On the Geometrical Structure of Electromagnetism

Enzo TONTI *

April 26, 1999

Contents

We show that the fundamental physical quantities of electromagnetism are naturally associated with geometrical elements of space-time such as points and instants (events), time-like lines (intervals), space-like lines, space-like surfaces, etc. This association requires a distinction between an inner and an outer orientations of the geometrical elements. This leads us to analyze physical quantities and equations in a discrete setting

*E. Tonti, "On the Geometrical Structure of Electromagnetism", in *Gravitation, Electromagnetism and Geometrical Structures*, for the 80th birthday of A. Lichnerowicz, Edited by G. Ferrarese, 1995, Pitagora Editrice Bologna, pp.281-308. Author's address: Department of Civil Engineering, University of Trieste, Piazzale Europa 1, 34127 Trieste, Italia. E-mail: tonti@univ.trieste.it

using a space-time cell complex and its dual, describing the distribution of physical variables by means of cochains. Afterwards, using a space-time coordinate system, one may pass to differential formulation. This association leads us to make use of the cohomology theory by means of cochains in a discrete setting and of exterior differential forms in the differential setting.

1 Discrete formulation

We propose to show that one may give a discrete formulation of electromagnetism using the elementary notions of algebraic topology and one may pass from it to the differential formulation via the exterior forms.

The mathematical description of physical phenomena rests upon the existence of quantitative attributes of physical systems that may be described by physical quantities. The descriptive and predictive power of a physical theory depends on the information content of physical quantities. We want to show that physical quantities contain more information than are considered normally. Among them:

- Many physical quantities are associated with four spatial elements, points (P), lines (L), surfaces (S) volumes (V) and with two temporal elements, time instants (I) and intervals (T). Considering space-time elements we may see that they are referred to eight possible combinations of space-time elements: events (PI), space-like lines (LI), time-like lines (PT), space-like surfaces (SI), time-like surfaces (LT), space-like volumes (VI), time-like volumes (ST) and hypervolumes (VT). This suggests an introduction of cell complexes in space-time and the study of physical laws in a discrete setting before going over to a differential setting.

- In this association the notion of inner and outer orientations of a space-time element plays a central role.

- In a discrete setting, if we use a cell complex and its dual, some physical quantities can naturally be associated with cells of a cell complex while others to its dual.

These properties reveal an underlying geometrical structure that pervades all physical theories.

Association with space-time elements. To illustrate further our viewpoint we quote Henry Lebesgue "les grandeurs de la physique directement measurables apparissent d'ailleurs toujours comme de fonctions de domaine; ... il peut s'agir de

domaines sur la droite, c'est a dire d'intervals, de domains plans ou de domains a plus de trois dimensions ..." [1]

This is the case of *global* quantities, i.e. those we usually obtain as line, surface, volume and time integrals of field functions. Table (1) lists the global variables of electromagnetism [2]. We may easily see that every global quantity on the left hand

Table 1: Global physical variables of electromagnetism. The measurable quantities are in boldface

first-kind variables *inner orientation*	*second-kind variables* *outer orientation*
gauge function χ	elec. charge prod. $Q^{prod} = \int_{\tilde{T}} \int_{\tilde{V}} \sigma \ \mathrm{d}\tilde{V} \ \mathrm{d}\tilde{t}$
elec. potential imp. $\mathcal{V} = \int_T V \ \mathrm{d}t$	**elec. charge content** $Q^{cont} = \int_{\tilde{V}} \rho \ \mathrm{d}\tilde{V}$
(no known name) $\pi = \int_L \mathbf{A} \cdot \mathrm{d}\mathbf{L}$	**elec. charge flow** $Q^{flow} = \int_{\tilde{T}} \int_{\tilde{S}} \mathbf{j} \cdot \mathrm{d}\tilde{\mathbf{S}} \ \mathrm{d}\tilde{t}$
elec. tension imp. $\mathcal{U} = \int_T \int_L \mathbf{E} \cdot \mathrm{d}\mathbf{L} \ \mathrm{d}t$	**elec. flux** $\Psi = \int_{\tilde{S}} \mathbf{D} \cdot \mathrm{d}\tilde{\mathbf{S}}$
magnetic flux $\Phi = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S}$	**magn. tens. imp.** $\mathcal{F} = \int_{\tilde{T}} \int_{\tilde{L}} \mathbf{H} \cdot \mathrm{d}\tilde{\mathbf{L}} \, \mathrm{d}\tilde{t}$
magn. charge flow $G^{flow} = \int_T \int_S \mathbf{k} \cdot \mathrm{d}\mathbf{S} \ \mathrm{d}t$	(no known name) $\alpha = \int_{\tilde{L}} \mathbf{T} \cdot \mathrm{d}\tilde{\mathbf{L}}$
magn. charge content $G^{cont} = \int_V g \ \mathrm{d}V$	magn. scalar pot. imp. $\mathcal{V}^{(m)} = \int_{\tilde{T}} V^{(m)} \, \mathrm{d}\tilde{t}$
magn. charge prod. $G^{prod} = \int_T \int_V \tau \, \mathrm{d}V \, \mathrm{d}t$	(no known name) η
SI unit: weber	*SI unit: coulomb*

side has the same physical dimensions as the dimension of the magnetic flux, and therefore the same unit (*weber* in the *SI* system). Also the global quantities of the right hand side have the same dimensions as the dimension of a charge, and therefore they may be all measured in the same unit (*coulomb* in the *SI* system). We remark that the product of the electric charge for the magnetic flux (that has the same dimensions of a magnetic charge) is the Action. The diagram of Tables (5) gives a classification of the global variables of electromagnetism in a discrete setting while Table (4) gives the classification of the usual field functions in the differential setting.

[1] "Physical quantities that are directly measurable arises moreover as functions of domain; ... it may be a line domain, i.e. an interval, a plane domain or a domain of more than three dimensions ..." Lebesgue [18, p.20].

[2] The tilde over the symbol representing the space-time elements denote outer orientation, as will be explained later.

The very fact that these global quantities are expressed by space and time integrals suggests that they have a natural association with the corresponding space-time elements, say points (P), lines (L), surfaces (S), volumes (V), time instants (I), time intervals (T) and various combinations of them.

Inner and outer orientation. A comparison of the integrals on the left and on the right columns of Table (1) shows that the same spatial elements have different kinds of orientations. Therefore, the magnetic flux Φ is associated with a surface element and *inner* orientation, i.e. with a prescribed direction along its boundary. On the contrary, the surfaces regarding the integrals of the right hand side such as the one of the electric flux Ψ, require an *outer* orientation, i.e. a specific direction across the surface element from one side to the other.

The same is for the electric and magnetic tension impulses [3]: the first is referred to lines endowed with an inner orientation, while the second with lines endowed with outer orientation. To prove the last statement, let us remark that in Ampère law the change of the current direction in a circuit implies a change of the magnetic tension sign: this means that an outer orientation of the line is involved in the definition of the magnetic tension. The association of global electromagnetic variables with space and time elements is shown in Fig.(3). The six quantities are:

Table 2: The six global variables of electromagnetism.

$\mathcal{U}(\boldsymbol{\tau}_n, \mathbf{l}_\alpha)$	electric tension impulse	weber
$\Phi(\mathbf{t}_n, \mathbf{s}_\beta)$	magnetic flux	weber
$\mathcal{F}(\tilde{\boldsymbol{\tau}}_n, \tilde{\mathbf{l}}_\alpha)$	magnetic tension impulse	coulomb
$\Psi(\tilde{\mathbf{t}}_n, \tilde{\mathbf{s}}_\beta)$	electric flux	coulomb
$Q^{flow}(\tilde{\boldsymbol{\tau}}_n, \tilde{\mathbf{s}}_\beta)$	electric charge flow	coulomb
$Q^{cont}(\tilde{\mathbf{t}}_n, \tilde{\mathbf{v}}_k)$	electric charge content	coulomb

The fact that global electromagnetic quantities are associated with different kind of orientation can be inferred from their operative definition illustrated in Fig.(1). As shown in a) and b), we first measure the force and then we evaluate the work W^* for every virtual displacement L. The electric tension U along the line L is then defined as the virtual work W^* per unit charge and the magnetic flux Φ as the virtual work per unit current [4],

$$U(L) \stackrel{\text{def}}{=} \frac{W^*}{q} \qquad\qquad \Phi(S) \stackrel{\text{def}}{=} \frac{W^*}{i} \tag{1}$$

[3] We use the term "tension" as synonymous of "voltage".

[4] See Langevin [16, p.494].

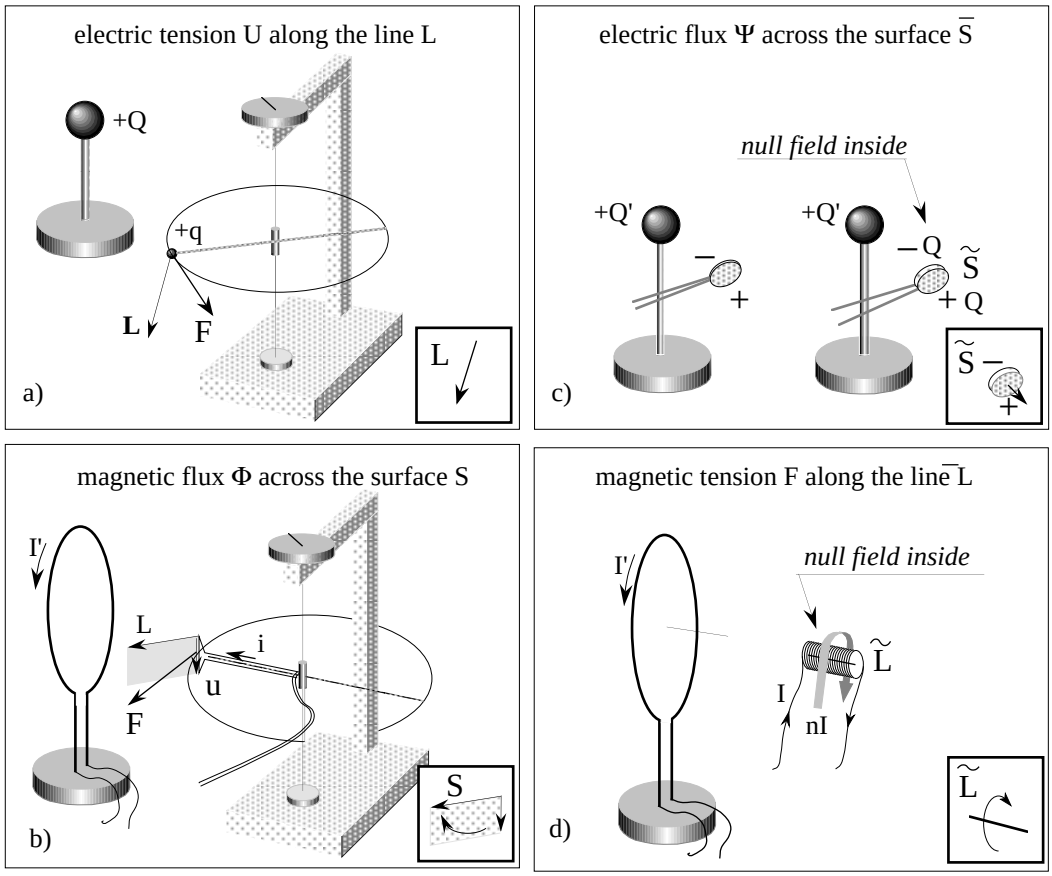

Figure 1: The operative definition of global electromagnetic quantities.

