Finite Formulation of Electromagnetic Field

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Abstract

The paper shows that the equations of electromagnetism can be obtained in a finite (=discrete) form directly, i.e. without passing throught the differential formulation. This finite formulation is the natural extension of network theory to electromagnetic field and it is convenient for computational electromagnetics.

1 Introduction

Computational electromagnetism requires the transformation of Maxwell's differential equations into algebraic equations. This is obtained using one of the many discretization methods like Finite Difference Method (FDM); Finite Difference in Time Domain (FDTD); Finite Element Method (FEM); Boundary Element Method (BEM); Edge Element Method (EEM), etc. Even in Finite Volume Method (FVM) and in Finite Integration Theory (FIT), that use an integral formulation, it is standard practice to use integrals of field functions; the last ones being indispensable ingredients of differential formulation. One can pose the following question: *is it possible to express the laws of electromagnetism directly by a set of algebraic equations, instead of obtaining them from a discretization process applied to differential equations?*

The answer is: *Yes, it is possible, it is easy and can be immediately utilized in computation.* What we now present is an alternative to differential formulation. To display field laws in a finite formulation we must introduce two classifications of physical quantities.

1.1 Configuration, source and energy variables

A first classification is based on the *role* that every physical variable plays in a theory: this leads to three classes of variables: *configuration*, *source* and *energy* variables as shown in Table (1). In every physical field one can find:

- *Configuration variables* that describe the configuration of the field or of the system. These variables are linked one to another by operations of sum, difference, limit, derivative and integral.
- *Source variables* that describe the sources of the field or the forces acting on the system. These variables are linked one to another by operations of sum, difference, limit, derivative and integral.
- *Energy variables* that are obtained as the product of a configuration for a source variable. These variables are linked one to another by operations of sum, difference, limit, derivative and integral.

1.2 Global and field variables

A second classification distinguish *global variables* from *field variables*. Global variables are those commonly called *integral variables*: we avoid the last expression because it refers to an integration process performed on field variables.

We must emphasize that physical measurements deal mainly with *global variables*: we measure directly voltages, fluxes, charge content and charge flows, not field vectors. Field variables are needed in a differential formulation because

Table 1: The three classes of variables of electromagnetism.

CONFIGURATION VARIABLES:	
gauge function	χ
e.m.f. (impulse)	$E, (\mathcal{E})$
electric field vector	$\mathbf{E}$
magnetic flux	Φ
electric potential (impulse)	$V, (\mathcal{V})$
magnetic vector potential	$\mathbf{A}$
magnetic induction	$\mathbf{B}$
SOURCE VARIABLES:	
electric charge content	Q^c
electric charge flow	Q^f
electric current density	$\mathbf{J}$
electric flux	Ψ
electric induction	$\mathbf{D}$
magnetic field strength	$\mathbf{H}$
m.m.f. (impulse)	$F_{\rm m}, (\mathcal{F}_m)$
magnetic scalar potential	$V_{\rm m}$
dielectric polarization	$\mathbf{P}$
magnetization	$\mathbf{M}$
ENERGY VARIABLES:	
work , heat	W,Q
electric energy density	$w_{\rm e}$
magnetic energy density	$w_{\rm m}$
Poynting vector	$\mathbf{S}$
electromagnetic momentum	$\mathbf{G}$
momentum density	$\mathbf{g}$
electromagnetic action	A

the very notion of derivative refers to point functions. On the contrary a global quantity refers to a system, to a space or time element, like lines, surfaces, volumes, intervals, i.e. is a *domain function.* Thus a flow meter measures the electric charge that crosses a given *surface* in a given *time interval.* A magnetic tensiometer measures the magnetic voltage impulse referred to a *line* and a *time interval.* The corresponding physical quantities are associated with extended space and time elements, not only with points and instants.

One fundamental advantage of global variables is that they are continuous through the separation surface of two materials while the field variables are discontinuous. This implies that the differential formulation is restricted to regions of material homogeneity: one must break the domain in subdomains, one for every material and introduce jump conditions. If one reflects on the great number of different materials present in a real device, one can see that the idealization required by differential formulation is too restrictive for the present technology.

This shows that *differential formulation imposes derivability conditions on field functions that are restrictive from the physical point of view.* Contrary to this, a direct finite formulation based on global variables accepts material discontinuities, i.e. does not add regularity conditions to those requested by the physical nature of the variable.

