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ON THE FORMAL STRUCTURE OF PHYSICAL THEORIES

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PREFACE

"The aim of Mathematical Physics
is not only to facilitate for the physicist
the numerical calculation of certain constants
or the integration of certain differential equations.
It is besides, it is above all, to reveal to him
the hidden harmony of things
in making him see them in a new way."

Henri Poincaré
The value of Sciences, Dover, 1958

This report collects the first results of a trial to make a synthesis in physics from the point of view of formal properties. Under the name "formal structure" we mean algebraic, topological, geometrical, analytical and others mathematical structures.

Perhaps the most concrete result is the construction of a classification scheme for the physical quantities and the equations of every physical theory. A number of such schemes are quoted in § 8. These schemes are all written using a single system of physical quantities, the rationalized M.K.S.A. system. Symbols and names are those recommended by the International Union of Pure and Applied Physics (S.U.N. Commission).

This collection of schemes, dealing with many physical theories, may form the core of an "Atlas of the physical theories" that under the name of many specialists of the various field of physics, may be printed in successive editions of increasing content.

Detailed comparisons between different physical theories have progressively displayed the existence of a common mathematical structure, that in turn, come up from a common topological

"back-cloth" of physical theories.

This comparison was made at first using concepts and tools of function spaces, like operators and bilinear forms (see reference [130]). Such powerful devices permit to go over the distinctions between one or more functions of one or more independent variables. The usual analogies between different theories are revealed by the fact that corresponding operators have the same structure. A typical adjointness relation between pairs of operators arising in every theory comes into appearence as the main mathematical property shared by physical theories. In fact it is exhibited by linear and non-linear theories, dealing with discrete or continuum systems, of classic, relativistic and quantum nature.

The existence of such unity of mathematical structure led us to construct a mathematical model (see reference [129]). It is dictated from the principle of economizing proofs in the single physical theories, showing the links between various properties commonly used in physical theories.

As the comparison was progressively extended to more and more theories we had the pleasant feeling that such mathematical structure has more deep roots.

The reason of such mathematical structure come into appearence when it was realized that in every physical theory there are physical variables that are naturally associated to simple geometrical and chronometrical elements like points, lines, volumes, surfaces, time instant and time intervals and combination of them.

At this point a paper of F.J. Branin jr. offered us the key to translate this association between physics and geometry

in mathematical language: this key is the algebraic topology, in particular the homology theory of cell-complexes. It is shown that with the language of "chains" and that of "coboundary" one can describe the typical operations of forming the various physical variables.

The adjointness between operators that was previously realized appeared as consequence of the adjointness of the boundary and coboundary operators, a well known property of homology of cell-complexes.

The point of view under taken in this paper is that of an experimentalist which tries to infer general conclusions from a body of experimental facts: these "facts" are here the existing physical theories. We have not tried to invent "new theories". On the contrary we have taken an inductive point of view.

A word about the style of the paper. It has been written in such way to be understandable by the largest audience: by experimental and theoretical physicists, by chemists, engineers, and also, by students of the last years of these faculties. In order to make possible this we have explained from the start the main algebraic and topological concepts giving an elementary and compact presentation of them and stressing how they arise quite naturally in the description of physical sciences. These mathematical tools are presented from a physicist point of view: this implies that sometime they may be lacking of mathematical rigour.

In dealing with this research the author was in an isolated position: our time is characterized by over-specialization

in all field of sciences. The author ask for contacts, collaboration and constructive criticism.

The content of this report will be pubblished in some review.

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INTRODUCTION

At the fundation of every physical theory we introduce some measurable physical quantities from wich, in subsequent stages, other physical quantities are derived. It is at this initial stage that one make the important remark that

a) in every physical theory there are basic physical quantities that are naturally referred to the most simple geometrical and chronometrical elements like points, lines, surface, volumes, hypervolumes, time instants and time intervals.

Let us give some examples: we speak of mass or electric charge contained into a volume of the probability of finding a particle into a volume. We speak of electric potential at a point, at a given time instant of the displacement of a point of a material continuum. We speak of radius vector of a particle at a time instant, of impulse given to a particle during a time interval, of displacement of a particle during a time interval. We speak also of electric flux through a surface, of energy and momentum flux through a surface during a time interval, or of internal energy produced into a volume during a time interval.

The remark a) leads to the following remark

b) in every physical theory there are basic physical laws that state that a physical quantity referred to a p-dimensional manifold ω, like lines, surfaces, volumes, time interval, etc. is equal to a physical quantity referred to its (p-1)-dimensional boundary $\partial\omega$.

Typical laws of this kind are those expressed by balance equations, in particular continuity equations, equilibrium equations, equations of motion; circuital equations, in particular compatibility equations; the equations that give the general solution of one of the proceeding equations; the equations defining the gradients; and so on. Exam-

ples of balance laws are: in magnetostatics the sum of the magnetic fluxes through the boundary of a volume vanishes; in continuum mechanics the law of equilibrium states that the sum of the forces acting on the boundary of a volume is equal to the sum of the external forces acting on that volume. The principle of conservation of energy states that the outgoing energy flux through the boundary of a volume during a time interval plus the energy stored into the volume in the same time-interval is equal to the energy produced into the volume in the time interval considered. This law can be restated, with reference to space-time, by saying that the outgoing energy flux through the three-dimensional boundary of a four-dimensional region is equal to the energy produced inside it.

An examples of circuital laws we mention the Ampère circuital law: the magnetomotive force along the boundary of a surface is equal to the current flowing through the surface. Often circuital equations arises as compatibility conditions of gradient-like equations. So in fluidynamics the condition that the circulation of the velocity vector along a closed line vanishes implies that the velocity vector is the gradient of a scalar function (velocity potential). In the thermodynamic configuration space the statment of the vanishing of the circulation of the vector field, whose components are the "intensive" variables, along a closed line (Maxwell reciprocity relations) amounts to the statment of the existence of the entropy.

Perhaps this connection between physics and geometry is well known from long time. But it does not seem that it has been realized how far one can go in the understanding of the formal structure of physical theories by exploring the consequence of this remarkable connection.

2 A PRELIMINARY CLASSIFICATION OF PHYSICAL VARIABLES

2.1. Configuration and source variables.

Since we propose to study the link between physical variables and basic geometrical elements we need a preliminary examination of the main physical variables and equations of a physical theory. This is a difficult task because of the greath difference in terminology of different physical theories: whatever name we will chose to denote a typical entity of a theory we are sure to be exposed to criticism. Since from some part we must start, we decide to use, ad far as possible, the terminology of mechanics, both analytical mechanics and continuum mechanics. The reason is that much part of physical theories is modelled on mechanics, among them field theories.

One of the basic notion of mechanics is that of "configuration" of a mechanical system. In discrete mechanical system the configuration is discribed by a set of n generalized coordinates, universally denoted by q^k. In continuum statics the configuration is described giving the displacement vector of every point of the continuum: this is almost universally denoted the letter $\underline{u}$. The coordinates q^k and the vector $\underline{u}$ will be called "configuration variables" of the respective mechanical system. In field theories the analogous of the mechanical configuration variables are the field functions (also called field variables). The variables that describe the configuration of a system or of a field will be called configuration variables.

So the configuration variable of an electromagnetic field is the vector potential A_μ ; the configuration variable of the gravitational field in the relativistic description in the metric tensor $g_{\mu\nu}$. The configuration variables of a thermodynamical system may be the intensive variables. In quantum mechanics the configuration variable of a quantum-mechanical system is the state vector ψ that give the probability amplitude.

Another basic entity of mechanics is the force concept. In continuum statics the body force is considered as a source of the change in configuration of the system. In particle mechanics the force is the source of the motion of the particle. The notion of "force" is replaced in other physical theories by that of source of a field. So the electric cha ges are the sources of the electrostatic field, the electric currents are the sources of the magnetic field. The variables that descri be the source of a field or of a phenomenon will be called source variables.

The choice of a set configuration variables and a set of source variables in a physical theory is not unique: reasons of convenience may lead to prefere one choice to another.

Typical pairs of configuration and source variables are the lagrangian coordinates q^k and the generalized forces Q_k of analytical mechanics; the electrostatic potential and the electric charge density in electrostatics; the displacement vector and the body force vector in continuum mechanics, the metric tensor $g_{\mu\nu}$ and the stress-energy-momentum tensor $T_\mu{}^\nu$ in the relativistic gravitational theory; the state vector ψ and the interaction term σ in quantum mechanics.

But other choices are possible: so in analytical mechanics one may choose the momenta p_k and the generalized velocities $\dot{q}^k$ as configuration and source variables respectively. In continuum mechanics one may choose the stress potentials as configuration variables and the dislocation tensor as source variable. In magnetostatics one may choose the pair vector potential A_μ-current density J^μ or the pair scalar potential χ-monopole charge density $\rho_{(m)}$ as configuration and source variables respectively.

In the general field theory a theory which essentially

rests upon formal properties common to many field theories, the configuration variables are the field functions, usually denoted by ψ_h . Usually no explicit mention is made of the source variables that, in the case of external sources interaction field are implicy included in the interaction lagrangian.

About the mathematical nature of the configuration and source variables we find the largest variaty of tipes: they may be real or complex numbers vectors, tensors, quaternions, motors, multivectors, Clifford numbers, matrices, spinors, operators, and so on.

2.2. Independent variables.

In continuons systems and fields the configuration and source variables depend from some indipendent variables like the time and space coordinates. In discrete physical systems and in those with lumped parameters the configuration variables may be considered at the same time indipendent variables. This is the case of the intensive variables of thermodynamics and of generalized coordinates in analytical statics. When the physical system under study exibits some symmetry (like plane symmetry, spherical symmetry, etc.) one may use a lower number of variables. In other cases one may treat one or more variables as parameters.

pendent variables as coordinates of some point of a space. When we say that in every physical theory there are some physical quantities that are naturally referred to the geometrical elements of a space we refer to the space of indipendent variables. Since the choice of indipendent variables is not

unique it follows that the same physical quantity may be referred to different geometrical elements in different spaces. So the electric potential φ will be referred to points of the three-dimensional physical space or to time lines of the four-dimensional space-time.

2.3. Configuration-kind variables.

In a physical theory, once we have chosen the configuration variables and the indipendent variables, we may consider those variables that are linked to the configuration variables by means of operations of sum and difference, of total or partial derivatives and total or partial integrals with respect to the indipendent variables. All this without the intervention of physical constants, material parameters, coupling constants, phenomenological coefficients and other parameters linked to the geometry or the physics of the system. Exception is made for the light speed in vacuo.

These variables, including the configuration variables themselves, will be called configuration-kind variables.

In continuum mechanics typical configuration-kind variables are the geometrical and kynematical variables like the displacement, strain, velocity, deformation gradient, angular velocity, vorticity, rate of deformation, etc.

2.4. Source-kind variables.

In analogous way let us consider those variables that are linked to the source variables by means of operations of sum and difference, of total or partial derivative and total or partial integrals with respect to the indipendent variables. All this without the intervention of physical constants, material parameters, coupling coefficients, phenomenological constants or any other parameter linked to the geometry or the physics of the

system, with the exception of the light speed in vacuo. All these variables, including the source variable themselves, will be called source-kind variables.

In continuum mechanics typical source-kind variables are statical and dynamical variables like the body force, body couple, momentum, angular momentum, stress, stress functions, stream functions, etc.

2.5. Phenomenological equations.

Configuration and source-kind variables of the same physical theory are linked by constitutive equations, also called material equations or equations of state. They include physical constants, geometrical and material parameters.

The interaction between two phenomena is described by giving the link between the source-kind variables of one theory and the configuration-kind variables of the other. These are the interaction equations that contain coupling coefficients. Both the constitutive equations and the interaction equations are of phenomenological kind and are usually called phenomenological equations.

2.6. Other physical variables.

In every physical theory one encounters other physical variables that are defined as function of the configuration-kind and source-kind variables. So in particle mechanics one define the potential U and the kinetic energy T respectively as

$$U \stackrel{def}{=} \int_0^{\mathbf{r}} \mathbf{f}(\mathbf{r}) \cdot d\mathbf{r} \qquad T \stackrel{def}{=} \int_0^{\mathbf{v}} \mathbf{p}(\mathbf{v}) \cdot d\mathbf{v}$$

that depend from the source-kind variables $\mathbf{f}, \mathbf{p}$ and from the configuration-kind variables $\mathbf{r}, \mathbf{v}$. Once the constitutive equations are given one may evaluate these physical variables. So when

$$\boldsymbol{p} = m\,\boldsymbol{v} \qquad \text{or} \qquad \boldsymbol{p} = \frac{m_0 \boldsymbol{v}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

we obtain

$$T = \frac{1}{2} m v^2 \qquad T = m_0 c^2 \left[\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right]$$

that are the typical expressions of the kinetic energy of classical and relativistic mechanics respectively. In electrostatics and in magnetostatics one introduce the variables

$$w_e \overset{def}{=} \int_0^{\boldsymbol{E}} \boldsymbol{D}(\boldsymbol{E}) \cdot d\boldsymbol{E} \qquad w_m \overset{def}{=} \int_0^{\boldsymbol{B}} \boldsymbol{H}(\boldsymbol{B}) \cdot d\boldsymbol{B}$$

that are defined as function of the source and configuration-kind variables of the respective theories. Once we specify the constitutive equations we obtain these new variables as function of the configuration-kind variables only. Another example of physical variable of the electrostatic field is the Maxwell tensor giving the stresses

$$t^h_{\ k} = D^h E_k - \frac{1}{2} (D^r E_r)\, \delta^h_k$$

This physical variable is function of the two variables $\boldsymbol{E}$ and $\boldsymbol{D}$. Once we specify the phenomenological equation $\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{E})$ the stress tensor becomes function of the configuration-kind variable $\boldsymbol{E}$ only.

2.7. Other equations.

The equations that link the various oonfiguration-kind variables of a single physical theory do not contain physical constants, neither material parameters, as we have said. These equations contain time or space derivatives and their linear combinations, algebraic sums and time or space integrals. Typical are the equations defining the gradients, those arising from circuital laws, and compatibility equations. Of this kind are the equations

$$\boldsymbol{B} = curl\, \boldsymbol{A} \qquad \boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} \qquad \varepsilon_{hk} = \partial_h u_k$$

Also the equations that link the various source-kind variables of a theory are of this kind. Thypical are the balance equations and the conservation laws, for ex. the equations

$$t^{hk}{}_{/k} = f^h \qquad \boldsymbol{f} = \boldsymbol{p}(t_2) - \boldsymbol{p}(t_1) \qquad \oint_{\partial w} \boldsymbol{B} \cdot \boldsymbol{n}\, dS = 0$$

Equations of this kind have not a particular name in the physical literature. Since in the following we shall deal mainly with this kind of equations we use tentatively a single name and call them equations of structure.

The equation of structure and the phenomenological equations do not exhaust, of course, the equatinns arising in a physical theory. But they are, so to speak, the building blocks of almost all equations of the physical theory. So the Poisson equation

$$div\,[\lambda\,(-grad\,\varphi)] = \rho$$

arises as a mixing of two equations of structure with one phenomenological equation. In general the wave equations and the field equations are obtainèd by combining phenomenological and structural equations.

All what we have said can be represented in a block diagram, see table 2.7.1.

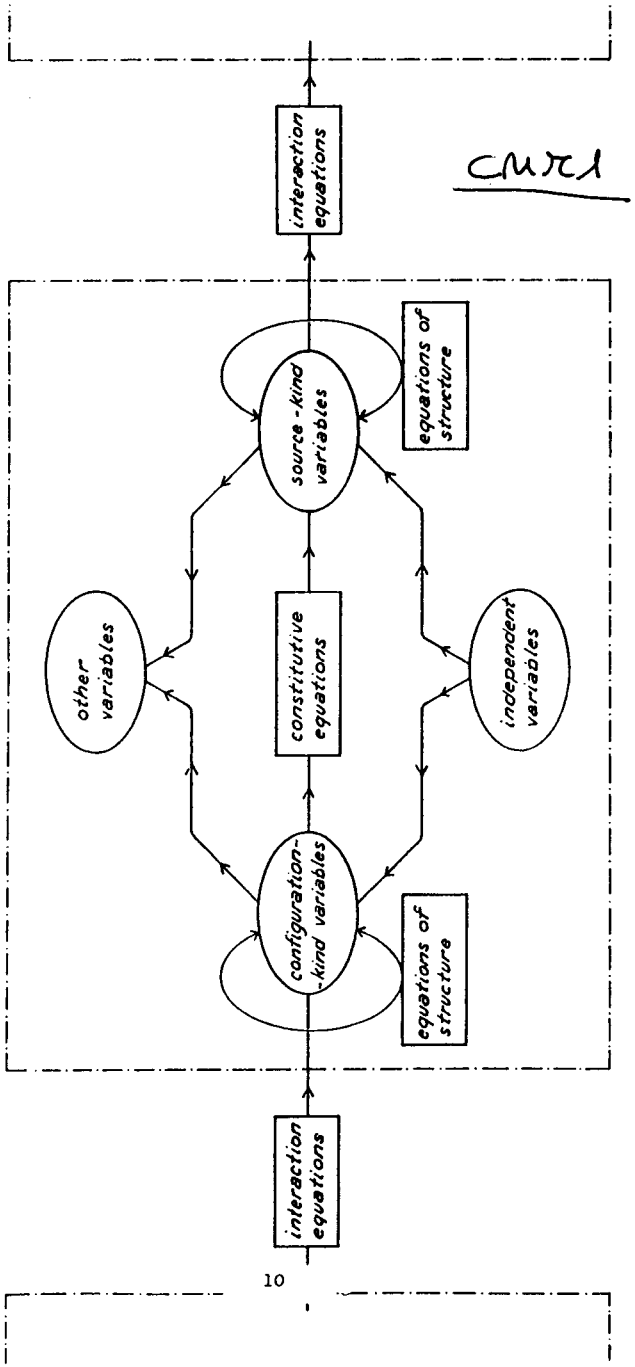

table 2.1 preliminary classification of physical variables of a physical theory

3 ALGEBRAIC TOPOLOGY: CHAINS AND COBOUNDARY

3.1. Introduction.

To point out the basic geometrical elements of a space it is useful to subdivide a region of the space in cells of whatever shape and dimension; in this way we construct a cell-complex. This is what we usually do in physics when we write balance equations in a local form: for this purpose we use a small parallelepiped that is understood as a cell of a cell-complex built up in the region in which the field or the continuous system is considered. Such a cell-complex exhibits vertices, edges, faces and volumes, i.e. four basic elements (in the three-dimensional space).

The study of cell-complexes is one of the branches of algebraic topology: this is essentially a method, initiated by Poincaré, to study the topological properties of a manifold (1).
Another branch of algebraic topology is the homotopy theory, also useful in the study of physical theories (Post, 1971). In the present paper we use only the first branch called the homology theory of cell-complexes. For the benefit of the reader we give here a brief survey of the homology theory of cell-complexes restricted to those concepts and tools that will be used in this paper (2).

(1) - A clear and elementary account of this method is given in the article of Alexandrov contained in the book: Mathematics. Its content, Methods and Meaning, edited by Alexandrov, Kolgomorov and Laurent'ev, MIT Press; 1963.

(2) - For further study the interested reader may see: Alexandrov (1956), (1961), Hocking-Young (1961), Franz (1968), Hilton-Wylie (1960), Patterson (1956), Wallace (1957).

3.2. Cell-complex.

Let us consider at first, a three-dimensional space $\mathbb{R}^3$ and a region Ω contained in it. Let us divide such region in three-dimensional cells. The vertices, edges, faces and three-dimensional cells are called respectively 0-cells, 1-cells, 2-cells and 3-cells. All these cells of various orders form a cell-complex that we denote K. The shapes and dimensions of the 3-cells may be arbitrary: nevertheless we shall find convenient to consider 3-cells delimited by the coordinate surfaces of a coordinate system, as is customary in physics.

More in general we shall consider n-dimensional spaces of the kind R^n and finite n-dimensional regions $\Omega \subset R^n$. Such regions may be covered by an atlas: this is a set of local coordinate systems that satisfy the two well known properties.
To construct a cell-complex in Ω we use the n-dimensional cells of local coordinate system, i.e. those n-dimensional regions of Ω that are delimited by the n families of (n-1) dimensional coordinate manifolds. The cell-complex so obtained enjoy the property that all its p-cells lie on p-dimensional coordinate manifold . Since for every point of R^n there pass $\binom{n}{p}$ p-dimensional coordinate manifolds we may divide the p-cells in $\binom{n}{p}$ families. So in R^3 we have $\binom{3}{0}$ families of 0-cells, $\binom{3}{1}$ families of 1-cells, $\binom{3}{2}$ families of 2-cells, $\binom{3}{3}$ families of 3-cells. Since we are particularly interested to spaces with $n<4$ we shall use special symbols for the cells of the various orders. So a 0-cell (point) will be denoted P, an 1-cell (line segment) L, a 2-cell (surface segment) S, a 3-cell (volume) V and a 4-cell (hypervolume) H.

For n=1 the space is stright line. A cell-complex then reduces itself to points, P, that are the 0-cells and to line segments, L, that are the 1-cells. A typical example is that of the time axis when the variable is the time: the 0-cells are the time instants and the 1-cells are the time intervals respectively. For n=2 the representative space is a plane. A cell-complex exhibits

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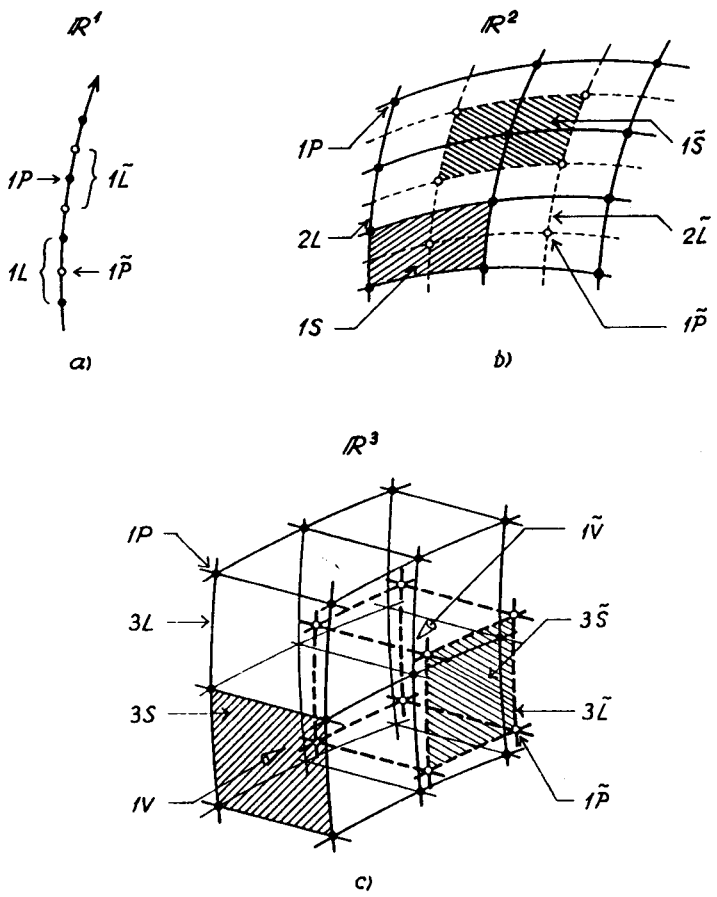


Fig. 3.2.1

points P, line segments L, and surface segments S. A 2-dimensional space R^2 arises, for ex., in the study of the longitudinal vibrations of a bar: in this case one variable is a spatial coordinate and the other is the time. Examples of cell-complex for n=1,2,3 are given in Fig. 3.2.1 (heavy lines). The simbols we have introduced to denote the p-cells and their possible physical meaning are:

0-cell: P- point, time instant, event.

1-cell: L- line segment, time interval.

2-cell: S- surface segment, line segment x time interval.

3-cell: V- volume, surface segment x time interval.

4-cell: H- hypervolume, volume x time interval.

The number of families of p-cells will be written before the symbol of the cell. So in space time we have one family of 0-cells, four families of 1-cells, six families of 2-cells, four families of 3-cells and one families of 4-cells: we shall write 1P, 4L, 6S, 4V, 1H respectively.

3.3. Dual cell-complex.

In physical theories we are lead to consider physical quantities referred to the baricentric points of the n-cells: these baricentric points may be considered as vertices of a second cell-complex that is called the dual of the initial cell-complex K and we shall denote it by $\tilde{K}$. The 0,1,2,3,4-cells of $\tilde{K}$ will be denoted by $\tilde{P}$, $\tilde{L}$, $\tilde{S}$, $\tilde{V}$, $\tilde{H}$ respectively. The dual cell-complex for n=1,2,3 is indicated in fig. 1 (dottes lines). From the way the dual cell-complex has been defined, it follows that to every p-cell of K there correspond a (n-p)-cell of $\tilde{K}$ and viceversa. All that we have said up to now permits to obtain a classification scheme of the basic geometrical elements of a cell complex and its dual as shown, in table 3.3.1.

3.4 Numeration of the cells

It will be useful to number all 0-cell of a cell-complex. Analogous numbering will be made for all 2-cell, 3-cells, etc. The numbering of the p-cells of K is completely indipendent from the numbering of the (p-1)-cells of K. We shall denote α_p the number of p-cells of K. But to number the cells of $\tilde{K}$ it will be useful to assign to a p-cell of $\tilde{K}$ the same number of the corresponding (n-p)-cell of K. We denote with α_p the number of p-cell of K and with β_q the number of q-cells of $\tilde{K}$. Then $\alpha_p = \beta_{n-p}$ (Frana, p.144).

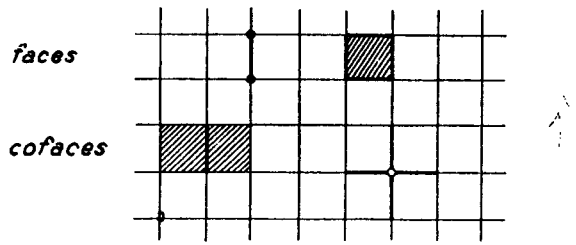
The faces of a p-cell are those (p-1)-cells that form the boundary of the p-cell. The cofaces of a p-cells are those (p+1)-cells that have the given p-cell as common face. Se if we consider a wall that separate two rooms we can say that the two rooms are the cofaces of the wall.

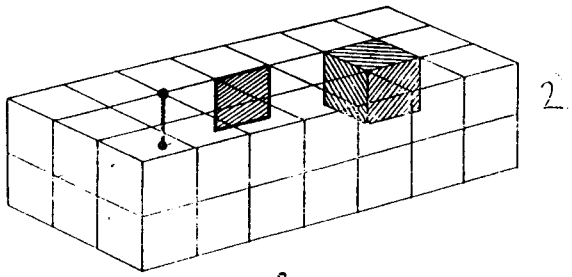
a classification scheme of the basic geometrical elements of a cell-complex K and its dual $\tilde{K}$ in R^n

$n=1$		$n=2$	
K	$\tilde{K}$	K	$\tilde{K}$
1P	$1\tilde{L}$	1P	$1\tilde{S}$
1L	$1\tilde{P}$	2L	$2\tilde{L}$
		1S	$1\tilde{P}$

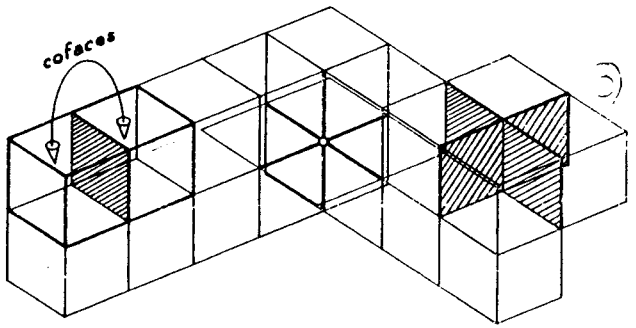
$n=3$		$n=4$	
K	$\tilde{K}$	K	$\tilde{K}$
1P	$1\tilde{V}$	1P	$1\tilde{H}$
3L	$3\tilde{S}$	4L	$4\tilde{V}$
3S	$3\tilde{L}$	6S	$6\tilde{S}$
1V	$1\tilde{P}$	4V	$4\tilde{L}$
		1H	$1\tilde{P}$

faces and cofaces





faces



cofaces

3.5. Orientation of the cells.

A polygon is said to be oriented when we have chosen a direction to go along its boundary. To give an orientation to a polygon it's sufficient to give an orientation to one of its sides and then to propagate the orientation to the adjacent sides according to the rule that if a side enters a vertex, the adjacent side leaves it (Fig. 3.5.1 a). Give two polygons with a common side, if the first one is oriented we can introduce and orientation on the second one with the analogous rule that the orientations induced by the two polygons on their common side be opposite (Fig. 3.5.1. b). This rule assures the compatibility of the orientation of a set at adjacent polygons (3)

A polyedron is oriented when one of its faces is oriented and the orientation is propagated by induction to all faces (Fig. 3.5.1 c). The two possible orientations of a face imply two possible orientations of the polyedron. Using the right-hand screw rule one can see that the two possible orientations of a polyedron correspond to the two possible choices of the inner or outher normals to its boundary. A cell-complex is said oriented when we have fixed an orientation to all its cells of the various orders.

An orientation of the p-cells of such a cell-complex is the one suggested by the following orientation of the coordinate lines (Fig. 3.5.2)

1-cells: the same orientation of the coordinate line x^k;

2-cells: we give an orientation to a 2-simplex (= triangle) of the coordinate manifold x^i, x^k taking the orientation OP_1P_2 and then propagate the orientation to the whole

(3) - Of course for two-sided surfaces: Mobius strips and Klein bottles are excluded from our considerations.

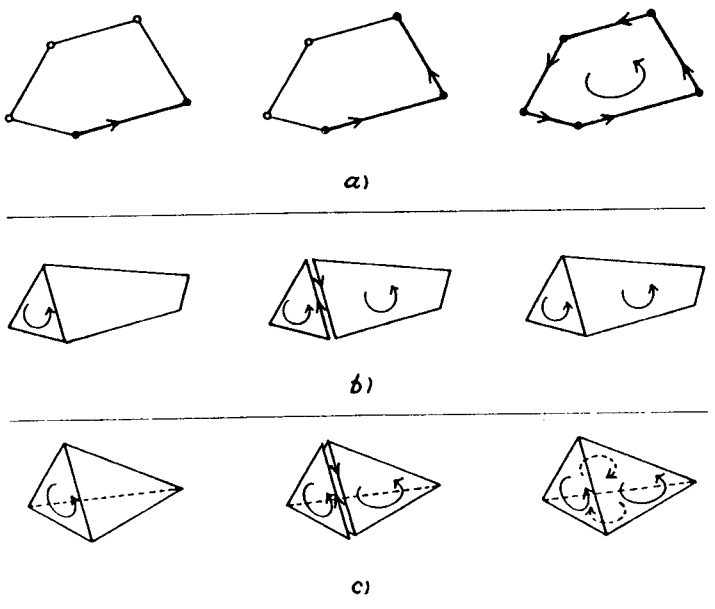


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Fig. 3.5.1

coordinate manifold. For the other coordinate manifolds x^2, x^3 and x^3, x^1 we choose the oriented simplexes OP_2P_3 and OP_3P_1 respectively;

3-cells: we give an orientation to the face $P_1P_2P_3$ taking the vertices in this order, propagating the orientation to the 3-simplex $OP_1P_2P_3$; then we propagate the orientation of the 3-simplex to all 3-cells;

and so on for the cells of higher order.

3.6. Chains.

Let us consider a cell-complex K built up in a three-dimensional region filled of a material continuum. With every 3-cell of K we may associate the amount of mass contained

into the cell: let V_h denotes the h-th 3-cell and m_h the corresponding mass. We can represent the mass distribution by the sequence

$$(m_1, m_2, \dots m_{\alpha_3}) \qquad (3.6.1)$$

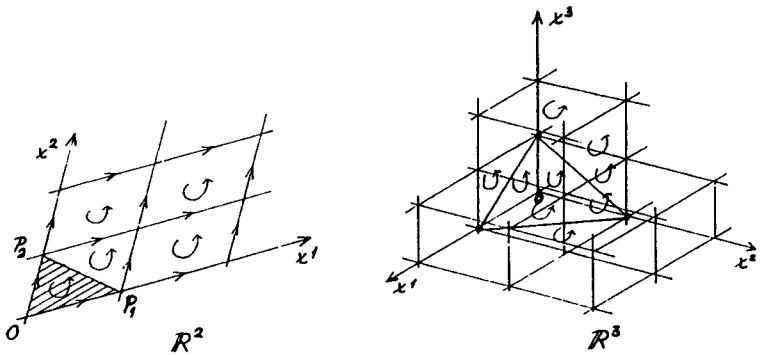


Fig. 3.5.2.