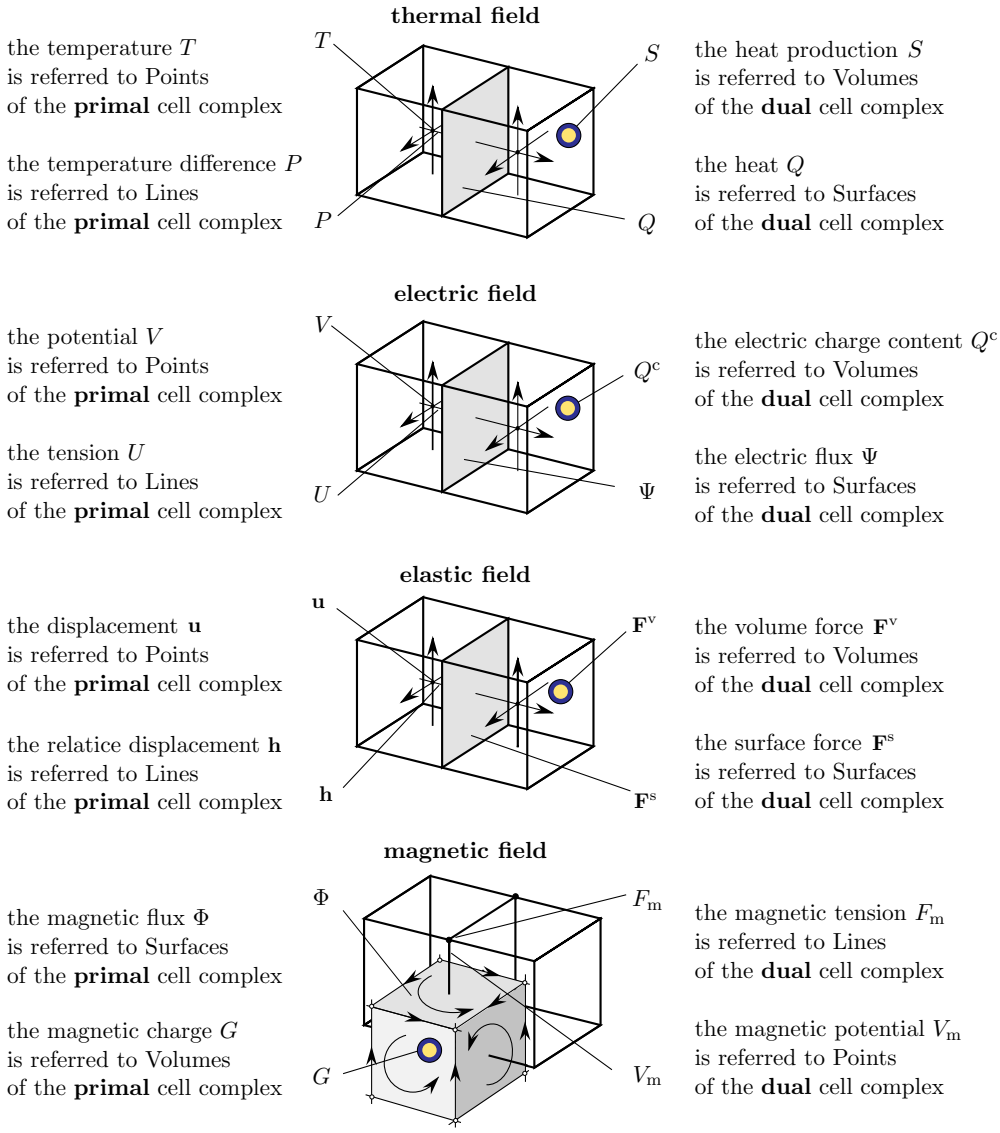

Figure 2: The root of analogies lies in the association of global variables to space and time elements. Heavy lines denote the primary complex, light lines the dual.

where q denotes the test charge and i the test current [5]. If we change the inner orientation of L then both W^* and U change sign. The corresponding definition of the magnetic tension and of the electric flux are, see Fig.(1) c) and d)

$$F(\tilde{L}) = n\, I \qquad\qquad \Psi(\tilde{S}) = Q, \tag{2}$$

where I denotes the current that nullifies the field inside the test coil [6] and Q denotes the charge induced on the positive face [7] of the surface element that nullifies the electric field inside the region bounded by two discs [8]. Here I and Q are measured, while in the Eq. (1) the corresponding quantities i and q are assigned and the virtual work is evaluated after the force has been measured.

It is important to note that some of the physical quantities are associated with cells of a given complex, while others are to its dual [9]. To explain this let us consider a thermal field. If the two cells of Fig.(2) are conceived as two rooms it is natural to assign the internal energy U to both rooms and the energy flow (heat) Q to the separating wall. It is natural also to associate the temperature of each room with their barycenters respectively. Consequently the temperature difference P is asssociated to the lines connecting the two barycenters. In doing so we notice that both cell complexes (primary and dual) are involved in the association of physical variables with the spatial elements. It is remarkable that the constitutive equation of Fourier links the variables Q and P that are associated with a pair of dual elements.

This property is common to many physical theories as may be seen in the other examples of Fig.(2).

The variables associated with a p-cell and to its dual (n-p)-cell are conjugated in the sense that their product gives energy (an action in a space-time cell complex). The prototype of conjugated variables are force and displacement. Force is a covector (or form) while displacement is a controvariant vector. Usually we consider force as an element of the dual space with respect to the space of displacements (tangent and cotangent spaces of differential geometry). We see now (see Fig.(2), last line) that the volume forces ar associated with volumes of the dual cell complex, while the displacements are associated with the points of the primary cell complex.

We find an interesting property: physical variables, which are dual in the sense of vector spaces are associated with dual cells when considered in the physical space or in space-time. Moreover, constitutive (or material) equations in physical space or in space-time, are relations between physical variables associated with pairs of dual cells. Such equations are usually described as mapping between a vector space and

[5] See [22, p.148].

[6] See Langevin [16, p.494]; Fouillé [9, p.224]; Pohl [25, p.66]; Schelkunoff [27, p.41].

[7] The arbitrariness of the choice of a face as positive is the same of the choice of a positive face in the definition of surface force in a material continuum.

[8] See [9, p.71]; Fleury-Mathieu [7, p.61]; Maxwell [21, p.47]; Rojansky [26, p.230]; Schelkunoff [27, p.25]; Jefimenko [14, p.80; p.225].

[9] See Branin [5].

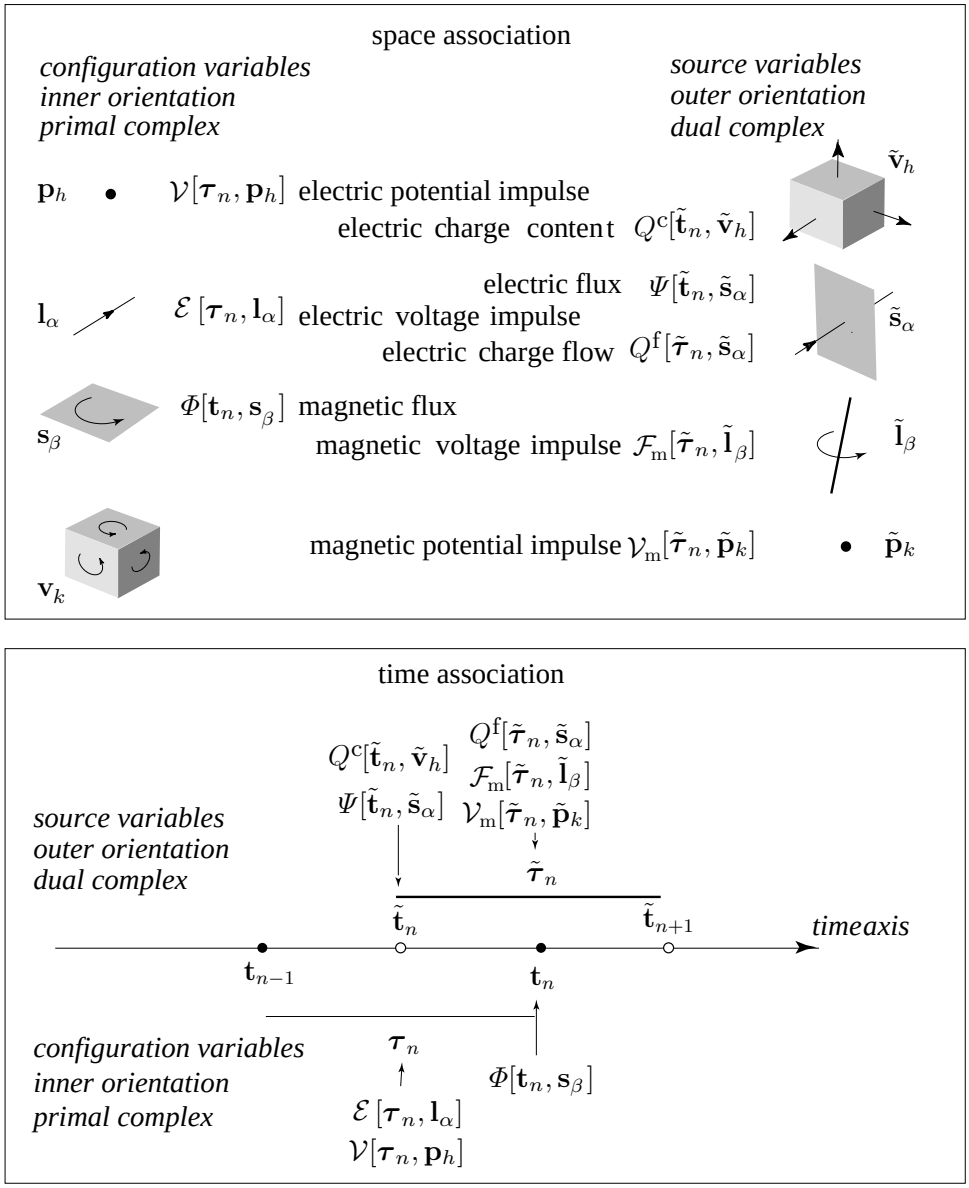

Figure 3: The six global variables of electromagnetism and the corresponding space and time elements.

its dual.