To help the reader, accustomed to think in terms of traditional field variables we first examine the global variables corresponding to traditional field functions:

$$\begin{array}{cccccc} \rho & \mathbf{J} & \mathbf{B} & \mathbf{D} & \mathbf{E} & \mathbf{H} \\ Q^{\rm c} & Q^{\rm f} & \Phi & \Psi & \mathcal{E} & \mathcal{F}_{\rm m} \end{array} \tag{1}$$

These are collected in Table (2). This table shows that integral variables arise by integration of field functions on space and time domains i.e. lines, surfaces, volumes and time intervals. The time integral of a physical variable, say E, will be called its *impulse* and will be denoted by the corresponding calligraphic letter, say $\mathcal{E}$.

It is remarkable that *global* configuration variables all have the dimension of a magnetic flux and that *global* source variables all have the dimension of a charge. The product of a global configuration variable and a global source variable has the dimension of an action (energy × time).

Table 2: Integral (=global) physical variables of electromagnetism and corresponding field functions.

configuration variables (weber)	*source variables (coulomb)*
$\mathcal{V} = \int_{\mathbf{T}} V \, \mathrm{d}t$	$Q^{\rm c} = \int_{\tilde{\mathbf{V}}} \rho \, \mathrm{d}V$
$p = \int_{\mathbf{L}} \mathbf{A} \cdot \mathrm{d}\mathbf{L}$	$Q^{\rm f} = \int_{\tilde{\mathbf{T}}} \int_{\tilde{\mathbf{S}}} \mathbf{J} \cdot \mathrm{d}\mathbf{S} \, \mathrm{d}t$
$\mathcal{E} = \int_{\mathbf{T}} \int_{\mathbf{L}} \mathbf{E} \cdot \mathrm{d}\mathbf{L} \, \mathrm{d}t$	$\Psi = \int_{\tilde{\mathbf{S}}} \mathbf{D} \cdot \mathrm{d}\mathbf{S}$
$\Phi = \int_{\mathbf{S}} \mathbf{B} \cdot \mathrm{d}\mathbf{S}$	$\mathcal{F}_{\rm m} = \int_{\tilde{\mathbf{T}}} \int_{\tilde{\mathbf{L}}} \mathbf{H} \cdot \mathrm{d}\mathbf{L} \, \mathrm{d}t$

1.3 Cell complexes

As is well known there is a strict link between physics and geometry. In spite of this it is not commonly stressed that global physical variables are naturally associated with space and time elements, i.e. points, lines, surfaces, volumes, instants and intervals.

In differential formulation a fundamental role is played by points: field functions are point functions. In order to associate points with numbers we introduce *coordinate systems*.

In finite formulation we need to consider not only points ($\mathbf{P}$) but also lines ($\mathbf{L}$), surfaces ($\mathbf{S}$) and volumes ($\mathbf{V}$). We shall call these *space elements*.

The natural substitute of coordinate systems are *cell complexes*. They exhibit vertices, edges, faces and cells. The latter are representative of the four spatial elements.

Once we have introduced a cell complex we can consider the *dual* complex. In a simplicial complex the commonst choices are either the barycentres of every simplex or the circumcentres (in 2D) and the circumspheres (in 3D): *in this paper we consider only circumcentres and circumspheres.* Since the straight line connecting the circumcentres of two adjacent simplexes in 2D is orthogonal to the common edge the dual polygon thus obtained has its sides orthogonal to the common edge. This is called *Voronoi polygon* in 2D and *Voronoi polyhedron* in 3D. The circumcentres have the disadvantage that for triangles with obtuse angles they lie outside the triangle. This is inconvenient when the circumcentre of one obtuse triangle goes beyond the one of the adjacent triangle with the common sides. This is avoided when the triangulation satisfies the *Delaunay condition*. This leads us to consider only Delaunay-Voronoi complexes, as we shall do in this paper. As in coordinate systems it is preferable to deal with orthogonal coordinate systems, so in a simplicial complex it is preferable to deal with a Delaunay complex and its associated Voronoi complex as dual, as shown in Figure (1 right).

1.4 Inner and outer orientation

The notions of inner and outer orientation of a space element play a pivotal role in electromagnetism as well as in all physical theories.