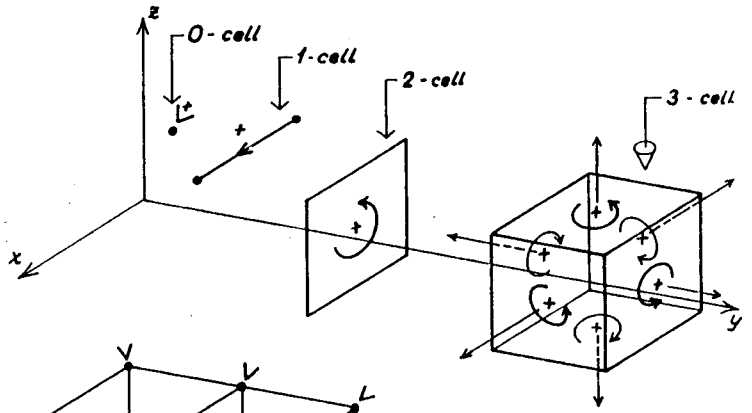
In analogous way we may consider the contract forces trasmitted through every 2-cell of K. Let S_h be the h-th 2-cell and $\bar{F}_h$ be the corresponding contract force (the orientation of the 2-cell is here essential). We may represent the distribution of contract forces by the sequence

$$(\bar{F}_1, \bar{F}_2, \dots \bar{F}_{\alpha_2}) \qquad (3.6.2)$$

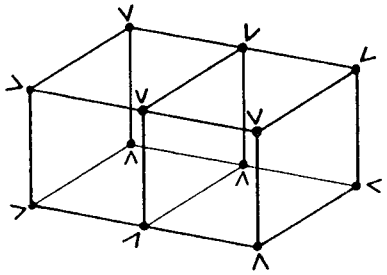
Sequences of this kind are called "chains". On account of the fact that the masses are referred to 3-cells the sequence (3.6.1) is called a three-dimensional chain or briefly 3-chain, while the sequence (3.6.2) is called a 2-chain.

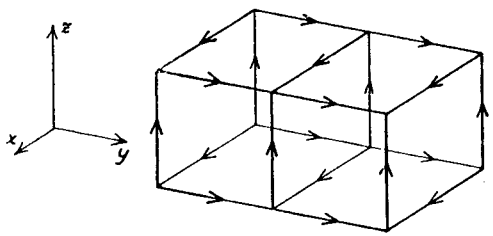
The masses m_h and the forces $\bar{F}_h$ are called the "coefficients" of

orientation of the cells : n = 3

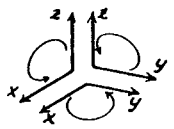


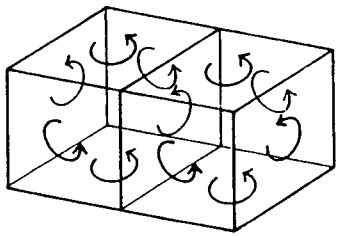
the 0-cells are positively oriented if the incoming lines are considered as positive. (This is the meaning of the symbol =>)





the 1-cells are oriented according with the coordinate lines





the 2-cells are oriented according with the cyclic permutation of pairs of coordinate lines

the corresponding chains. We emphasize the fact that in the first case the coefficients are numbers, while in the second case they are vectors.

When the coefficients are vectors, and not merely numbers, they are applied to different space points. If we wont to conceive them as elements of a single vector space we must transport them at a common space point. No problem arises if the space is of euclidean kind, because the notion of parallel transport is the customary one, largerly used in physics. Nevertheless the introduction of an euclidean metric is not necessary. It suffices that the region Ω be equipped with an absolute parallelism (or teleparallelism). This means that the connection be such that the curvature tensor vanishes. In what follow, we assume that the n-dimensional region Ω of $\mathbb{R}^n$ be flat.

After these preliminary considerations we return to the general case. Let us consider an n-dimensional region Ω of linear space $\mathbb{R}^n$: let us cover the region Ω with a cell-complex K that will be oriented and numbered. Let us consider a set $\mathcal{C}$ of elements which an operation of sum is defined, i.e. with the structure of an additive abelian group. For the purposes of physical theories this may be the group R of reals, the group C of complex numbers, the groups $\mathbb{R}^m$, $\mathbb{C}^m$ of m-ples of real complex numbers, the group of mxm matrices, the group of quaternions or that of Clifford numbers, of spinors, of infinite matrices of of operators acting on some linear space, etc. If we associate with every oriented p-cell σ_h^p of K an element $a_h \in \mathcal{C}$ we have defined a p-chain with coefficients in the group $\mathcal{C}$. (Hocking-Young, 225; Alexandrov, 286).

Then a p-chain is a mapping that assigns to every p-cell an element of $\mathcal{C}$. A p-chain can be denoted with the symbol

$$a^{(p)} = (a_1, a_2, \cdots a_{\alpha_p}) \qquad (3.6.3)$$

Two p-chains can be summed according with the rule

$$a^{(p)} + b^{(p)} \stackrel{def}{=} (a_1 + b_1, a_2 + b_2, \cdots, a_{\alpha_p} + b_{\alpha_p}) \qquad (3.6.4)$$

Moreover one can introduce the null p-chain, we denote it by $0^{(p)}$, whose coefficients are all equals to the null elements of the group $\mathcal{C}$. In this way the set of all p-chains on K has the structure of a group called the chain group.

3.7. Incidence matrices.

Given a p-cell, one may consider those (p-1) cells that are incident with it: these are called the faces of the p-cell. Moreover one may consider the (p+1) cells that are incident with the p-cell: these are called the cofaces of the p-cell.

To every pair formed by a p-cell and a (p+1)-cell we may assign an incidence number that is zero if the two cells are not incident and is + 1 or - 1 according if they are incident with compatible orientation or with opposite orientation. The incidence number of the k-th p-cell with the h-th (p+1)-cell will be denoted by $e_{hk}^{(p+1,p)}$.

The fact that the incidence numbers of the p-cells with the (p+1)-cells have two indices permits to consider them as entries of an incidence matrix

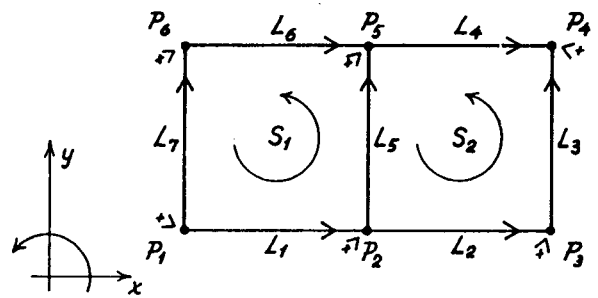
$$E^{(p+1,p)} \stackrel{def}{=} \| e_{hk}^{(p+1,p)} \| \qquad (3.7.1)$$

For a n-dimensional space we have n incidence matrices. So in R^3 we have the three matrices

$$E^{(3,2)} \quad E^{(2,1)} \quad E^{(1,0)} \qquad (3.7.2)$$

A look to an incidence matrix permits to detect the cofaces of a given cell alongside with their mutual orientation. If given a region $\Omega \subset \mathbb{R}^n$ the cell-complex covers the whole region Ω then the connectivity properties in the large of the region are re-

incidence matrices of a complex K



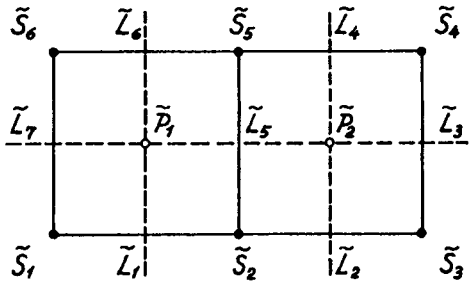
E^0 points

lines	1	2	3	4	5	6
1	-1	+1	0	0	0	0
2	0	-1	+1	0	0	0
3	0	0	-1	+1	0	0
4	0	0	0	+1	-1	0
5	0	-1	0	0	+1	0
6	0	0	0	0	+1	-1
7	-1	0	0	0	0	+1

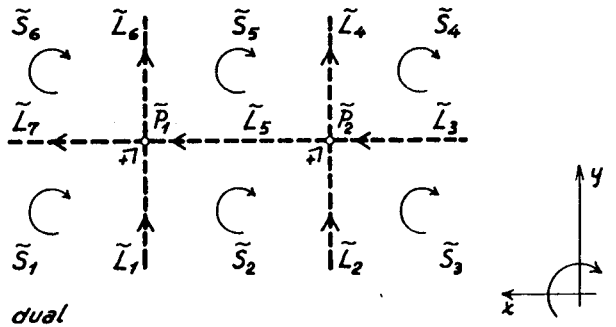
E^1 lines

surfaces	1	2	3	4	5	6	7
1	+1	0	0	0	+1	-1	-1
2	0	+1	+1	-1	-1	0	0

dual cell complex $\tilde{K}$



incidence matrices of the dual complex $\tilde{K}$



$F^{(1,0)}$ dual points

dual lines	1	2
1	+1	0
2	0	+1
3	0	+1
4	0	-1
5	+1	-1
6	-1	0
7	-1	0

$F^{(2,1)}$ dual lines

dual surfaces	1	2	3	4	5	6	7
1	-1	0	0	0	0	0	-1
2	+1	-1	0	0	-1	0	0
3	0	+1	-1	0	0	0	0
4	0	0	+1	+1	0	0	0
5	0	0	0	-1	+1	+1	0
6	0	0	0	0	0	-1	+1

flected by the incidence matrices of the various orders: (Hocking-Young, 224) more precisely they can be inferred from the ranks of these matrices (Patterson, 1966).

3.8. Coboundary of a chain.

Let us consider a p-chain $a^{(p)}$: we perform the following two-steps process to obtain a (p+1)-chain

1) for every p-cell we consider the set of its cofaces. Then we transfer the mathematical entity associated with the p-cell to every coface, with the same or opposite sign according to the fact that the orientation of the coface agrees or not with that induced from the p-cell;
2) for every (p+1)-cell we sum the amounts of the mathematical entities that are transferred to it from its faces. The sum so obtained is naturally referred to the (p+1)-cell.

In this way we have constructed a (p+1)-chain that is called the coboundary of the given p-chain. The two-step process is shown in Fig. 3.8.1. In symbols we write:

$$c^{(p+1)} = \delta a^{(p)} \qquad (3.8.1)$$

where δ denotes the coboundary operator. A matrix representation of this operator is possible using the incidence matrices. Let

$$a^{(p)} = (a_1, a_2, \dots a_{\alpha_p}) \qquad c^{(p+1)} = (c_1, c_2, \dots c_{\alpha_{p+1}}) \qquad (3.8.2)$$

then eq. (3.8.1) becomes

$$c_h = \sum_{1}^{\alpha_p} {}_k \, e_{hk}^{(p+1,\,p)} a_k \qquad (3.8.3)$$

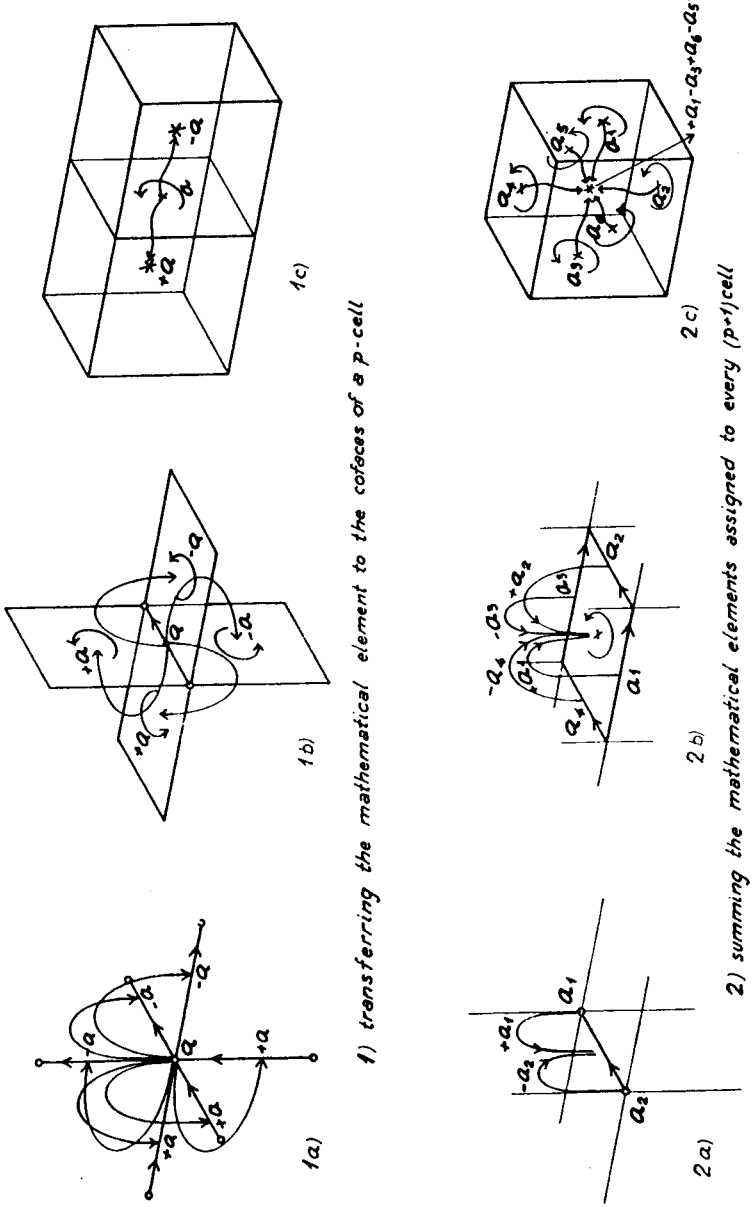
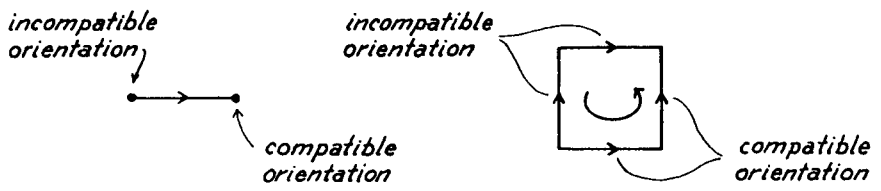


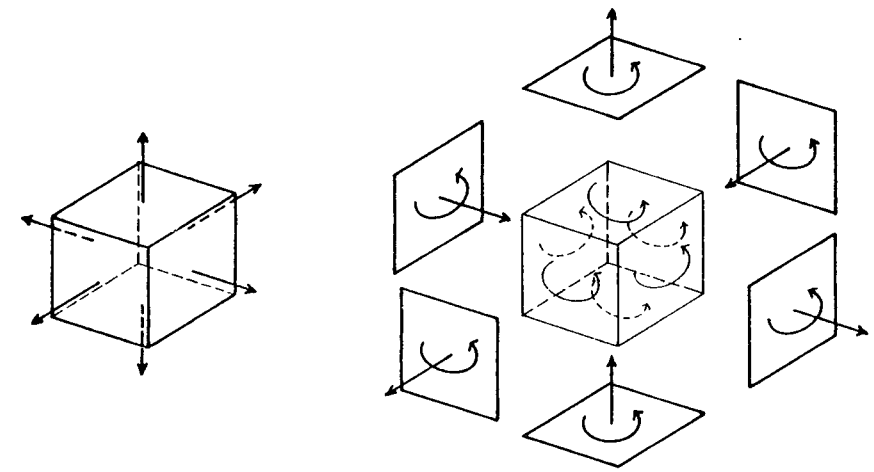
Fig. 3.8.1 the process of forming the coboundary

1) transferring the mathematical element to the cofaces of a p-cell

2) summing the mathematical elements assigned to every (p+1)cell

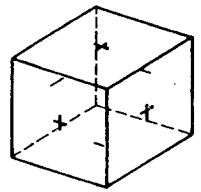
mutual orientation of a cell with its faces





orientation of a 3-cell

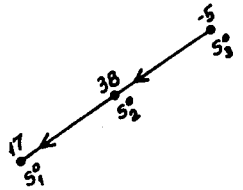
orientation of its faces



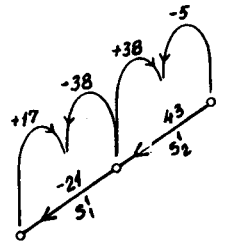
\+ compatible orientation

\- incompatible orientation

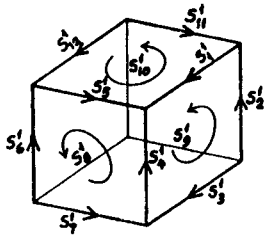
examples of coboundary of chains



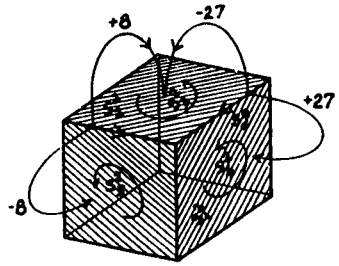
$c^0 = (17, 38, -5)$



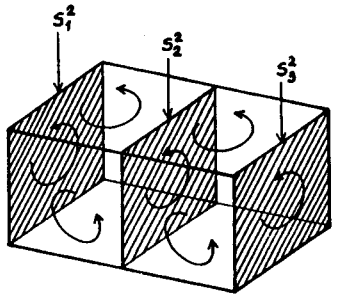
$\delta c^0 = (17-38, 38+5) = (-21, 43)$



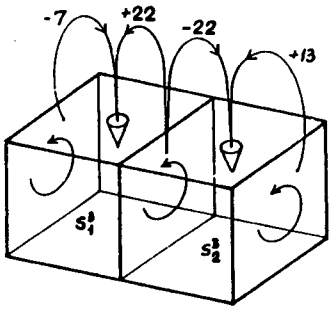
$c^1 = (27,0,0,0,8,0,0,0,0,0,0,0)$



$\delta c^1 = (0, 0, -27, +8, +27, -8, 0) =$
$= (0, 0, -19, +27, -8, 0)$



$c^2 = (7, 22, 13, 0, 0, \ldots 0)$



$\delta c^2 = (-7+22, -22+13) = (15, -9)$

3.9. Poincaré lemma

One car easily check by examples that repeating two times in sequence the process of forming the coboundary of a p-chain the (p+2)-chain so obtained is the null chain $0^{(p+2)}$. In symbols

$$\delta\,\delta\, a^{(p)} = 0^{(p+2)} \qquad (3.9.1)$$

Stated in words: the coboundary of the coboundary of a p-chain vanishes identically. This fundamental property of the coboundary process is known as the Poincarè lemma. As we shall see later (§7) it is the origin of many identities largerly used in physical theories like the identities curl grad $\varphi = 0$, div curl $\mathbf{v} = 0$. It is also the key to find the compatibility conditions and the general solutions of an equation (7). Stated in terms of the incidence matrices the Poincarè lemma becomes

$$E^{(2,1)}E^{(1,0)} = 0 \quad , \quad E^{(3,2)}E^{(2,1)} = 0 \quad , \quad etc. \qquad (3.9.2)$$

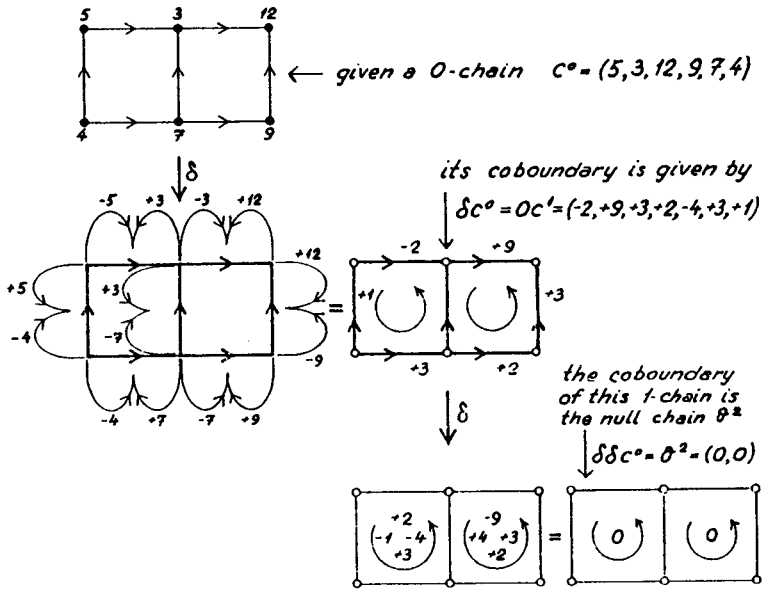
3.10. Particular chains

Among the p-chains there are some whose coboundary vanishes, i.e.

$$\delta c^{(p)} = 0^{(p+1)} \qquad (3.10.1)$$

These are called cocycles (Franz, 1968, p. 44). A p-chain that is the coboundary of a (p-1)-chain is called a p-coboundary. From the Poincarè lemma follows that the coboundary of a chain is a cocycle but the inverse is not generally true, i.e. a cocycle is

example illustrating the property $\delta\delta c^0 = 0^2$



not necessarily a coboundary. Those cocycles that are also coboundaries are called bounding cocycles: the existence of non bounding cocycles is linked to the multiply connected nature of the region Ω covered by the cell-complex.

3.11 Homologous chains

Two p-chains $a^{(p)}$ and $b^{(p)}$ are said homologous if their difference is a coboundary, i.e.

$$a^{(p)} - b^{(p)} = \delta c^{(p-1)} \qquad (3.11.1)$$

To denote that $a^{(p)}$ and $b^{(p)}$ are homologous one writes $a^{(p)} \sim b^{(p)}$. Two homologous p-chains have the same coboundary: this fact is reflected in physical theories by the notion of gauge invariance. The notion of homologous chains provides a justification of the name "homology theory of cell-complexes" given to this branch of algebraic topology.

3.12. Chain spaces

The space of all p-chains with coefficients in the linear space $\mathcal{C}$ defined on a cell-complex K is denoted by $C^p(K,\mathcal{C})$. It is a linear space of the kind R^{α_p} because its elements are the ordered sets of α_p elements $a_1, a_2, \ldots a_{\alpha_p}$ where $a_k \in \mathcal{C}$. The p-cocycles form a subspace denoted by $Z^p(K,\mathcal{C})$. The bounding p-cocycles form another subspace, contained in the previous one, that is denoted by $B^p(K,\mathcal{C})$. The coboundary operator δ is a linear operator whose null space is Z^p and whose range is B^{p+1}. We show in Fig.3.12.1 the various chain spaces pertinent to a cell-complex that covers a region of a three-dimensional space.

The chains of the various dimensions of K and $\tilde{K}$ in R^3 are represented in table 3.12.1. The 0, 1, 2, 3-chains are pertinent to the geometrical elements P,L,S,V respectively. In order to obtain a simple notation in this and in the following schemes we denote by the same letter, say $b^{(1)}$, both an arbitrary 1-chain and those 1-chains that are the coboundary of a 0-chain. So $b^{(1)}$ in the second frame, L, is an arbitrary 1-chain while $b^{(1)}$ appearing in the upper rectangular frame, $b^{(1)} = \delta a^{(0)}$, denotes a 1-chain that is the coboundary of a 0-chain.

The sequences of fig. 6 clarify this point: when we consider the coboundary of those 1-chains $b^{(1)}$ that are in turn the coboundary of a 0-chain $a^{(0)}$ we obtain the null-chain $0^{(2)}: \delta\delta a^{(0)} = 0^{(2)}$. There may be other 1-chains $b^{(1)}$ that are not of the kind $b^{(1)} = \delta a^{(0)}$ but nevertheless are such that $\delta b^{(1)} = 0^{(2)}$. Lastly there are 1-chains $b^{(1)}$ such that $\delta b^{(1)} \neq 0^{(2)}$. The fact

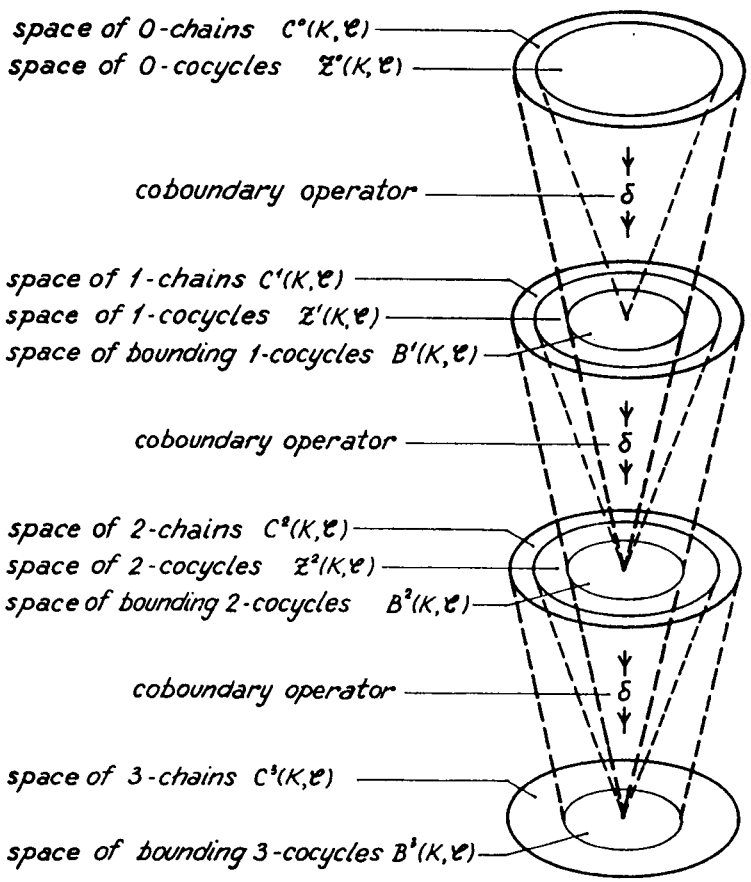
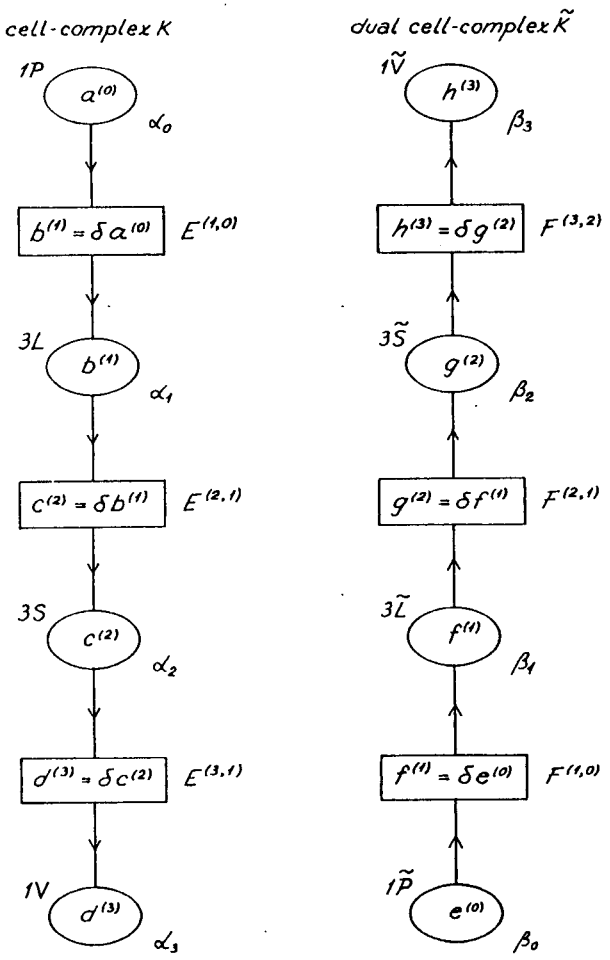


Fig. 3.12.1

table 3.12.1 *a classification scheme for the chains of K and $\tilde{K}$ in $\mathbb{R}^3$*



of using the same symbol $b^{(1)}$ to denote both a general 1-chain and a bounding 1-chain may appear at the beginning rather confusing.

But once we remind the distrinction we obtain a very great semplification in all subsequent schemes.

3.13. Generalized Stokes' theorem

Let us consider as a preliminary examples the two-dimensional cell complex on Fig. 3.13.1. Let $a^{(1)} = (a_1, a_2, \ldots a_7)$ be a 1-chain and $b^{(2)} = (b_1, b_2)$ its coboundary Since

$$\begin{cases} b_1 = (+1)a_6 + (+1)a_4 + (-1)a_1 + (-1)a_7 \\ b_2 = (+1)a_5 + (+1)a_3 + (-1)a_2 + (-1)a_4 \end{cases} \qquad (3.13.1)$$

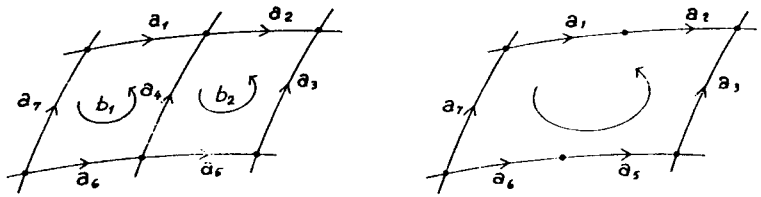


Fig. 3.13.1

adding the two equations we obtain the relationship

$$b_1 + b_2 = (+1)a_6 + (+1)a_5 + (+1)a_3 + (-1)a_2 + (-1)a_1 + (-1)a_7 \qquad (3.13.\ )$$

This identity states that the sum of the mathematical entities associated with the 2-cells of a two-dimensional manifold by the coboundary process is equal to the sum of the mathematical entities associated with the 1-cells that form the boundary of the manifold. This is the essence of the celebrated Stokes theorem:it is a simple consequence of the process of making the coboundary

of a 1-chain. It is also apparent that one can obtain analogous theorems starting from p-chains. To prove it let us consider a (p-1)-chain $a^{(p-1)}$ and its coboundary $b^{(p)} = \delta a^{(p-1)}$.
This means that

$$b_h = \sum_{1}^{\alpha_p}{}_k \, e_{hk}^{(p,p-1)} a_k \qquad (3.13.3)$$

An usual a_k and b_h are elements of a linear space $\mathcal{C}$. Let us chose an arbitrary set of p-cells and denote by ω_p the p-dimensional manifold covered by them. Let us sum the mathematical entities b_h of the various cells of ω_p : we obtain

$$\sum_{\omega_p}{}_h \, b_h = \sum_{\omega_p}{}_h \left(\sum_k e_{hk}^{(p,p-1)} a_k \right) = \sum_k \left(\sum_{\omega_p}{}_h \, e_{hk}^{(p,p-1)} a_k \right) \qquad (3.13.4)$$

In the last sum many terms vanish: first of all vanish all terms pertinent to those (p-1)-cells that are not faces of one of the p-cells that belong to ω_p (their incidence numbers vanish) and secondly vanish all pairs of terms that correspond to those (p-1)-cells that are common faces of two adjacent p-cells (the sum of their incidence numbers vanishes). At the end only those terms survive that are pertinent to the oriented boundary $\partial\omega_p$ of the p-dimensional manifold ω_p, with the sign plus or minus according if the orientation of the (p-1)-cell agrees or not with that of the boundary $\partial\omega_p$.
Finally we can write

$$\sum_{\omega_p}{}_h \, b_h = \sum_{\partial\omega_p}{}_k (\pm) a_k \qquad (3.13.5)$$

This relationship expresses the generalized Stokes'Theorem (Franz, 1968, p. 46). It is "generalized" in two distinct ways: first because it is valid for a p-dimensional manifolds instead of the usual Stokes' theorem valid for p=2 and second because is valid whatever the mathematical nature of a_k and b_h may be, while the ordinary Stokes theorem is usually proved under the hypotesis that a_k and b_h are numbers (or at most tensors). In particular

when they are numbers we have
for p = 3 Gauss theorem
for p = 2 Stokes' theorem
for p = 1 fundamental theorem of the integral calculus.

The proof we have given shows that the generalized Stokes' theorem is a simple and immediate consequence of the coboundary process. Since no metrical properties are involved in this process the non metrical nature of Stokes' theorem is apparent.(Synge-Schild, 1956, p. 267).

4 CHAINS AND COBOUNDARIES IN PHYSICAL THEORIES

Among the physical quantities of a physical theory, there are some that are naturally referred to the geometrical objects of the space of the indipendent variables. If we introduce a cell-complex in this space the distribution of these physical quantities may be described by chains. To single out these physical quantities we observe that they are usually global quantities, i.e. in field theories they arise by integration of the field variables on lines, surfaces, volumes, hypervolumes, etc.

4.1. Examples of chains

We give here a number of examples. The magnetic flux, given by the integral

$$\phi = \iint_S B^h n_h \, dS \tag{4.1.1}$$

is naturally referred to the surfaces (S). In particle mechanics the displacement vector given by

$$s = \int_{t_0}^{t_1} v \, dt \tag{4.1.2}$$

is naturally referred to the time intervals (L); in continuum mechanics the displacement vector u^h is naturally referred to the points (P). Always in continuum mechanics the stress tensor $p_h{}^k$ gives rise by integration on a surface to the contract force

$$f_h = \iint_S p_h{}^k n_k \, dS \tag{4.1.3}$$

that is naturally referred to surfaces (S). The electromagnetic potential A_α give rise, by integration on a space-time line, to a circulation

$$C = \int_L A_\alpha \, dL^\alpha \tag{4.1.4}$$

that is naturally referred to lines (L). The stress-energy-momentum tensor give rise, by integration on a volume, to the energy-momentum vector

$$P_\mu = \iiint_V T_\mu{}^\nu n_\nu \, dV \qquad (4.1.5)$$

of space-time that is naturally referred to volumes (V).

In quantum mechanics the probability π to find a particle into a volume at a given time instant is naturally referred to the volumes (V); the probability production $\dot{\pi}$ is naturally referred to the product of a volume by a time interval and then to an hypervolume (H).

As we have said al these physical quantities give rise to chains: the coefficients of these chains are the amount of the physical quantity associated with the corresponding cells. They can be real or complex numbers, vectors, tensors, matrices, operators, etc. So in quantum mechanics, according to the Heisemberg formulation, to every time instant (P) we may associate an infinite dimensional matrix $\mathbf{Q}$ whose entries are (Born-Jordan, 1967)

$$q_{mn}(t) = a_{mn} e^{2\pi i \nu_{nm} t} \qquad (4.1.6)$$

Then we have a 0-chain with coefficients in the linear space $\mathcal{C}$ of the infinite dimensional matrices. In the mechanics of polar continua with every 3-cell two vectors are associated: the external force and the external couple. The aggregate formed by a polar and an axial vector is called "motor" (Schaefer, 1967) (Brand, 1947). It comes out that a motor is a particular Clifford number, like the quaternions, the complex numbers and the spinors.