All these remarks bring us to give a precise meaning to the notion of inner and outer orientation of a spatial element. Before doing this we must ask ourselves if there is any natural framework that may emphasize the spatial elements and their orientations. Such a natural framework exists if we use the notion of cell complexes from algebraic topology, in particular homology theory [10].

2 Cell complexes

To analyze the geometrical background of physical quantities we find useful to work in a discrete setting using a cell complex: this will play the role of the coordinate system used in the differential setting. To do this we consider a domain of the physical space as being an affine space and subdivide it into subdomains in contiguous cells of whatever shape and dimensions. Vertices, edges, faces and volumes are called respectively 0-cells, 1-cells, 2-cells and 3-cells. The edges and the faces may be straight or curved: we shall consider cell complexes with straight edges and plane faces. In Fig.(4) we show two cell complexes in a two-dimensional space. While in algebraic topology one consider simplicial complexes, in physics it is preferable to use cell complexes formed by parallelotopes because they make easy the passage to the differential formulation. For the numerical solution of physical problems, on the contrary, simplicial complexes are more flexible because they permits to fit better the boundaries of the region and to do local refinements in the mesh.

A simplicial complex in two dimensions is said a Delaunay complex if the circle passing from the three vertices of a simplex does not contain any other vertices of the complex. This condition assures that the circumcenters of two adjacent simplexes, even if not contained in their respective cells, are not inverted. Analogous definition is valid in $\mathbb{E}^3$ taking the circumsphere. If in a Delaunay complex we choose as dual vertices the circumcenters of its simplexes, the corresponding 1-cells connecting these vertices are orthogonal to the $(n-1)$ cells of the Delaunay complex. One obtain in this way a couple of Delaunay-Voronoi complexes [11], as shown in Fig.(**??**right) that are, in a discrete setting, the equivalent of an orthogonal coordinate systems in the differential setting.

The notion of inner orientation is of combinatorial nature: it arises when we consider the simplex of $\mathbb{E}^n$ and the possible ordering of its vertices. So if A, B, C denote the vertices of the simplex of $\mathbb{E}^2$, the arrangements $\langle A, B, C\rangle$, $\langle B, C, A\rangle$ and $\langle C, A, B\rangle$, which differ for an *even* number of permutations of the sequence, define an (inner) orientation while all arrangements that differs for an odd number of permutations define the opposite orientation.

[10] The idea of considering the association of physical variables with the cells of a cell complex and its dual was introduced by Branin, an IBM engineer, in 1966 [5].

[11] See Frey and Cavendish [11].

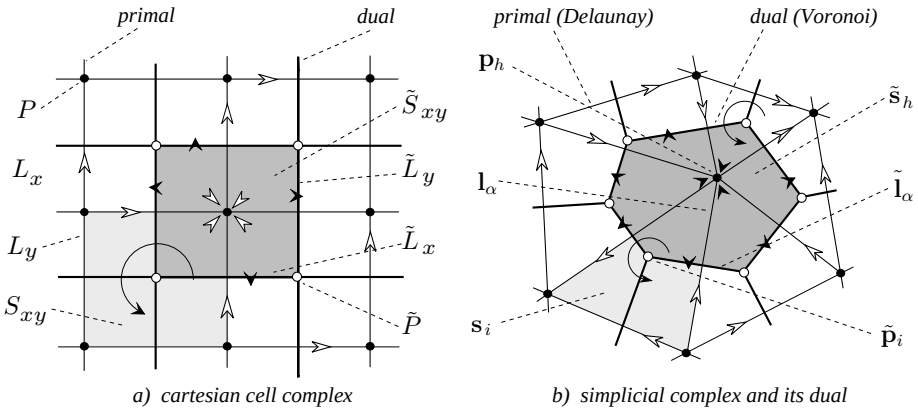

Figure 4: A two-dimensional cell complex (thick lines) and its dual (thin lines). In the triangular complex the vertices of the dual complex are the intersections of the three axes of the primary 1-cells, instead of barycenters used in algebraic topology. This gives the advantage that 1-cells of dual are orthogonal to primary 1-cells.

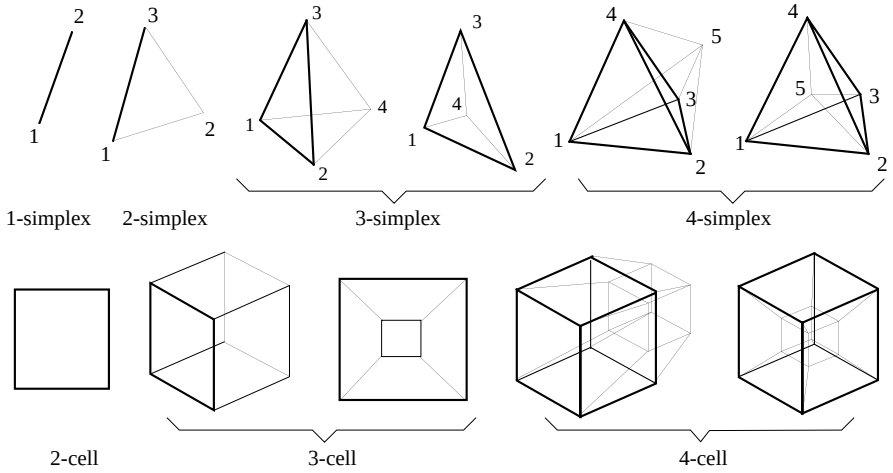

Figure 5: The p-cell of a simplicial complex (above) and of a cell complex (below): the last two pictures of every line shows the three-dimensional projection of a four-dimensional simplex and cube respectively.

In order to define the orientation of the p-cell we decompose the p-cell into p-simplexes, we orient one simplex and then we propagate the orientation to the other simplexes of the cell. The 0-cells are oriented when all are considered "sources" or "sinks". The notion of source and sink, borrowed from electromagnetism and fluid dynamics, may be used to define an inner orientation of points because it permits to maintain the notion of incidence number from a $(p+1)$-cell and a p-cell also when $p = 0$.

In an affine space if one introduce a rectangular coordinate system one may consider as 3-cells the regions delimited by the coordinate planes. The edges of such cells have as natural orientation the one of coordinate lines and the faces have the orientation of the coordinate planes, say xy, yz, zx as shown in Fig.(6).

In a cell complex all cells of a same dimension p are numbered according with some convenience criterion.

2.1 Dual cell complex

Given a cell complex in a 3-dimensional space we are led to introduce the dual cell complex defined as follows. We start by considering a point inside every 3-cell: in particular one may consider its barycenter or its circumcenter. The 1-cells of the dual are the lines connecting the 0-cells contained in two adjacent 3-cells. The 2-cells of the dual are the surfaces delimited by the 1-cells of the dual. The 3-cells of the dual are the volumes delimited by the 2-cells of the dual. Then for every p-cell of the primary complex there is a corresponding (n-p)-cell of the dual. The correspondence consists in the fact that a primary p-cell contains or crosses or is contained in an (n-p)-cell of the dual. This one-to-one mapping permits us to assign the same label to a p-cell and the dual (n-p)-cell.

The one-to-one mapping between primary p-cells and (n-p)-cells of the dual permits us to define the notion of outer orientation. Following Veblen and Whitehead [12] we call *outer orientation* of a p-cell of the dual cell complex $\tilde{K}$ the inner orientation of the corresponding (n-p)-cell of the primary complex K. With this definition we have a systematic way of defining outer orientation of the p-cells of the dual. In particular we note that points are usually oriented as *sinks*. This is never explicitly stated but may be inferred from the fact that the space differences of a point function between two points P and Q is given by $(+1)f(Q) + (-1)f(P)$. This means that the line segment PQ, oriented from P to Q, is positively incident in Q (incidence number +1) and negatively incident in P (incidence number -1). In other words: the signs of a difference may be interpreted as incidence numbers between the orientation of the line segment and those of its terminal points.

In physics the fact that points are orientes as sinks conflicts with the traditional convention of outgoing normals for volumes and this justifies the omnipresent minus sign in front of the gradient, as in the relation of electrostatics $\mathbf{E} = -\nabla\varphi$. In fact we

[12] See [34, p.56]

have defined as outer orientation of a p-cell of the dual complex the inner orientation of the corresponding $(n\text{-}p)$-cell of the primary. If the 0-cells of the primal (points) are oriented as sinks and the corresponding 3-cell of the dual (volume) is oriented with outwards normals, then the outer orientation of the 3-cell is *opposite* to the inner orientation of the 0-cell of the primal.

A cell complex and its dual permit us to classify the spatial elements of a manifold, as shown in Fig.(6).

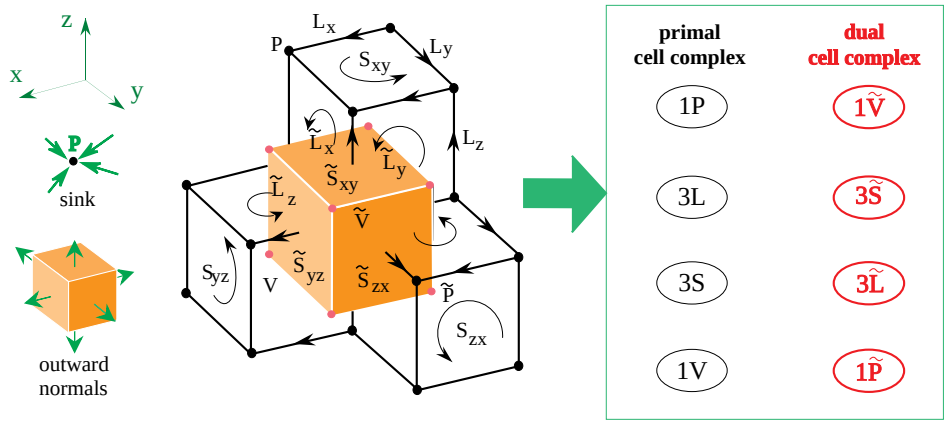

Figure 6: A classification of the spatial elements of a cell complex and its dual in a three-dimensional affine space. The numbers refers to the families: so $3L$ stands for (L_x, L_y, L_z) while $3S$ stands for (S_{xy}, S_{yz}, S_{zx}).