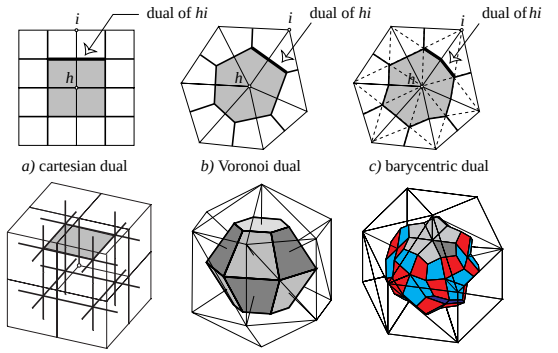

Figure 1: Three kind of dual: those of the first and second columns satisfy the orthogonality condition, between dual elements.

Inner orientation. Points can be oriented as "sources" or "sinks". The notion of source and sink, borrowed from fluid dynamics, can be used to define an inner orientation of points because it permits us to maintain the notion of incidence number from lines and points. A line is endowed of inner orientation when a direction has been chosen on the line. A surface is endowed with inner orientation when its boundary has an inner orientation. A volume is endowed with inner orientation when its boundary is so. The four space elements endowed with inner orientation will be denoted $\mathbf{P}, \mathbf{L}, \mathbf{S}, \mathbf{V}$.

Outer orientation. To write a balance we need a notion of exterior of a volume, because we speak of charge *contained* in the volume. This is usually done by fixing outwards or inwards normals to its boundary. A surface is equipped with outer orientation when one of its faces has been chosen as positive and the other negative: this is equivalent of fixing the direction of an arrow crossing the surface from the negative to the positive face. We need the outer orientation of a surface when we consider a flow crossing the surface. A line is endowed with outer orientation when a direction of rotation *around* the line has been defined: think to the rotation of the plane of polarization of a ligth beam. A point is endowed with outer orientation when all line segments with origin in the point have an outer orientation. Think, for example, to the sign of the scalar magnetic potential of a coil at a point: its sign depends on the direction of the current in the coil.

The four space elements endowed with outer orientation will be denoted $\tilde{\mathbf{P}}, \tilde{\mathbf{L}}, \tilde{\mathbf{S}}, \tilde{\mathbf{V}}$.

A cell complex and its dual enjoy a peculiar property: once the vertices, edges, faces and cells of the primal complex has been endowed with inner orientation, this induces an outer orientation on the cells, faces, edges and vertices of its dual. It follows that a pair formed by a cell complex and its dual is the natural frames to exhibit all space elements with the two kind of orientations.

1.5 Cell complex in time

Let us consider a given interval of the time axis and divide it into small intervals, as shown in Fig. (2). A primal instant $\mathbf{I}$ is oriented as sink, such as space points. A primal interval $\mathbf{T}$ will be

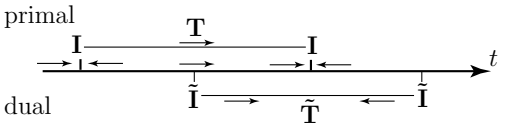

Figure 2: Cell complex on time axis and its dual.

endowed with *inner* orientation, i.e. it is oriented towards increasing time. If we choose an instant inside every interval we obtain a dual instant $\tilde{\mathbf{I}}$ that is authomatically endowed with *outer* orientation. The interval $\tilde{\mathbf{T}}$ between two dual instants is a dual interval and is authomatically endowed with *outer* orientation.

In this fashion every instant of the primal complex there corresponds an interval of the dual and to every interval of the primal there corresponds an instant of the dual. Thus we have the correspondence $\mathbf{I} \leftrightarrow \tilde{\mathbf{T}}$ and $\tilde{\mathbf{I}} \leftrightarrow \mathbf{T}$ and this is a duality map.

