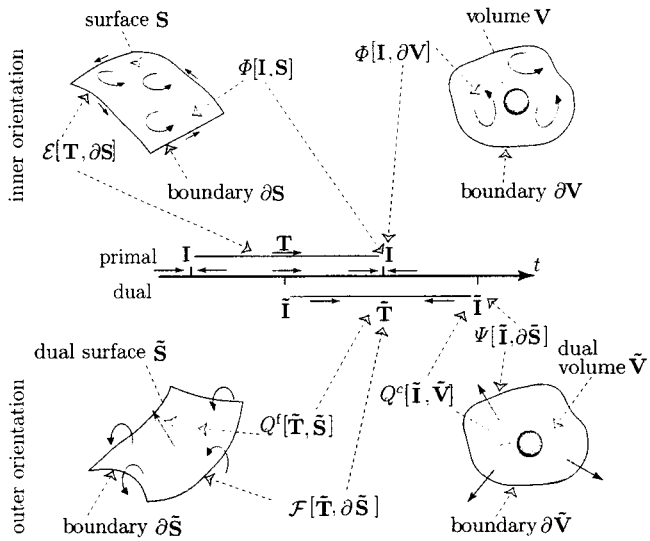


Fig. 3. Field variables and field laws are related to space and time elements.

The field laws of electromagnetism satisfy these principles. Since configuration variables are associated with space elements endowed with inner orientation, as shown in Fig. 3, it follows that they can be associated with the vertices, edges, faces, and cells of the *primal complex.* Moreover, since source variables are associated with space elements endowed with outer orientation, it follows that they can be associated with cells, faces, edges, and vertices of the *dual complex.*

II. Finite Field Equations

The four field laws of electromagnetism in global form are

$$\begin{cases} \Phi[\mathbf{I}, \partial\mathbf{V}] = 0 \\ \mathcal{E}[\mathbf{T}, \partial\mathbf{S}] = \Phi[\mathbf{I}^-, \mathbf{S}] - \Phi[\mathbf{I}^+, \mathbf{S}] \\ \Psi[\tilde{\mathbf{I}}, \partial\tilde{\mathbf{V}}] = Q^c[\tilde{\mathbf{I}}, \tilde{\mathbf{V}}] \\ \mathcal{F}[\tilde{\mathbf{T}}, \partial\tilde{\mathbf{S}}] = \Psi[\tilde{\mathbf{I}}^+, \tilde{\mathbf{S}}] - \Psi[\tilde{\mathbf{I}}^-, \tilde{\mathbf{S}}] + Q^f[\tilde{\mathbf{T}}, \tilde{\mathbf{S}}] \end{cases} \quad (1)$$

These equations are valid for whatever shape of the closed lines and of the closed surfaces involved for whatever material inside

the region. For these reasons, field equations are *topological* equations and describe the structure of electromagnetism. The three constitutive equations of electromagnetism are

$$\begin{cases} \dfrac{\Psi[\tilde{\mathbf{I}}, \tilde{\mathbf{S}}]}{\tilde{S}} = \varepsilon \dfrac{\mathcal{E}[\mathbf{T}, \mathbf{L}]}{TL}, & \text{when } \tilde{\mathbf{S}} \perp \mathbf{L} \\ \dfrac{\Phi[\mathbf{I}, \mathbf{S}]}{S} = \mu \dfrac{\mathcal{F}[\tilde{\mathbf{T}}, \tilde{\mathbf{L}}]}{\tilde{T}\tilde{L}}, & \text{when } \mathbf{S} \perp \tilde{\mathbf{L}} \\ \dfrac{Q^{\mathrm{f}}[\tilde{\mathbf{T}}, \tilde{\mathbf{S}}]}{\tilde{T}\tilde{S}} = \sigma \dfrac{\mathcal{E}[\mathbf{T}, \mathbf{L}]}{TL}, & \text{when } \tilde{\mathbf{S}} \perp \mathbf{L}. \end{cases} \tag{2}$$

The constitutive equations in the form (2) are valid only in regions of *uniform* field, for *homogeneous* and *isotropic* materials, are valid only when $\mathbf{S}$, $\tilde{\mathbf{S}}$ are plane surfaces and $\mathbf{L}$, $\tilde{\mathbf{L}}$ are straight lines *orthogonal* to them. The equations in (2) that require orthogonality can be applied to Cartesian meshes or to Delaunay–Voronoi meshes. Equations (1) and (2) are the laws of electromagnetism in the finite formulation we were searching for [1].