In the study of a physical theory, say thermodynamics, electromagnetism, elasticity, fluid dynamics, etc. if one introduces a cell complex and its dual one may associate the global variables of such theory with the cells of various orders of the cell complex, i.e. to the 0-cells (points, nodes, vertices), 1-cells (lines, edges) to the 2-cells (surfaces, faces) to the 3-cells (volumes, elements, cells).

So in space we have

potentials (like temperature)	are associated with 0-cells
tensions (like voltage)	are associated with 1-cells
fluxes (like energy flux)	are associated with 2-cells
contents (like entropy content)	are associated with 3-cells.

while in space-time we have

functions (like gauge function)	are associated with 0-cells (events)
potential impulses (like $\int V\,\mathrm{d}t$)	are associated with time-like 1-cells
circulations (like $\int \mathbf{A}\cdot\mathrm{d}\mathbf{L}$)	are associated with space-like 1-cells
tension impulses (like $\int U\,\mathrm{d}t$)	are associated with time-like 2-cells
fluxes (like magnetic flux)	are associated with space-like 2-cells
flows (like energy flow)	are associated with time-like 3-cells
contents (like mass content)	are associated with space-like 3-cells
productions (like entropy prod.)	are associated with time-like 4-cells.

It follows that the variables associated with inner oriented elements are naturally associated with the cells of the given cell complex K while those associated with outer oriented elements are naturally associated with the cells of the dual cell complex $\tilde{K}$. These remarks lead to a general classification diagram of the physical quantities of whatever physical theory as Tables (5) and (4).

To analyze the geometrical content of physical equations we summarize the notions of *chain, boundary, cochain and coboundary*

2.2 Incidence numbers

The (p-1)-faces of a p-cell are those (p-1)-cells that are incident on the given p-cell. Given a p-cell consider all (p+1)-cells which have a p-cell as common face. These will

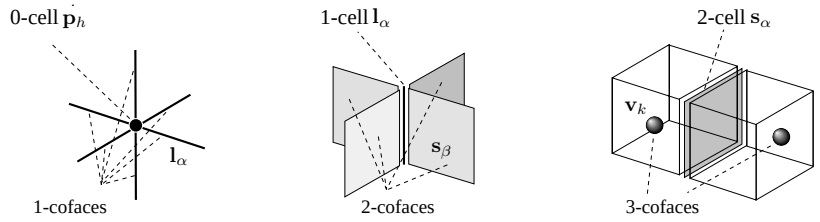

Figure 7: The coboundary of a p-cell defined as the set of all cofaces.

be called *cofaces* of the p-cell as shown in Fig.(7). Since the set of all faces of a p-cell form the *boundary* of a p-cell, therefore the set of all cofaces form its *coboundary*.

We are now in a position to define the *incidence number* of a p-cell $\mathbf{c}_h$ with a (p-1)-cell $\mathbf{b}_k$. This is a relative integer $\eta_{hk} = [\mathbf{c}_h : \mathbf{b}_k]$ whose values are:

- +1 if $\mathbf{b}_k$ is a face of $\mathbf{c}_h$ and the orientations of $\mathbf{b}_k$ and $\mathbf{c}_h$ are compatible;
- -1 if $\mathbf{b}_k$ is a face of $\mathbf{c}_h$ and the orientation of $\mathbf{b}_k$ and $\mathbf{c}_h$ are not compatible;
- 0 if $\mathbf{b}_k$ is not a face of $\mathbf{c}_h$.

We remark that the order of cells in the square brackets is irrelevant, i.e. $[\mathbf{c}_h : \mathbf{b}_k] = [\mathbf{b}_k : \mathbf{c}_h]$, while in the notation η_{kh} *the first index refers to the cell of greatest*

dimension [13]

In the three-dimensional space there are three matrices which we shall denote as $\mathbf{G}, \mathbf{C}, \mathbf{D}$ for the primary complex K and three matrices $\tilde{\mathbf{G}}, \tilde{\mathbf{C}}, \tilde{\mathbf{D}}$ for the dual complex $\tilde{K}$.

From Fig.(8) one may see an important fact that *the incidence number between a p-cell and a (p-1)-cell of the primary cell complex is equal to the incidence number between the corresponding dual cells*. Note that the indices of the matrix elements $\tilde{d}_{h\sigma}$ and $g_{\sigma h}$ are reversed and then the corresponding matrices are adjoint to one another. We have

$$\begin{cases} g_{\alpha h} \stackrel{\text{def}}{=} [\mathbf{l}_\alpha : \mathbf{p}_h] = [\tilde{\mathbf{s}}_\alpha : \tilde{\mathbf{v}}_h] = \tilde{d}_{h\alpha} & \mathbf{G} = \tilde{\mathbf{D}}^{\mathrm{T}} \\ c_{\beta\alpha} \stackrel{\text{def}}{=} [\mathbf{s}_\beta : \mathbf{l}_\alpha] = [\tilde{\mathbf{l}}_\beta : \tilde{\mathbf{s}}_\alpha] = \tilde{c}_{\alpha\beta} & \mathbf{C} = \tilde{\mathbf{C}}^{\mathrm{T}} \\ d_{k\beta} \stackrel{\text{def}}{=} [\mathbf{v}_k : \mathbf{s}_\beta] = [\tilde{\mathbf{p}}_k : \tilde{\mathbf{l}}_\beta] = \tilde{g}_{\beta k} & \mathbf{D} = \tilde{\mathbf{G}}^{\mathrm{T}} \end{cases} \tag{3}$$

Since the usual convention about points is that they are oriented as sinks while volumes are oriented as sources it follows that $\tilde{d}_{h\alpha} = -g_{\alpha h}$. This fact is the basis of all adjointness relations between the differential operators of field theories. In particular it may be shown that the adjointness relation between the divergence and the minus gradient, i.e.

$$\int_V f \ \mathrm{div}\ \mathbf{v}\ dV = \int_V \mathbf{v} \cdot (-\mathrm{grad} f)\ dV + \int_{\partial V} \frac{\partial f}{\partial n}\ \mathbf{v} \cdot \mathbf{n}\ dS \tag{4}$$

is the differential analog of the algebraic relation $\tilde{d}_{h\alpha} = -g_{\alpha h}$.

2.3 Chains

Let us consider an aggregate formed by a set of p-cells $\mathbf{c}_k$, each with a definite orientation and with a definite multiplicity, and a set of number n_k describing the orientation and multiplicity of the cell $\mathbf{c}_k$ in the set. One may represent the members of this aggregate by couples $(\mathbf{c}_k, n_k)$ and call every couple an *elementary chain*. This aggregate is an algebraic entity called *p-dimensional algebraic chain* with integer coefficients. Briefly one call this a *p-dimensional chain*, or *p-chain*. It is an essential requirement of a chain that if the orientation of one cell of the chain is inverted then the corresponding coefficient in the chain is changed to its opposite, i.e.

$$\text{if} \qquad \mathbf{c}_k \to -\mathbf{c}_k \qquad \text{then} \qquad n_k \to -n_k \tag{5}$$

This property is called the *oddness condition*.

[13] See Alexandrov [2, p.275]; Franz [10, p.30]; Patterson [23, p.103]; Hocking and Young [13, p.223]; Lefschetz [20, p.99]. The reader should be aware that some authors use the opposite convention for the order of indices in η_{hk}.

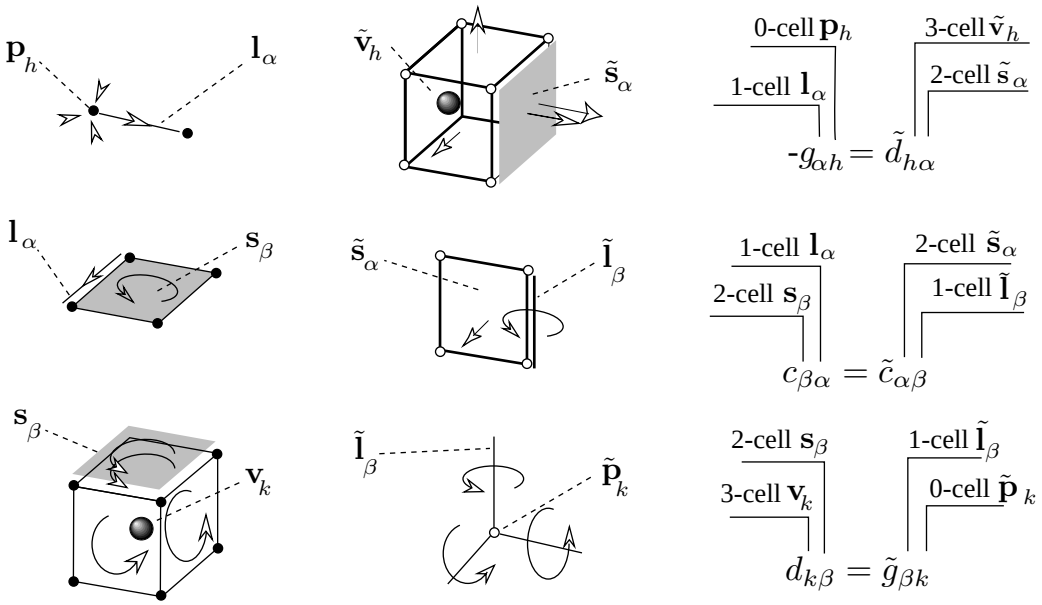

Figure 8: The incidence numbers of a pair of cells are equal to those of the dual pair.

Denoting with $\mathbf{c}_k$ the k-*th* p-dimensional cell of K and with $\tilde{\mathbf{c}}_k$ the k-*th* p-dimensional cell of $\tilde{K}$ a p-chain may be written as a formal linear combination of the p-cells, [14].

$$\mathbf{C}_p = \sum_k n_k \mathbf{c}_k^p \qquad \tilde{\mathbf{C}}_p = \sum_k n_k \tilde{\mathbf{c}}_k^p \tag{6}$$

according to whether the cells belong to the primary or dual complex. We shall distinguish chains on a cell complex from those on its dual by putting a tilde over the cell symbol and the chains symbol of the dual [15]. An oriented p-cell with associated coefficient $+1$ may be considered as the smallest p-chain and it is called an *elementary p-chain* [16].

While the p-cells are point sets and then topological entities, an elementary p-chain is an algebraic entity [17]. Point sets cannot be added, elementary chains can be added. The sum of two p-chains is defined as the p-chain whose coefficients are the sum of the corresponding coefficients of the given p-chains. In particular, the p-chain whose coefficients are all vanishing is called a *null* p-chain denoted as Θ_p.

[14] Seifert [29, p.61]) Franz [10, p.31], Hilton [12, p.56] Hocking [13, p.226]; Patterson [23, p.117] Alexandrov [1, p.18]; Alexandrov [2, p.264]; Wallace [36, p.105]; Patterson [23, p.117]; Hilton [12, p.58].