1.6 Global variables and space-time elements

From the analysis of a great number of physical variables of classical fields one can infer that: [3], [4]

> First Principle: *Global configuration variables are associated with space and time elements endowed with inner orientation while global source variables are associated with space and time elements endowed with outer orientation.*

Table 3: Global variables of electromagnetism.

global physical variable	*symbol*
electric charge content	$Q^{\mathrm{c}}[\tilde{\mathbf{I}}\,\tilde{\mathbf{V}}]$
electric charge flow	$Q^{\mathrm{f}}[\tilde{\mathbf{T}}\,\tilde{\mathbf{S}}]$
e.m.f. impulse	$\mathcal{E}[\mathbf{T}\mathbf{L}]$
m.m.f. impulse	$\mathcal{F}_{\mathrm{m}}[\tilde{\mathbf{T}}\,\tilde{\mathbf{L}}]$
magnetic flux	$\Phi[\mathbf{I}\,\mathbf{S}]$
electric flux	$\Psi[\tilde{\mathbf{I}}\,\tilde{\mathbf{S}}]$
electric potential impulse	$\mathcal{V}[\mathbf{T}\,\mathbf{P}]$
magnetic potential impulse	$\mathcal{V}_{\mathrm{m}}[\tilde{\mathbf{T}}\,\tilde{\mathbf{P}}]$

This principle gives the reason of the use of differential forms in electromagnetism [5]. The reason for associating source variables with outer orientation is that they are used in balance equations and a balance require a volume with outer orientation (outwards or inwards normals). In short:

configuration variables	source variables
inner orientation	outer orientation

This principle offers a rational criterion to associate global variables of every physical theory to space and time elements and, as such, it is useful in computational electromagnetism. Figure (3) shows this association for physical variables of electromagnetism.

It is important to note that each one of these six variables of Eq.(1) admits an operational definition [7]. One can say that *the role of the dual complex is to form a reference structure to which source variables can be referred.*

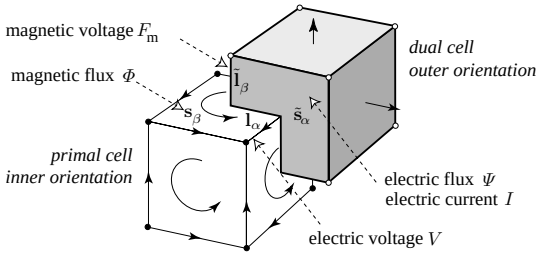

Figure 3: Global physical variables are associated with elements of a cell complex and its dual.

The space and time association of global electromagnetic variables is summarized in Table (3).

1.7 Physical laws and space-time elements.

The first Principle states that global physical variables refer to the oriented space and time elements. From the analysis of a great number of physical variables of classical fields one can infer [3], [4]:

> SECOND PRINCIPLE: *In every physical theory there are physical laws that link global variables referred to an oriented space-time element, say* $\mathbf{\Omega}$*, with others referred to its oriented boundary, say* $\partial\mathbf{\Omega}$*.*

The fundamental laws of electromagnetism satisfy this principle. This principle motivates the role of exterior differential on differential forms [5].

1.8 Field laws

Experiments lead us to infer the following four laws of electromagnetism (refer to Fig. (4)):

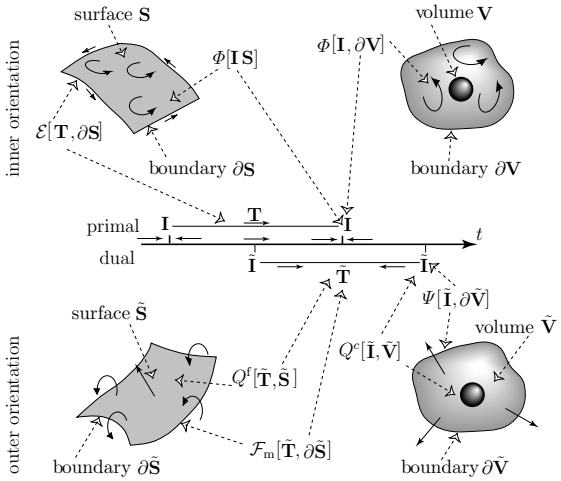

Figure 4: Field laws are associated with space and time elements.

- The *magnetic flux* Φ referred to the *boundary* of a volume endowed with inner orientation at any instant vanishes (Gauss).
- The *electromotive force impulse* $\mathcal{E}$ referred to the *boundary* of a surface endowed with inner orientation during a time interval is opposite to the *magnetic flux* Φ variation across the surface in the same interval (Faraday).
- The *electric flux* Ψ across the *boundary* of a volume endowed with outer orientation at any instant is equal to the *electric charge* Q^{c} contained inside the volume at that instant (Gauss).
- The *magnetomotive force impulse* $\mathcal{F}_{\mathrm{m}}$ referred to the *boundary* of a surface endowed with outer orientation in a time interval is equal to the sum of the *electric charge flow* Q^{f} across the surface in that time interval and the *electric flux* Ψ variation across the surface in that interval (Ampère-Maxwell).