The space of coefficients is the linear space of Clifford numbers. The amplitude ψ of the Schrödinger field give rise to a 0-chain with coefficients in the complex field $\mathbf{C}$. The operators of quantum mechanics will be referred to the same geometrical objects to which the corresponding classical observable are referred.

4.2. Examples of coboundaries

We now propose to show that there are some typical equations of every physical theory that can be described by the coboundary process. Of this kind are the balance equations, in particular the conservation laws, the equilibrium equations, the statical balances (e.g. the Gauss theorem of electrostatics). Of this kind are also the equations that deal with a circuital law (e.g. Ampére law of currents) the compatibility equations, in particular those that states the irrotational nature of a vector field. Another class is that of the equations that define the gradients of a scalar, vector or tensor field, those that define the velocity (that will be considered as a time gradient). Of this kind are also the general solutions of the balance equations as, for ex. those that introduce the stress potentials in continuum mechanics or the dual electromagnetic potential in electromagnetism, etc.

4.3. Balance equations

The typical structure of a balance equation

$$\left\{\begin{array}{l}\text{amount of a}\\ \text{physical quantity}\\ \text{PRODUCED inside}\\ \text{a volume during}\\ \text{a time interval}\end{array}\right\} = \left\{\begin{array}{l}\text{amount of the}\\ \text{physical quantity}\\ \text{STORED inside}\\ \text{the volume during}\\ \text{the time interval}\end{array}\right\} + \left\{\begin{array}{l}\text{amount of the}\\ \text{physical quantity}\\ \text{OUTFLOWED}\\ \text{from the boundary}\\ \text{during the time in}\\ \text{terval}\end{array}\right\}$$

We shall consider a balance equation in a local form: to this we consider a 3-cell of a cell-complex in R^3 bounded by six coordinate surfaces and we orient the 3-cell

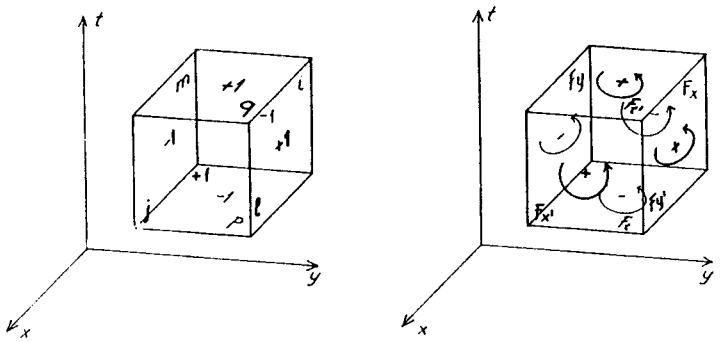


Fig. 4.3.1

chosing the outher normal to its boundary. Also the time axis will be divided in 1-cells. It will be useful to consider a four dimensional space-time, as the space of the four variables t, x^1, x^2, x^3 [8] and to consider a space-time cell-complex. Let F be a physical quantity: we shall denote by $F^{(P)}$ the amount of F produced in side a 3-cell during a time interval $(t'-t)$: then $F^{(P)}$ is referred to an hypervolume (4-cell) of R^4. Let F_t the amount of F contained into the 3-cell at the time instant t: the amount of F stored in the time interval $(t'-t)$ is $F^{(S)} = F_{t'} - F_t$: it is naturally referred to those 3-cells that are of spatial kind (x,y,z). Now let us denote by F_x the amount of F that outflow through the face yz during the time interval $(t'-t)$: then F_x is naturally referred to those 3-cells that are of mixed kind, i.e. the cells (y,z,t). Let us denote by $F_{x'}$ the corresponding amount pertinent to the opposite surface. The balance equation

$$F^{(P)} = F^{(S)} + F^{(O)} \qquad (4.3.1)$$

(8) - This is a kinematical space that need not to have a metrical structure, in particular it is not necessarily the space-time of relativity.

can be written

$$F^{(P)} = (F_{t'} - F_t) + (F_{x'} - F_x) + (F_{y'} - F_y) + (F_{z'} - F_z) \qquad (4.3.2)$$

An useful picture can be obtained ignoring one space coordinate, say z as shown in Fig.4.3.1. The balance equation (4.3.2) must be valid for every 4-cell of the cell-complex K in the space-time. Then if we consider the h-th 4-cell, and refer at a first time to Fig.4.3.1 for notations, we may write eq. (4.3.2) as follows

$$\begin{aligned} F_h^{(P)} &= (+1)F_q + (-1)F_p + (+1)F_j + (-1)F_i + (+1)F_t + (-1)F_m = \\ &= e_{hq}F_q + e_{hp}F_p + e_{hj}F_j + e_{hi}F_i + e_{ht}F_t + e_{hm}F_m \end{aligned} \qquad (4.3.3)$$

where e_{hk} denote the incidence numbers of the h-th 3-cell with the k-th 2-cell. Returning to the general case we may write

$$F_h^{(P)} = \sum_1^{\alpha_3}{}_k \, e_{hk}^{(4,3)} F_k \qquad (h = 1,2,3,\ldots \alpha_4) \qquad (4.3.4)$$

Is this sum the physical quantities F_k that are referred to those 3-cells that are not faces of the h-th 4-cell does not appear, because their incidence numbers vanish. Since the F_k describe a 3-chain on K and the $F_h^{(P)}$ describe a 4-chain we may write

$$\begin{aligned} F^{(3)} &= (F_1, F_2, \ldots F_{\alpha_3}) \\ G^{(4)} &= (F_1^{(P)}, F_2^{(P)}, \ldots F_{\alpha_4}^{(P)}) \end{aligned} \qquad (4.3.5)$$

and then eq.(4.3.4) is a realization of the equation

$$G^{(4)} = \delta F^{(3)} \qquad (4.3.6)$$

i.e. the balance equation (4.3.2) asserts that a given a 4-chain is the coboundary of a given 3-chain.

We emphasize that the mathematical nature of the physical quantity F (scalar, vector, matrix, operator, etc.) is completely arbitrary: a balance is essentially indipendent from the nature of the physical quantity on which the balance is made. This is

reflected in the language of the homology theory of cell-complex by the fact that the coboundary process is indipendent of the nature of the space of the coefficients $\mathcal{C}$ of the chains on which it is performed. The universal nature of the balance equations is evidentiated by the natural association of the measurable quantities with the geometrical objects. Balance equations do not use metrical properties of the space; they do not depend on the shape of the space-time region on which the balance is made, neither on the measure of the areas, volumes and time intervals. So the displacement of a particle in space, during a time interval, is associated with that time interval and does not depend from the measure of the time interval itself. Perhaps this is difficult to accept because we are used to assign time intervals giving the measure of it with reference to a conventional time interval.

4.4. Conservation laws

When a physical quantity F is not produced, i.e. it is conserved, the balance becomes a conservation law.

The equation (4.3.6) becomes

$$\delta F^{(3)} = 0^{(4)} \tag{4.4.1}$$

and this states that the 3-chain $F^{(3)}$ is a cocycle. Then a conservation law can be stated asserting that a given 3-chain is a cocycle. This point links the notion of cocycle of the homology theory with the conservation laws of physical theories.

4.5. Equilibrium equations

Let us consider a material continuum and, in it, a three-dimensional region. The equilibrium condition asserts that the sum of the contract forces transmitted through the boundary of the region is equal to the external force acting on the matter contained in the region. If we consider as usual a 3-cell and call F_h the exterior force acting on the h-th cell, f_m the contract force

acting on the m-th 2-cell we may consider the two chains

$$\begin{aligned} F^{(3)} &= (F_1, F_2, \ldots F_{\alpha_3}) \\ f^{(2)} &= (f_1, f_2, \ldots f_{\alpha_2}) \end{aligned} \tag{4.5.1}$$

with reference to Fig.4.5.1 the balance equation can be written

$$F_h = +f_j - f_i + f_l - f_m + f_q - f_p \tag{4.5.2}$$

or

$$F_h = \sum_{1}^{\alpha_2} {}_k \, e_{hk}^{(3,2)} f_k \qquad (h = 1, 2, \ldots \alpha_3) \tag{4.5.3}$$

i.e.

$$F^{(3)} = \delta f^{(2)} \tag{4.5.4}$$

Then an equilibrium equation states that a given 3-chain is the coboundary of a given 2-chain.

4.6. Circuital laws

Let us consider a small parallelogram as indicated in Fig. 4.5.1 and a physical quantity f referred to the lines.

A typical circuital law has the form

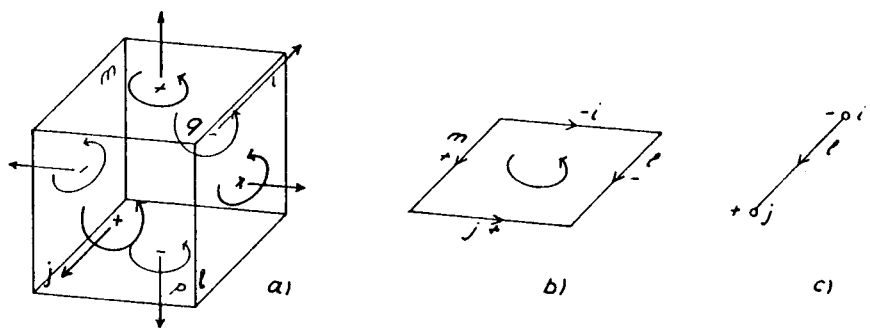


Fig. 4.5.1

$$F = +f_j - f_i + f_\ell - f_m \qquad (4.6.1)$$

where F is the amount of f associated with the 2-cell. This law can be written

$$F_h = \sum_{1\,k}^{\alpha_1} e_{hk}^{(2,1)} f_k \qquad (4.6.2)$$

or

$$F^{(2)} = \delta f^{(1)} \qquad (4.6.3)$$

where we have put

$$f^{(1)} = (f_1, f_2, \ldots f_{\alpha_1}) \qquad F^{(2)} = (F_1, F_2, \ldots F_{\alpha_2}) \qquad (4.6.4)$$

Then a circuital law is a statement that a given 2-chain is the coboundary of a given 1-chain.

4.7. Irrotationality conditions

The typical condition for a vector field to be irrotational is that the circulation of the vector along every reducible closed line vanishes. Since with the circulation we may define a 1-chain $c^{(1)}$ the irrotationality condition becomes

$$\delta c^{(1)} = O^{(2)} \qquad (4.7.1)$$

i.e. if a vector field is irrotational the corresponding 1-chain of the circulations is an 1-cocycle.

4.8. The gradients

Let us consider the process of forming the gradient of a scalar, vector or tensor field. The first act is that of forming the differences between the physical quantities f_i and f_j referred to two points P_i and P_j (see Fig. 4.5.1). This difference

$$F_h = f_j - f_i \qquad (4.8.1)$$

is associated with the line segment: we may write

$$F_h = \sum_{1\,k}^{\alpha_0} e_{hk}^{(1,0)} f_k \qquad (4.8.2)$$

or

$$F^{(1)} = \delta f^{(0)} \qquad (4.8.3)$$

Then we see that the construction of the gradients of a physical quantity associated with the points is equivalent to the process of forming the coboundary of a 0-chain.

4.9. The compatibility conditions

Given an equation of the kind

$$b^{(p+1)} = \delta a^{(p)} \qquad (4.9.1)$$

we may consider $b^{(p+1)}$ as given and ask for a chain $a^{(p)}$ that solves it. We may ask for the conditions on the (p+1)-chain $b^{(p+1)}$ in order that a solution exists. A necessary condition is easily found as consequence of the Poincarè lemma:

$$\delta b^{(p+1)} = O^{(p+2)} \qquad (4.9.2)$$

i.e. $b^{(p+1)}$ must be a cocycle. This condition is sufficient if the region Ω covered by the cell-complex is simply connected to the order (p+1) (i.e. all closed (p+1)-dimensional manifolds are reducible). When this do not happens the condition (4.9.2) must be supplemented with the condition

$$\sum_{\gamma_{p+1}\,k} b_k = O \qquad (4.9.3)$$

where γ_{p+1} is an arbitrary closed (p+1)-dimensional manifold formed by (p+1)-cells, the index k assumes all the values corresponding to the (p+1)-cells that form γ_{p+1}. Lastly O denotes the null element of $\mathcal{G}$. This is the first de Rham theorem for chains.

4.10. The general solution

Given an equation of the kind

$$b^{(p+1)} = \delta a^{(p)} \qquad (4.10.1)$$

with a given (p+1)-chain $b^{(p+1)}$ we may ask for the class of p-chains $a^{(p)}$ that satisfy the equation. If $a^{(p)}$ is a solution of the equation and $c^{(p-1)}$ denotes an arbitrary (p-1)-chain also the p-chain

$$\bar{a}^{(p)} = a^{(p)} + \delta c^{(p-1)} \qquad (4.10.2)$$

is solution of the equation. From the definition of homologous chains we see that the all solutions of eq.(4.10.1) are homologous chains. This fact establishes the link between the general solution of an equation and the theory of homology between chains.

4.11. Gauge invariance

The statement that $a^{(p)}$ and $\bar{a}^{(p)}$ are both solutions of the equation (4.10.1) means that $b^{(p+1)}$ considered as defined by the eq.(4.10.1) does not change if to $a^{(p)}$ we add the coboundary of an arbitrary (p-1)-chain. This is the well-known gauge invariance for the eq. (4.10.1) . Then the gauge invariance is linked with the notion of homologous chains.

The fact that so a large number of equations typical of all physical theories can be expressed by a single process of performing the coboundary of a chain indicates the existence of a formal structure common to physical theories.

5. A CLASSIFICATION SCHEME: DISCRETE CASE

In § 4. we have shown that those physical quantities that are referred to geometrical elements of the space of the indipendent variables give rise to chains and that many physical laws can be described by the coboundary process.
In this section we show that if the chains described by the configuration-kind variables are defined on a cell-complex K then the chains described by the source-kind variables are naturally defined on the dual cell-complex $\tilde{K}$. Moreover we show that the various chains of K and $\tilde{K}$ and their connecting equations can be inserted into a scheme that is the same for many physical theories (table 5.3.1).

To do this we take as usual an experimental point of view and analyze two simple theories: electrostatics and classical particle dynamics. The results of such an analysis will indicate the way to obtain a similar classification for other theories.

5.1 Electrostatics

Let us consider an electrostatic field in a region Ω of the three-dimensional space: this region will be covered a cell complex K and its dual $\tilde{K}$.

Let us consider the electric potential φ that describes the configuration of the electric field; it is associated with the points, then its distribution in the region Ω is described by a 0-chain. Since it is immaterial to consider 0-cells of K or of $\tilde{K}$ for this assignement, we decide to choose those of the primal cell complex K .Then we can write

$$\varphi^{(0)} = (\varphi_1, \varphi_2, \dots \varphi_{\alpha_0}) \quad \begin{pmatrix} 0\text{-chain} \\ \text{of } K \end{pmatrix} \qquad (5.1.1)$$

The electric voltages V_h are naturally referred to the couples of points of K ,i.e. with the 1-cells of K .Then we can

write

$$v^{(1)} = (v_1, v_2, \dots v_{\alpha_1}) \qquad \text{(1-chain of } K\text{)} \qquad (5.1.2)$$

The relation $v_h = \varphi_j - \varphi_i$ implies that

$$v^{(1)} = \delta \varphi^{(0)} \qquad (5.1.3)$$

i.e. the 1-chain of the electric voltages is the coboundary of the 0-chain of the electric potential.

The electric charge Q, the source of the electric field, is naturally referred to three-dimensional regions: it follows that are the charge distribution in the region Ω is described by the 3-chains that assigns to every 3-cell the amount of charge contained inside the cell.

It is natural to refer the electric charge to the 3-cells of $\tilde{K}$.

$$Q^{(3)} = (Q_1, Q_2, \dots Q_{\beta_3}) \qquad \text{(3-chain of } \tilde{K}\text{)} \qquad (5.1.4)$$

The electric flux ϕ is naturally referred to the surfaces: it follows that the electric flux distribution is described by a 2-chain that assigns to every 2-cell the amount of flux through the cell. Now we are no longer free to assign the fluxes to the 2-cells of K or of $\tilde{K}$ but we are constrained to assign them to the 2-cells of $\tilde{K}$. In fact Gauss theorem requires a balance between the charge contained inside a three-dimensional region and the flux through its boundary. Then we can write

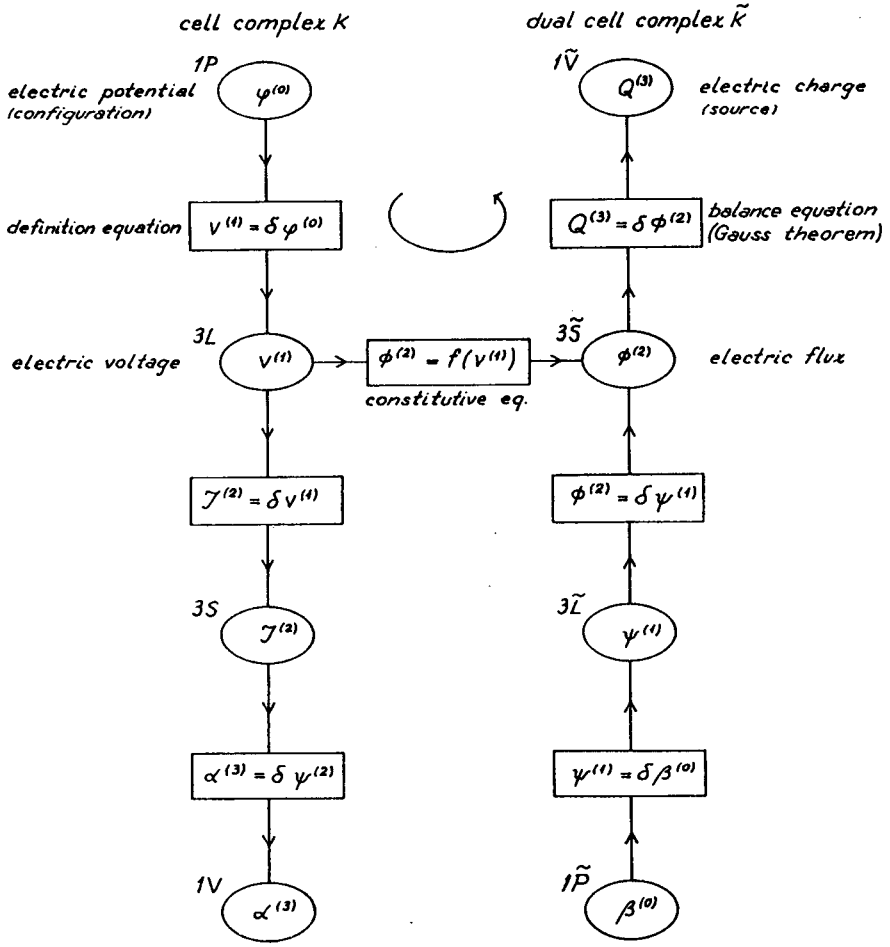
$$\phi^{(2)} = (\phi_1, \phi_2, \dots \phi_{\beta_2}) \qquad \text{(2-chain of } \tilde{K}\text{)} \qquad (5.1.5)$$

For what we have said in § 3.9 Gauss theorem can be written as

$$\delta \phi^{(2)} = Q^{(3)} \quad \text{(balance equation: Gauss theorem)} \qquad (5.1.6)$$

This can be stated as follows: the 3-chain of the electric charges is the coboundary of the 2-chain of the electric fluxes.

table 5.1.1 the classification scheme of electrostatics



dary of the 0-chain $\varphi^{(0)}$ and then in this case its coboundary may do not vanish

$$\delta\bar{v}^{(1)} = j^{(2)} \qquad (5.1.11)$$

This 2-chain so obtained may have a physical meaning.

The scheme indicates the possible chains: only a part of them may be filled of physical content. The remaining chains may be useful only for the mathematical treatment of the theory or may acquire a physical meaning with the progress of the knowledge in the theory.

5.2 Particle dynamics.

As a second example let us consider the motion of a point particle. The physical variables like the radius vector the velocity, the momentum, the force, etc. are all functions of the time. Then we consider the time axis as the space of the indipendent variable. On it we choose a time interval (0,T) in which the motion is studied and we cover it with a cell-complex. The 0-cells (P) are the time instants and the 1-cells (L) are the time intervals. The dual cell-complex is formed by the time instants ($\tilde{P}$) in the middle of the time intervals L and by the corresponding time intervals ($\tilde{L}$) (see fig. 5.2.1)

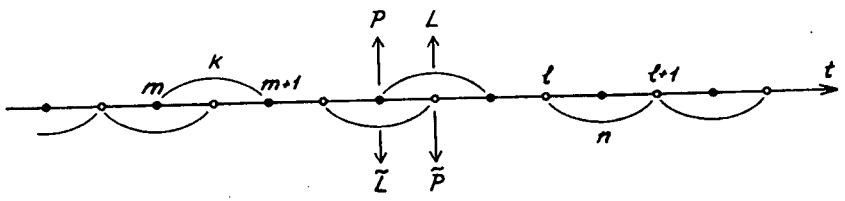


Fig. 5.2.1

The configuration of the particle is described by the radius vector r : it is natural to associate it with the instants of the primal cell complex K . The time dependence of r is then described by the 0-chain of K

$$r^{(0)} = (r_1, r_2, \dots r_{\alpha_0}) \qquad \text{(0-chain of } K\text{)} \qquad (5.2.1)$$

The displacement s of the particle during a time interval is described by the 1-chain of K

$$s^{(1)} = (s_1, s_2, \dots s_{\alpha_1}) \qquad \text{(1-chain of } K\text{)} \qquad (5.2.2)$$

Since the displacement is linked to the radius vector by the definition equation (see Fig. 5.2.1 for notations)

$$s_k = r_{m+1} - r_m \qquad (5.2.3)$$

we have the relation

$$s^{(1)} = \delta r^{(0)} \qquad (5.2.4)$$

As source variable we take the impulse h , a physical quantity referred to the time intervals. It is natural to refer them to the time intervals of $\tilde{K}$. Hence the time distribution of the impulses given to the particle is described by an 1-chain:

$$h^{(1)} = (h_1, h_2, \dots h_{\beta_1}) \qquad \text{(1-chain of } \tilde{K}\text{)} \qquad (5.2.5)$$

on the dual cell-complex $\tilde{K}$. The momentum p is then naturally associated with the dual time instants $\tilde{P}$ and its time distribution is described by the 0-chain of $\tilde{K}$

$$p^{(0)} = (p_1, p_2, \dots p_{\beta_0}) \qquad \text{(0-chain of } \tilde{K}\text{)} \qquad (5.2.6)$$

The link between the impulse and the momentum is expressed by the balance equation (see fig. 5.2.1 for notations)

$$h_n = p_{e+1} - p_e \qquad (5.2.7)$$

and then it is expressed by the relation

$$h^{(1)} = \delta p^{(0)} \qquad \text{(balance equation-Newton's equation)} \qquad (5.2.8)$$

At this point we consider the constitutive equations. In the nonrelativistic mechanics the link between momentum and displacement is given by

$$p_k = m \frac{s_k}{T_k} \qquad \text{(constitutive equation)} \qquad (5.2.9)$$

where m is the mass of the particle and T_k is the measure of the time interval to which the displacement is referred. Once more we see that the constitutive equations require the intervention of the geometry because we need the extension of the 1-cells. Another constitutive equation exists when the impulse is linked with the radius vector (and perhaps with the displacement) as in the elastic restoring force with viscous damping described by the relation

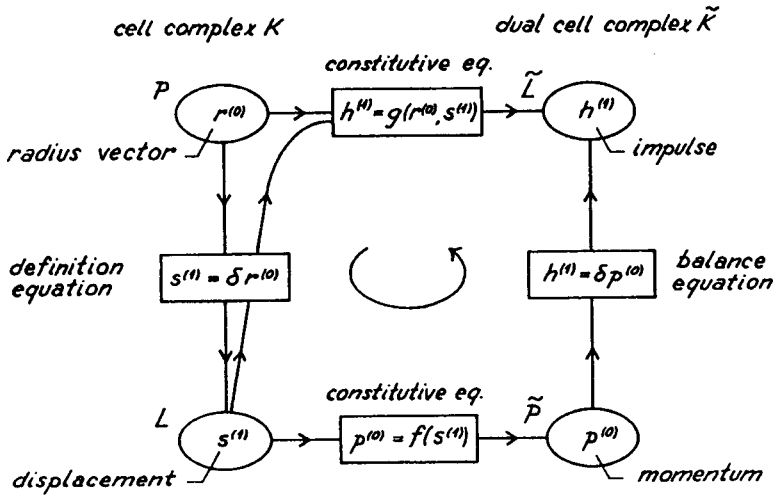
$$h_k = -\mu T_k r_k - \nu s_k \qquad \text{(constitutive equation)} \qquad (5.2.10)$$

The two constitutive equations (5.2.9) and (5.2.10) can be restated as mappings between chains of K and $\tilde{K}$ as follows

$$\begin{cases} p^{(0)} = f(s^{(1)}) \\ h^{(0)} = g(r^{(0)}, s^{(1)}) \end{cases} \qquad (5.2.11)$$

All these things may be collected in a classification scheme as shown in Tab. 5.2.1. The two previous examples suggest the construction of a classification scheme for the physical quantities and equations for every physical theory.

table 5.2.1 : the classification scheme of particle dynamics



5.3 Classification scheme

In order to construct a classification scheme for a physical theory, the first thing to do is to choose independent variables on which the other physical quantities depend. This choice is largerly a matter of convenience. Moreover once we have found a set of variables we are free to choose only some of them as independent variables and to consider the remaining as parameters.

So if $\varphi(x,y,z,t)$ is a function of space and time variables, one may be interested to the time evolution of φ at a given point: in this case we may consider the time t as independent variable and the remaining space variables as parameters. In this case the space of the independent variables is one dimensional (the time axis).

But one can also be interested to the space distribution of φ at a given time instant t: then we may consider the three space variables as independent variables and the time as a parameter. In this case the space is three-dimensional.

If one spatial dimension, say z, can be neglected or if the function has constant value along $\vec{z}$, one may omit $\vec{z}$ and consider x,y,t, as independent variables. And so on.

After these premises the construction of the scheme proceede along the following steps:

1) We choose a set of n independent variables, and consider the n-dimensional space $\mathbb{R}^n$ of the independent variables. Since every independent variables has a domain of variability, se are interested not the whole $\mathbb{R}^n$ but to a region Ω whose points have admissible coordinates.

2) The region Ω will be covered by a cell-complex K and by its dual $\tilde{K}$ whose 0-cells are the baricentric points of the n-cells of K .

3) At this point we remember the preliminary classification

of physical variables we have given in § 2. Those configuration-kind variables (§2.3) that are referred to the geometrical objects of $\mathbb{R}^n$ will be described by chains of K, while those source-kind variables (§2.4) that are referred to the geometrical objects of $\mathbb{R}^n$ will be described by chains of $\tilde{K}$.

4) The various chains of K and $\tilde{K}$ are then inserted in the scheme of table 5.3.1.

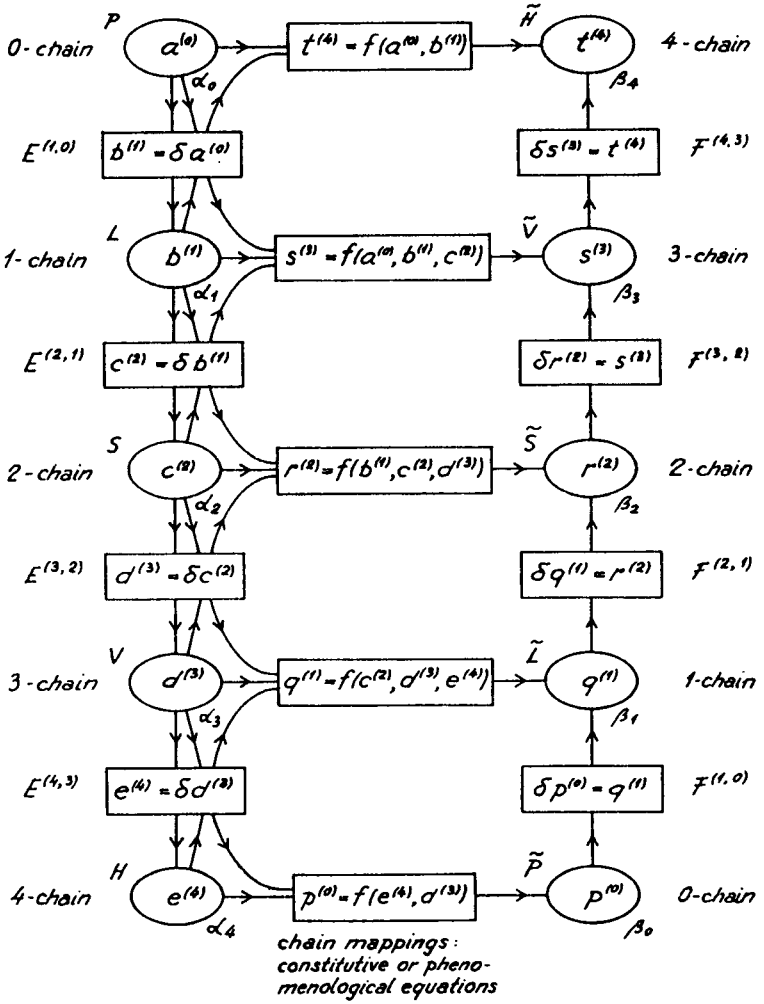
5) The constitutive equations are represented by mappings between the chains of the two columns.

6) The equations of structure are described by the coboundary process, i.e. by mappings between the chains of a same column.

We remark that only some of the chains of the scheme may have a physical content. The remaining ones may be useful for the mathematical treatment of the theory or may acquire a physical meaning as consequence of the description of new phenomena.

table 5.3.1

n = 4 classification scheme: discrete systems



6. MULTIVECTOR CALCULUS

6.1 Introduction

In order to pass from a discrete cell-complex to the continuum we must find an algebraic tool to describe a p-cell. Like an infinitesimal oriented 1-cell may be described by an infinitesimal vector, so an infinitesimal oriented 2-cell may be described by the geometrical object formed by two infinitesimal vectors with a common origin called "bivector". In general an infinitesimal oriented p-cell may be described by a set of P infinitesimal vectors with the same origin i.e. by an infinitesimal "p-vector". This leads us to use the so-called multivector calculus: it comes out that the multivector calculus is a corner-stone of many mathematical tools used in physics. It is that natural generalization of the vector calculus that retains two main features:the geometrical content and the syntetic notation.

We shall show in a later chapter that the multivector theory lead to the concept of Grassmann number conceived as the aggregate formed by a scalar, a vector, a bivector, an n-vector, where n is the dimension of the space. On such aggregates one define naturally the notion of exterior product and obtain in this way an algebra, the Grassmann Algebra. With the introduction of another product, the Clifford product one comes to the Clifford algebra. It turns out that Grassmann and Clifford algebras contain as particular cases, or is related to,many algebraic tools, used in physics as indicated in table.

Is this chapter we deal with the multivector calculus. Since this powerful tool is not easily found in the books of algebra (it is a matter of multilinear algebra) and is completely absent in physical books, we shall give a short survey of it, manteining our exposition as simple and intuitive as possible.We start with the three-dimensional euclidean space:later in this chapter

we give a more general version taking in mind particularly the needs of space-time.

6.2 Bivector

Two vectors with a common origin not alligned, taken in a definite order can be considered as a unit, a single object, called bivector.

As a vector is a natural tool to describe oriented line segments and physical quantities related to them, like forces and velocities, so a bivector is a natural tool to describe plane oriented areas and physical quantities related to them, like couples and angular velocities.

The area of parallelogram formed by the two vectors of a bivector is called the measure of the bivector. The plane of the bivector will be called the support of the bivector (Schouten, 1951 p.12). The order in which the two vectors are taken defines the orientation of the bivector. A bivector formed by the two vectors u and v will be denoted provisionally $b(u,v)$.
Two bivectors are said equal when they have the same support, the same origin, the same orientation and the same measure. This does not imply that they are formed by the same vectors.

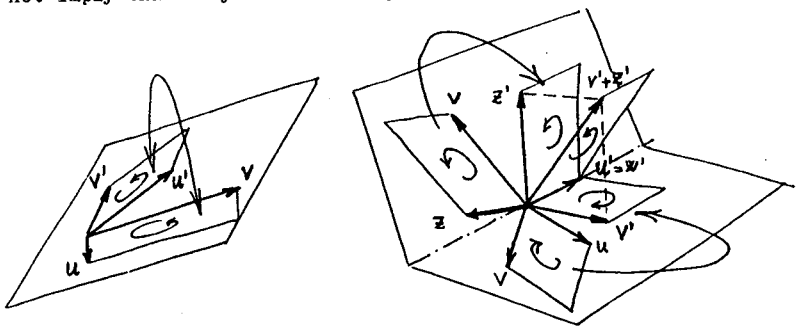


Fig. 6.2.1

If two bivectors $b(u,v)$ and $b(u',v')$ are equal their vectors are linked by the relation

$$\begin{cases} u' = \alpha u + \beta v \\ v' = \gamma u + \delta v \end{cases} \qquad (6.2.1)$$

with $\alpha\delta - \beta\gamma = 1$. In facts u', v' have the same support of u, v the same origin; their parallelogram has the same oriented area of the parallelogram formed by u and v, as can easily be shown.

The role of eq. (6.2.1) can be reversed and it can be used to define the equality of two bivectors without mentioning the area of the parallelogram and then without using a metrical property.