[15] This notation is new because books on algebraic topology do not consider chains on the dual cell complex.

[16] Franz [10, p.31]; Hocking [13, p.297].

[17] Alexandrov [2, p.20].

As a consequence of the definition of the sum of (algebraic) chains, the p-chains on a cell complex K form an additive group denoted $\mathcal{C}_p(K)$, called the *p-dimensional chain group*. Since there are two cell complexes, the primary K and the dual $\tilde{K}$, there are two p-dimensional chain groups $\mathcal{C}_p(K)$ and $\mathcal{C}_p(\tilde{K})$. Since the elements of these groups may also be multiplied by relative integers, the groups have a richer structure. They are moduli over the ring of integers $\mathbf{Z}$. Observe that they are not vector spaces because the integers do not form a field as reals do.

Remark. One may also give a more general definition of a p-chain as a function that assign to every oriented p-cell an element of a commutative group (additively written) [18]. When the coefficients belong to an arbitrary additive group the intuitive geometrical content is lost. If one wants to maintain a geometrical content, as we do, the group must be that of integers [19]. For this reason we shall consider only the relative integers as coefficient group.

2.4 Boundary of a chain

The boundary of a cell is a point set of one dimension lesser. An analogous notion may be defined on a chain. Let us denote with

$$\mathbf{C}_p = \sum_k n_k \mathbf{c}_k \qquad \text{and} \qquad \mathbf{B}_{p-1} = \sum_h m_h \mathbf{b}_h \tag{7}$$

two chains of dimensions p and (p-1) respectively. With reference to Fig.(9) we may define the *boundary* of a p-chain C_p as the (p-1)-chain B_{p-1} whose coefficients m_h are the sum of the coefficients n_k of the cofaces of the (p-1)-cell $\mathbf{b}_k$, each multiplied by the corresponding incidence number

$$m_h \stackrel{\text{def}}{=} \sum_k n_k \, [\mathbf{c}_k : \mathbf{b}_h]. \tag{8}$$

This is represented symbolically as

$$\mathbf{B}_{p-1} = \partial \mathbf{C}_p. \tag{9}$$

The process of forming the boundary is called the *boundary process* and the operator ∂ is called the *boundary operator*.

We see that the boundary operator is a linear mapping with integer coefficients between the group of p-chains and that of (p-1)-chains, i.e. $\partial : \mathcal{C}_p \to \mathcal{C}_{p-1}$.

A closed line is a line with no boundary; its algebraic description is a closed chain, i.e. a chain such that $\partial \mathbf{C} = \Theta_{p-1}$. This is called p-dimensional *cycle*. The obvious fact that the boundary of a surface is a closed line may be restated as "the boundary of a surface has no boundary". In the same way "the boundary of a volume has no

[18] Hocking [13, p.225].

[19] Hilton [12, p.58]; Bourgin [4, p.22]; Wallace [37, p.6].

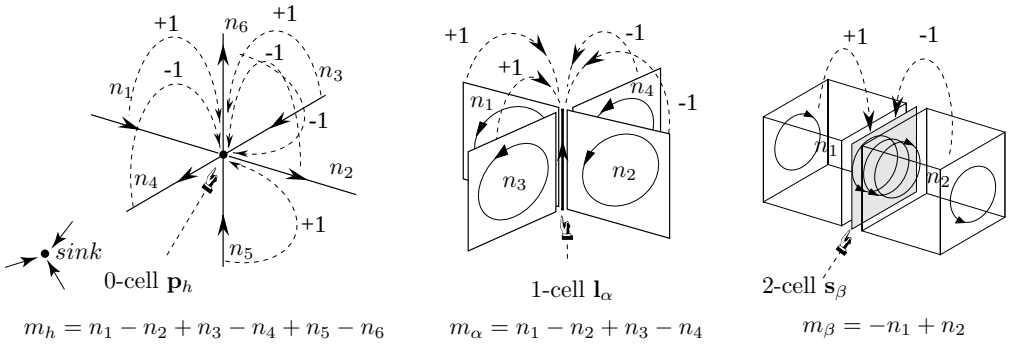

Figure 9: The geometric description of the boundary process on p-dimensional chain.

boundary". In the language of chains it is said that *the boundary of the boundary of a chain is the null chain*

$$\partial\partial\mathbf{C}_p \equiv \Theta_{p-2} \tag{10}$$

where Θ_{p-2} is the null $(p\text{-}2)$-chain. This peculiar identity plays a central role in the theory of chains, and is the algebraic formulation of a primitive topological property. The identity (10) may be expressed by the relations [20]

$$\sum_k [\mathbf{d}_i : \mathbf{c}_k][\mathbf{c}_k : \mathbf{b}_h] \equiv 0. \tag{11}$$

Referring to Fig.(8) and remembering Eq.(3), we may write

$$\begin{cases} \sum_\alpha [\mathbf{s}_\beta : \mathbf{l}_\alpha][\mathbf{l}_\alpha : \mathbf{p}_h] = \sum_\alpha c_{\beta\alpha}\ g_{\alpha h} \equiv 0 \\ \sum_\beta [\mathbf{v}_k : \mathbf{s}_\beta][\mathbf{s}_\beta : \mathbf{l}_\alpha] = \sum_\beta d_{k\beta}\ c_{\beta\alpha} \equiv 0 \end{cases} \tag{12}$$

or

$$\mathbf{CG} = \mathbf{0} \qquad \mathbf{DC} = \mathbf{0}. \tag{13}$$

As we shall show these relations are the discrete counterpart of the differential identities

$$\text{curl grad } f \equiv 0 \qquad \text{div curl } \mathbf{v} \equiv 0 \tag{14}$$

and it is for this reason that we have used the letters $\mathbf{G}, \mathbf{C}, \mathbf{D}$ that are the initials of the words "gradient", "curl" and "divergence".

2.5 Cochains

The fact that global physical variables are associated with p-dimensional submanifolds of an n-dimensional manifold (say potentials to points, tensions to lines, fluxes to

[20] Hocking [13, p.224].

surfaces, contents to volumes in the three-dimensional space) implies that with every p-cell and every p-chain one may associate a certain value of the global variable. This mapping gives the distribution of the physical variable on the submanifolds.

Such a mapping is called *p-cochain* in algebraic topology. So the potential of a vector field gives rise to a 0-dimensional cochain; the circulation of a vector to a 1-dimensional cochain; the flux to a 2-dimensional cochain and the contents to a 3-dimensional cochain.

The notion of a p-cochain of a global physical variable is the generalization of the notion of field function used in the differential setting. As a field function associates an element of a vector space with every point, so a 0-dimensional cochain is a function that associates an element of a vector space to every 0-dimensional cell. A p-cochain is a set function.

Thus pressure is a scalar field function while its corresponding global variable, i.e. the normal surface force, is a 2-dimensional vector valued cochain defined on the dual cell complex because it is associated with the outer orientation of the surface.

A p-cochain f^p on a cell complex is a function that with every p-chain $\mathbf{C}_p$ of a cell complex, associates an element of an additive group $\mathbf{G}$ i.e. one that satisfies the additive property [21]

$$f^p\left(\mathbf{C}_p + \mathbf{C}'_p\right) = f^p(\mathbf{C}_p) + f^p(\mathbf{C}'_p). \tag{15}$$

One may say that a p-dimensional cochain is an homomorphism of the group of chains $\mathcal{C}_p$ on the group $\mathbf{G}$ [26] or a linear mapping with integer coefficients of the group $\mathcal{C}_p$ on the group $\mathbf{G}$, i.e. $f^p : \mathcal{C}_p \mapsto \mathbf{G}$ [27]. In particular, the group $\mathbf{G}$ may be a vector space of real or complex numbers, a function space, or it may be an algebra, like matrix and Clifford algebras.

Denoting with $\mathbf{f}$ the value of the p-cochain f^p on the chain $\mathbf{C}_p$, we may write

$$\mathbf{f} = f^p(\mathbf{C}_p). \tag{16}$$

A peculiar property of a p-cochains is that the value $\mathbf{f}$ associated with a p-chain changes its sign when its orientation is inverted. This comes from Eq. (15) by putting $\mathbf{C}'_p = -\mathbf{C}_p$. This is the *oddness condition* of p-cochains that corresponds to the analogous condition on chains. We then have

$$f^p(-\mathbf{C}_p) = -f^p(\mathbf{C}_p). \tag{17}$$

Moreover, to the null p-chain Θ_p there corresponds the null element Θ of the group $\mathbf{G}$.

[21] There are many notations for cochains in books of algebraic topology: the most common are $c^p(d_p)$ [22] and $c^p \cdot d_p$ [23]; (d_p, c^p) [24]; (c^p, d_p) and $(c_p)c^p$ [25].

[26] Franz [10, p.42-46].

[27] Dubrovin [6, p.32]; Hocking [13, p.300].

A p-cochain is assigned when we allot to every elementary p-chain $\mathbf{c}_k$ the quantity $\mathbf{f}_k = f^p(\mathbf{c}_k)$. From Eq. (15) it follows that

$$\mathbf{f} = f^p(\mathbf{C}_p) = f^p\left(\sum_k n_k \mathbf{c}_k\right) = \sum_k n_k f^p(\mathbf{c}_k) = \sum_k n_k \mathbf{f}_k \tag{18}$$

i.e. the value $\mathbf{f}$ of a p-cochain f^p on a p-chain $\mathbf{C}_p$ is the sum of the values $\mathbf{f}_k$ of f^p on the cells, each multiplied by the coefficient of the cell in the chain.

The introduction of a cell complex in the space considered permits us to select some p-dimensional domains, formed by a set of p-cells. The algebraic entity that describe them are p-chains. Corresponding to points and coordinate systems used in traditional physics the cells and their labels are used in our description.

In space-time, *circulations* (line integrals) are described by 1-cochains; *fluxes* and *tension impulses* by 2-cochains; *flows* are described by 3-cochains; *productions* (e.g. entropy production) by 4-cochains.

2.6 Coboundary of a p-cochain

The *coboundary process* on a p-cochain generates a $(p+1)$-cochain. It is remarkable that this process plays a key role in physics because balance, circuital and difference equations may be expressed by the coboundary process performed on 3-, 2-, 1-dimensional cochains, respectively. This process in the differential setting is the exterior differentation on exterior p-forms and leads to typical operators such as *grad*, *curl*, *div*, along with time derivatives [28].