In formulae:

$$\begin{aligned}
\Phi[\mathbf{I},\partial\mathbf{V}] &= 0\\
\mathcal{E}[\mathbf{T},\partial\mathbf{S}] &= \Phi[\mathbf{I}^{-},\mathbf{S}] - \Phi[\mathbf{I}^{+},\mathbf{S}]\\
\Psi[\tilde{\mathbf{I}},\partial\tilde{\mathbf{V}}] &= Q^{\mathrm{c}}[\tilde{\mathbf{I}},\tilde{\mathbf{V}}]\\
\mathcal{F}_{\mathrm{m}}[\tilde{\mathbf{T}},\partial\tilde{\mathbf{S}}] &= \Psi[\tilde{\mathbf{I}}^{+},\tilde{\mathbf{S}}] - \Psi[\tilde{\mathbf{I}}^{-},\tilde{\mathbf{S}}] + Q^{\mathrm{f}}[\tilde{\mathbf{T}},\tilde{\mathbf{S}}].
\end{aligned} \tag{2}$$

Equations (2) are the four laws of electromagnetism in a finite formulation we are searching for [7]. These algebraic equations enjoy the following properties:

- they link physical variables of the same kind, i.e. configuration variables with configuration variables and source variables with source variables;
- they are valid *in whatever medium* and then are free from any material parameter;
- they are valid for *whatever surface* and *whatever volume* and then are valid in the large as in the small;
- they *do not involve metrical notions*, i.e. lengths, areas, measures of volumes and durations are not required.

These properties show that field equations do not require infinitesimal space elements and then they are not responsible of differential formulation.

Comparison with other methods. Since the distinction between the two classes of physical variables is not commonly done and since it is not recognized that two kind of orientations are needed, it follows that differential formulation uses only one kind of *infinitesimal* cells. It follows that FEM, arising from a discretization of differential equations, ignorate the need of two dual complexes and only one complex is used whose cells are the *elements*. In 1966 Yee [10], using a cartesian complex and with an appropriate choice of points at which the various field components are to be evaluated, opened the way to the introduction of a pair of dual complexes done by Weiland [8], the *electric* and *magnetic* grids. Nevertheless the two complexes were not justified by physical considerations but only by computational advantages. In the realm of differential forms the two kind of orientation gives rise to *normal* and *twisted* forms [1, p.183].

Local formulation. In order to obtain a set of algebraic equations we must introduce a cell complex and its dual. All elements must be labelled. Let $\mathbf{l}_\alpha, \mathbf{s}_\beta$ the edges and faces of the primal complex respectivly; $\tilde{\mathbf{l}}_\beta$ and $\tilde{\mathbf{s}}_\alpha$ the same for the dual complex, $c_{\alpha\beta}$,$d_{k\beta}$, etc the incidence numbers.

When equations (2) are applied to the corresponding cells of the two complexes, we obtain a *local* form of the field equations of the electromagnetic field in a finite setting. We can write

$$\begin{aligned}
&\sum_\alpha c_{\beta\alpha}\, \mathcal{E}[\boldsymbol{\tau}_{n+1}, \mathbf{l}_\alpha] + \Phi[\mathbf{t}_{n+1}, \mathbf{s}_\beta] - \Phi[\mathbf{t}_n, \mathbf{s}_\beta] = 0 \\
&\sum_\beta d_{k\beta}\, \Phi[\mathbf{t}_n, \mathbf{s}_\beta] = 0 \\
&\sum_\beta \tilde{c}_{\alpha\beta}\, \mathcal{F}_{\mathrm{m}}[\tilde{\boldsymbol{\tau}}_n, \tilde{\mathbf{l}}_\beta] - \Psi[\tilde{\mathbf{t}}_{n+1}, \tilde{\mathbf{s}}_\alpha] + \Psi[\tilde{\mathbf{t}}_n, \tilde{\mathbf{s}}_\alpha] \\
&\qquad\qquad = Q^{\mathrm{f}}[\tilde{\boldsymbol{\tau}}_n, \tilde{\mathbf{s}}_\alpha] \\
&\sum_\alpha \tilde{d}_{h\alpha}\, \Psi[\tilde{\mathbf{t}}_n, \tilde{\mathbf{s}}_\alpha] = Q^{\mathrm{c}}[\tilde{\mathbf{t}}_n, \tilde{\mathbf{v}}_h]
\end{aligned} \tag{3}$$