The notion of sum of two bivectors can be inferred from that of sum of two couples or of two angular velocities, just as the notion of sum of two vectors was inferred from the sum of two forces (Varignon parallelogram law) and from the sum of two velocities (Galileo parallelogram law). To define the sum we transform every bivector in an equal one such that their first sides lie on the intersecting line of the two supports. The bivector formed by this common vector (taken as the first vector) and by the sum of the two remaining vectors is called the sum of the two bivectors. In symbols if $b(u,v)$ and $b(w,z)$ are two bivectors if we construct two bivectors

$$b(u',v') = b(u,v) \qquad b(u',z') = b(w,z) \qquad (6.2.2)$$

then $$b(u,v) + b(w,z) = b(u',v') + b(u',z') = b(u', v'+z')$$

The product of a bivector by a number λ is defined as the bivector with the same support the same origin and a measure that be λ time that of the originary bivectors.

The bivector $b(v,u)$ is called the opposite of $b(u,v)$ and we write $b(v,u) = -b(u,v)$.

6.3 Trivector

Three linearly indipendent vectors u, v, w, with a common origin taken in a definite order can be considered as a unit, as a single geometrical object called a trivector and denoted provisionally by $t(u,v,w)$.

A trivector is a natural tool to describe oriented volumes and physical quantities related to them, like electric charge and mass. Two trivectors are said equal when they have the same origin, enclose the same volume and have the same orientation i.e. are congruent.

The volume enclosed (by) the parallelogram is called the measure of the trivector. Two trivectors are said equals if they have the same origin, the same measure and the same orientation. These conditions are equivalent to the condition that the vectors u, v, w of the first trivector and u', v', w' of the second be linked by the relation

$$\begin{cases} u' = \alpha u + \beta v + \gamma w \\ v' = \delta u + \varepsilon v + \zeta w \\ w' = \eta u + \vartheta v + \lambda w \end{cases} \qquad \begin{vmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \zeta \\ \eta & \vartheta & \lambda \end{vmatrix} = +1 \qquad (6.3.1)$$

The sum of two trivectors can be defined as follows: we reduce the two trivectors to other two with their first two vectors in common.

The sum is the trivector formed by the common two vectors and the sum of the two remaining vectors.

In general we speak of multivector to denote trivectors, bivectors, vectors and scalars One may also use the name p-dimensional vector or briefly p-vector.

The dimensionality of a p-vector refers to the dimensionality of the support of the p-vector. The order of a multivector, say p refers to the dimensionality of its support. So the support of a 2-vector (≡ bivector) is a plane (dimension two). A scalar is referred to a point and then it is a 0-vector.

Since in a threedimensional space we cannot have more that three linearly indipendent vectors the trivectors are the multivectors of higher order we can consider. Two multivectors are called orthogonal when their supports are orthogonals.

6.4 The exterior product

A bivector b (u, v) can be considered as the result of an operation performed on the two vectors u and v. Such operation is called exterior product of the two vectors and is denoted with the symbol " $\wedge$ " called "wedge" or "hook". Then

$$u \wedge v \stackrel{def}{=} b(u,v) \tag{6.4.1}$$

Then the exterior product of two vectors with the same origin is the bivector formed by the two vectors taken in the given order.

On account of the property (6.2.4) we have

$$v \wedge u = b(v,u) = -b(u,v) = -u \wedge v \tag{6.4.2}$$

that shows the skew-symmetric property of the exterior product of two vectors. This reflect the fact that a plane area may have two opposite orientations.

The exterior product of a bivector $b(u,v)$ for a vector w with the same origin is defined as the trivector $t(u,v,w)$ formed by them

$$b(u,v) \wedge w \stackrel{def}{=} t(u,v,w) \tag{6.4.3}$$

We require that the exterior product be associative so that

$$(u \wedge v) \wedge w = u \wedge (v \wedge w) \tag{6.4.4}$$

In this way we may omit the brackets and denote the trivector formed by $u\ v\ w$ simply as

$$u \wedge v \wedge w \tag{6.4.5}$$

Of particular importance is the trivector formed by the three base vectors e_1, e_2, e_3 of a coordinate system. It is denoted by ϵ and it is called the base trivector

$$\epsilon \stackrel{def}{=} e_1 \wedge e_2 \wedge e_3 \tag{6.4.6}$$

The exterior product of two multivectors with the same origin gives a multivector of greater order. More precisely the exterior product of a p-vector for a q-vector with $p+q \leq 3$ gives a (p+q)-vector. The exterior product of two multivectors whose supports have a common part of dimension greatest then zero, vanishes.

6.5 Scalar product of two p-vectors

The scalar product of two bivectors is the number obtained by multipication of their measures by the cosinus of the angle between their oriented supports

$$(u \wedge v) \cdot (w \wedge z) = [meas.(u \wedge v)][meas.(w \wedge z)][cosinus\ angle] \tag{6.5.1}$$

In particular the scalar product of two bivectors vanishes when their supports are orthogonals.

It can be shown that

$$(u \wedge v) \cdot (w \wedge z) = \begin{vmatrix} u \cdot w & u \cdot z \\ v \cdot w & v \cdot z \end{vmatrix} \tag{6.5.2}$$

The scalar product of two trivectors is defined as the product of their measure with the plus or minuus sign according to the fact that the two trivectors have or not the same orientation. This is equivalent to define the scalar product of two trivectors by the formula

$$(u\wedge v\wedge w)\cdot(a\wedge b\wedge c)=\begin{vmatrix} u\cdot a & u\cdot b & u\cdot c\\ v\cdot a & v\cdot b & v\cdot c\\ w\cdot a & w\cdot b & w\cdot c\end{vmatrix} \qquad (6.5.3)$$

6.6 Inner products

The scalar product of two p-vectors is a number i.e. a 0-vector. One may introduce a kind of inner product for multivectors of different rder according with the following definition: if p denotes a p-vector, q a q-vector with $p\leq q$ and r a (q-p) vector then the left inner product of p and q is a $(q-p)$-vector s denoted

$$s = p \lrcorner q \qquad (6.6.1)$$

definite implicitly by the formula

$$(p\lrcorner q)\cdot r \overset{def}{=} q\cdot(r\wedge p) \qquad (6.6.2)$$

To make esplicit this left inner product we consider particular cases so if $p=u$ and $q=v\wedge w$

$$[u\lrcorner(v\wedge w)]\cdot z=(v\wedge w)\cdot(z\wedge u) \qquad (6.6.3)$$

performing the scalar product of the second member one find the explicit formula

$$u\lrcorner(v\wedge w) = -w(u\cdot v)+v(u\cdot w) \qquad (6.6.4)$$

This shows that the left inner product of a vector for a bivector is a vector that lies in the support of the bivector, is orthogonal to the projection of the vector on the plane of the bivector and its measure is the product of the measure of the bivector for that of the projection of u

Alongside with the left inner product one may define a right inner product of a p-vector p and a q-vector q for which $p\geq q$. It is defined implicitly by the relation

$$(q\wedge r)\cdot p = r\cdot(p\llcorner q) \qquad (6.6.5)$$

Proceeding along the same way one may show that

$$(v\wedge w)\llcorner u = w(u\cdot v)-v(u\cdot w) \qquad (6.6.6)$$

More in general it can be shown that

$$p\llcorner q \equiv (-1)^{q(p-q)}\, q\lrcorner p \qquad (6.6.7)$$

The following identity is often useful

$$p\llcorner(q\wedge r)\equiv(p\llcorner q)\llcorner r \qquad (6.6.8)$$

In particular putting $r=q$ one deduce that <u>the inner product of two multivectors is orthogonal to the one of lower order.</u>

The various inner products between a p-vector and a q-vector of E^3 are summarized in the following scheme

	s	v	b	t
s'	$s'\cdot s$	$s'\lrcorner v$	$s'\lrcorner b$	$s'\lrcorner t$
v'	$v'\llcorner s$	$v'\cdot v$	$v'\lrcorner b$	$v'\lrcorner t$
b'	$b'\llcorner s$	$b'\llcorner v$	$b'\cdot b$	$b'\lrcorner t$
t'	$t'\llcorner s$	$t'\llcorner v$	$t'\llcorner b$	$t'\cdot t$

Table 6.6.1

6.7 The supplementary multivector

The classical notion of cross product of two vectors in vector calculus is transferred in the notion of supplementary of a bivector. The supplementary of a bivector b is defined as the vector, denoted $b^{\perp}$, that is orthogonal to the support of the bivector, whose norm is equal to the measure of the bivector and whose orientation is such that the trivector $b \wedge b^{\perp}$ has the same orientation of the trivector ϵ. The vector $b^{\perp}$ coincides with the customary cross product (*), of the two vector u and v that form the bivector

$$u \times v = (u \wedge v)^{\perp} \qquad (6.7.1)$$

The supplementary of a vector u is defined as a bivector, denoted $u^{\perp}$ whose support is orthogonal to u whose measure is equal to the measure of u and such that $u \wedge u^{\perp}$ has the same orientation of ϵ. The supplementary of a trivector t denoted $t^{\perp}$ is defined as the scalar equal to the volume enclosed by t end equipped with the plus or minus sign according if t has the same or opposite orientation of ϵ. One may extend this notion of supplementary to a scalar considered as a 0-vector:the supplementary of a 0-vector is a trivector whose measure is equal to the absolute value of the 0-vector, and whose orientation is the same or opposite to that of ϵ according if the 0-vector is positive or negative. The various supplementaries are collected in the following scheme

multivector	supplementary
0-vector (scalar)	3-vector (trivector)
1-vector (vector)	2-vector (bivector)
2-vector (bivector)	1-vector (vector)
3-vector (trivector)	0-vector (scalar)

(*) Historically the introduction of the cross product, made possible by the three-dimensional nature of our every day space, originated a great development of the vector calculus to the expense of the multivector calculus.

From the properties of the inner product of two multivectors it can be shown that the supplementary of a p-vector in E^3 can be written as follows

$$p^{\perp} = \epsilon \llcorner p \qquad (6.7.2)$$

The left and right inner products occur frequently in physics if Ω denotes the angular velocity bivector ($\Omega = \omega^{\perp}$) and r the radius vector, the velocity v of a point of a rigid body with a fixed point can be written as

$$v = \Omega \llcorner r \qquad (6.7.3)$$

Moreover if b denotes the magnetic induction bivector ($b = B^{\perp}$) and j the current density, the force density is given by

$$f = j \lrcorner b \qquad (6.7.4)$$

The passage form the usual notation to the proceeding one is easily done by means of the indentity

$$(p>q) \qquad \epsilon \llcorner (p \wedge q) \equiv (\epsilon \llcorner p) \llcorner q \qquad (6.7.5)$$

So, for ex. using eq. 6.7.1

$$v = \omega \times r = (\omega \wedge r)^{\perp} = \epsilon \llcorner (\omega \wedge r) = (\epsilon \llcorner \omega) \llcorner r = \Omega \llcorner r$$

The equivalence between the usual operation of vector calculus and

the corresponding ones of the multivector calculus is the following

vector product $\quad \boldsymbol{u} \times \boldsymbol{v} = (\boldsymbol{u} \wedge \boldsymbol{v})^{\perp}$

scalar triple product $\quad (\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} = (\boldsymbol{u} \wedge \boldsymbol{v} \wedge \boldsymbol{w})^{\perp}$ (6.7.7)

vector triple product $\quad (\boldsymbol{u} \times \boldsymbol{v}) \times \boldsymbol{w} = (\boldsymbol{u} \wedge \boldsymbol{v}) \llcorner \boldsymbol{w}$

6.8 Components of multivector in E^3

The following p-vectors formed by the base vectors $\boldsymbol{e}_\kappa$ are called base p-vectors:

(6.8.1)

$\boldsymbol{e}_1 \quad \boldsymbol{e}_2 \quad \boldsymbol{e}_3$	three base 1-vectors
$\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \quad \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 \quad \boldsymbol{e}_3 \wedge \boldsymbol{e}_1$	three base 2-vectors
$\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3$	one base 3-vectors

A given p-vector can be written as the linear combination of the base p-vectors as follows

$$\begin{cases} \boldsymbol{v} = v^1 \boldsymbol{e}_1 + v^2 \boldsymbol{e}_2 + v^3 \boldsymbol{e}_3 = \frac{1}{1!} v^\kappa \boldsymbol{e}_\kappa \\ \boldsymbol{b} = b^{12} \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 + b^{23} \boldsymbol{e}_1 \wedge \boldsymbol{e}_3 + b^{31} \boldsymbol{e}_3 \wedge \boldsymbol{e}_1 = \frac{1}{2!} b^{h\kappa} \boldsymbol{e}_h \wedge \boldsymbol{e}_\kappa \\ \boldsymbol{t} = t^{123} \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 = \frac{1}{3!} t^{hki} \boldsymbol{e}_h \wedge \boldsymbol{e}_k \wedge \boldsymbol{e}_i \end{cases} \qquad (6.8.2)$$

The coefficients of these linear combinations are called the components of the vector, bivector and trivector respectively. More precisely these are the controvariant components of the p-vector: the exhibit the typical skew-symmetric character

$$b^{h\kappa} = -b^{\kappa h} \qquad t^{hki} = -t^{khi} = -t^{hik} \qquad (6.8.3)$$

This establishes the fundamental link between p-vectors and skew--symmetric tensors or rank p : the components of a p-vector form a skew symmetric tensor of rank p . The components of a bivector $\boldsymbol{b} = \boldsymbol{u} \wedge \boldsymbol{v}$ are linked to those of the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ by the relation

$$b^{h\kappa} = u^h v^\kappa - u^\kappa v^h = \begin{vmatrix} u^h & u^\kappa \\ v^h & v^\kappa \end{vmatrix} = \begin{vmatrix} \delta^h_r & \delta^\kappa_r \\ \delta^h_s & \delta^\kappa_s \end{vmatrix} u^r v^s \qquad (6.8.4)$$

as may be easily proved. In analogous way we obtain for $\boldsymbol{t} = \boldsymbol{u} \wedge \boldsymbol{v} \wedge \boldsymbol{w}$ the components

$$t^{hki} = u^h v^\kappa w^i + u^\kappa v^i w^h + u^i v^h w^\kappa - u^\kappa v^h w^i - u^h v^i w^\kappa - u^i v^\kappa w^h =$$

$$= \begin{vmatrix} u^h & u^\kappa & u^i \\ v^h & v^\kappa & v^i \\ w^h & w^\kappa & w^i \end{vmatrix} = \begin{vmatrix} \delta^h_r & \delta^\kappa_r & \delta^i_r \\ \delta^h_s & \delta^\kappa_s & \delta^i_s \\ \delta^h_t & \delta^\kappa_t & \delta^i_t \end{vmatrix} u^r v^s w^t \qquad (6.8.5)$$

We may introduce the generalized Kronecker deltas (Synge-Schild, 1956, P. 242) (Brand, 1947, p. 353)

$$\delta^{h\kappa}_{rs} \overset{def}{=} \begin{vmatrix} \delta^h_r & \delta^\kappa_r \\ \delta^h_s & \delta^\kappa_s \end{vmatrix} \qquad \delta^{hki}_{rst} \overset{def}{=} \begin{vmatrix} \delta^h_r & \delta^\kappa_r & \delta^i_r \\ \delta^h_s & \delta^\kappa_s & \delta^i_s \\ \delta^h_t & \delta^\kappa_s & \delta^i_s \end{vmatrix} \qquad (6.8.6)$$

that form a mixed tensor. They assume the value 0 when two indices of the same variance are equal, the value + 1 (−1) when the upper indices are an even (odd) permutation of the lower ones.

With the Kronecker deltas we can write the components of the bivectors and trivectors as follows

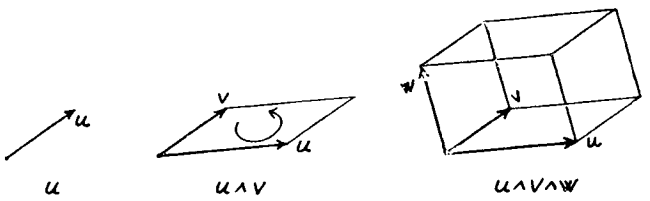
$$\boldsymbol{b} = \boldsymbol{u} \wedge \boldsymbol{v} \qquad b^{hk} = \frac{1}{1!}\frac{1}{1!}\,\delta^{hk}_{rs}\,u^r v^s \tag{6.8.7}$$

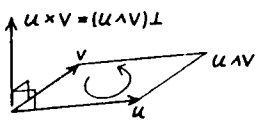
$$\boldsymbol{t} = \boldsymbol{b} \wedge \boldsymbol{w} \qquad t^{hki} = \frac{1}{2!}\frac{1}{1!}\,\delta^{hki}_{rst}\,u^r v^s w^t$$

Using the components of a multivector one may show that the inner products of two multivectors can be written as follows

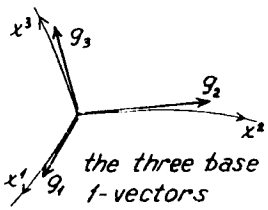
$$\begin{aligned}
&\boldsymbol{b} = \boldsymbol{t} \llcorner \boldsymbol{u} && b^{hi} = \frac{1}{1!}\,t^{khi}\,u_k \\
&\boldsymbol{v} = \boldsymbol{b} \llcorner \boldsymbol{u} && v^h = \frac{1}{1!}\,b^{kh}\,u_k \\
&\boldsymbol{v} = \boldsymbol{t} \llcorner \boldsymbol{b} && v^h = \frac{1}{2!}\,t^{ikh}\,b_{ik} \\
&\boldsymbol{s} = \boldsymbol{t} \llcorner \bar{\boldsymbol{t}} && s = \frac{1}{3!}\,t^{hki}\,\bar{t}_{hki}
\end{aligned} \tag{6.8.8}$$

table : multivectors

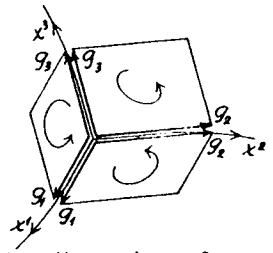




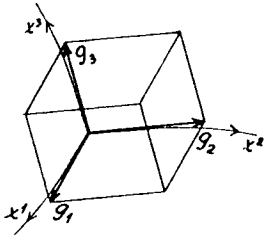
the supplementary of a bivector



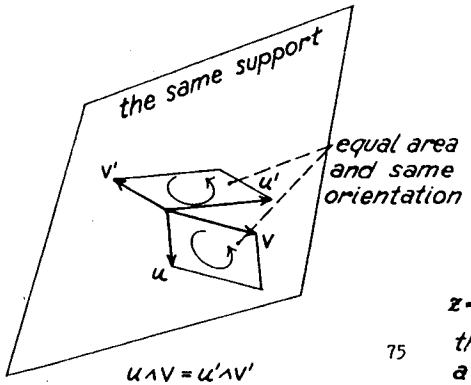
the three base 1-vectors

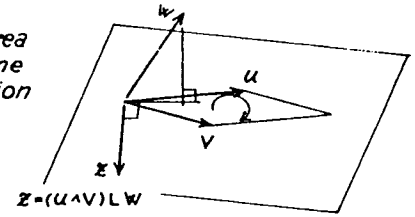


the three base 2-vectors



the base 3-vector





the (right) interior product of a bivector for a vector

6.9 Multivectors in finite dimensional spaces

The theory of multivectors as summarized in the previous sections is restricted to a three-dimensional space with euclidean metric. Metrical and non-metrical properties were mixed togheter in order to have an elementary and intuitive approach. It is of greath interest for the analysis of the formal structure of physical theories to single out those definitions and properties of the multivector calculus that are of non metrical nature, i.e. that are independent from the kind of distance chosen in the space. To this aim we sketch here the essential facts about multivectors in an n-dimensional space without metric.

Let us consider a linear space U of finite dimension n. We call simple p-vector the geometrical object formed by p linearly indipendent vectors taken in a definite order. If we denote $u_1, u_2, \dots u_p$ the p-vectors, the corresponding p-vector will be denoted $p(u_1, u_2, \dots u_p)$. The linear space spanned by these p vectors is called the support of the p-vector.

6.10 Equal p-vectors

Two simple p-vectors

$$p(u_1, u_2, \dots u_p) \quad \text{and} \quad \pi(\bar{u}_1, \bar{u}_2, \dots \bar{u}_p) \tag{6.10.1}$$

whose vectors are linked by a linear relation

$$\bar{u}_h = L_h^k u_k \tag{6.10.2}$$

such that

$$det \| L_h^k \| = +1 \tag{6.10.3}$$

are said equal (Bourbaky, 1958, p. 96). In particular two simple p-vectors that are formed by the same vectors but taken in an order that differs only by an even number of inversions, are equal.

Two simple p-vectors whose vectors are linked by the relation (6.10.2) but with

$$det \| L_h^k \| = -1 \tag{6.10.4}$$

are said opposite. In particular two p-vectors formed by the same vectors taken in an order that differs by an odd number of inversions are opposite.

The simple p-vector formed by p vectors that are linearly dependent is called null p-vector. Since in an n-dimensional space all p-vectors with $p>n$ are linearly dependent it follows that the p-vectors with $p>n$ are all equal to the null p-vector.

6.11 Exterior product

The exterior product of two simple multivectors $p(u_1, \dots u_p)$ and $q(v_1, \dots v_q)$ is defined as the multivector $r(u_1, \dots u_p, v_1, \dots v_q)$ of order $r = p+q$. In symbols

$$p \wedge q = r \tag{6.11.1}$$

In one or more vectors v_k are linearly dependent from some u_h the multivector r is the null multivector and then the exterior product vanishes. From eq.6.10.3 and 6.10.4 follows the connection rule

$$p \wedge q = (-1)^{p1}\, q \wedge p \tag{6.11.2}$$

In particular

$$u \wedge v = - v \wedge u \tag{6.11.3}$$

The exterior product is associative, i.e.

$$w \wedge (u \wedge v) = (w \wedge u) \wedge v \tag{6.11.4}$$

This property makes possible to write simply $w \wedge u \wedge v$ instead of $w \wedge (u \wedge v)$ or $(w \wedge u) \wedge v$.

The definition of exterior product between two multivectors must be completed for the case that one of the two is a scalar according with the following deninitions (Bourbaky, 1958, pag. 96) (Kowalsky, 1963,pag. 101)

$$s \wedge p \overset{def}{=} sp \qquad p \wedge s \overset{def}{=} ps \qquad s \wedge s \overset{def}{=} ss$$

6.12 Sum of two multivectors

In a three-dimensional space the sum of two bivectors is defined using the intersection of the corresponding linear manifolds: one reduce both bivectors to other two with the same first vector.

This process cannot be used, in general, in spaces with n>3 because two-dimensional linear spaces may do not intersect along a one-dimensional linear space. This is the case of space-time where the system

$$\begin{cases} a_1 x^0 + b_1 x^1 + c_1 x^2 + d_1 x^3 = 0 \\ a_2 x^0 + b_2 x^1 + c_2 x^2 + d_2 x^3 = 0 \\ a_3 x^0 + b_3 x^1 + c_3 x^2 + d_3 x^3 = 0 \\ a_4 x^0 + b_4 x^1 + c_4 x^2 + d_4 x^3 = 0 \end{cases}$$

admits in general only the null solution i.e. the two linear subspaces intersect only at the origin. In space-time we are then forced to introduce the concept of compound bivector as the aggregate of two simple bivectors whose linear manifolds do not intersect. The two linear manifolds are called the blades of the compound bivector (Schouten, 1949, pag. 15). The compound bivector formed by the two bivectors $u \wedge v$ and $w \wedge z$ will be written

$$b = u \wedge v + w \wedge z$$

Now the sum of two compound bivectors in space time say

$b = u \wedge v + w \wedge z$ and $\bar{b} = \bar{u} \wedge \bar{v} + \bar{w} \wedge \bar{z}$

is defined as follows: we consider the intersection of the support of say, $\bar{u} \wedge \bar{v}$ with one of the two blades of b. Let us suppose that

the intersected blade be that of $w \wedge z$. We may sum the two simple bivectors $\bar{u} \wedge \bar{v}$ and $w \wedge z$ to obtain a simple bivector $u' \wedge v'$. Analogous sum will be made between the bivectors $\bar{w} \wedge \bar{z}$ and $u \wedge v$ the result will be a simple bivector $w' \wedge z'$. The sum of two compound bivectors b and $\bar{b}$ will be by definition

$$b + \bar{b} = u' \wedge v' + w' \wedge z'$$

An analogous procedure may be used to define the sum of two compound multivector on a linear space U.

6.13 Simple and compound bivectors

Since a compound bivector of space-time is formed by two blades it cannot exist a vector orthogonal to the whole bivector: then a bivector b is simple if there exist a vector u such taht

$$b \lrcorner u = 0 \qquad \text{or} \qquad b_{hk} u^h = 0$$

this means that

$$\det \| b_{hk} \|'' = 0$$

So the electromagnetic bivector F of the electromagnetic field in space-time is compound (Fokker, 1965, pag. 125) while the orbital angular momentuum $L = r \wedge p$ is simple.

6.14 Base p-vectors

If we introduce a base $e_1, e_2, \dots e_n$ in the linear space U we can consider the $\binom{n}{2}$ simple bivectors $e_h \wedge e_k$ with $h<k$ and call them base bivectors. In an analogous way one can consider the $\binom{n}{p}$ p-vectors $e_h \wedge e_k \wedge \dots \wedge e_r$ with $h<k<\dots<r$ and call them base p-vectors. Of particular interest is the simple n-vector

$$\mathcal{E} \overset{def}{=} e_1 \wedge e_2 \wedge \dots \wedge e_n \qquad (6.14.1)$$

that plays a pivotal rôle in the multivector calculus.

6.15 Components of a multivector

Using the decomposition of a vector $u = u^k e_k$ we can obtain an analogous decomposition of a simple bivector as follows

$$u \wedge v = (u^h e_h) \wedge (v^k e_k) = u^h v^k e_h \wedge e_k = \frac{1}{2!}(u^h v^k - u^k v^h) e_h \wedge e_k \qquad (6.15.1)$$

the numbers

$$b^{hk} = -b^{kh} = u^h v^k - u^k v^h \qquad (6.15.2)$$

for which $h<k$ are called the essential components of the bivector $u \wedge v$[ ]. The numbers b^{hk} without the restriction $h<k$ are called the components of $u \wedge v$.

In analogous way one can define the components of a simple p-vector.

The components of a compound p-vector are defined as the sum of the components of the same index of the simple p-vectors that form the p-vector. So if b and $\bar{b}$ are simple bivectors, the compound bivector $\beta = b + \bar{b}$ has for components

$$\beta^{hk} = b^{hk} + \bar{b}^{hk} \qquad (6.15.3)$$

6.16 The tensorial nature of the components of a multivector

Let us consider a p-vector that remains fixed for a change of the base it can be shown that its components: form a controvariant tensor of order p. But if we consider p-vectors that change for a change of the base then their components do not form a tensor. So the base n-vector $\mathcal{E}$ given by eq. (6.14.1) in a new base becomes

$$\bar{\mathcal{E}} = \bar{e}_1 \wedge \bar{e}_2 \wedge \dots \wedge \bar{e}_n \qquad (6.16.1)$$

and is, in general, different from $\mathcal{E}$. If we put

$$\bar{e}_h = \ell_h^{\ k} e_k \qquad e_k = L_k^{\ j} \bar{e}_j \qquad (6.16.2)$$

it is easily shown that

$$\bar{\mathcal{E}} = D\,\mathcal{E} \qquad D = \det \| \ell_h^{\ k} \| \qquad (6.16.3)$$

The components of the n-vector $\overset{e}{\mathcal{E}}$ change as follows: since

$$\bar{\mathcal{E}} = \frac{1}{n!} \bar{\varepsilon}^{ij\dots k} (\bar{e}_i \wedge \bar{e}_j \wedge \dots \wedge \bar{e}_k)$$
$$\bar{\mathcal{E}} = \frac{1}{n!} \varepsilon^{pq\dots s} (e_p \wedge e_q \wedge \dots \wedge e_s) \qquad (6.16.4)$$
$$= \frac{1}{n!} \varepsilon^{pq\dots s} L_p^{\ i} L_q^{\ j} \dots L_s^{\ k} (\bar{e}_i \wedge \bar{e}_j \wedge \dots \bar{e}_k)$$

remembering eq (6.16.3) will be

$$\bar{\varepsilon}^{ij\dots k} = D L_p^{\ i} L_q^{\ j} \dots L_s^{\ k} \varepsilon^{pq\dots s} \qquad (6.16.5)$$

and then the components of $\mathbf{e}$ transform as a <u>controvariant tensor density</u> (Synge-Schild, 1949, p. 244). $e^{pq\cdots s}$ is the permutation symbol whose values, in all the coordinate systems, are + 1, - 1, when the indexes $h,k,\ldots s$ are an even or odd permutation of 1, 2, ..., n and are 0 when two indices are equal.

This point is of importance because the ordinary presentation of the multivector calculus rests upon the definition of a p-vector as a completely skew-symmetric tensor of rank p and, as such, does not include tensor- densities. The presentation we have given of the multivector calculus has a more geometrical character, does not rest upon the tensor calculus and permits to treat those multivector, like the base p-vectors, that are not fixed but change with the change of the basis.

6.17 Components of the exterior product

Remembering the definition of the generalized Kronecker symbol one can easily show the identities

$$\frac{1}{p!}\,\delta^{hk\ldots l}_{rs\ldots t}\,p^{rs\ldots t} \equiv p^{hk\ldots l} \tag{6.17.1}$$

$$\frac{1}{p!}\,\delta^{hk\ldots l}_{rs\ldots t}\,\mathbf{e}_h\wedge\mathbf{e}_k\wedge\ldots\mathbf{e}_l \equiv \mathbf{e}_r\wedge\mathbf{e}_s\wedge\ldots\mathbf{e}_t \tag{6.17.2}$$

These idendities are of large use. So the components of the exterior product of a bivector and a trivector are obtained as follows

$$\mathbf{r} = \mathbf{b}\wedge\mathbf{t} = \left(\frac{1}{2!}\,b^{hk}\mathbf{e}_h\wedge\mathbf{e}_k\right)\wedge\left(\frac{1}{3!}\,t^{pqr}\mathbf{e}_p\wedge\mathbf{e}_q\wedge\mathbf{e}_r\right) = \tag{6.17.3}$$

$$= \frac{1}{2!}\frac{1}{3!}\,b^{hk}t^{pqr}\mathbf{e}_h\wedge\mathbf{e}_k\wedge\mathbf{e}_p\wedge\mathbf{e}_q\wedge\mathbf{e}_r$$

In this equation is h ≠ k and p ≠ q ≠ r. But since every index runs between 1 and n there are some values of h and k that are equal to some values of p, q, r. The corresponding terms in the sum vanish. Then if a ≠b ≠c ≠d ≠e are a set of five indices we may write:

$$\mathbf{e}_h\wedge\mathbf{e}_k\wedge\mathbf{e}_p\wedge\mathbf{e}_q\wedge\mathbf{e}_r = \frac{1}{5!}\,\delta^{abcde}_{hkpqr}\,\mathbf{e}_a\wedge\mathbf{e}_b\wedge\mathbf{e}_c\wedge\mathbf{e}_d\wedge\mathbf{e}_e \tag{6.17.4}$$

Then

$$\mathbf{r} = \frac{1}{5!}\left(\frac{1}{2!}\frac{1}{3!}\,\delta^{abcde}_{hkpqr}\,b^{hk}t^{pqr}\right)\mathbf{e}_a\wedge\mathbf{e}_b\wedge\mathbf{e}_c\wedge\mathbf{e}_d\wedge\mathbf{e}_e \tag{6.17.5}$$

It follows that are components of the exterior product $\mathbf{r} = \mathbf{b}\wedge\mathbf{t}$ are given by

$$r^{abcde} = \frac{1}{2!}\frac{1}{3!}\,\delta^{abcde}_{hkpqr}\,b^{hk}t^{pqr} \tag{6.17.6}$$

More in general if $\mathbf{p}$ and $\mathbf{q}$ are two multivectors the exterior product $\mathbf{r} = \mathbf{p}\wedge\mathbf{q}$ has a components

$$r^{ab\ldots} = \frac{1}{p!}\frac{1}{q!}\,\delta^{ab\ldots}_{hk\ldots\,rs\ldots}\,p^{hk\ldots}q^{rs\ldots} \tag{6.17.7}$$

In particular when $\mathbf{p}$ and $\mathbf{q}$ are vectors, denoted by $\mathbf{u}$ and $\mathbf{v}$ respectively, we have

$$r^{ab} = \frac{1}{1!}\frac{1}{1!}\,\delta^{ab}_{hr}\,u^h v^r = u^a v^b - u^b v^a \tag{6.17.8}$$

as we have already seen.

The exterior product of three vectors can be obtained as follows: if

$$\mathbf{r} = \mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w} = (\mathbf{u}\wedge\mathbf{v})\wedge\mathbf{w} \tag{6.17.9}$$

then

$$r^{abc} = \frac{1}{2!}\frac{1}{1!}\delta^{abc}_{hkr}\left(\frac{1}{1!}\frac{1}{1!}\delta^{hk}_{pq}\, u^p v^q\right) w^r = \\ = \delta^{abc}_{pqr}\, u^p v^q w^r \qquad (6.17.10)$$

where we have used the property

$$\frac{1}{2!}\delta^{abc}_{hkr}\,\delta^{hk}_{pq} \equiv \delta^{abc}_{pqr} \qquad (6.17.11)$$

6.18 Dual space

One of the main concepts of modern mathematics is that of considering alongside with every linear space U another linear space V and to define a bilinear form, that associates with every pair of elements $u \in U$ and $v \in V$ a real number, denoted $\langle v, u\rangle$, and called the scalar product of v and u. It is a common requirement that the bilinear form be nondegenerate or separating, i.e. if $\langle v_0, u\rangle = 0$ for every $u \in U$ then v_0 is the null element of the space V; and inversely if $\langle v, u_0\rangle = 0$ for every $v \in V$ this implies that u_0 is the null element of the space U. The two spaces U and V are then said to have been put in duality or to form a pair of dual spaces (Schaefer, 1971, p. 123) (Bourbaky, 1966, p. 88).