We present the coboundary process in a geometrical language which makes it very simple as compared to the corresponding analytical definition of exterior derivative on p-forms. With reference to Fig.(10), we shall introduce the following notion:

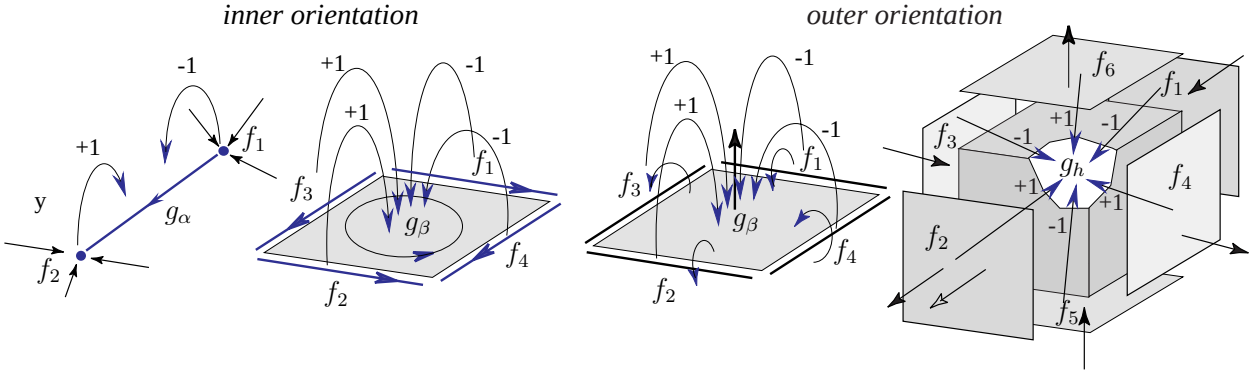

Figure 10: The geometric description of the coboundary process on p-dimensional cochains.

[28] Hilton [12, p.72].

Given a (p -1)-cochain f^{p-1}, we may obtain a p -cochain g^p by the following process: we associate with every p -cell $\mathbf{c}_k$ the sum of the products of the quantities $f^{p-1}(\mathbf{b}_h)$ associated with its (p-1)-faces each multiplied by the relative incidence number. In this way we associate with every p -cell the quantity $g^p(\mathbf{c}_k)$ given by

$$g^p(\mathbf{c}_k) \stackrel{\text{def}}{=} \sum_h [\mathbf{c}_k : \mathbf{b}_h]\ f^{p-1}(\mathbf{b}_h). \tag{19}$$

represented symbolically as

$$g^p = \delta f^{p-1}. \tag{20}$$

The process so performed is called the coboundary process and the operator δ is called the coboundary operator.

Let us consider a chain $\mathbf{C}_p$ and its boundary $\mathbf{B}_{(p-1)} = \partial \mathbf{C}_p$. Remembering Eq. (8) we have the following fundamental relation

$$\begin{aligned}(\delta f)^p(\mathbf{C}) = g^p(\mathbf{C}) = g^p\left(\sum_k n_k \mathbf{c}_k\right) = \sum_k n_k g^p(\mathbf{c}_k) = \sum_k \left\{ n_k \sum_h [\mathbf{c}_k : \mathbf{b}_h] f^{p-1}(\mathbf{b}_h)\right\} \\ = \sum_h \left\{ \sum_k n_k [\mathbf{c}_k : \mathbf{b}_h]\right\} f^{p-1}(\mathbf{b}_h) = \sum_h m_h f^{p-1}(\mathbf{b}_h) = f^{p-1}\left(\sum_h m_h \mathbf{b}_h\right) = f^{p-1}(\partial \mathbf{C}).\end{aligned} \tag{21}$$

This means that as the boundary process on a chain lowers the degree of the chain by one unit so the coboundary process on a cochain raises the degree of the cochain by one unit.

As an example, with reference to Fig.(10), the coboundary of a 1-, 2- and 3-cochain assigns to a 1-, 2- and 3-cell respectively the quantities

$$\mathbf{g} = +\mathbf{q} - \mathbf{p} \qquad \mathbf{g} = (\mathbf{p} - \mathbf{r}) + (\mathbf{q} - \mathbf{s}) \qquad \mathbf{g} = (\mathbf{p} - \mathbf{s}) + (\mathbf{q} - \mathbf{t}) + (\mathbf{r} - \mathbf{u}) \tag{22}$$

This example is very important because it corresponds to three common processes of physics, those of forming the difference of a potential between two points, the circulation of a vector along a closed line and the sum of the fluxes across a closed boundary.

2.7 Discrete Stokes theorem

Equation (21) may be rewritten as

$$(\delta f)^p(\mathbf{C}) = f^{p-1}(\partial \mathbf{C}). \tag{23}$$

Stated in words: *the value of the coboundary of a (p -1)-dimensional cochain on a p -dimensional chain is equal to the value of the cochain on the (p -1)-dimensional boundary of the chain* [29].

[29] Dubrovin [6].

Relation (23) is the algebraic form of Stokes theorem and is called "generalized" or "combinatorial" form of Stokes theorem [30]. This shows that the definition of coboundary process is such that Stokes theorem be an immediate consequence of it.

This also shows that Stokes theorem *is a purely topological relation* [31] and that the various continuity and differentiability conditions usually required in its proof depend on the fact that one uses the field functions. This is a typical situation in physics: *many continuity and differentiability requirements are not ineherent physical laws but are imposed by the differential tools used in their description.*

The notion of coboundary operator which raises the degree of a cochain by one unit, corresponds to the notion of exterior differential which raises the degree of an exterior differential form by one unit [32].

From the identities (23) and (10) we obtain the important identity

$$(\delta\delta f)^{p+1}(\mathbf{C}_{p+1}) \equiv (\delta f)^{p}(\partial\mathbf{C})_{p} \equiv f^{p-1}(\partial\partial\mathbf{C})_{p-1} \equiv f^{p-1}(\Theta_{p-1}) \equiv 0. \tag{24}$$

It means that when the coboundary process is performed twice in sequence it gives rise to the null element of group $\mathbf{G}$. This is the algebraic root of the differential identities, $\nabla\times\ \nabla \equiv 0$ and $\nabla\cdot\ \nabla\times \equiv 0$. Physical laws in their original form, as inferred from experiments, naturally involve *global variables* associated with spatial and temporal elements and *not* field functions. The forming of densities and rates and then the passage to the limit to form the field functions that is typical of field and continuum theories, deprives physical variables of their geometrical content. As far as physics is concerned *it appears natural to utilize in a discrete setting the p-dimensional cochains to describe the global variables associated with p-dimensional manifolds.* It is enough to cover the space-time region with a cell complex K and its dual $\tilde{K}$ and to approximate the p-dimensional manifolds with p-dimensional chains. In this way, an amount of the global physical quantity is associated with every p-chain and then a p-dimensional cochain is obtained. The association of an amount of a physical variable with every p-cell gives rise to a *set function* that is the natural extension of *point functions* used in physics.

3 Topological laws

According with Table (1) the Maxwell's equations may be written as follows

$$\begin{cases} \Phi(I,\partial V) = 0 \\ \mathcal{U}(T,\partial S) + \Phi(\partial T, S) = 0 \end{cases} \qquad \begin{cases} \Psi(\tilde{I},\partial\tilde{V}) = Q^{cont}(\tilde{I},\tilde{V}) \\ \mathcal{F}(\tilde{T},\partial\tilde{S}) = Q^{flow}(\tilde{T},\tilde{S}) + \Psi(\partial\tilde{T},\tilde{S}). \end{cases} \tag{25}$$

[30] Franz [10, p.43]; Hocking [13, p.301].
[31] Synge [30, p.267].
[32] Hilton [12, p.72].

where I stands for *instant* and T for *interval* . This is a compact notation equivalent to the usual integral formulation. So the last equation is

$$\int_{\tilde{T}} \int_{\partial \tilde{S}} \mathbf{H} \cdot \mathrm{d}\tilde{\mathbf{L}} \ \mathrm{d}\tilde{t} = \int_{\tilde{T}} \int_{\tilde{S}} \mathbf{j} \cdot \mathrm{d}\tilde{\mathbf{S}} \ \mathrm{d}\tilde{t} + \tilde{\Delta}_t \int_{\tilde{S}} \mathbf{D} \cdot \mathrm{d}\tilde{\mathbf{S}}. \tag{26}$$

The symbols Δ_t and $\tilde{\Delta}_t$ refer to time differences between the values of a quantity referred to a primary and dual time cell complex as shown in Fig.(3):

$$\begin{array}{ll} \Delta_t f = f(t_n) - f(t_{n-1}) & \text{backward difference} \\ \tilde{\Delta}_t g = g(\tilde{t}_{n+1}) - g(\tilde{t}_n) & \text{forward difference} \end{array} \tag{27}$$

It is evident that these relations relate global quantities of the same kind and do not involve metrical notions: length, areas, measures of volumes and durations are not required. It is for this reason that we call them *topological equations*.

In Fig.(3) we have presented six global variables (in space) of electromagnetism and the corresponding spatial elements. This leads to introduce three space-time cochains defined as follows

$$\begin{array}{ll} \text{2-cochain of magnetic fluxes } \Phi^{(2)} & \left\{ \begin{array}{ll} \mathcal{U}^{(2)}(\boldsymbol{\tau}_n, \mathbf{l}_\alpha) & \text{electric tension impulses} \\ \Phi^{(2)}(\mathbf{t}_n, \mathbf{s}_\beta) & \text{magnetic fluxes} \end{array} \right. \\ \text{2-cochain of electric fluxes } \Psi^{(2)} & \left\{ \begin{array}{ll} \mathcal{F}^{(2)}(\tilde{\boldsymbol{\tau}}_n, \tilde{\mathbf{l}}_\alpha) & \text{magnetic tension impulses} \\ \Psi^{(2)}(\tilde{\mathbf{t}}_n, \tilde{\mathbf{s}}_\beta) & \text{electric fluxes} \end{array} \right. \\ \text{3-cochain of electric charges } \mathrm{Q}^{(3)} & \left\{ \begin{array}{ll} Q^{(3)flow}(\tilde{\boldsymbol{\tau}}_n, \tilde{\mathbf{s}}_\beta) & \text{electric charge flow} \\ Q^{(3)cont}(\tilde{\mathbf{t}}_n, \tilde{\mathbf{v}}_h) & \text{electric charge content} \end{array} \right. \end{array} \tag{28}$$

We may write Maxwell's equations in space-time in the form:

$$\delta \Phi^{(2)} = 0 \qquad\qquad \delta \Psi^{(2)} = \mathrm{Q}^{(3)} \tag{29}$$

That is the discrete analogous of the two equations (A, B, C, P, Q denote tensorial indices in space-time notation)

$$\partial_A F_{BC} + \partial_B F_{CA} + \partial_C F_{AB} = 0 \qquad\qquad \partial_Q G^{PQ} = J^P \tag{30}$$

used in the differential formulation. When equations (29) are applied to the cells of the two complexes, we obtain a "local" form of Maxwell's equations in a discrete setting, i.e.

$$\left\{ \begin{array}{l} \sum\limits_{\alpha} d_{h\alpha} \, \Phi_\alpha = 0 \\ \sum\limits_{\beta} c_{\alpha\beta} \, \mathcal{U}_\beta + \Delta_t \Phi_\alpha = 0 \end{array} \right. \qquad\qquad \left\{ \begin{array}{l} \sum\limits_{\alpha} \tilde{d}_{h\alpha} \, \Psi_\alpha = Q_h^{cont} \\ \sum\limits_{\beta} \tilde{c}_{\alpha\beta} \, \mathcal{F}_\beta - \tilde{\Delta}_t \Psi_\alpha = Q_\alpha^{flow}. \end{array} \right. \tag{31}$$

These equations are to be used for numerical solution of electromagnetic problems.