Let us introduce a Delaunay–Voronoi pair of cell complexes and let us denote by l_α and s_β an arbitrary edge and face of the primal complex. Let us denote with $\tilde{l}_\beta$ and $\tilde{s}_\alpha$ an arbitrary edge and face of the dual complex, with $c_{\beta\alpha}$ and $\tilde{c}_{\alpha\beta}$ the corresponding incidence numbers. Inserting the constitutive equations in the two circuital equations we obtain [1]

$$\begin{cases} E_\alpha^{n+1/2} = E_\alpha^{n-1/2} + \dfrac{l_\alpha \tilde{\tau}}{\varepsilon \tilde{s}_\alpha} \left[\displaystyle\sum_\beta \tilde{c}_{\alpha\beta} F_\beta^n - I_\alpha^n \right] \\ F_\beta^{n+1} = F_\beta^n - \dfrac{\tilde{l}_\beta \tau}{\mu s_\beta} \left[\displaystyle\sum_\alpha c_{\beta\alpha} E_\alpha^{n+1/2} \right] \end{cases} . \tag{3}$$

The electric current is the sum of an impressed current $\overline{I}_\alpha^n$ and a conduction current

$$I_\alpha^n = \overline{I}_\alpha^n + \sigma \frac{\tilde{s}_\alpha}{l_\alpha} \frac{E_\alpha^{n+1/2} + E_\alpha^{n-1/2}}{2}. \tag{4}$$

If we put

$$a \stackrel{\text{def}}{=} 1 + \frac{\sigma\tilde{\tau}}{2\varepsilon} \qquad b \stackrel{\text{def}}{=} 1 - \frac{\sigma\tilde{\tau}}{2\varepsilon} \tag{5}$$

the first equation, (3) becomes

$$E_\alpha^{n+1/2} = \frac{b}{a} E_\alpha^{n-1/2} + \frac{1}{a} \frac{l_\alpha \tilde{\tau}}{\varepsilon \tilde{s}_\alpha} \left[\sum_\beta \tilde{c}_{\alpha\beta} F_\beta^n - I_\alpha^n \right]. \tag{6}$$

If we use a Cartesian cell complex and its dual [see Fig. 2(a)], like the one used in finite difference in time domain (FDTD), these equations can be directly applied. In this case, one obtains the same equations of FDTD [5, p. 303]. What is important is that the same equations can be applied to a couple of Delaunay–Voronoi complexes [see Fig. 2(b)] because the orthogonality condition requested in (2) is satisfied. *This means that we have obtained a natural extension of the FDTD formulation to unstructured meshes.* Equation (3) can be compared with [6, p. 939] and [7, p. 547].

A numerical comparison between classical FDTD technique and the present extension for a resonant cavity has been done in [8]. The discrete method arising from this formulation, which uses space global variables and a couple of cell complexes, has been called *cell method.*

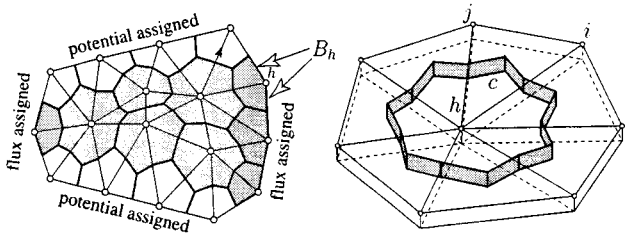


Fig. 4. Cell complexes for the electrostatic problem.

III. Second-Order Convergence

A. Delaunay–Voronoi Complexes

With reference to Fig. 2(a) and (b), we can give in a straigthforward way a discrete formulation of Poisson's equation

$$V_\alpha = \sum_i g_{\alpha i} \varphi_i \qquad \Psi_\alpha = \varepsilon \frac{\tilde{S}_\alpha}{L_\alpha} V_\alpha \qquad \sum_\beta \tilde{d}_{h\beta} \Psi_\beta = Q_h + B_h \tag{7}$$

where $g_{\alpha i}$ is the incidence number of the primal node i with the primal edge α and $\tilde{d}_{h\beta}$ is the incidence number of the dual face β with the dual cell h, Q_h is the charge content of the dual cell h, and B_h is the electric flux related to the external boundary of broken cells, as shown in Fig. 4. These relations are valid both for a Cartesian cell complex and a simplicial complex of the kind Delaunay–Voronoi. They are valid both for two-dimensional (2-D) and three-dimensional (3-D) problems. Composing these three relations and performing a little change of notation, we obtain

$$-\varepsilon \left[\left(\sum_{k \in \mathcal{J}(h)} \frac{\tilde{S}_{hk}}{L_{hk}} \right) \varphi_h - \left(\sum_{k \in \mathcal{J}(h)} \frac{\tilde{S}_{hk}}{L_{hk}} \varphi_k \right) \right] = Q_h + B_h \tag{8}$$

where $\mathcal{J}(h)$ denotes the set of nodes that are connected with the node h by an edge. The first member of this equation *coincides* with the one obtained with the finite-element method (FEM): it is remarkable that it has been obtained in elementary way avoiding variational formulation, Galerkin method, shape functions, and weight functions. The order of convergence is two.

B. Barycentric Subdivision

If one uses the barycentric subdivision instead of the Delaunay–Voronoi one, the constitutive equations in (2) cannot be used in this form, but must be put in the traditional vectorial form.