A vector $v \in V$ and a vector $u \in U$ are said orthogonal if

$$\langle v, u\rangle = 0 \qquad (6.18.1)$$

The notion of orthogonality is easily extended to two subspaces as follows: a linear variety A of the V-space is said orthogonal to a linear variety B of the U-space if every vector $v \in A$ is orthogonal to every vector $u \in B$.

6.19 Dual basis

If we choose a base $e_1, e_2, \ldots e_n$ in the U-space and a base $G^1, G^2, \ldots G^n$ in the V-space we can consider the scalar products

$$\langle G^h, e_k\rangle = a^h_k \qquad (6.19.1)$$

In particular we can always construct a base $\bar{e}^1, \bar{e}^2, \ldots \bar{e}^n$ in the V-space such that

$$\langle \bar{e}^h, e_k\rangle = \delta^h_k \qquad (6.19.2)$$

where δ^h_k is the Kronecker symbol. Such a base is called dual of the base $e_1, e_2, \ldots e_n$ of the U-space. In other words: requiring that $a^h_k = \delta^h_k$ we associate with a base in U a well defined base in V. This link between the two basis implies that if we change the first base also the dual base changes. So if

$$\bar{e}_r = \ell_r^{\ k}\, e_k \qquad \bar{\bar{e}}^s = L^s_k\, \bar{e}^k \qquad (6.19.3)$$

are two linear transformations of the base vectors of the U-space and of the V-space respectively in order to satisfy eq. (6.19.2) it must be

$$\ell_r^{\ k}\, L^s_k = \delta_r^{\ s} \qquad (6.19.4)$$

The the matrix L is the ~~transpose~~ transpose of the inverse of the matrix ℓ (Gel'fand, 1961). Now for a change of the base vectors e_k the components of a vector u change according to the law

$$\bar{u}^r = L^r_s\, u^s \qquad (6.19.5)$$

that involves the transpose of the inverse of ρ . For this reason it is said that the components u^s are <u>contro-variant</u> (with respect to base vectors e_k). In analogous way the components v_k of a vector $v \in V$ are contro-variant with respect to a change of the base of the V-space i.e.

$$\bar{v}_r = \rho_r^s \, v_s \tag{6.19.6}$$

But since they have the same transformation law of the base vectors e_r of the space U wa can say that they are <u>covariant</u> with respect to the base vectors e_r . In this way the variance of the components of u and of v are referred to the variance of the base of the space U . This opposite variance is memorized putting the indexes at the bottom (covariance) and at the top (controvariance). One can see that

forms
$$v_k = \langle v, e_k \rangle$$
$$u^h = \langle \hat{e}^h, u \rangle \tag{6.19.7}$$

Then the scalar product of two vectors can be computed using the components pertinent a dual base by the equation

$$\langle v, u \rangle = v_h \, u^h \tag{6.19.8}$$

6.20 <u>Scalar product</u>

The notion of scalar product can be extended to a simple bivector $u_1 \wedge u_2$ on the U-space and a simple bivector $v_1 \wedge v_2$ on the V-space by the defining relation

$$\langle v_1 \wedge v_2 \, , \, u_1 \wedge u_2 \rangle_2 \overset{def}{=} \begin{vmatrix} \langle v_1, u_1 \rangle & \langle v_1, u_2 \rangle \\ \langle v_2, u_1 \rangle & \langle v_2, u_2 \rangle \end{vmatrix} \tag{6.20.1}$$

More in general (Flanders, 1963, p. 14) - (Bourbaky, 1958, p.102) (Witney, 1957, p. 48) - (Greub, 1967, p. 104)

$$\langle v_1 \wedge v_2 \wedge \dots v_p \, , \, u_1 \wedge u_2 \wedge \dots u_p \rangle_p \overset{def}{=} \det \| \langle v_h, u_k \rangle \| \tag{6.20.2}$$

In particular the scalar product of two base p-vectors gives

$$\langle \hat{e}^h \wedge \hat{e}^k \wedge \dots \, , \, e_r \wedge e_s \wedge \dots \rangle_p = \delta^{hk\dots}_{rs\dots} \tag{6.20.3}$$

where use has been made of eq. (6.19.2). The natural countepart of the n-vector σ on U is the n-vector e on V defined by

$$\varepsilon \overset{def}{=} \hat{e}^1 \wedge \hat{e}^2 \wedge \dots \wedge e^n \tag{6.20.4}$$

From eq. (6.20.3) one can see that (Kowalsky, 311)

$$\langle e, \hat{e} \rangle_n \, \langle e, \hat{e} \rangle_n = +1 \tag{6.20.5}$$

Analogous definition can be used for the product of two compound p-vectors. Moreover it can be shown that if π is a p-vector on the V-space and p a p-vector on the U-space is

$$\langle \pi, p \rangle_p = \sum_{<} \pi_{hk\dots} \, p^{hk\dots} = \frac{1}{p!} \, \pi_{hk\dots} \, p^{hk\dots} \tag{6.20.6}$$

(Greub, 1967, p. 106). The components of a p-vector can be calcu lated by the relation

$$p^{hk\cdots} = \langle e^h \wedge e^k \wedge \cdots , p \rangle_p$$

$$\pi_{rs\cdots} = \langle \pi, e_r \wedge e_s \wedge \cdots \rangle_p \qquad (6.20.7)$$

In particular for the n-vectors ϵ and e we find

$$\epsilon^{hk\cdots} = \langle e^h \wedge e^k \wedge \cdots , e_1 \wedge e_2 \wedge \cdots e_n \rangle_n = \begin{cases} +1 \\ 0 \\ -1 \end{cases}$$

$$e_{rs\cdots} = \langle e^1 \wedge e^2 \wedge \cdots e^n , e_r \wedge e_s \wedge \cdots \rangle_n = \begin{cases} +1 \\ 0 \\ -1 \end{cases} \qquad (6.20.8)$$

showing that the $\epsilon^{hk\cdots}$ are the components of <u>controvariant ten sor-density</u> and the $e_{rs\cdots}$ that of a <u>covariant tensor-density.</u> (Schouten-Kulk, 1949, p.19).

6.21 <u>Inner products</u>

The notion of scalar product of two p-vectors permits to in troduce a new operation called the inner product of a p-vector p on U with a q-vector q on V. For $p \leq q$ we define the <u>left inner product</u> of p and q as that (q-p)-vector r on V, deno- ted $p \lrcorner q$ that is defined implicitly by the relation (Mal'cev, 1973, p. 279) - (Kawalsky, 1963, p. 310) - (Bourbaky, 1958,p.105).

$$\langle p \lrcorner q, s \rangle_{q-p} \stackrel{def}{=} \langle q, s \wedge p \rangle_q \qquad \left(\begin{matrix} \text{for every} \\ s \in \Lambda^{q-p} U \end{matrix} \right) (6.21.1)$$

$\langle \sigma, \alpha \lrcorner q \rangle \stackrel{def}{=} \langle \sigma \wedge \alpha, q \rangle$

$\lrcorner$ libri gli indici di sinistra
$\llcorner$ " " " " destra

The components of the left inner product, are given by

$$r = p \lrcorner q \qquad (6.21.2)$$

$$r_{rs\cdots} = \frac{1}{p!} q_{rs\cdots hk\cdots} p^{hk\cdots} \qquad (6.21.3)$$

that shows the conne x ion with the usual tensor composition. The main identity relating the left inner product with the exterior product is the following: denoting with a and b two multivec- tors on the U-space and with c a multivector on the V-spa- ce such that $c \geq a+b$ is

$$(a \wedge b) \lrcorner u = a \lrcorner (b \wedge u) \qquad (6.21.4)$$

and those obtained by iteration as

$$(a \wedge b \wedge c) \lrcorner u = a \lrcorner [b \lrcorner (c \lrcorner u)] \qquad (6.21.5)$$

When $p \geq q$ we can define the <u>right inner product</u> by the implicit relation (Graeub, 1958, p. 102)

$$\langle q \wedge r, p \rangle_p \stackrel{def}{=} \langle r, p \llcorner q \rangle_{p-q} \qquad (6.21.6)$$

where p is a p-vector on U and q is a q-vector on V. The right inner product is a (p-q)-vector on the U-space. Its ex- pression in components is the following: putting

$$t = p \llcorner q \qquad (6.21.7)$$

we obtain

$$t^{rs\cdots} = \frac{1}{q!} \qquad q_{hk\cdots} p^{hk\cdots rs\cdots} \qquad (6.21.8)$$

(Graeub, 1958, p. 103)

$\langle \alpha \wedge \beta, u \wedge v \rangle = \langle (\alpha \wedge \beta) \llcorner u, v \rangle$

We remark the following mnemonic rule: in the left inner product the indices not summed are on the left, and in the right inner product the indices not summed are on the right.

We have the identity

$$(p \llcorner q) \llcorner z \equiv p \llcorner (q \wedge z) \tag{6.21.9}$$

$$\langle p, u \wedge v \wedge w \rangle \equiv \langle p \llcorner (u \wedge v), w \rangle$$
$$\equiv \langle p \llcorner u, v \wedge w \rangle \equiv \langle (p \llcorner u) \llcorner v, w \rangle$$

from which by iteration we obtain

$$p \llcorner (a \wedge b \wedge c) \equiv [(p \llcorner a) \llcorner b] \llcorner c \tag{6.21.10}$$

A p-vector p on U and q-vector q on V are said orthogonal when their left or right inner product vanish.

We emphasize that while the exterior product of two multivectors on the same space is a multivector, on the same space, the inner product involves a multivector on one space and a multivector on its dual space. The result is a multivector on the space of the multivector of greatest order.

When $p=q$ the left and right inner products coincide and are equal to the scalar product. We see that while in the ordinary vector calculus we have only the scalar product, (also called inner product) in the multivector calculus we have a scalar product for multivectors of the same order and two kind of inner products for multivectors of different orders.

The right inner products is sometime denoted as follows

$$p \llcorner q = i(q) p \tag{6.21.11}$$

(Bourbaky, 1959, p. 155) (Goldberg, 1962, p. 171) - (Greub, 1967 p. 118). The exterior product is sometimes denoted

$$p \wedge q = \mu(p) q \tag{6.21.12}$$

(Greub, 1967, p. 116) - (Goldberg, 1962, p. 96). The two operators $i(u)$ and $\mu(u)$ with $u \in U$ are used in quantum field

theory where are interpreted as "absorption" and "ommission" operators for fermion fields (Kastler, 1961, p. 298).

Among the identities involving the inner and exterior products there are the following

$$(p \wedge q) \llcorner u = (p \llcorner u) \wedge q + (-1)^p p \wedge (q \llcorner u) \tag{6.21.13}$$

(Soriau, 1964, p. 230) and

$$u \lrcorner (p \wedge q) \equiv (u \lrcorner p) \wedge q + (-1)^p p \wedge (u \lrcorner q) \tag{6.21.14}$$

(Hermann, 1973, p. 30) whose proof is not easily found in the literature.

6.22 Supplementary multivector

The notion of inner product permits to define the right-supplementary of a p-vector p on V as an (n-p)-vector $p^\perp$ on U by means of the relation (Kowalsky, 1963, p. 311)-(Witney, 1957, p.47) (Mal'cev, 1963, p. 280) - (Bourbaky, 1958, p. 108)

$$p^\perp = \epsilon \llcorner p \tag{6.22.1}$$

that in components becomes

$$(p^\perp)^{hk\ldots} = \frac{1}{p!} \delta^{rs\ldots hk\ldots}_{rs\ldots} p_{rs\ldots} \tag{6.22.2}$$

In analogous way the left supplementary of a q-vector q on U is defined as the (n-q)-vector ${}^\perp q$ on V by means of the relation

$${}^\perp q \overset{def}{=} q \lrcorner \epsilon \tag{6.22.3}$$

that in components becomes

$$({}^\perp q)_{rs\ldots} = \frac{1}{q!} \epsilon_{rs\ldots hk\ldots} q^{hk\ldots} \tag{6.22.4}$$

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It can be shown that

$$({}^{\perp}q)^{\perp} \equiv q \qquad {}^{\perp}(p^{\perp}) \equiv p \tag{6.22.5}$$

In fact

$$[({}^{\perp}q)^{\perp}]^{ij\cdots} \equiv \frac{1}{(n-q)!}\,\epsilon^{rs\cdots ij\cdots}\left(\frac{1}{q!}\,e_{rs\cdots hk\cdots}\,q^{hk\cdots}\right) \equiv$$
$$\equiv \frac{1}{q!}\,\delta^{ij\cdots}_{hk\cdots}\,q^{hk\cdots} \equiv q^{ij\cdots} \tag{6.22.6}$$

that prove the first identity: analogous proof for the second one.

The supplementary of a multivector is orthogonal to the multivector: in fact $(\epsilon \llcorner p)\llcorner p \equiv \epsilon \llcorner (p \wedge p) \equiv \epsilon \llcorner 0 \equiv 0$.

There are many identities relating the exterior and inner products with the supplementary. So

$$\langle \pi, p \rangle_p \equiv (-1)^{p(n-p)}\,\langle {}^{\perp}p, {}^{\perp}\pi \rangle_{n-p} \tag{6.22.7}$$

$$(p \lrcorner q)^{\perp} \equiv p \wedge q^{\perp} \tag{6.22.8}$$

(Kowalsky, 1963, p. 314) - (Bourbaky, 1958, p. 109)

$${}^{\perp}(p \llcorner q) \equiv {}^{\perp}p \wedge q \tag{6.22.9}$$

(Witney, 1957, p. 48).

The theory of multivectors we have summarized here has been developped without recourse to any metric in the U-space: U and V are a pair of dual spaces that satisfy anly the requirement of linearity.

6.23 Grassmann algebra

We now show how the multivector calculus may be structured in order to obtain an algebra. Let us consider the bivectors on a real linear space U of finite dimensions: they can be added and multiplied by a real number. Then the set of all bivector on U is in turn, a linear space that is called the space of bivectors and is denoted $\Lambda^2 U$.

In analogous way one can consider the set of all p-vectors on U and this is a linear space also: it is denoted $\Lambda^p U$ .Then the p-vectors can be considered either as compound objects of a given linear space U or as elementary objects of another linear space $\Lambda^p U$. .

The idea arises to consider the aggregate formed by a scalar (0-vector), a 1-vector, a 2-vector, ... an n-vector on U

Such aggregate, we shall denote by a can be represented (for ex. for $n=3$) as follows

$$a = (s, v, b, t) \tag{6.23.1}$$

exactly as a complex number can be denoted by $z = (x, y)$. But just as a complex number can be denoted also $z = x + iy$ so the aggregate a can be written as a formal sum

$$a = s + v + b + t \tag{6.23.2}$$

This is an hypercomplex number Using the base multivectors one may write

$$a = s + v^k e_k + \frac{1}{2!}\,b^{hk}\,e_h \wedge e_k + \frac{1}{3!}\,t^{hk\ell}\,e_h \wedge e_k \wedge e_\ell \tag{6.23.3}$$

The numbers $s, v^k, b^{hk}, t^{hk\ell}$ etc. will be called the components of the hypercomplex number.

The set of all aggregates a has the structure of a linear space when we define the two operations

$$\begin{cases} a + \bar{a} \stackrel{def}{=} (s+\bar{s}) + (v+\bar{v}) + (b+\bar{b}) + \cdots \\ \lambda a \stackrel{def}{=} \lambda s + \lambda v + \lambda b + \cdots \end{cases} \qquad (6.23.4)$$

The new linear space so constructed is formed by the aggregate of an element of $\Lambda^0 U$ (a scalar), one of $\Lambda^1 U$ (a vector)... one of $\Lambda^n U$ (an n-vector): it is denoted ΛU. Now the reason for the introduction of the aggregate of multivectors of different orders lies in that, if we introduce the exterior products of two aggregates a and $\bar{a}$ as the aggregate denoted $a \wedge \bar{a}$ and defined by (for ex. for $n=3$)

$$\begin{aligned} a \wedge \bar{a} &= (s+v+b+t) \wedge (\bar{s}+\bar{v}+\bar{b}+\bar{t}) = \\ &= s\wedge\bar{s} + s\wedge\bar{v} + s\wedge\bar{b} + s\wedge\bar{t} + \\ &\quad + v\wedge\bar{s} + v\wedge\bar{v} + v\wedge\bar{b} + \\ &\quad + b\wedge\bar{s} + b\wedge\bar{v} + \\ &\quad + t\wedge\bar{s} \end{aligned} \qquad (6.23.5)$$

Then the result is again an element of ΛU. The linear space equipped with this product becomes an algebra that is called the exterior or Grassmann algebra. (Mal'cev, 1963) (Witney, 1957, p. 42).

On account of the associative property of the exterior product among multivectors it follows that

and then the Grassmann algebra is associative. Every aggregate is called a Grassmann number. (6.23.2) show that a Grassmann number has $\binom{n}{0} + \binom{n}{1} + \cdots \binom{n}{n} = 2^n$ components.

We remark that this is also the number of coordinate manifolds of the various orders passing through a point of $\mathbb{R}^n$.

6.24 Scalar and inner products

All that we have said up to now can be repeated for the dual space ΛV : then we can construct the Grassmann algebra over ΛV. The space ΛV can be put in duality with the space introducing the bilinear form (Mal'cev, 1963, p. 278)

$$\begin{aligned} \langle \bar{a}, a \rangle &\stackrel{def}{=} \langle \bar{s}, s \rangle_0 + \langle \bar{v}, v \rangle_1 + \langle \bar{b}, b \rangle_2 + \langle \bar{t}, t \rangle_3 + \cdots \\ &= \bar{s}s + \bar{v}_h v^h + \frac{1}{2!} \bar{b}_{hk} b^{hk} + \frac{1}{3!} \bar{t}_{hkl} t^{hkl} + \cdots \end{aligned} \qquad (6.24.1)$$

The scalar product is useful to pik-up the components of a Grassmann number. So

$$\langle a, e^h \wedge e^k \rangle = \langle \frac{1}{2!} b^{rs} e_r \wedge e_s , e^h \wedge e^k \rangle_2 = \frac{1}{2!} b^{rs} \delta_{rs}^{hk} = b^{hk} \qquad (6.24.2)$$

In particular the scalar part of a Grassmann number is given by

$$s = \langle a, 1 \rangle \qquad (6.24.3)$$

The definition of (left and right) inner products between multivectors we have given can be restated for Grassmann numbers as follows (Mal'cev, 1963, p. 279) - (Bourbaky, 1958, p. 105).

$$\langle x \lrcorner a, c \rangle \stackrel{def}{=} \langle a, c \wedge x \rangle \qquad (6.24.4)$$

$$\langle y \wedge a, d \rangle \stackrel{def}{=} \langle d, c \llcorner y \rangle \qquad x, c, d \in \wedge U \quad y, a \in \wedge V$$

To make explicit this definition we perform the left inner product of two Grassmann numbers of E^3

$$\begin{aligned} \bar{a} \lrcorner a = (\bar{s} + \bar{v} + \bar{b} + \bar{t}) \lrcorner (s + v + b + t) = \\ = \bar{s} \lrcorner s + \bar{s} \lrcorner v + \bar{s} \lrcorner b + \bar{s} \lrcorner t + \\ + \bar{v} \lrcorner v + \bar{v} \lrcorner b + \bar{v} \lrcorner b + \\ + \bar{b} \lrcorner b + \bar{b} \lrcorner t + \\ + \bar{t} \lrcorner t \qquad (6.24.5) \\ = \left[\frac{1}{0!} \bar{s} s + \frac{1}{1!} \bar{v}_h v^h + \frac{1}{2!} \bar{b}_{hk} b^{hk} + \frac{1}{3!} \bar{t}_{hke} t^{hke} \right] + \\ + \left[\frac{1}{0!} 3 v^k + \frac{1}{1!} \bar{v}_h b^{hk} + \frac{1}{2!} \bar{b}_{he} t^{khe} \right] e_k + \\ + \left[\frac{1}{0!} \bar{s} b^{hk} + \frac{1}{1!} \bar{v}_e t^{hke} \right] e_h \wedge e_k + \\ + \left[\frac{1}{0!} \bar{s} t^{hke} \right] e_h \wedge e_k \wedge e_e . \end{aligned}$$

The result is then Grassmann number whose p-vectors are obtained summing all possible inner products that gives as result a p-vector.

7. EXTERIOR FORMS AND EXTERIOR DIFFERENTIALS IN PHYSICAL THEORIES

7.1 The rise of exterior forms

To pass from the discrete to the continuum, one must consider a cell-complex K formed with infinitesimal cells. Every infinitesimal oriented p-cell can be described by an infinitesimal simple p-vector. In this way the infinitesimal physical quantity ρ that is associated with an infinitesimal p-cell with origin at a point M becomes associated with the corresponding infinitesimal p-vector dp with origin at M. We may exspress this dependence writing

$$\rho (M, dp) \qquad (7.1.1)$$

where $M \in \Omega$, $dp \in \Lambda^p U$, $\rho \in \mathcal{E}$. In we choose a basis in the space $\Lambda^p U$ we may write

$$dp = \frac{1}{p!} dp^{hk \cdots s} e_h \wedge e_k \wedge \cdots \wedge e_s \qquad (7.1.2)$$

Now a part from points in which the physical properties of the medium are discontinuous we may suppose that ρ depends in a continuous way from the components $dp^{hk \cdots s}$. Then we put

$$\rho (M, dp) = \frac{1}{p!} R_{hk \cdots s} (M) \, dp^{hk \cdots s} \qquad (7.1.3)$$

where the coefficients $R_{hk \cdots s} (M)$ are elements of the

of the coefficient's space $\mathcal{C}$. In the coefficient space is a linear space, one may choose a basis E_A in it to write

$$\boldsymbol{\rho}(M, d\boldsymbol{p}) = \rho^A(M, d\boldsymbol{p})\, \boldsymbol{E}_A \qquad (7.1.4)$$

where the index A may denote either a single index or a collective index. Then every single component ρ^A (a number) may be written as follows

$$\rho^A(M, d\boldsymbol{p}) = \frac{1}{p!}\, R^A{}_{hk\ldots s}(M)\, dp^{hk\ldots s} \qquad (7.1.5)$$

We have in this way evidentiated two sets of indices h, k, ...s. that are pertinent to the p-vectors and are dummy indices in the eq.(7.1.5) they will be called geometric indices, because they are related to the geometrical element described by the p-vector dp. On the contrary the index A, that may stay for a collective indices, is pertinent to the mathematical nature of the physical quantity $\boldsymbol{\rho}$ and will be called internal index. (Anderson, 1967, p. 37).

The form so obtained is called an exterior differential form. The physical quantity $\boldsymbol{\rho}$ is called the value of the exterior differential form (Lichnerowicz, 1972, p. 43). The forms we use are in general vector valued differential forms, i.e. their valued are elements of an arbitrary vector space. (Lichnerowicz, 1972, p. 42).

The functions $R^A{}_{hk\ldots s}$ have a mixed character: they are skew-symmetric tensors with respect to the geometric indices (h, k,... s) and have an a priori unprecised tensor nature with respect to the internal index A: such tensorial behaviour depends from the mathematical nature of $\boldsymbol{\rho}$. These functions describe a

field and will be called field functions relative to the physical quantity $\boldsymbol{\rho}$. The field functions may be considered as components of a "multivector-object".

The knoweledge of the field functions permits to specify the amount of the corresponding physical quantity $\boldsymbol{\rho}$ for every point M and for every infinitesimal p-vector .

We then see that the exterior differential p-form is the continuous analogous of a p-chain, because it contains the amount of informations of the latter. But it contain more, because it permits us to know the amount of $\boldsymbol{\rho}$ associated with an arbitrary infinitesimal p-vector that, may be or may be not a p-cell of the given cell-complex.

Let us give some examples. Let us consider a magnetic field: the magnetic flux ϕ through an infinitesimal oriented surface dS described by the infinitesimal bivector $d\boldsymbol{S}$ will be:

$$\phi(M, d\boldsymbol{S}) = \frac{1}{2!}\, B_{hk}(M)\, dS^{hk} \qquad (7.1.6)$$

The skew-symmetric tensor B_{hk} is the magnetic induction tensor that describes the density of the magnetic flux. It has no internal indices because ϕ is a scalar, and has two geometric indices because of the two-dimensional character of $d\boldsymbol{S}$.

In continuum statics let us consider the contact force acting on an oriented surface: the infinitesimal amount $\boldsymbol{f}$ acting on an infinitesimal oriented 2-dimensional surface will be:

$$\boldsymbol{f}(M, d\boldsymbol{S}) = \frac{1}{2!}\, \boldsymbol{P}_{hk}(M)\, dS^{hk} \qquad (7.1.7)$$

or, in components:

$$f^A(M, dS) = \frac{1}{2!} p^A{}_{hk}(M)\, dS^{hk} \qquad (7.1.8)$$

The $p^A{}_{hk}$ are components of a tensor of rank three that may be called the dual stress tensor. It has one internal index on account of the vector nature of the contract force and two geometric indices on account of the two dimensional nature of surfaces element dS .

Let us consider as a third example the parallel displacement of a vector along an infinitesimal closed line a manifold V^n (fig. 7.1.1). The infinitesimal plane oriented area delimited by the line may be described by an infinitesimal bivector dS . The vector v that has

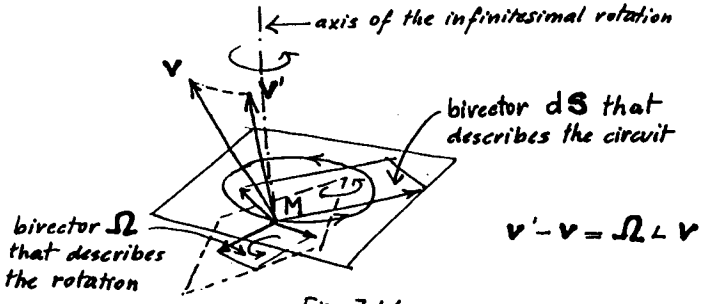


Fig. 7.1.1

experienced the parallel transport around the closed line may be expressed as follows

$$v'^A = v^A + \Omega^A{}_B\, v^B \qquad (7.1.9)$$

where the component v'^A, v^B are relative to a basis at the initial point M . Now the matrix Ω whose entries are $\Omega^A{}_B$ depend from the point M and from the bivector dS :then using a base in the space of bivectors we write: .

$$\Omega(M, dS) = \frac{1}{2!} R_{hk}(M)\, dS^{hk} \qquad (7.1.10)$$

If we use a base in space GL (n) of all square matrices of order n, we may write

$$\Omega^A{}_B(M, dS) = \frac{1}{2!} R^A{}_{Bhk}(M)\, dS^{hk} \qquad (7.1.11)$$

The field functions $R^A{}_{Bhk}(M)$ are the components of the curvature tensor of the manifold. If one take into account the fact that the bivector dS is simple one may write

$$dS = du \wedge dv \qquad (7.1.12)$$

and then the last equation may be written

$$\Omega^A{}_B(M, dS) = \frac{1}{2!} R^A{}_{Bhk}(M)\left(du^h dv^k - du^k dv^h\right) \qquad (7.1.13)$$

(Cartan, 1951, pag. 182).

Tensorial symmetries in the density fields are possible only for pairs formed by an internal and a geometrical indices, or by two internal ones. For ex. the stress tensor linked to the surface force f_A by the relation

$$f^A(M, {}^{\perp}dS) = p^{Ar}(M)\, dL_r \qquad (7.1.14)$$

where $dL = {}^{\perp}dS$, may enjoy the symmetry $p^{Ar} = p^{rA}$. So the curvature tensor, may have the symmetry

$$R_{ABhk} = R_{hkAB} \qquad (7.1.15)$$

7.2 Dual forms

Since an infinitesimal p-dimensional manifold may be described either by the infinitesimal p-vector dp or by its supplementary $dq = {}^{\perp}dp$ we may write

$$\rho^A(M, dq) = \frac{1}{(n-p)!} S^{Ars\cdots}(M)\, dq_{rs\cdots} \qquad (7.2.1)$$

This procedure is common in physical theories where we are accustomed to describe an infinitesimal oriented area dS in the three-simensional space by the orthogonal vector $n\,dS$ and an oriented volume dV by the supplementary scalar dV. So to make economy of indices in place of the formulae

$$\phi(M, dS) = \frac{1}{2!} B_{hk}(M) dS^{hk} \qquad Q(M, dV) = \frac{1}{3!} \rho_{hkl}(M) dV^{hkl}$$

we use

$$\phi(M, n\,dS) = B^r(M)\, n_r\, dS \qquad Q(M, dV) = \rho(M)\, dV$$

In continuum mechanics the same process leads to the relation

$$f^A(M, n\,dS) = p^{Ar}(M)\, n_r\, dS \qquad (7.2.4)$$

where p^{Ar} is the customary second rank stress tensor.

The set of functions $S^{Ars\cdots}$ are called <u>dual field functions</u>. To find the link between the field functions and its dual we observe that, on account of eq. (6.22.5) is $dp = dq^{\perp}$ and then

$$\rho^A(M, dp) = \frac{1}{p!} R^A{}_{hk\cdots} \left[\frac{1}{(n-p)!} \in^{rs\cdots hk\cdots} dq_{rs\cdots}\right] = \qquad (7.2.5)$$

$$= \frac{1}{(n-p)!} \left[\frac{1}{p!} \in^{rs\cdots hk\cdots} R^A{}_{hk\cdots}\right] dq_{rs\cdots}$$

It follows that

$$S^{Ars\cdots} = \frac{1}{p!} \in^{rs\cdots hk\cdots} R^A{}_{hk\cdots} \qquad (7.2.6)$$

In particular

$$B^r = \frac{1}{2!} \in^{rhk} B_{hk} \qquad (7.2.7)$$

$$\rho = \frac{1}{3!} \in^{hkl} \rho_{hkl} \qquad (7.2.8)$$

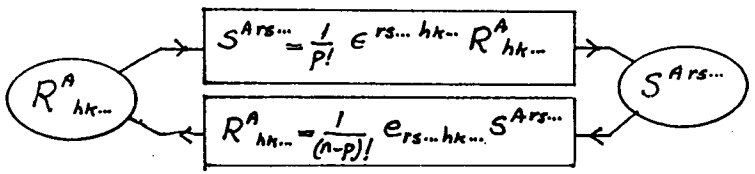
$$p^{Ar} = \frac{1}{2!} \in^{rhk} p^A{}_{hk} \qquad (7.2.9)$$

as is well known. Inversely one finds

$$R^A{}_{hk\cdots} = \frac{1}{(n-p)!} e_{rs\cdots hk\cdots} S^{Ars\cdots} \qquad (7.2.10)$$

We emphasize the fact that the same physical quantity ρ may be written in two ways: or as the value of a p-form or as a value of an (n-p)-form called its <u>dual</u>.

From the present point of view a differential form or its dual are expressions that give the same infinitesimal amount of a physical quantity associated with an infinitesimal region of a dimensional manifold.



$$\rho^A = \frac{1}{p!} R^A{}_{hk\cdots}(M)\, dp^{hk\cdots} = \frac{1}{(n-p)!} S^{Ars\cdots}(M)\, dq_{rs\cdots}$$

7.3 Connexions

In order to find the continuous analogous of the coboundary process we must analize carefully the process of "transferring" a mathematical object from one point in space to another. In § 3.6 we have supposed that the curvature and the torsion of the connection vanish, i.e. space be flat. This is the case of euclidean and pseudo-euclidean spaces. This means that the process of "transferring" a mathematical object to a common space point is independent from the line of parallel transport (zero curvature implies teleparallelism) and this permits us to consider all mathematical objects associated with the various p-cell as elements of a single space $\mathcal{C}$. In this way the "transferring" is given once and for all: we need only to perform the algebraic sums of the mathematical objects of the space $\mathcal{C}$ referred to the boundary of a p-cell as indicated by the coboundary process.

We are now at position to remove this restriction on the geometry of the space. Instead to consider a single linear space $\mathcal{C}$ we consider a linear space for every point $M \in \Omega$: we shall denote these spaces with $\mathcal{C}_M$ .All spaces $\mathcal{C}_M$ are supposed to be of the same mathematical nature: in particular of the same dimension when are finite dimensional spaces. This assures that any pair of spaces $\mathcal{C}_M$ and $\mathcal{C}_{M'}$ be isomorphic.

To give some examples one may consider the linear spaces $\mathcal{C}_M$ to be of the kind R^m with $m \geq n$ (n= dimension of the space of the independent variables). $\mathcal{C}_M$ may be the space of mxm matrices or that quaternions, of spinors, of linear operators on some linear space, etc.

The "transfer" of a mathematical object $r' \in \mathcal{C}_{M'}$ from the point M' to an infinitely near point M means that we must associate with every element $r' \in \mathcal{C}_{M'}$ an element $r'_{//} \in \mathcal{C}_M$. This process is called the <u>parallel transport</u> of r' from $\mathcal{C}_{M'}$ to $\mathcal{C}_M$. When the parallel transport has been prescribed we have fixed a

<u>connexion</u>. (see fig. 7.3.1).