Some authors use chains with coefficients on the Reals instead of cochains to describe the association of physical quantities to p-cells.

We now show that the use of chains instead of cochains implies that one cannot describe a fundamental feature of physical quantities i.e. their additive property.

The homomorphism of p-cochains to p-chains expressed in Eq. (18) is an essential property for global physical variables because the amount of a physical quantity that corresponds to a domain is the sum of the amounts corresponding to its parts. For a better understanding of this property let us remark that in physics one defines an extensive quantity as one that is additive on volumes. It is not commonly stressed (even if it is commonly used) that quantities associated with lines and surfaces are additive. So the electric tension along a line is the sum of the electric tensions on its parts; the magnetic flux associated with a surface is the sum of the fluxes associated with its parts. Additivity allows expression of global variables as line, surface and volume integral of field functions. Global quantities in physics are always additive as consequence of their own definition. This addittivity cannot be expressed by the notion of chain with coefficient domain in a group $\mathbf{G}$ instead of integers. This is because the notion of sum of two p-cells is lacking. On the contrary it can be expressed by the cochain notion because there is the notion of sum of two p-chains with integer coefficients and this permits an algebraic treatement of p-dimensional manifolds formed by p-cells.

With reference to the use of chains or cochains in physics Post [33], said "Epistemologically it is better if the cohomology of fields is given a primary physical role than the homology of chains over wich these fields are being integrated."

Some authors justify the choice of chains instead of cochains due to the fact that in a finite-dimensional cell complexes one may establish a one-to-one mapping between p-chains and p-cochains with the same coefficient group $\mathbf{G}$. Hilton and Wylie wrote: "This does not justify us in concealing the distinction between these two notions, for the correpondence depends on a choice of basis of $\mathcal{C}_p$. The situation in vector space theory is entirely analogous. Although a finite-dimensional vector space is isomorphic to its dual, the elements of the dual are linear functions and it is essential that the two spaces be not confused" [34].

Since a p-dimensional domain can be described by a p-chain with integer coefficients, it follows that cochains are a natural tool to describe set functions.

In the same book, Lebesgue quoted as saying: [35] "Si pourtant, on parle peu de ces fontions [les fonctions d'ensemble], c'est que les mathématiciens n'ont pas encore créé l'Algèbre et l'Analyse des fonctions de domaine. " [36] We see that the notion of

[33] Post [24, p.517].

[34] Hilton [12, p.67].

[35] Lebesgue [18, p.293].

[36] "Thus, one talks less of a set of function, it is because mathematicians have not invented Algebra and Analysis of set-functions."

cochain and the theory of cohomology are the tools that Lebesgue was talking about.

4 Metrical laws

In Fig.(11) we see that the three constitutive equations of electromagnetism link global variables associated with dual cells. In such a link the notion of orthogonality is essential; moreover the length and area of line and a surface elements are indispensable. This shows that the constitutive equations, contrary to the topological equations, require metrical notions. Constitutive equations are valid also in a finite

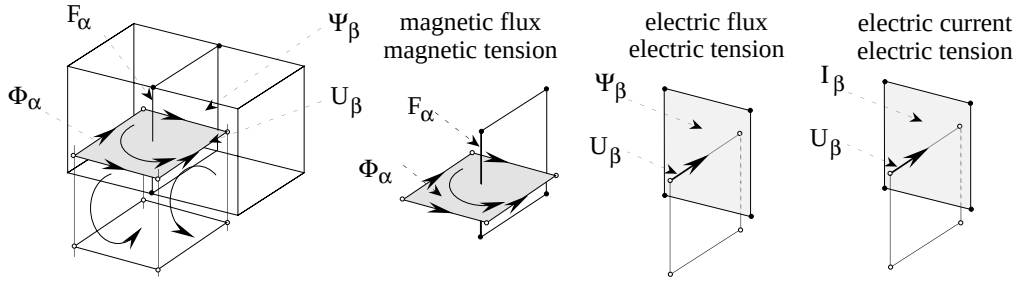

Figure 11: The constitutive equations require metrical notions.

setting, but only in a region of *uniform field*. It is important to notice that constitutive equations are experimentally determined in a laboratory on a specimen under the condition of *uniform* field. The three constitutive equations of electromagnetism are

$$\begin{array}{rcl} \text{second-kind variables} & & \text{first-kind variables} \\ \dfrac{\Psi(\tilde{\mathbf{t}}_n, \tilde{\mathbf{s}}_\alpha)}{\tilde{s}_\alpha} & \overset{\text{law}}{=} & \epsilon\, \dfrac{\mathcal{U}(\boldsymbol{\tau}_n, \mathbf{l}_\alpha)}{\tau_n\, l_\alpha} \\ \dfrac{\mathcal{F}(\tilde{\boldsymbol{\tau}}_n, \tilde{\mathbf{l}}_\beta)}{\tilde{\tau}_n\, \tilde{l}_\beta} & \overset{\text{law}}{=} & \dfrac{1}{\mu}\, \dfrac{\Phi(\mathbf{t}_n, \mathbf{s}_\beta)}{s_\beta} \\ \dfrac{Q^{flow}(\tilde{\boldsymbol{\tau}}_n, \tilde{\mathbf{s}}_\alpha)}{\tilde{\tau}_n\, \tilde{s}_\alpha} & \overset{\text{law}}{=} & \sigma\, \dfrac{\mathcal{U}(\boldsymbol{\tau}_n, \mathbf{l}_\alpha)}{\tau_n\, l_\alpha} \end{array} \tag{32}$$

in which $\tau_n, \tilde{\tau}_n, l_\alpha, \tilde{l}_\beta, s_\beta, \tilde{s}_\alpha$ are the extensions of the corresponding cells. These equations are valid if cells are orthogonal parallelotopes (parallelepipeds, rectangles), or if the simplicial complex is a *Delaunay* complex and its dual a *Voronoi* complex. Only in these cases, 1-cells of the dual are orthogonal to the primary 2-cells and viceversa. As consequence of the natural association of physical variables with the space-time elements endowed with inner or outer orientations, we have obtained a natural separation between non metrical and metrical equations of electromagnetism as obtained Kottler [15], Heargraves and Van Dantzig [33].

It is remarkable that *the constitutive equations of every physical theory are links between a variable associated with a cell and another associated with its dual.*

5 Conclusion

To conclude we quote some sentences taken from a paper of Bohm, Hiley and Stuart [37] "The laws of electrodynamics were first expressed in terms of integrals of fields over cycles of varying dimensionality, e.g. Ampère's law, Faraday's law, Gauss law, etc. It is only from the extrapolation of these integrals to infinitely small cycles that one obtain Maxwell's equations. Thus these equations go considerably beyond-what can be inferred from observations alone. The relative ease of the mathematical application of the differential form of Maxwell's equations has made this approach attractive. However the infinities that arise in the indefinite extension of this form, both classically and quantum mechanically imply that it may be appropriate to go back to the integral form in spite of the possibility of greater mathematical difficulty. The appropriate mathematics for doing this is the theory of complexes of chains and cochains that we have described earlier".

Acknowledgments. The author is grateful to Gianni Landi and Alain Bossavit for their critical reading of the manuscript.

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[37] Bohm, Hiley, Stuart[3, p.180].

Table 3: The correspondence between the discrete and the differential formulation of the electromagnetic equations.