Remembering that constitutive equations are experimented in regions of uniform field, we do the approximation of considering the field *uniform* inside every simplex. In this case, the potential ϕ inside every simplex can be approximated with an affine function $\phi(\xi, \eta) = p + E_\xi \xi + E_\eta \eta$ being ξ and η the areal coordinates of the triangle and E_ξ and E_η of the two components of electric field vector $\mathbf{E}$. From these, one can evaluate

the Cartesian components E_x and E_y. The three coefficients can be expressed in terms of the three nodal values ϕ_h, ϕ_i, ϕ_j, as shown in Fig. 4. We remark that the vector so introduced does not require a limit process because, say, E_x is the ratio between the voltage referred to a line segment in the x direction to the length. For anisotropic materials, we can write

$$\begin{Bmatrix} D_x \\ D_y \end{Bmatrix}_c = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_{yy} \end{bmatrix}_c \begin{Bmatrix} E_x \\ E_y \end{Bmatrix}_c . \tag{9}$$

With the vector $\mathbf{D}$ so obtained, one can evaluate the electric flux Ψ across the faces of the dual polygon contained inside the triangle. If $\mathbf{A}$ denotes the area vector of a face of the dual cell, the electric flux is given by

$$\Psi = (A_x \quad A_y) \begin{Bmatrix} D_x \\ D_y \end{Bmatrix}_c . \tag{10}$$

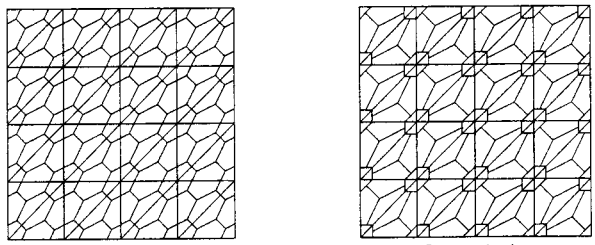
In this way, we can evaluate the electric fluxes for every face of the dual polygon and to insert them in the Gauss equation. Doing so, we obtain the discrete formulation of the fundamental equation of electrostatics for an anisotropic medium using the barycentric dual (see details in [9]). Every dual polygon acts as a tributary area for the internal node. The flux boundary conditions can be directly inserted into balance equations. The fundamental matrix coincides with the one of FEM and the order of convergence is two.

IV. Fourth-Order Convergence

Since we introduced shape functions to deal with the barycentric subdivision, we opened the way to obtain higher order approximations using quadratic interpolation polynomials, like in FEM. If we consider the midpoints of the triangle edges as three supplementary nodes, we can choose quadratic interpolating functions. Doing so, the field vector $\mathbf{E}$ is no longer uniform, but becomes an affine function and the same is for the vector $\mathbf{D}$. On account of the linear behavior of the field, in order to evaluate the electric flux Ψ across every face of the dual polygon, it is enough to evaluate the vector $\mathbf{D}$ in the barycenter of the face. The scalar product of such vector times the area-vector of the face gives the required flux. Then, even in the case of quadratic interpolation, *no integration is required.*

To give an example, we shall refer to a square divided into small squares, each one divided into two triangles. How to choose the dual polygon? Here resides the main difference with FEM: *in the cell method, the tributary area of each node can be chosen at will.*

A first choice is the barycentric subdivision, shown in Fig. 5(a): the order of convergence is slightly greater than two (2.3) and then is lower than the one of FEM that, for a quadratic shape function, is three [10, p. 85]. However, a second choice for the tributary area is the one obtained using two Gauss points on every edge as shown in Fig. 5(b): the order of convergence is now four, as shown in Fig. 5(c). We remark that Abaqus has a fourth-order convergence because it uses the points of superconvergence (the library used is DC2D6). Contrary to FEM, here the fundamental matrix is not symmetric. We remark that in FEM "the system is symmetric which is a consequence of the choice of the Galerkin weighting and the symmetry form of Poisson equation" [11, p. 251].



a) barycentric dual b) Gauss dual

c)

N cells	cell method	Abaqus	cells-Abaqus
4	3,77E-03	3,77E-03	0,00E+00
16	3,76E-04	3,76E-04	0,00E+00
64	2,49E-05	2,49E-05	0,00E+00
256	1,56E-06	1,56E-06	1,00E-15
1024	9,68E-08	9,68E-08	4,00E-15
4096	6,02E-09	6,02E-09	1,40E-14
16384	3,75E-10	3,75E-10	-5,10E-14

Fig. 5. With a proper choice of the tributary region of every node, as shown in (b), the order of convergence is four.

V. Conclusion

We have shown that, instead to discretize Maxwell's equations, it is possible to give a direct finite formulation of electromagnetic laws. This formulation can be immediately used in computational electromagnetism. When applied to simplicial complexes (triangles in 2-D and tetrahedra in 3-D), the finite formulation constitutes an extension of FDTD to unstructured meshes. The use of interpolation functions permits to obtain orders of convergence larger than the second one.

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