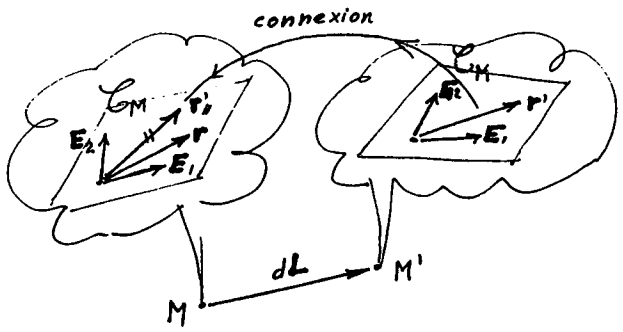


Fig. 7.3.1

To define a parallel transport we choose a frame at all spaces $\mathcal{C}_M$. Now if $E_A(M)$ is a base of $\mathcal{C}_M$ and $E_A(M')$ is a base of $\mathcal{C}_{M'}$ putting

$$\begin{aligned} r &= r^A E_A(M) \qquad (A = 1,2,\dots m) \\ r' &= r'^A E_A(M') \\ r'_{//} &= r'^A_{//} E_A(M) \end{aligned} \tag{7.3.1}$$

a connexion is defined giving a set of $m^2 \times n$ functions such that

$$r'^A_{//} = r'^A + L^A_{Bk}(M)\, r'^B\, dL^k \tag{7.3.2}$$

(Willmore, 1959, pag. 218) (Favard, 1957, pag. 469).
The functions $L^A_{Bk}(M)$ are called <u>connexion coefficients</u>. They have two internal indices A and B and a geometrical index k and

then can be considered as field functions of a mathematical object ω associated with the lines: the object ω is a matrix whose entries are ω^A_B given by

$$\omega^A_B (M, dL) = L^A_{Bk} (M) dL^k \tag{7.3.3}$$

One may write also

$$\omega (M, dL) = L_k (M) dL^k \tag{7.3.4}$$

This is a 1-form called the 1-form of the connexion. We remember that A and B may be a collective of indices. So if r is a bivector on an n-dimensional space we may write

$$r = \frac{1}{2!} r^{\alpha\beta} e_\alpha \wedge e_\beta \tag{7.3.5}$$

instead of writing

$$r = r^A E_A \tag{7.3.6}$$

Then

$$r'^{\alpha\beta}_{//} = r'^{\alpha\beta} + L^{\alpha\beta}_{\mu\nu k} (M) r'^{\mu\nu} dL^k \tag{7.3.7}$$

The functions $L^{\alpha\beta}_{\mu\nu k} (M)$ are the components of a tensor connexion (Bompiani, 1946). These funcions are linked to the connexion functions $L^\alpha_{\mu k}$ for vectors by the relation

$$L^{\alpha\beta}_{\mu\nu k} = L^\alpha_{\mu k} \delta^\beta_\nu + L^\beta_{\nu k} \delta^\alpha_\mu . \tag{7.3.8}$$

If we consider a mixed tensor r'^α_β we have

$$r'^\alpha_{//\beta} = r'^\alpha_\beta + L^{\alpha\mu}_{\beta\nu k} (M) r'^\nu_\mu dL^k \tag{7.3.9}$$

where

$$L^{\alpha\mu}_{\beta\nu k} = L^\alpha_{\nu k} \delta^\mu_\beta - L^\mu_{\beta k} \delta^\alpha_\nu \tag{7.3.10}$$

In general if we consider a mixed tensor r'^A_B where A and B are two set of indices we may write

$$r'^A_{//B} = r'^A_B + L^{AC}_{BDk} (M) r'^D_C dL^k \tag{7.3.11}$$

where

$$L^{AC}_{BDk} = L^A_{Dk} \delta^C_B - L^C_{Bk} \delta^A_D \tag{7.3.12}$$

Only after the introduction of a parallel transport we may perform the difference between an element $r' \in \mathcal{C}_{M'}$ and an element $r \in \mathcal{C}_M$. Then while the difference $(r'-r)$ has no meaning because $r' \in \mathcal{C}_{M'}$ the difference $(r'_{//} - r)$ with $r'_{//} \in \mathcal{C}_M$ is meaningfull and is called the absolute differential or covariant differential (Willmore, 1959, pag. 213) (Schouten, 1954, pag. 125). The difference between the mathematical objects (say two vectors) located at M and M' respectively means that we must perform a parallel transport of the mathematical object referred to M' from M' to M and then to make the difference with the mathematical object referred to M.

This way of doing emphasize the role of the parallel transport, i.e. the choice of a connexion. This permit us to depart from the customary euclidean connexion when appear useful to describe physical facts.

So in continuum mechanics one may consider the displacement vector of every point M of the material continuum. The "difference" between displacement $u(M')$ and $u(M)$ is usually meant as that obtained performing the euclidean parallel displacement of the vector $u(M')$ to M and then deducing $u(M)$. The difference so obtained does not give indication of the strain on the continuum because also a simple rigid body rotation give rise a di-

splacement difference. Heuce if one wont to depurate this displa cement difference from the rigid rotation part one is lead to a connexion different from the customary euclidean one, i.e. one that take into account the rigid body rotation. This is what has been done by Schaefer (1967, p. 319), in the realm of oriented continua.

In analogous way one may describe the interaction between two field considering the perturbing field as changing the connexion of the underlying space: this is the case of the electromagnetic field (Hehl-Heyde, 1973, p. 179) (Trautmann, 1973, p.3).

7.4 The exterior differential

We are now at position to perform on differential forms the analogous of the coboundary process on chains. To this end let us consider an infinitesimal p-vector: it describes an infinitesimal p-cell that may be considered either as a member of a cell-complex or a single cell. The coboundary process amounts to transfer to the p-cell the mathematical entity associated with every face of the p-cell with the same or opposite sign as usual and then summing all these amounts. When this process is performed starting with a p-form we obtain a (p+1)-form that is called the exterior differential of the given p-form. We shall consider in detail the process starting with a 0-form, 1-form and 2-form respectively.

Let us consider a physical quantity $\boldsymbol{a}$ referred to the points: it defines a 0-form $\boldsymbol{a}(M)$. If we consider an infinitesimal 1-cell described by an infinitesimal 1-vector $d\boldsymbol{L}$ (see fig. 7.4.1). In order to perform the

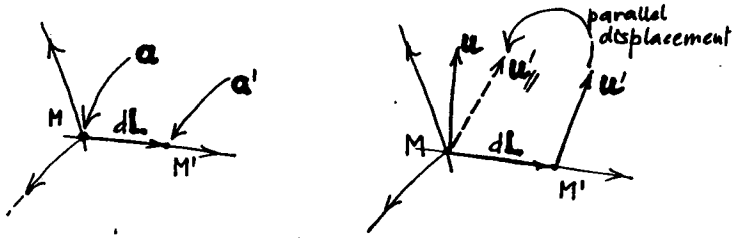


Fig. 7.4.1

coboundary process we must evaluate the difference between the physical quantity $\boldsymbol{a}'$ at M' and the physical quantity $\boldsymbol{a}$ at M. This means that we must consider the difference between the mathematical object $\boldsymbol{a}'_{//}$ at M and $\boldsymbol{a}$. So if $\boldsymbol{a}$ is a vector, denoted by $\boldsymbol{u}$, we must transfer by parallel displacement $\boldsymbol{u}'$ from M' to M and then perform the difference $\boldsymbol{u}'_{//} - \boldsymbol{u}$ (see fig. 7.4.1 right). Then putting

$$\boldsymbol{a} = a^A \boldsymbol{E}_A \tag{7.4.1}$$

will be

$$\begin{aligned} \bar{\beta}^A(M, d\boldsymbol{L}) &= a'^A_{//} - a^A = a'^A - a^A + L^A_{Bk}\, a'^B\, dL^k \\ &= (\partial_k a^A + L^A_{Bk}\, a^B)\, dL^k = \\ &= (\nabla_k a^A)\, dL^k \end{aligned} \tag{7.4.2}$$

where we have put

$$\nabla_k a^A \overset{def}{=} \partial_k a^A + L^A_{Bk}\, a^B \tag{7.4.3}$$

The expression $\nabla_k a^A$ is called the covariant derivative of a $a^A(M)$. Then the coboundary process performed on the 0-form $a^A(M)$ gives rise to a 1-form $\bar{\beta}^A(M, d\boldsymbol{L})$, exactly as the coboundary process on a 0-chain give rise to a 1-chain.

Now let us consider a 1-form

$$\beta^A(M, dL) = b^A_k(M)\, dL^k \qquad (7.4.4)$$

If we consider an infinitesimal 2-cell (fig. 7.4.2)

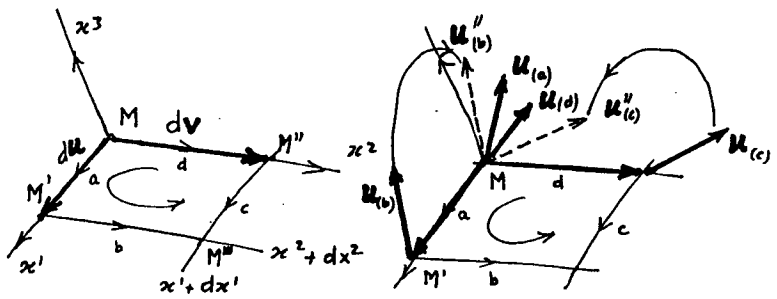


Fig. 7.4.2

to construct the coboundary we must perform the difference

$$\bar{\gamma}^A(M, dS) = \beta^A_{(a)} - \beta^{A\,//}_{(c)} + \beta^A{}''_{(b)} - \beta^A_{(d)} \qquad (7.4.5)$$

This can be best visualized when β is a vector, call it u (see fig. 7.4.2. right). Now

$$\beta^A_{(a)} = b^A{}_h(M)\, du^h \qquad \beta^A{}_{(b)} = b^A_k(M')\, dv^k$$
$$\beta^A_{(c)} = b^A_k(M'')\, du^k \qquad \beta^A{}_{(d)} = b^A_k(M)\, dv^k \qquad (7.4.6)$$

where we have used the property that the infinitesimal vectors $du = MM'$ and $M''M'''$ have the same components with respect to the natural basis because M''' and M^* lie on a same coordinate line as M' and M do. Now

$$\beta^{A\,//}_{(c)} = \beta^A_{(c)} + L^A_{Bk}(M)\beta^A_{(c)}\, dv^k =$$
$$= \left[b^A_h(M'') + L^A_{Bk}(M) b^A_h(M'') dv^k\right] du^h \qquad (7.4.7)$$
$$\beta^{A\,//}_{(b)} = \beta^A_{(b)} + L^A_{Bh}(M)\beta^A_{(b)}\, du^h =$$
$$= \left[b^A{}_k(M') + L^A_{Bh}(M)\, b^A_k(M')du^h\right] dv^k$$

Then

$$\bar{\gamma}^A(M, dS) = -\left[b^A_h(M'') - b^A_h(M) + L^A_{Bk}(M) b^B_h(M'')dv^k\right] du^h +$$
$$+\left[b^A_k(M') - b^A_h(M) + L^A_{Bh}(M)\, b^B_k(M')\, du^h\right] dv^k =$$
$$= -\left[\partial_k b^A_h + L^A_{Bk} b^B_h\right] du^h dv^k + \left[\partial_h b^B_k + L^A_{Bh} b^B_k\right] du^h dv^k =$$
$$= \frac{1}{2!}\left[\left(\partial_h b^A_k + L^A_{Bh} b^B_k\right) - \left(\partial_k b^B_h + L^A_{Bk} b^B_h\right)\right]\left(du^h dv^k - du^k dv^h\right)$$
$$= \frac{1}{2!}\left[\bar{\nabla}_h b^A_k - \bar{\nabla}_k b^A_h\right] dS^{hk}$$

where we have put

$$\bar{\nabla}_h b^A_k \overset{def}{=} \partial_h b^A_k + L^A_{Bh} b^B_k \qquad (7.4.9)$$

This expression will be called the <u>internal covariant derivative</u> of b^A_k because refers only to the internal index A. In differential geometry it is usual to consider the <u>covariant derivative</u>.

$$\nabla_h b^A_{\ k} \overset{def}{=} \partial_h b^A_{\ k} + L^A_{Bh}\, b^B_{\ k} - \Gamma^j_{kh}\, b^A_{\ j} \qquad (7.4.10)$$

where Γ^j_{kh} are the connexion coefficients of the tangent spaces. Using the covariant derivative the final expression of (7.4.8) can be written

$$\bar\nabla_h b^A_{\ k} - \bar\nabla_k b^A_{\ h} = \nabla_h b^A_{\ k} - \nabla_k b^A_{\ h} + T^j_{hk}\, b^A_{\ j} \qquad (7.4.11)$$

where we have put

$$T^j_{hk} \overset{def}{=} \Gamma^j_{hk} - \Gamma^j_{kh} \qquad (7.4.12)$$

that is known as the torsion tensor of the connexion Γ^j_{hk}. (Willmore, 1959, pag. 219) (Schouten, 1954, pag. 126). We emphasize the fact that if one distinguish from the start between internal indices and geometrical ones and introduce the "internal" covariant derivative one avoid to introduce the connexion on the tangent spaces and the torsion of this connexion.

Now let us consider a 2-form

$$\gamma^A(M, d\mathbf{S}) = \frac{1}{2!}\, c^A_{hk}(M)\, dS^{hk} \qquad (7.4.13)$$

If we consider an infinitesimal 3-cell (fig. 7.4.3)

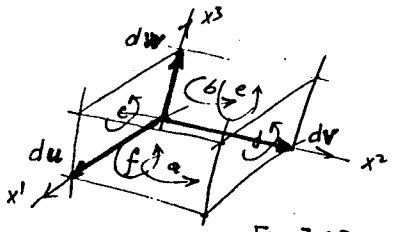


Fig. 7.4.3

the coboundary process performed on the 2-chain is

$$\bar\delta^A(M, d\mathbf{V}) = \left(\gamma^{A\,\prime\prime}_{(b)} - \gamma^A_{(a)}\right) + \left(\gamma^{A\,\prime\prime}_{(d)} - \gamma^A_{(c)}\right) + \left(\gamma^{A\,\prime\prime}_{(f)} - \gamma^A_{(e)}\right)$$

Since

$$\begin{aligned}
\gamma^A_{(b)}{}'' &= \gamma^A_{(b)} + L^A_{Be}\, \gamma^B_{(b)}\, dw^e \\
\gamma^A_{(d)}{}'' &= \gamma^A_{(d)} + L^A_{Be}\, \gamma^B_{(d)}\, dv^e \qquad (7.4.15) \\
\gamma^A_{(f)}{}'' &= \gamma^A_{(f)} + L^A_{Be}\, \gamma^B_{(f)}\, du^e
\end{aligned}$$

it follows

$$\begin{aligned}
\bar\delta^A(M, d\mathbf{V}) = &\frac{1}{2!}\left[\partial_e c^A_{hk} + L^A_{Be}\, c^B_{hk}\right] dw^e \left(du^h dv^k - du^k dv^h\right) + \\
&+ \frac{1}{2!}\left[\partial_e c^A_{hk} + L^A_{Be}\, c^B_{hk}\right] dv^e \left(dw^h du^k - dw^k du^h\right) + \\
&+ \frac{1}{2!}\left[\partial_e c^A_{hk} + L^A_{Be}\, c^B_{hk}\right] du^e \left(dv^h dw^k - dv^k dw^h\right)
\end{aligned}$$

With the position analogous to (7.4.9)

$$\bar\nabla_e c^A_{hk} \overset{def}{=} \partial_e c^A_{hk} + L^A_{Be}\, c^B_{hk} \qquad (7.4.16)$$

and observing that the sum of the terms in round brackets is the developement of the determinant that gives the components of the trivector $d\mathbf{V} = du \wedge dv \wedge dw$, we may write

$$\bar\delta^A(M, d\mathbf{V}) = \frac{1}{2!}\left(\bar\nabla_e c^A_{hk}\right) dV^{ehk}$$

Since the coefficient in round brackets is skew-symmetric in the indices h,k only, in order to obtain the skew-symmetry also with respect to 1,h and 1,k we write:

$$\begin{aligned}
\bar\delta^A(M, d\mathbf{V}) &= \frac{1}{3}\frac{1}{2!}\left[\bar\nabla_e c^A_{hk}\, dV^{ehk} + \bar\nabla_h c^A_{ke}\, dV^{hke} + \bar\nabla_k c^A_{eh}\, dV^{keh}\right] = \\
&= \frac{1}{3!}\left[\bar\nabla_e c^A_{hk} + \bar\nabla_h c^A_{ke} + \bar\nabla_k c^A_{eh}\right] dV^{ehk} \qquad (7.4.17)
\end{aligned}$$

on account of the relations $dV^{ehk} = dV^{hke} = dV^{keh}$. Collecting the results obtained by the coboundary process on the 0,1,2-forms we may write

$$\begin{aligned}
a &\longrightarrow \bar\beta = \frac{1}{1!}\left(\frac{1}{0!}\,\delta^r_h\, \bar\nabla_r a\right) dL^h \\
\beta &\longrightarrow \bar\gamma = \frac{1}{2!}\left(\frac{1}{1!}\,\delta^{rs}_{hk}\, \bar\nabla_r b_s\right) dS^{hk} \qquad (7.4.18) \\
\gamma &\longrightarrow \bar\delta = \frac{1}{3!}\left(\frac{1}{2!}\,\delta^{rst}_{hke}\, \bar\nabla_r c_{st}\right) dV^{hke}
\end{aligned}$$

and in general if

$$\rho = \frac{1}{p!} R_{hk\ldots}\, dp^{hk\ldots} \qquad (7.4.19)$$

$$\bar{q} = \frac{1}{(p+1)!} Q_{hk\ell}\, dp^{hk\ell\ldots}$$

is

$$Q_{hk\ell\ldots} = \frac{1}{p!} \delta^{rst\ldots}_{hk\ell\ldots} \bar{\nabla}_r R_{st\ldots} \qquad (7.4.20)$$

Remembering the expression of the exterior product if we put

$$D\wedge\ldots \overset{def}{=} (dx^r \bar{\nabla}_r)\wedge\ldots = (dx^r \partial_r)\wedge\ldots + (dx^r L_r)\wedge\ldots - \ldots \wedge (dx^r L_r)$$
$$= d\wedge\ldots + \omega\wedge\ldots - \ldots\wedge\omega$$

we may write

$$\bar{q} = D\wedge\rho \qquad (7.4.22)$$

(Cartan, 1951, pag. 209) (Willmore, 1959, pag. 219).
We have the sequence

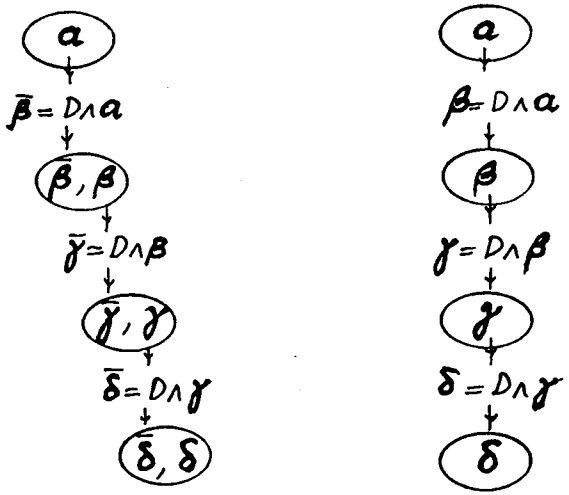


fig. 7.4.4

The expression $D\wedge\rho$ is called the absolute exterior differential of the form ρ (Cartan, 1951, pag. 209).
From now on we omit the bar over the form obtained by the absolute exterior differential, say $\bar{\beta}$, and use the same letter, say β, to denote either a 1-form obtained by the absolute exterior differential of a 0-form, say a, or an arbitrary 1-form. This may be a source of misleading at the beginning has the great inerit of simplifing all subsequent schemes.
When we pass from a flat space to a space with torsion different from zero the Poincaré lemma cease to be valid (Guggenheimer, 1963 p.358): then in general $D\wedge(D\wedge\rho) \neq 0$.

Summarizing we can say that the continuum analogous of the coboundary operator on chains in the exterior differential on forms.

Since we can write $(dq = {}^{\perp}dp\ ,\ dr = {}^{\perp}ds)$

$$\rho = \frac{1}{p!} R_{hk\ldots}\, dp^{hk\ldots} = \frac{1}{(n-p)!} S^{rst\ldots}\, dq_{rst\ldots} \qquad (7.4.23)$$

$$q = \frac{1}{(p+1)!} Q_{hk\ell\ldots}\, ds^{hk\ell\ldots} = \frac{1}{(n-p-1)!} T^{rs\ldots}\, dr_{rs\ldots}$$

the relation between $T^{rs\ldots}$ and $S^{rst\ldots}$ will be

$$T^{rs\ldots} = \frac{1}{(p+1)!} \epsilon^{rs\ldots hk\ell\ldots} Q_{hk\ell\ldots} =$$
$$= \frac{1}{(p+1)!} \epsilon^{rs\ldots hk\ell\ldots} \left(\frac{1}{p!} \delta^{mnp}_{hk\ell} \bar{\nabla}_m R_{np\ldots}\right) = \qquad (7.4.24)$$
$$= \frac{1}{p!} \epsilon^{rs\ldots mnp\ldots} \bar{\nabla}_m \left(\frac{1}{(n-p)!} \epsilon_{ab\ldots np\ldots} S^{ab\ldots}\right) =$$
$$= \bar{\nabla}_m \left[\frac{1}{(n-p)!} \delta^{rs\ldots m}_{ab\ldots} S^{ab\ldots}\right] =$$
$$- \bar{\nabla}_m S^{rs\ldots m}$$

then

$$T^{rs\cdots}(M) = \overline{\nabla}_m S^{rs\cdots m}(M) \qquad (7.4.25)$$

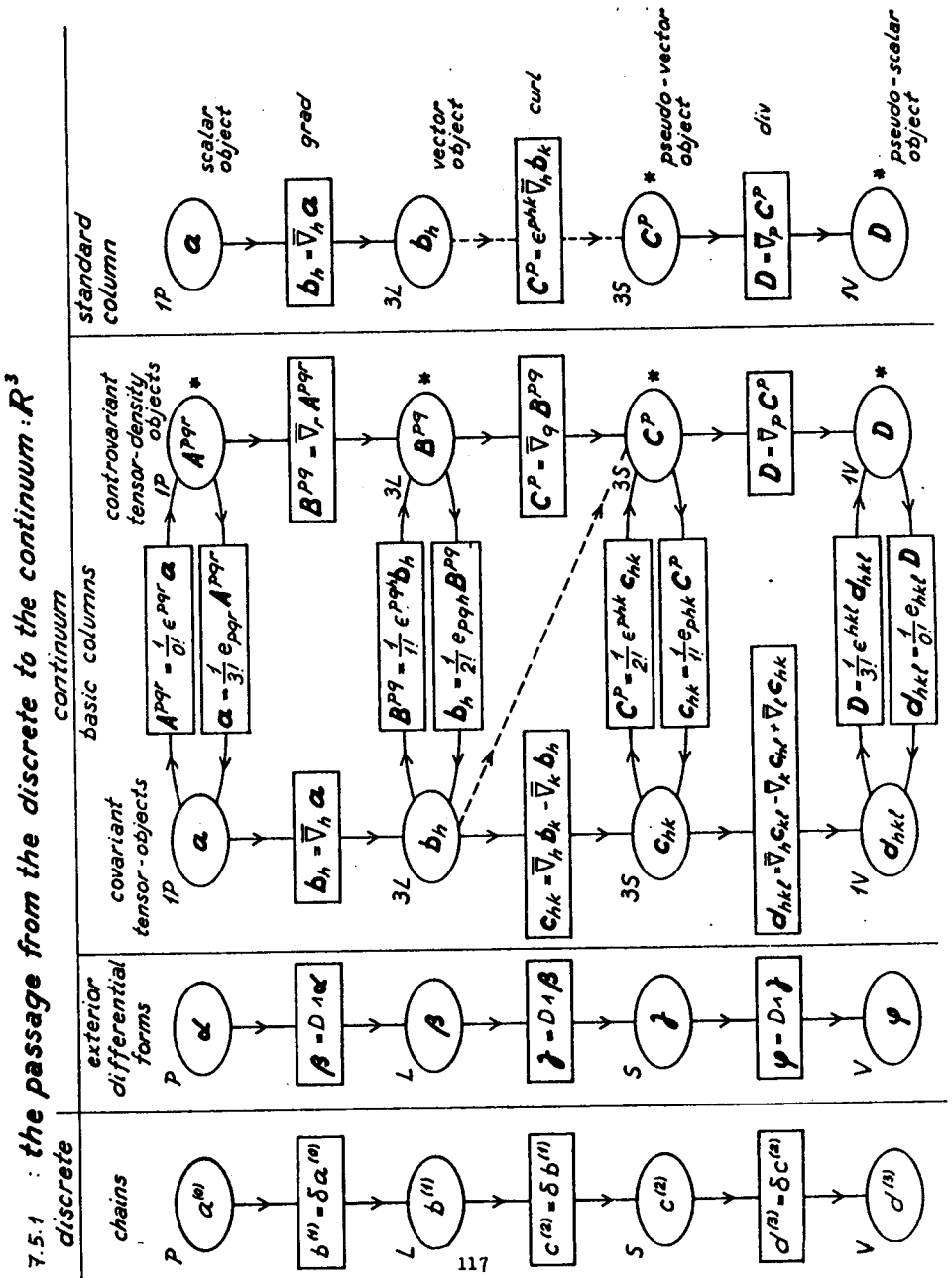
All these relations are summarized in the following tables. We remark the important fact that in all these relations no metrical properties appear, i.e. they are valid for euclidean, pseudoeuclidean and riemannian spaces.

7.5 Basic and standard columns.

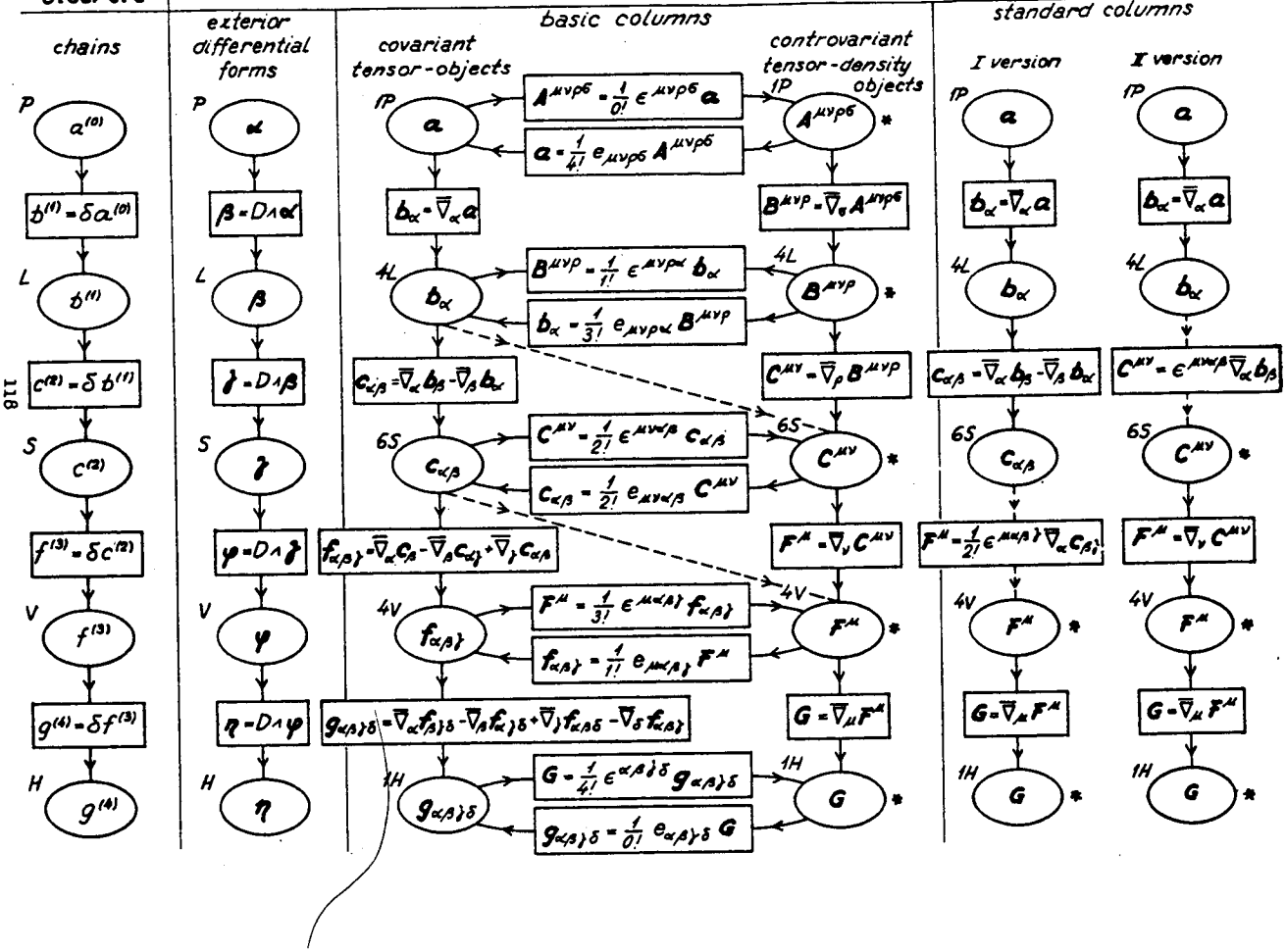
The passage from the discrete to the continuum is summarized in table (7.5.1) for $n=3$ and in table (7.5.2) for $n=4$. In the first column we represent the p-chains and the coboundary operator. In the second column we represent the corresponding exterior differential forms with the exterior derivative. This column is then splitted into two new columns we call basic columns where we represent the corresponding density fields and their duals alongside with their connecting equations. Once we decide to use one tensor-object for every row, the primal or the dual, according with the criterion to economize indices, we come to a single column that will be referred to as standard column. The result is that the first column dealing with chains and coboundary operator is transformed in the last column dealing with tensor objects and typical differential operators.

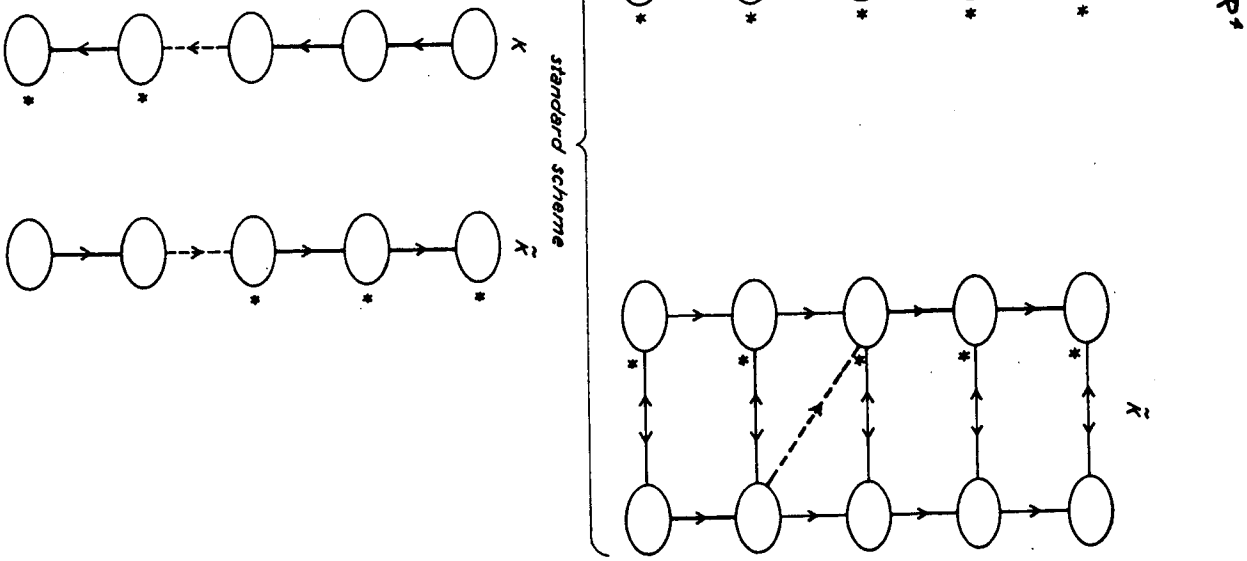
Since we have a standard column for every one of the cell-complex K and $\widetilde{K}$ we may consider the pair formed by two standard columns as the equivalent in the continuum of the classification scheme of § 5 . This scheme will be called the standard scheme. All schemes contained in § 8 are standard schemes.