	discrete formulation *global variables*	*differential formulation* *field functions*
1	charge conservation $\sum_\alpha \tilde{d}_{h\alpha}\, Q^{flow}(\tilde{\boldsymbol{\tau}}_n, \tilde{\mathbf{s}}_\alpha)$ $+\Big[Q^{cont}(\tilde{\mathbf{t}}_{n+1}, \tilde{\mathbf{v}}_h) - Q^{cont}(\tilde{\mathbf{t}}_n, \tilde{\mathbf{v}}_h)\Big] = 0$	$\begin{cases} \text{div}\, \mathbf{j} + \partial_t \rho = 0 \\ \mathbf{n}\cdot(\mathbf{j}^+ - \mathbf{j}^-) = 0 \end{cases}$
2	Gauss' law $\sum_\beta d_{k\beta}\, \Phi(\mathbf{t}_n, \mathbf{s}_\beta) = 0$	$\begin{cases} \text{div}\, \mathbf{B} = 0 \\ \mathbf{n}\cdot(\mathbf{B}^+ - \mathbf{B}^-) = 0 \end{cases}$
3	Faraday's law $\sum_\alpha c_{\beta\alpha}\, \mathcal{U}(\boldsymbol{\tau}_n, \mathbf{l}_\alpha) + \Big[\Phi(\mathbf{t}_n, \mathbf{s}_\beta) - \Phi(\mathbf{t}_{n-1}, \mathbf{s}_\beta)\Big] = 0$	$\begin{cases} \text{curl}\, \mathbf{E} + \partial_t \mathbf{B} = 0 \\ \mathbf{n}\times(\mathbf{E}^+ - \mathbf{E}^-) = 0 \end{cases}$
4	electrostatic induction $\sum_\alpha \tilde{d}_{h\alpha}\, \Psi(\tilde{\mathbf{t}}_n, \tilde{\mathbf{s}}_\alpha) = Q^{cont}(\tilde{\mathbf{t}}_n, \tilde{\mathbf{v}}_h)$	$\begin{cases} \text{div}\, \mathbf{D} = \rho \\ \mathbf{n}\cdot(\mathbf{D}^+ - \mathbf{D}^-) = \sigma \end{cases}$
5	Maxwell-Ampère's law $\sum_\beta \tilde{c}_{\alpha\beta}\, \mathcal{F}(\tilde{\boldsymbol{\tau}}_n, \tilde{\mathbf{l}}_\beta) - \Big[\Psi(\tilde{\mathbf{t}}_{n+1}, \tilde{\mathbf{s}}_\alpha) - \Psi(\tilde{\mathbf{t}}_n, \tilde{\mathbf{s}}_\alpha)\Big]$ $= Q^{flow}(\tilde{\boldsymbol{\tau}}_n, \tilde{\mathbf{s}}_\alpha)$	$\begin{cases} \text{curl}\, \mathbf{H} - \partial_t \mathbf{D} = \mathbf{j} \\ \mathbf{n}\times(\mathbf{H}^+ - \mathbf{H}^-) = \mathbf{K} \end{cases}$
6	electric constitutive law $\Psi(\tilde{\mathbf{t}}_n, \tilde{\mathbf{s}}_\alpha) \overset{\text{law}}{=} \epsilon \dfrac{\tilde{s}_\alpha}{\tau_n l_\alpha}\, \mathcal{U}(\boldsymbol{\tau}_n, \mathbf{l}_\alpha)$	$\mathbf{D} \overset{\text{law}}{=} \epsilon \mathbf{E}$
7	magnetic constitutive law $\mathcal{F}(\tilde{\boldsymbol{\tau}}_n, \tilde{\mathbf{l}}_\beta) \overset{\text{law}}{=} \dfrac{1}{\mu} \dfrac{\tilde{\tau}_n \tilde{l}_\beta}{s_\beta}\, \Phi(\mathbf{t}_n, \mathbf{s}_\beta)$	$\mathbf{H} \overset{\text{law}}{=} \dfrac{1}{\mu}\mathbf{B}$
8	Ohm's constitutive law $Q^{flow}(\tilde{\boldsymbol{\tau}}_n, \tilde{\mathbf{s}}_\alpha) \overset{\text{law}}{=} \sigma \dfrac{\tilde{\tau}_n \tilde{s}_\alpha}{\tau_n l_\alpha}\, \mathcal{U}(\boldsymbol{\tau}_n, \mathbf{l}_\alpha)$	$\mathbf{j} \overset{\text{law}}{=} \sigma \mathbf{E}$
9	general solution of Gauss' equation $\Phi(\mathbf{t}_n, \mathbf{s}_\beta) = \sum_\alpha c_{\beta\alpha}\, \Pi(\mathbf{t}_n, \mathbf{l}_\alpha)$	$\begin{cases} \mathbf{B} = \text{curl}\, \mathbf{A} \\ \mathbf{n}\times(\mathbf{A}^+ - \mathbf{A}^-) = 0 \end{cases}$
10	general solution of Faraday's equation $\mathcal{U}(\boldsymbol{\tau}_n, \mathbf{l}_\alpha) = \sum_h g_{\alpha h} \mathcal{V}(\boldsymbol{\tau}_n, \mathbf{p}_h)$ $-\Big[\Pi(\mathbf{t}_n, \mathbf{l}_\alpha) - \Pi(\mathbf{t}_{n-1}, \mathbf{l}_\alpha)\Big]$	$\begin{cases} \mathbf{E} = -\text{grad}\, V - \partial_t \mathbf{A} \\ V^+ - V^- = 0 \end{cases}$

Table 4: The differential structure of electromagnetism

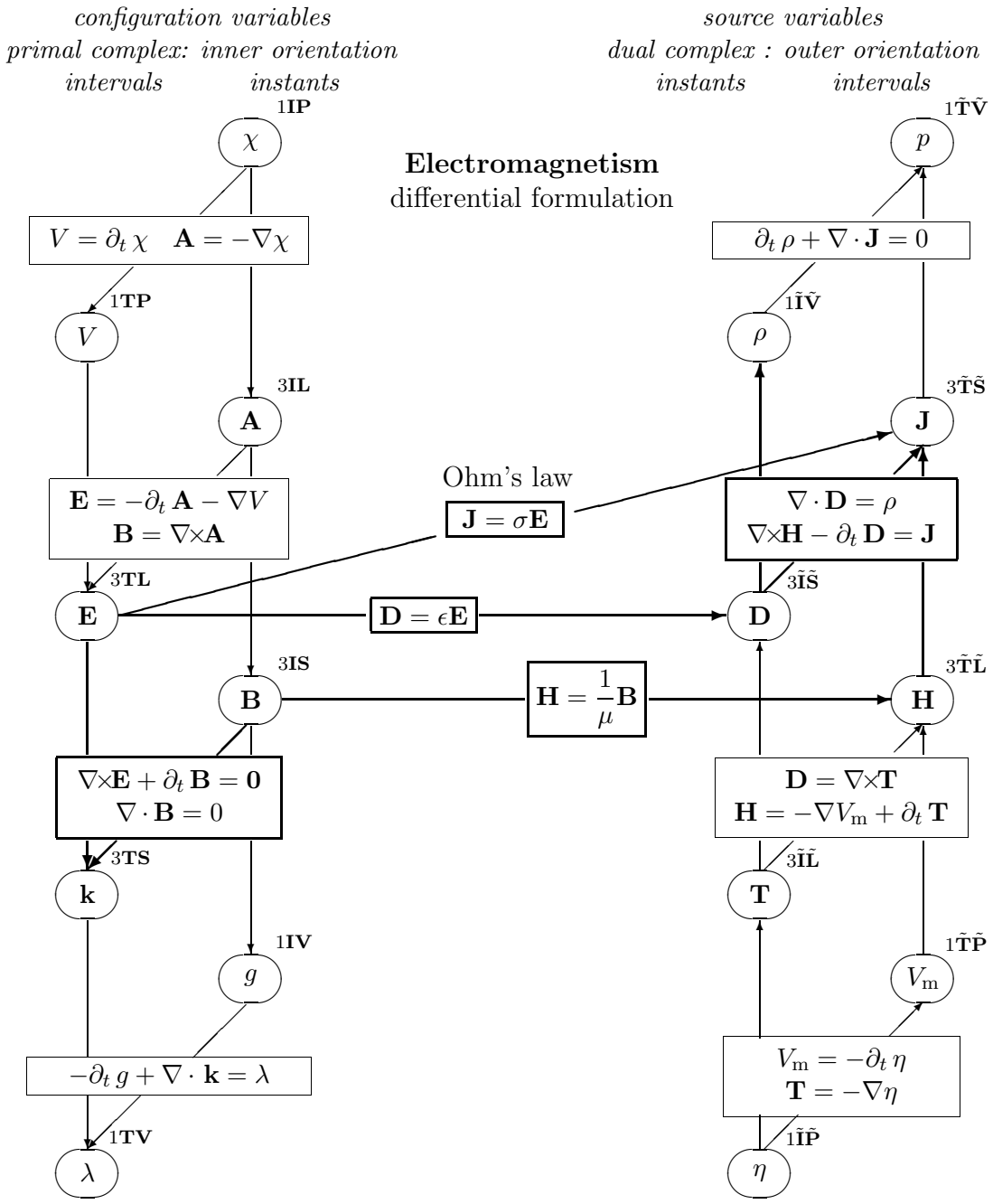
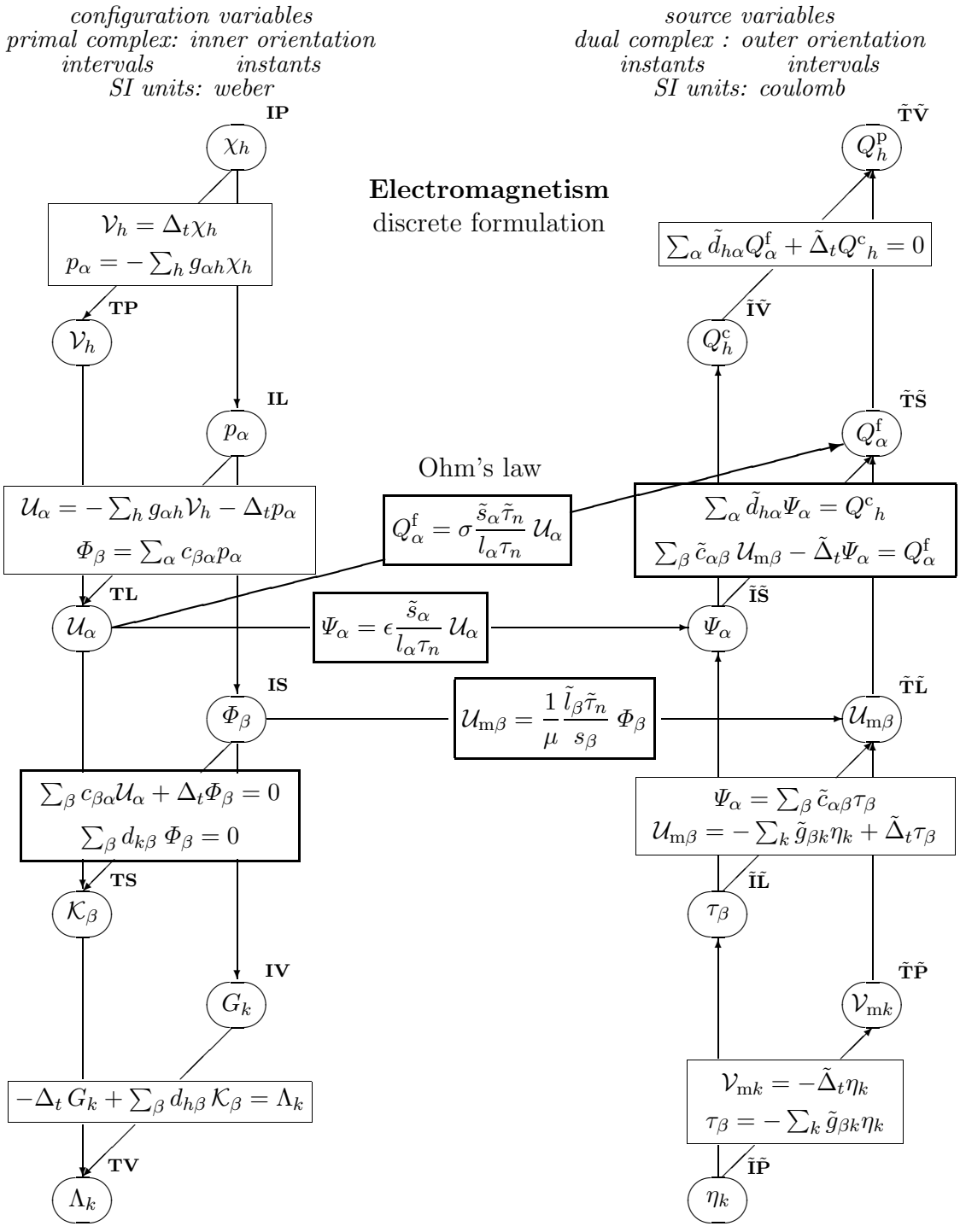


Table 5: The discrete structure of electromagnetism



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