For computational purposes it is useful to make the following changes of symbols: $t_n \to n$; $\tilde{t}_n \to n + 1/2$; $\Phi[\mathbf{t}_n, \mathbf{s}_\beta] \to \Phi^n_\beta$; *etc.* In particular the two evolution equations can be written as ($\tilde{c}_{\alpha\beta} = c_{\beta\alpha}$, see [7])

$$\begin{aligned}
\Phi^{n+1}_\beta &= \Phi^n_\beta - \sum_\alpha c_{\beta\alpha}\, \mathcal{E}^{n+1/2}_\alpha \\
\Psi^{n+1/2}_\alpha &= \Psi^{n-1/2}_\alpha + \sum_\beta c_{\beta\alpha}\, (\mathcal{F}_{\mathrm{m}})^n_\beta - (Q^{\mathrm{f}})^n_\alpha.
\end{aligned} \tag{4}$$

This gives rise to the *leapfrog algorithm*.

1.9 Material laws

The equations that link the source variables with configuration ones are *material* or *constitutive* equations. In a region of *uniform* field the three material equations of electromagnetism in finite form and *for orthogonal duals* $\tilde{\mathbf{s}}_\alpha \perp \mathbf{l}_\alpha$ are

$$\begin{aligned}
\frac{\Psi[\tilde{\mathbf{t}}_n, \tilde{\mathbf{s}}_\alpha]}{\tilde{s}_\alpha} &= \epsilon\, \frac{\mathcal{E}[\boldsymbol{\tau}_n, \mathbf{l}_\alpha]}{\tau_n\, l_\alpha} \\
\frac{\Phi[\mathbf{t}_n, \mathbf{s}_\beta]}{s_\beta} &= \mu\, \frac{\mathcal{F}_{\mathrm{m}}[\tilde{\boldsymbol{\tau}}_n, \tilde{\mathbf{l}}_\beta]}{\tilde{\tau}_n\, \tilde{l}_\beta} \\
\frac{Q^{\mathrm{f}}[\tilde{\boldsymbol{\tau}}_n, \tilde{\mathbf{s}}_\alpha]}{\tilde{\tau}_n\, \tilde{s}_\alpha} &= \sigma\, \frac{1}{2} \left(\frac{\mathcal{E}[\boldsymbol{\tau}_n, \mathbf{l}_\alpha]}{\tau_n\, l_\alpha} + \frac{\mathcal{E}[\boldsymbol{\tau}_{n+1}, \mathbf{l}_\alpha]}{\tau_{n+1}\, l_\alpha} \right)
\end{aligned} \tag{5}$$

in which $\tau_n, \tilde{\tau}_n, l_\alpha, \tilde{l}_\beta, s_\beta, \tilde{s}_\alpha$ are the extensions of the corresponding elements. To explain the particular form of Ohm's law let us remark that while the electric charge flow Q^{f} is referred to dual intervals, the electric tension impulse $\mathcal{E}$ is referred to primal ones. These equations are valid if cells are cubes or belong to a Delaunay-Voronoi complex, as is shown in Fig.(1). In these cases 1-cells of the dual are orthogonal to the primal 2-cells and viceversa. It is possible to avoid the orthogonality condition and then the Voronoi complexes using barycenters [2].

The main properties of material laws are:

- They are valid in regions in which the field is *uniform* because they are tested under such conditions;
- They link a variable referred to a p-cell of a complex with the dual $(n-p)$-cell of the dual complex. This geometrical property is not apparent in differential formulation.
- They contain *material parameters*.
- They require *metrical notions* such as length, areas, volumes and orthogonality.

While field equations in finite form describe the corresponding physical laws *exactly*, the material ones in finite form describe the corresponding physical laws *approximately* because they are tested only in regions of uniform field.

The material equations became approximate only when applied to a non uniform field.

At this point we have two alternatives:

1. To perform the limit process on the dimensions of the cells in order to obtain uniform field at the limit. This is the traditional way that leads to differential formulation.

2. To assume that the field is uniform inside every cell. This is the only moment in which we perform an approximation. In this way we arrive to obtain an algebraic system of equations useful for computation.

The last choice is the one offered in this paper.

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