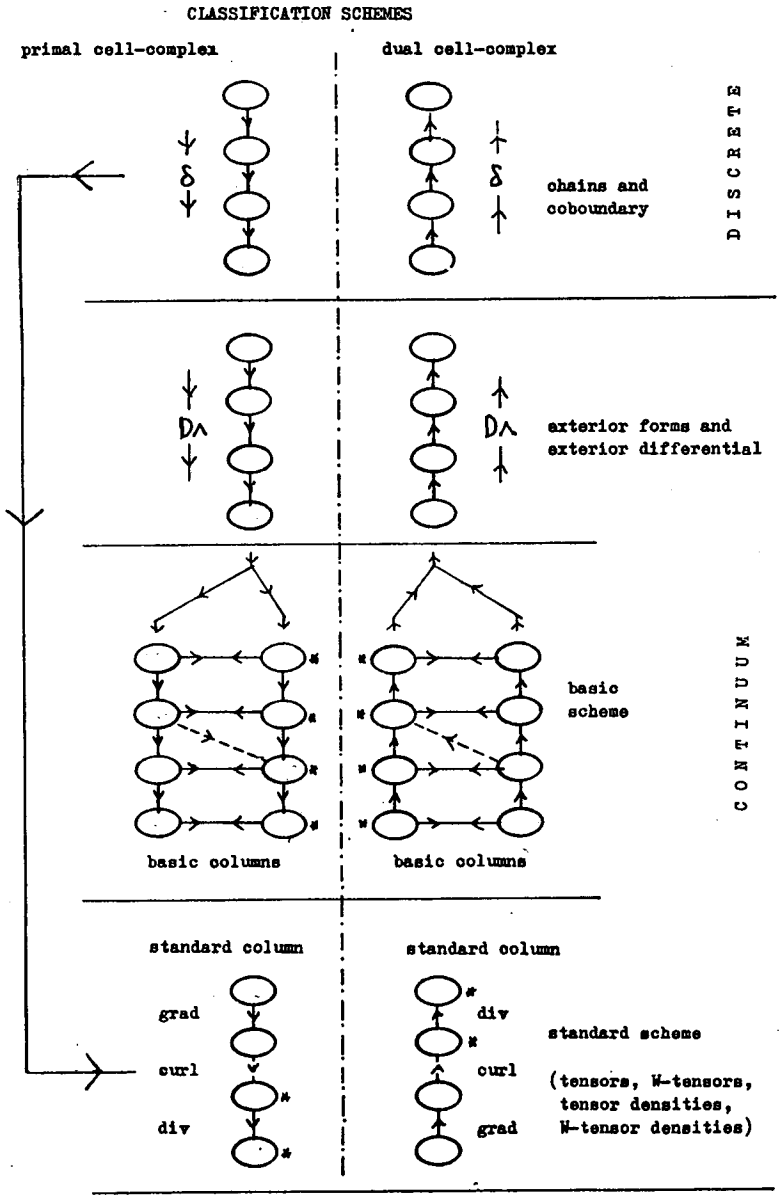
7.5.1 : the passage from the discrete to the continuum : R^3



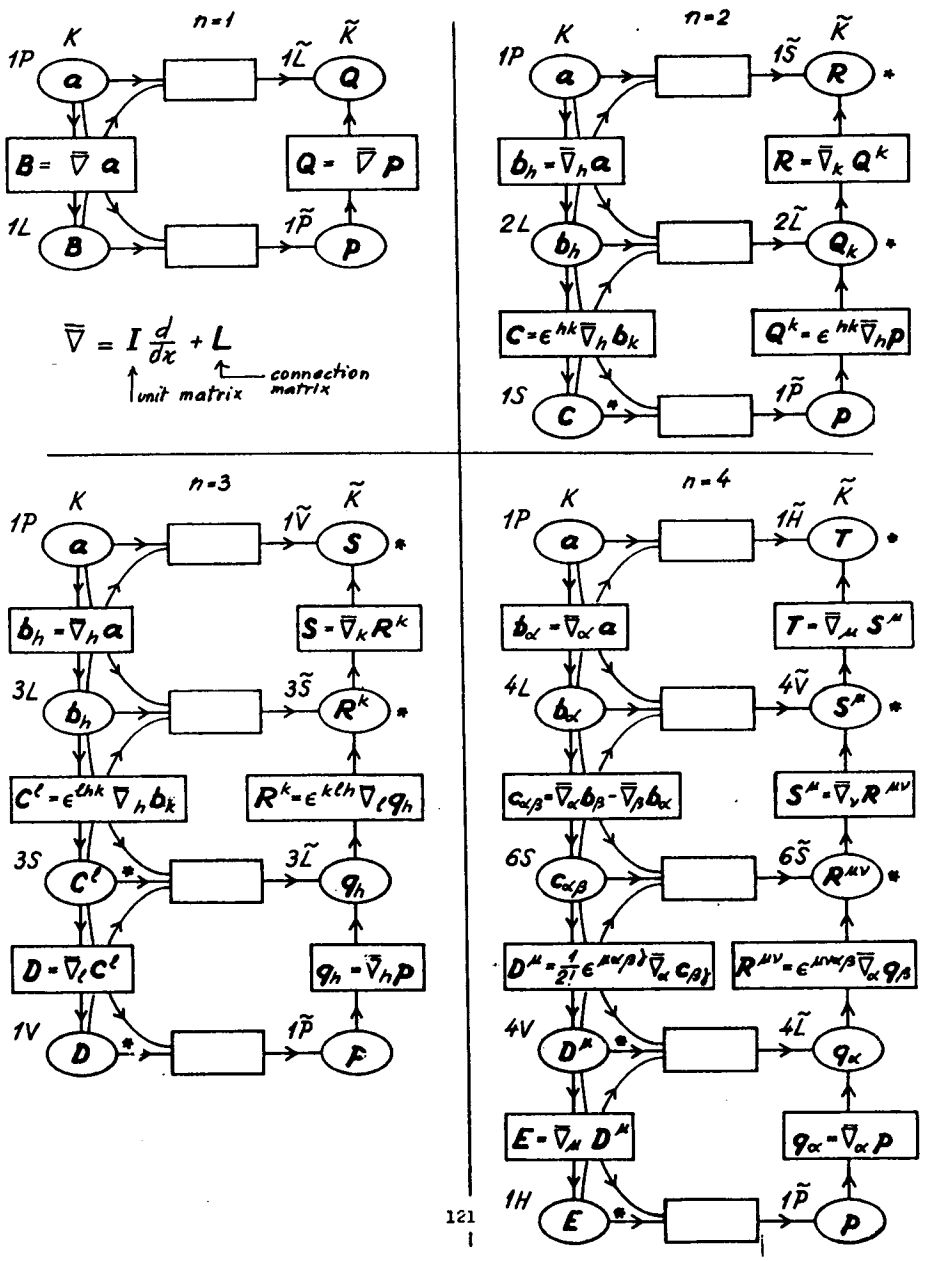
the passage from the discrete to the continuum: $\mathbb{R}^4$



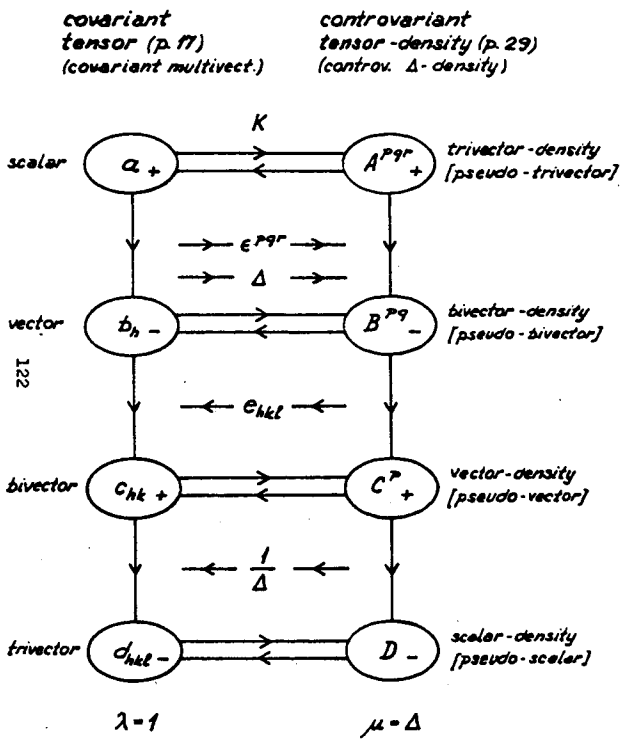


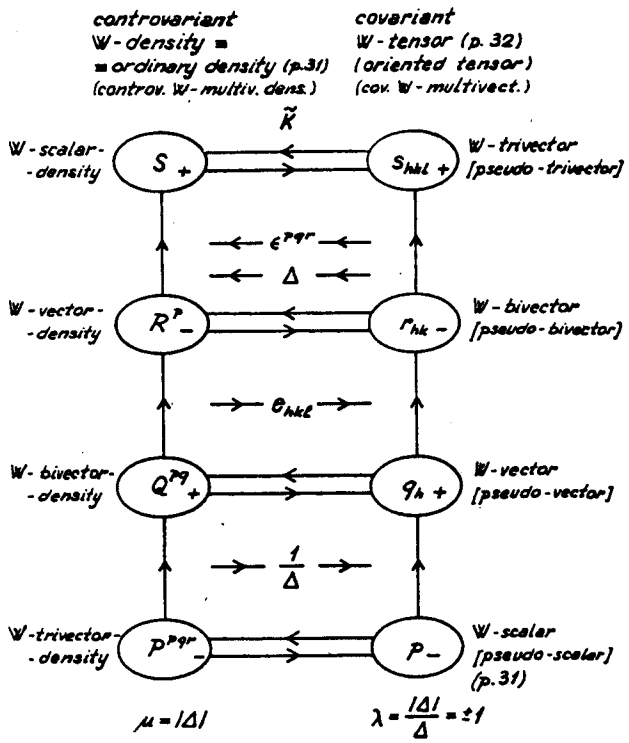


7.5.3 *the standard schemes for n=1,2,3,4.*



7.5.4 *the therminology used is that of J.A.Schouten, Tensor analysis for physicists, Oxford, 1951*
the names in square brackets denotes the current physical nomenclature





$\rho^k_h L^p_k = \delta^p_h \qquad \rho^k_h = \frac{\partial x^k}{\partial \bar{x}^h}$

$\Delta = J$

e_h = base vectors $\quad h = 1,2,3$

$\bar{e}_h = \rho^k_h e_k \qquad e_k = L^p_k \bar{e}_p$

$x^k = x^k(\bar{x}^h)$

$\Delta = \det \|\rho^k_h\| = \det \left\| \frac{\partial x^k}{\partial \bar{x}^h} \right\|$

$\bar{\rho} = |\Delta| \rho$ ordinary density

$\bar{\rho}\, d\bar{V} = \frac{|\Delta|}{\Delta} \rho\, dV \rightarrow \bar{Q} = \frac{|\Delta|}{\Delta} Q$ oriented tensor

$\bar{\rho}\,(\text{mv. of pos}) = \frac{\bar{Q}}{\sqrt{\bar{a}}} = \frac{-Q}{-\sqrt{a}} = \rho$

general

$\bar{\alpha}_{rs\ldots} = \lambda \rho^p_r \rho^q_s \ldots \alpha_{pq\ldots}$

$\bar{\alpha}^{rs\ldots} = \frac{1}{\lambda} L^r_p L^s_q \ldots \alpha^{pq\ldots}$

$\mathcal{E}^{pqr}$ = controv. tensor-density $\left(\frac{1}{\Delta}\right)$

$\mathcal{E}_{hkl}$ = cover. tensor-density (Δ)

$\sqrt{\bar{a}} = |\Delta| \sqrt{a} \qquad (|\Delta|)$

$d\bar{V} = \frac{1}{\Delta} dV \qquad \left(\frac{1}{\Delta}\right)$

Ricci $\varepsilon^{pqr} = \frac{1}{\sqrt{a}} \varepsilon^{pqr}$ W-tensor

$\bar{\varepsilon} = \frac{|\Delta|}{\Delta} \varepsilon$ fermut type

B^h_a density ? (Brillouin)

H^p_p W-tensor (Sch., Van D., Post.?)

general Kron. Symbol $\delta^h_p, \delta^{hs}_{pq}, \delta^{hkl}_{pqr}$ tensor $\lambda = 1$

ε^{pqr} (perm. symbol) $\sqrt{a}$ tensor/density $\lambda = \Delta$ tensor capacity $\frac{1}{\Delta}$

ordinary tensor density $\lambda = |\Delta|$ ordinary tensor capacity $\frac{1}{|\Delta|}$

ε^{hkr} (Ricci) oriented tensor $\lambda = \frac{|\Delta|}{\Delta}$

ε^{hkr} (Ricci)

$\varepsilon^{hkr}\sqrt{a}\, dV$

$\frac{1}{\sqrt{a}}$

"It is only through the progress of science in recent times that we have become acquainted with so large a number of physical quantities that a classification of them is desiderable" "...I think that the progress of science, both in the way of discovery, and in the way of diffusion, would be greatly aided if more attention were paid in a direct way to the classification of quantities"

J. Clerk-Maxwell, 1871

8. A CLASSIFICATION SCHEME: CONTINUOUS SYSTEMS AND FIELDS.

We have whown in § 7 that those physical quantities that are referred to the geometrical objects are described in the continuum by the corresponding field functions. It follows that a classification scheme for the physical quantities of a continuous system or of a field imply a classification of these field function. The natural criterion for this classification is that of referring a field function to the same geometrical object to which the corresponding extensive physical quantity is referred.

So since the magnetic flux ϕ is referred to the 2-cells of K its density fields $B_{hh}(M)$ or its dual $B^r(M)$ will be referred also to the 2-cells of K. In analogous way since the displacement s of a particle is referred to time intervals of the primal cell-complex K then its time densits, the velocity v, will be referred also to the time intervals of K. In this way we may obtain the classification scheme for the main physical theories: these are shown in the tables that follow.

8.1 How to read the schemes.

Notations. Symbols and names are those recommended be the International Union of Pure and Applied Physics. In order to make easier the comparison among different theories all schemes are written in the same system of unities: the system chosen is the MKSA razionalized system. The space-time variables are denoted $x^0 = ct, x^1, x^2, x^3$ the signature of the metric is + − − − .

Stars. * The star on the right side of a frame denotes a dual density field that is formed by a tensor-density object.

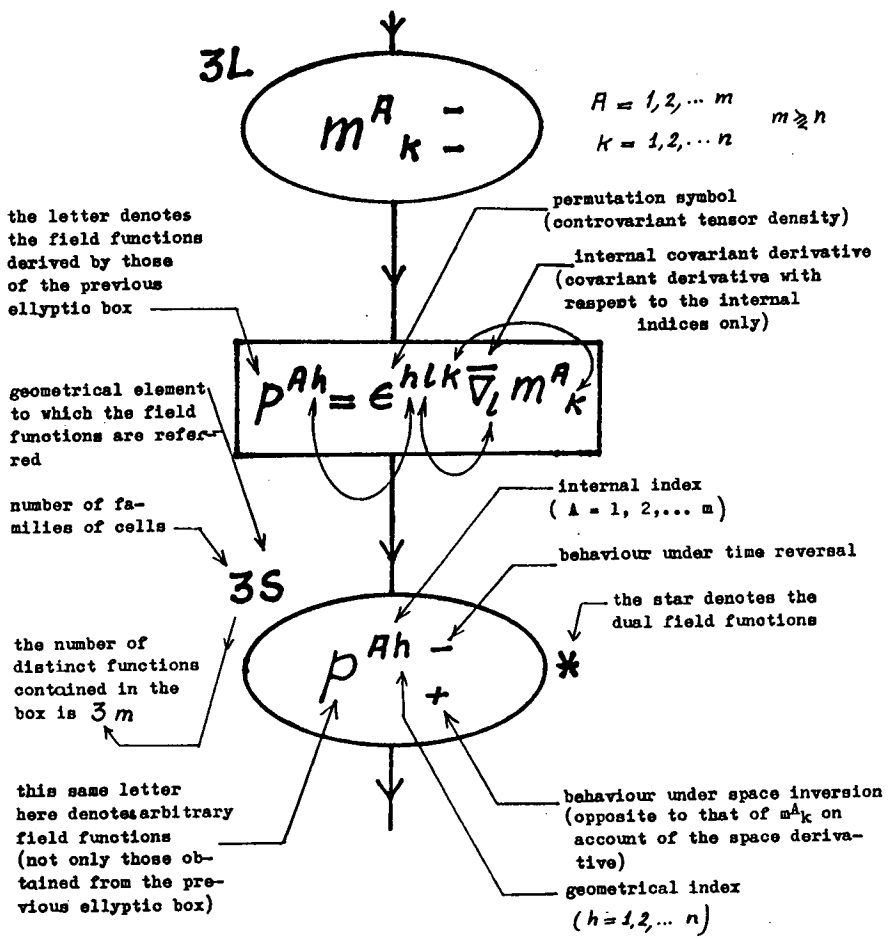
We remark that the pseudo-tensor nature for space-time is referred to the inversion of space-time axes (in relativistic quantum mechanics it is customary to speak of pseudo-tensor referring to space inversion only).

Space and time inversions. The behaviour of the field functions under time or space inversions is indicated with a plus or minus sign to the right of every letter. The upper sign denotes the behaviour of the functions under a time inversion: so f_h^+ means that $f_h(-t,x) = +f_h(t,x)$. The lower sign denotes the behaviour under space inversion: so T_{hk-} means that $T_{hk}(t,-x) = -T_{hk}(t,x)$.

One of the interesting features exhibited by the schemes is the regular alternation of the signs along the same vertical line.

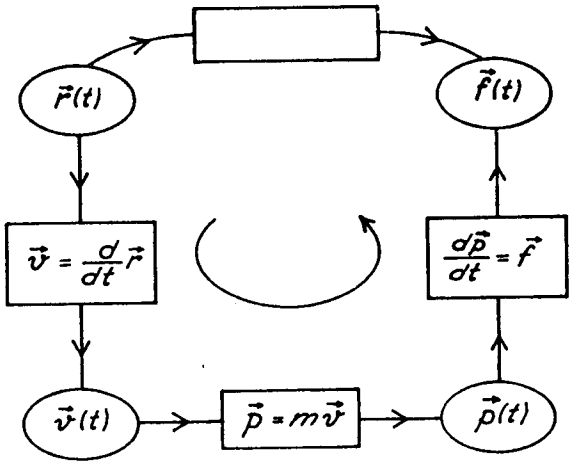
Physical dimensions. The physical dimension of the physical quantity that lies in the first ellyptic box of every standard column gives the dimensions of all the following ones dividing by L or T at every step according if there are space or time derivatives.

Elaborated schemes. From page 207 to page 212 there are some scheme obtained from the standard ones after some elaborations. So one may take the symmetric part of the displacement gradient of page 143 to obtain the usual strain tensor of p. 207. Many properties of the standard schemes are lost in these elaborated schemes: so the number of distinc functions contained in a box is no longer the product of the number of the families of cells by the number m of distinct functions contained in the first box. The regular alternation of signs in space inversions is lost. There appear second order derivatives between two successive boxes: see p. 207. This phenomenon is explained in the tables of page 206 (compare also with p. 145).

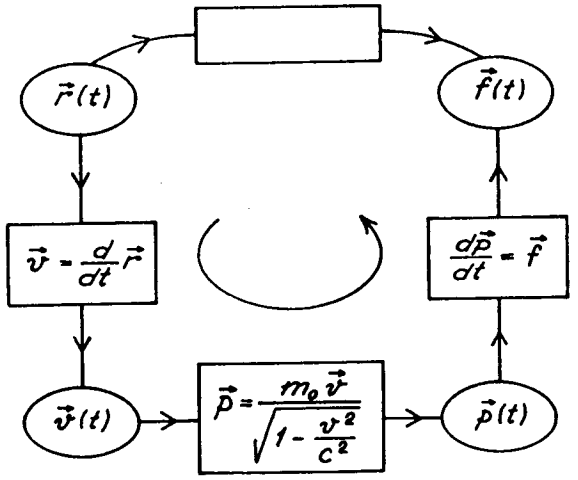


Remark on the symbol $\overline{\nabla}_h$. The internal covariant derivative reduces to the partial derivative in euclidean spaces, referred to cartesian coordinates and when the connexion is the euclidean one. We emphasize that the space may be euclidean, the coordinates may be cartesians but the connexion may be not euclidean. The reason is that the connexion coefficients L^A_{Bk} of the coefficient spaces $\mathcal{C}_M$ are not necessarily the same of the tangent spaces, i.e. Γ^r_{sh} See for ex. the schemes of polar continua p.145 and that of the bending of beams p.137 and 206.

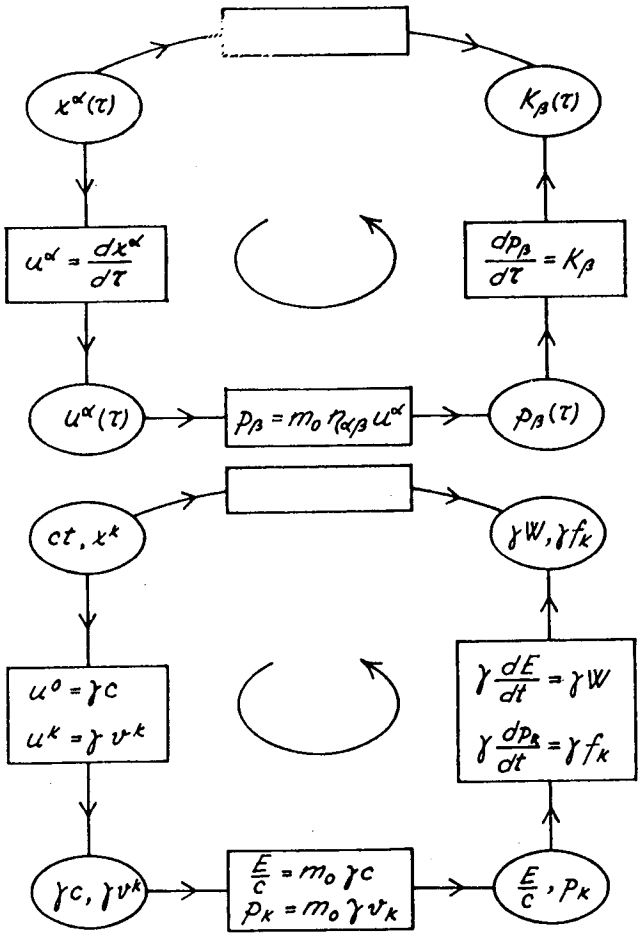
classical particle dynamics



relativistic particle dynamics

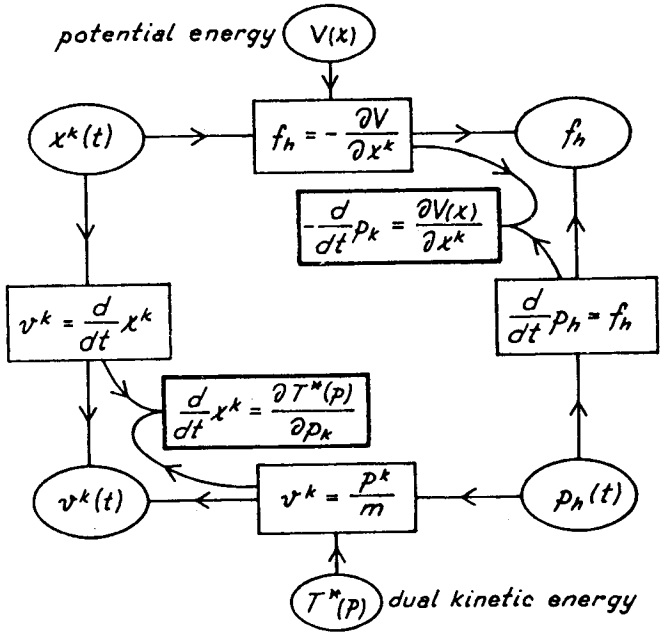


relativistic particle dynamics
(space - time formulation)



$W = f_k v^k$ = power $E = m_0 c^2 \gamma$ = total energy

classical particle dynamics
the canonical equations



analytical mechanics

variable: t

particle dynamics

variable: t

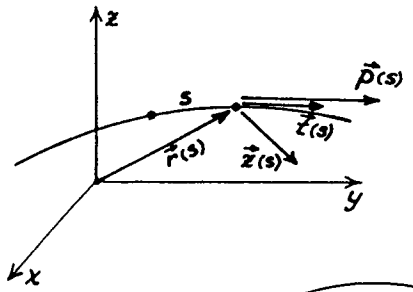
classical particle dynamics

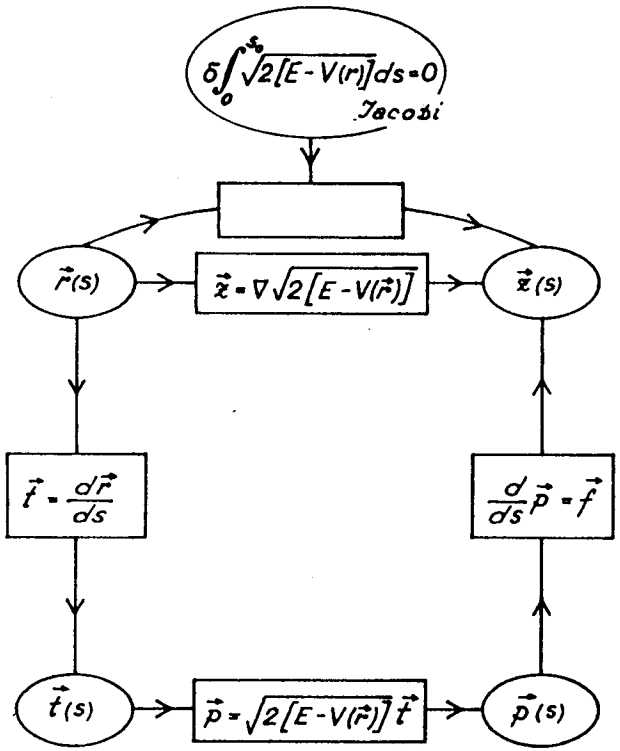
Hamilton-Jacobi equation

$$-H(p_k, q^k, t) = \frac{\partial S}{\partial t}$$
$$p_k = \frac{\partial S}{\partial q^k}$$
$$\longrightarrow \quad +H\left(\frac{\partial S}{\partial q^k}, q^k, t\right) + \frac{\partial S}{\partial t} = 0$$

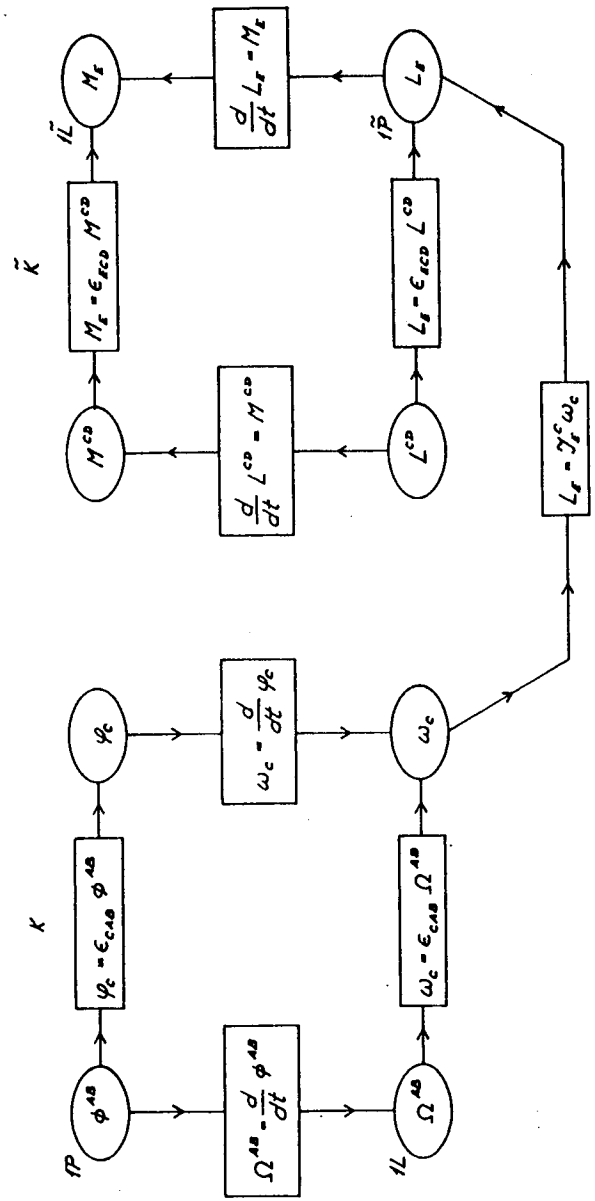
Hamilton-Jacobi

dynamics of a particle in a conservative field
trajectory equation

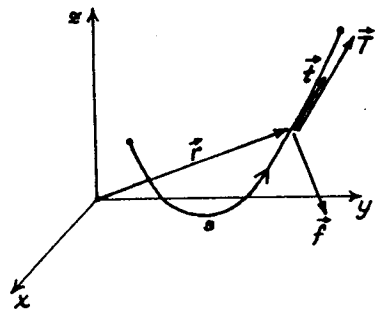


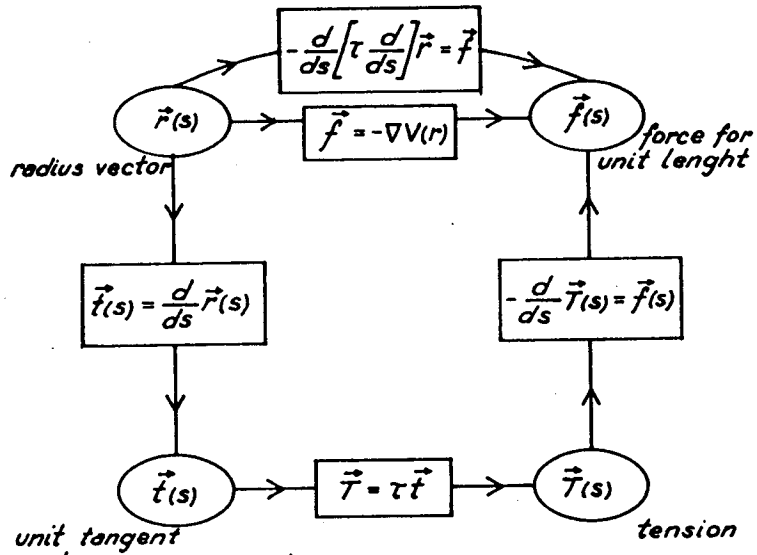


rigid body motion with a fixed point: the basic columns



statics of strings



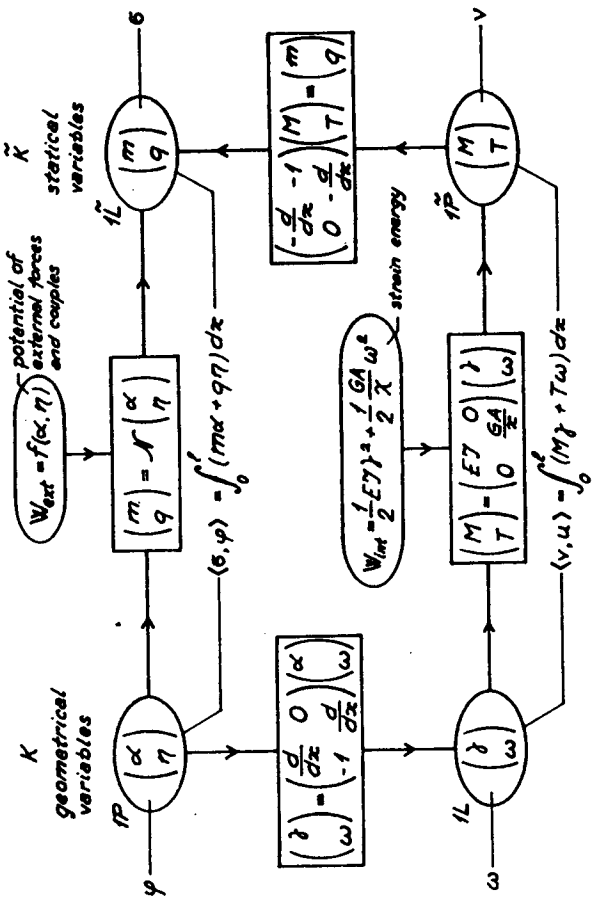


$$\langle f,r\rangle \overset{def}{=} \int_0^1 \vec{f}(s)\cdot\vec{r}(s)\frac{ds}{d\lambda}d\lambda \qquad s=s(\lambda)$$

$$0\le\lambda\le 1$$

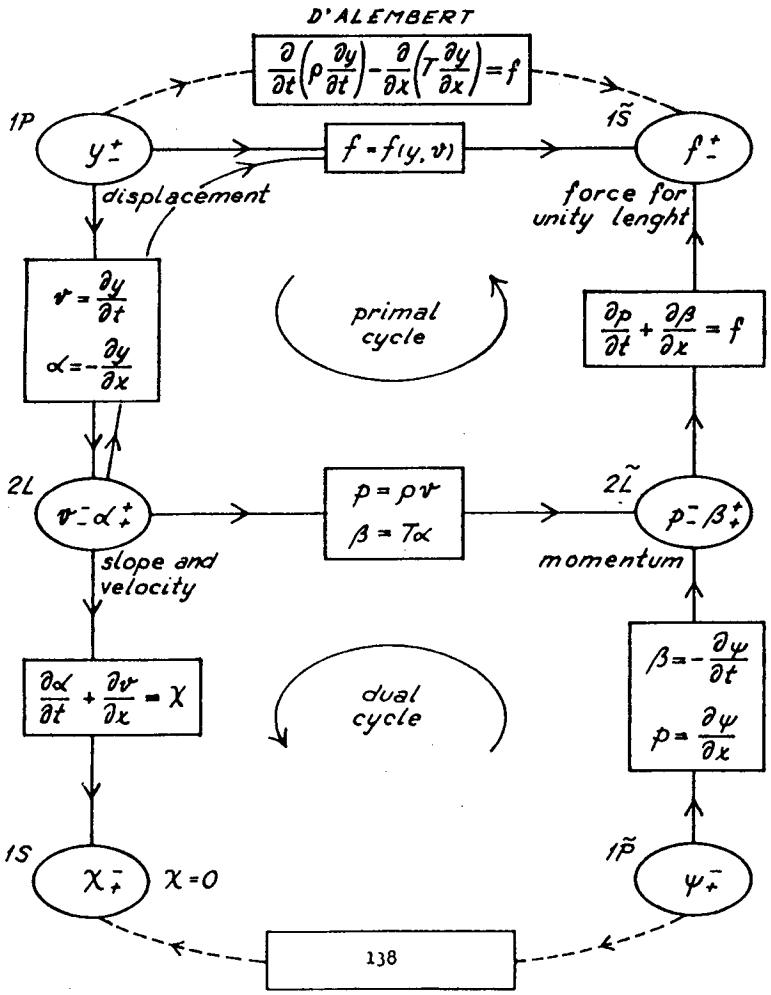
$$\langle T,t\rangle \overset{def}{=} \int_0^1 \vec{T}(s)\cdot\vec{t}(s)\frac{ds}{d\lambda}d\lambda \qquad \begin{cases} s(0)=0 \\ s(1)=L \end{cases}$$

bending of beams



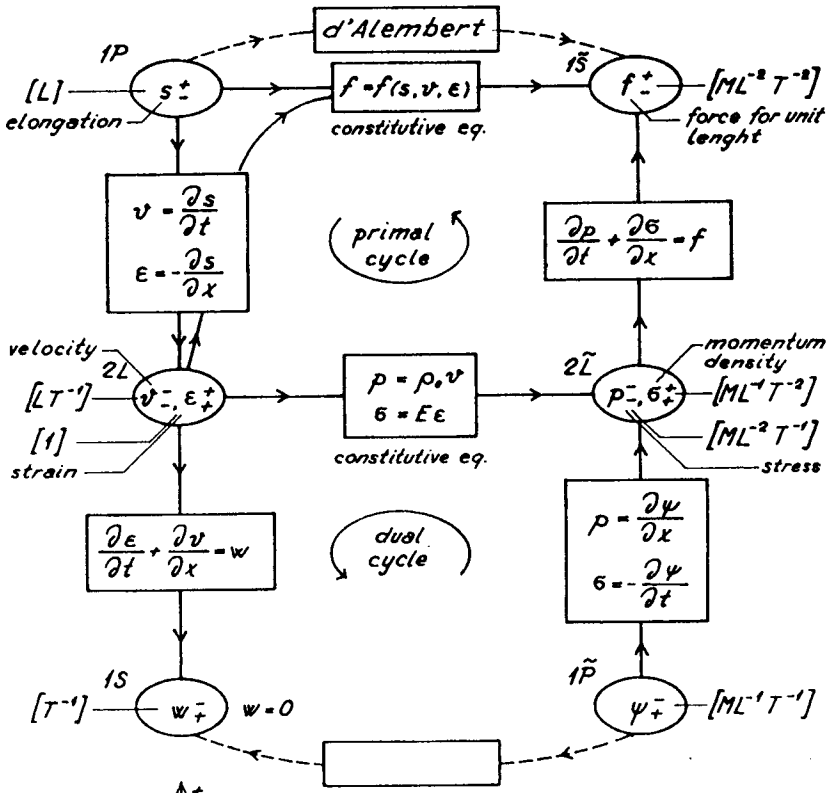
vibrations of a string

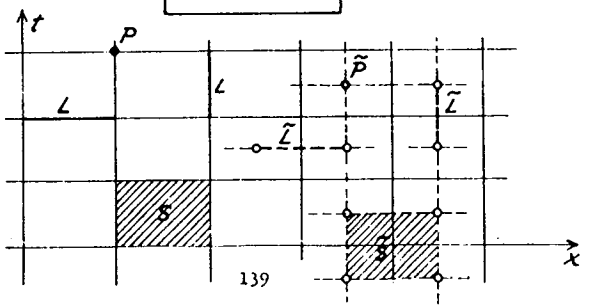
variables: t, x



longitudinal vibrations of a rod

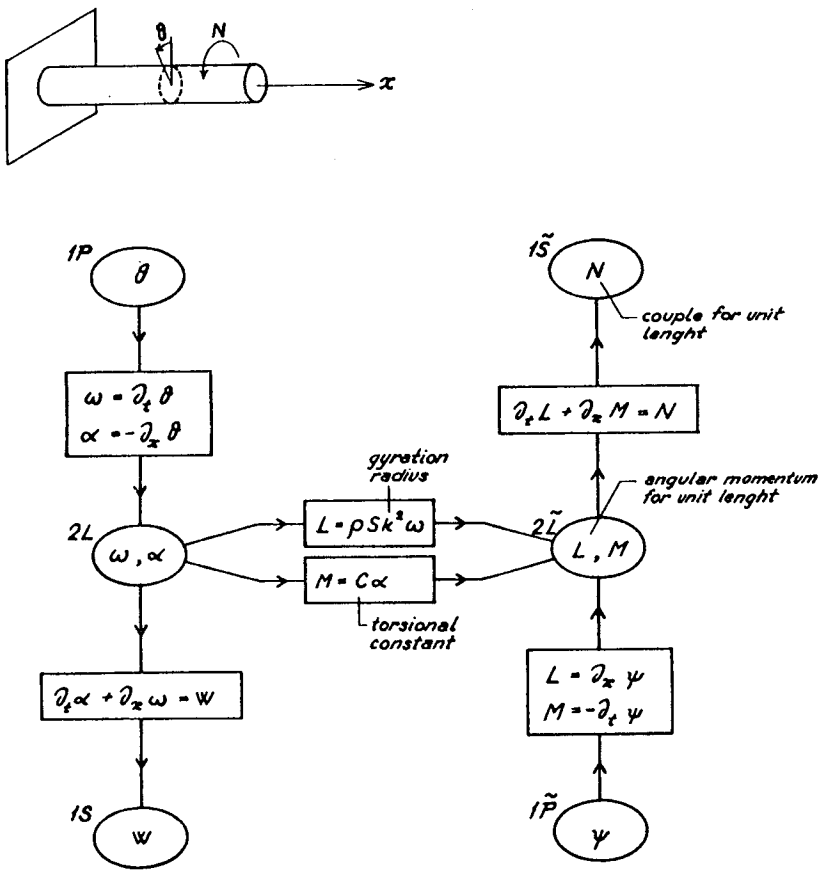
variables t, x





torsional vibration of a rod

(variables x, t)



Soap films

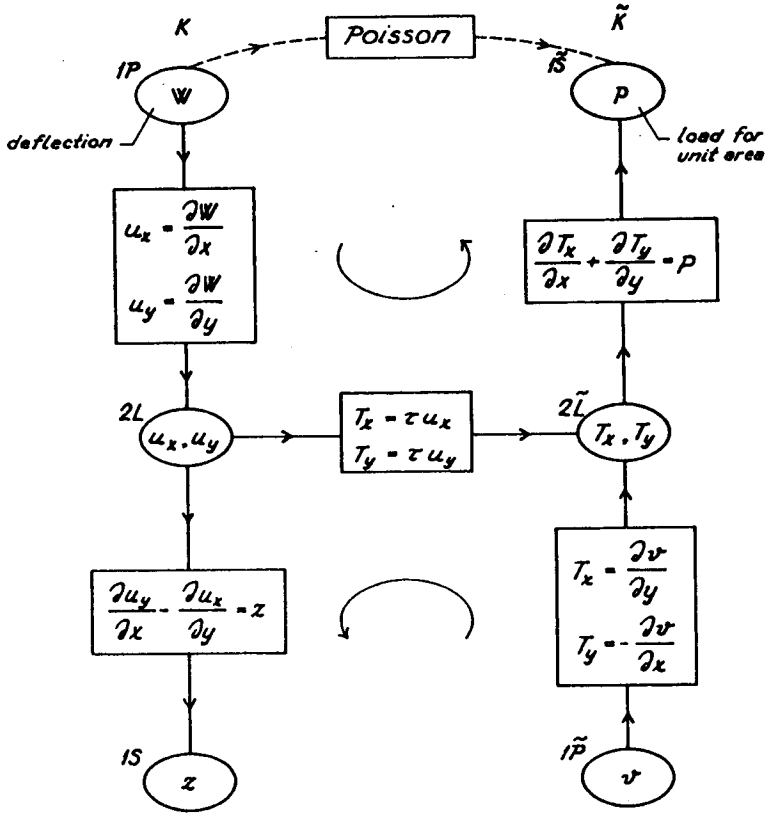
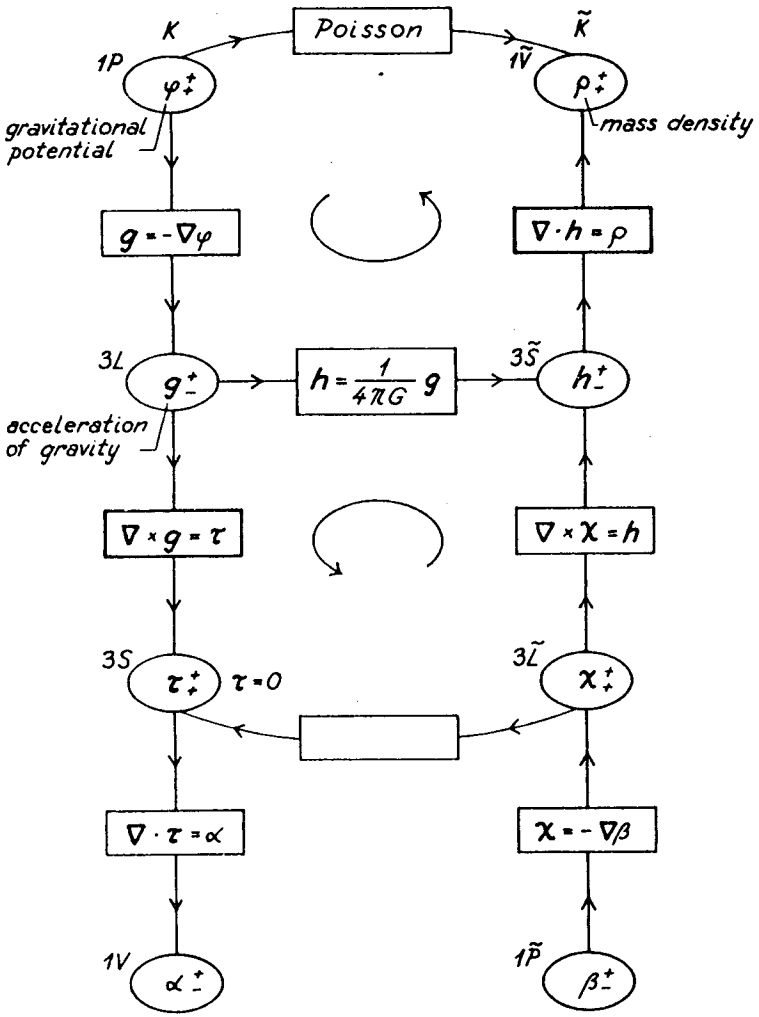


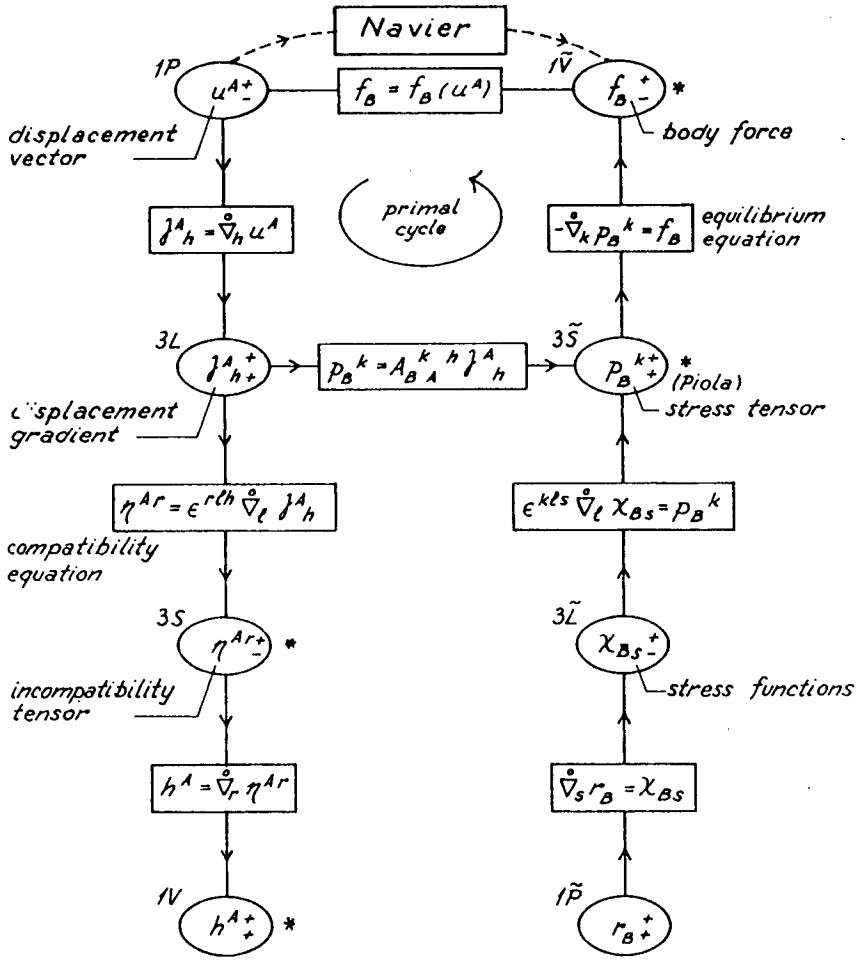
table : classical gravitational field



continuum statics: large elastic deformations

$d\mathring{s}^2 = \mathring{a}_{hk}\, dx^h dx^k \qquad ds^2 = a_{hk}\, dx^h dx^k$

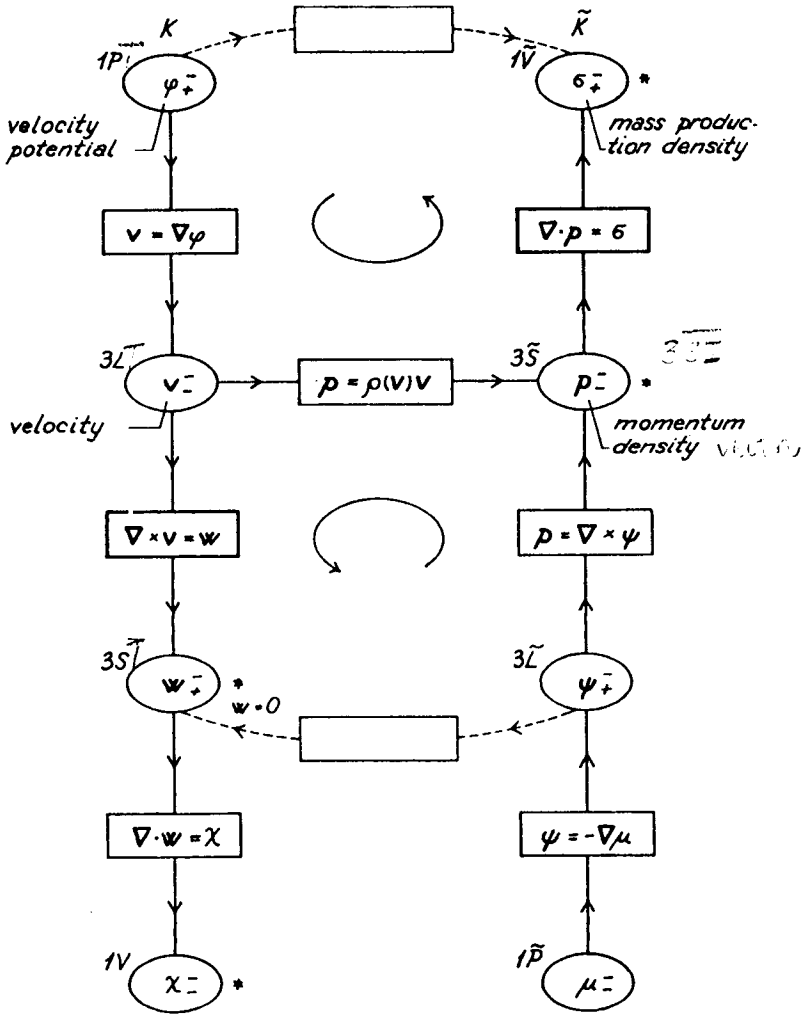
$a_{hk} = \mathring{a}_{hk} + (\gamma_{hk} + \gamma_{kh} + \gamma^{A}{}_{h}\, \gamma_{Ak}) \qquad \gamma_{hk} = \mathring{a}_{hA}\, \gamma^{A}{}_{k}$



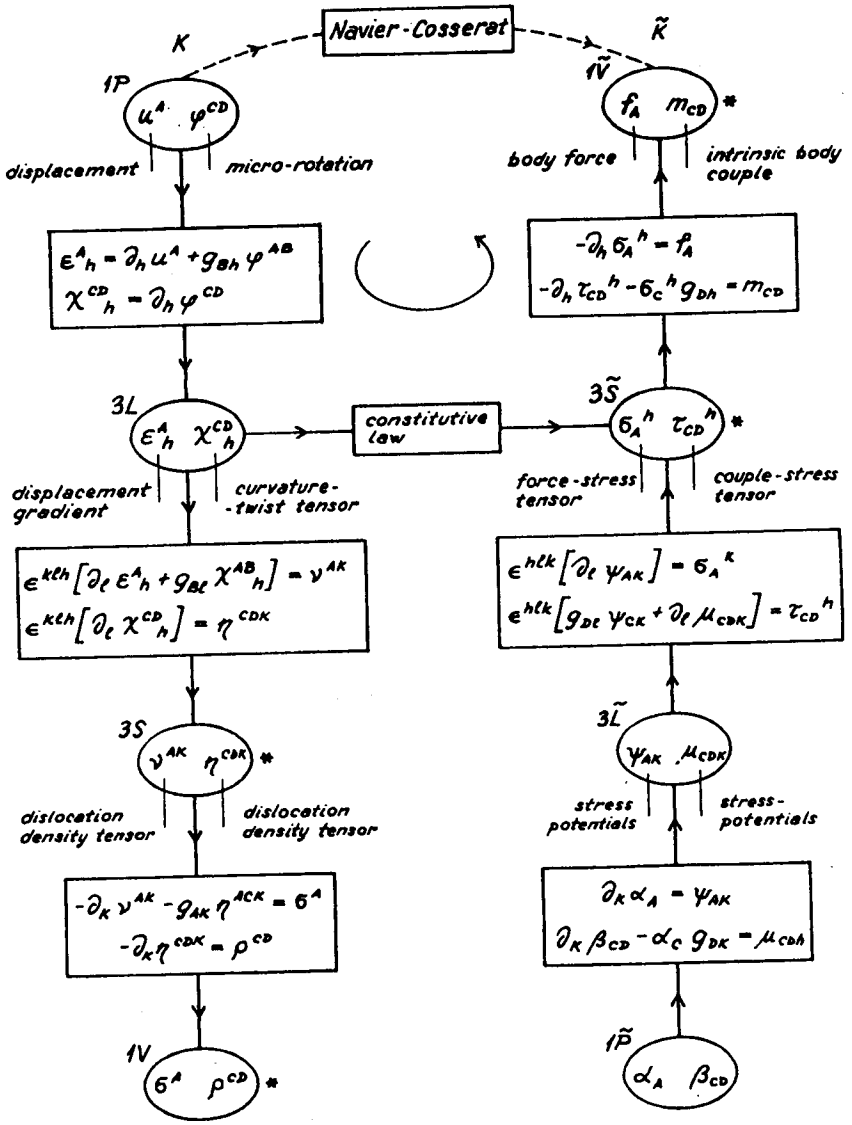
Ref: Fung, 1965, p. 441

perfect fluid motion

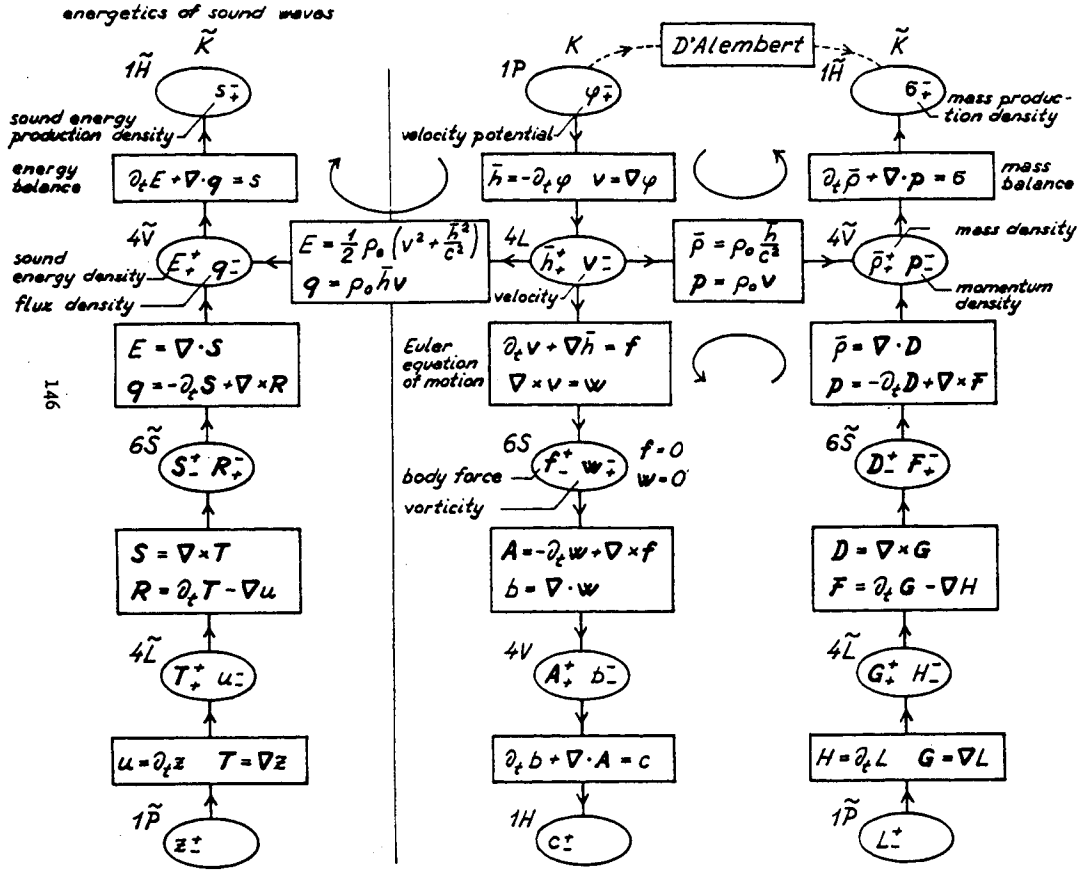
(barotropic, irrotational, stationary)

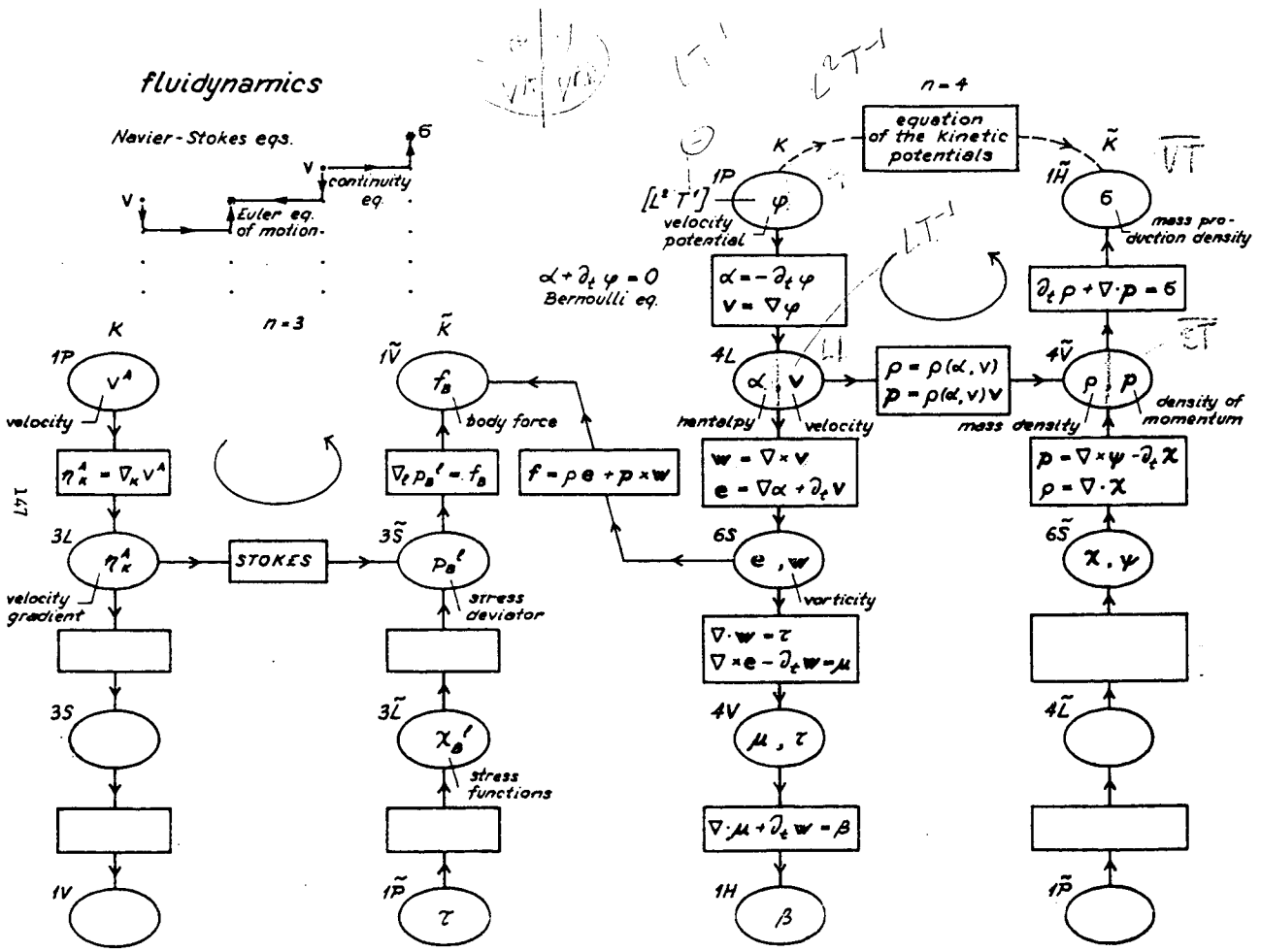


mechanics of polar continua

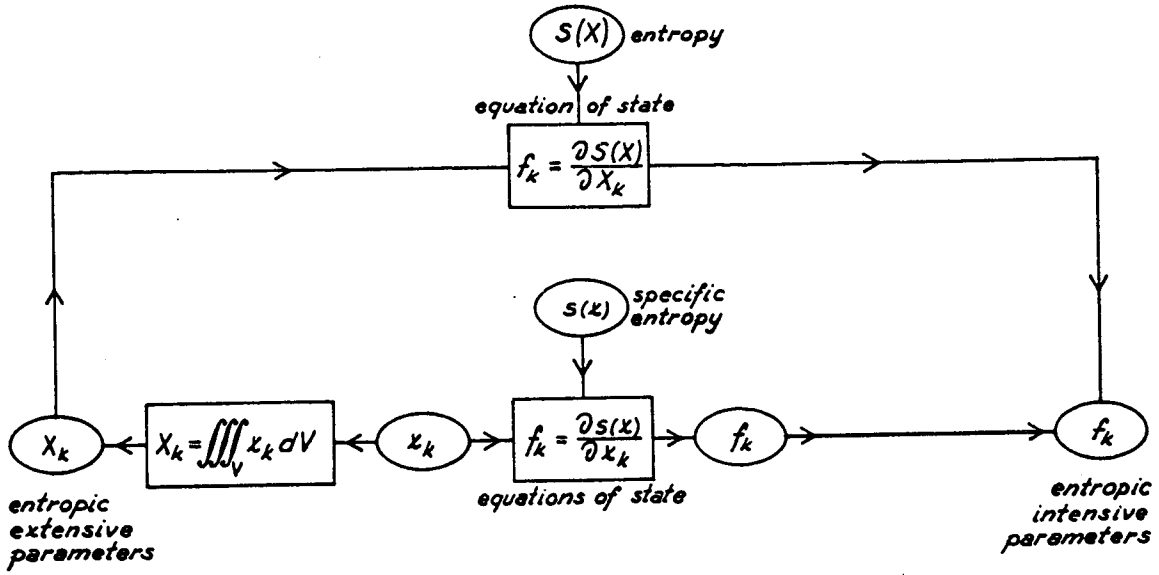


linear acoustics

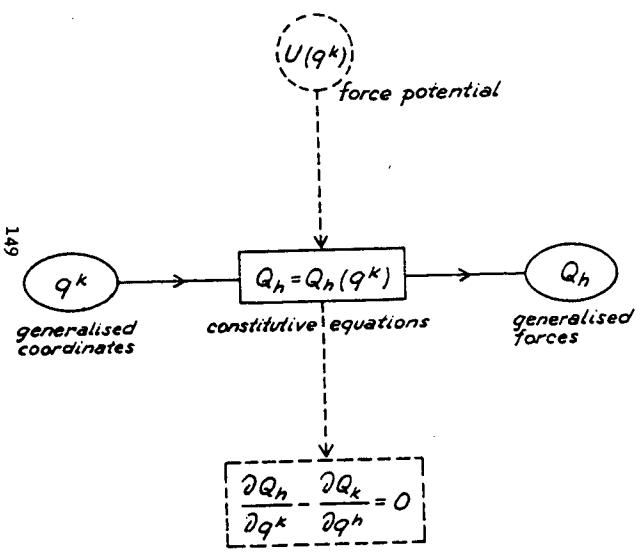




thermostatics
(entropic representation)

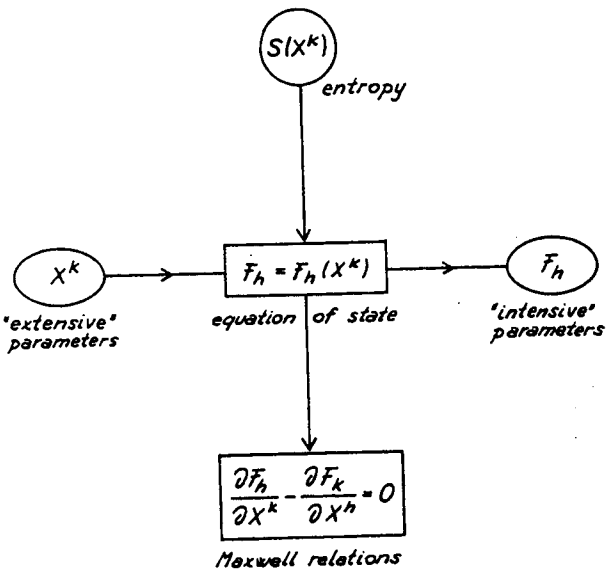


analytical statics



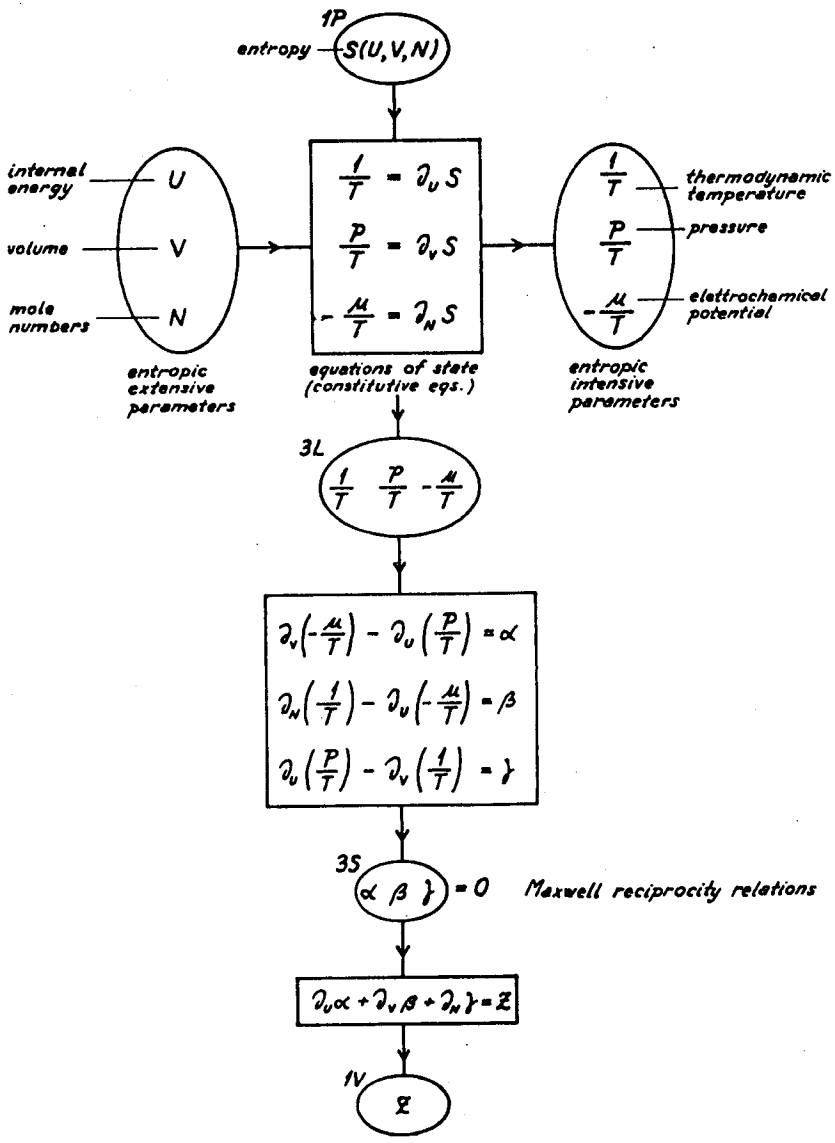
$\delta L = Q_k \delta q^k \; (= \delta U)$

thermostatics
(entropic representation)



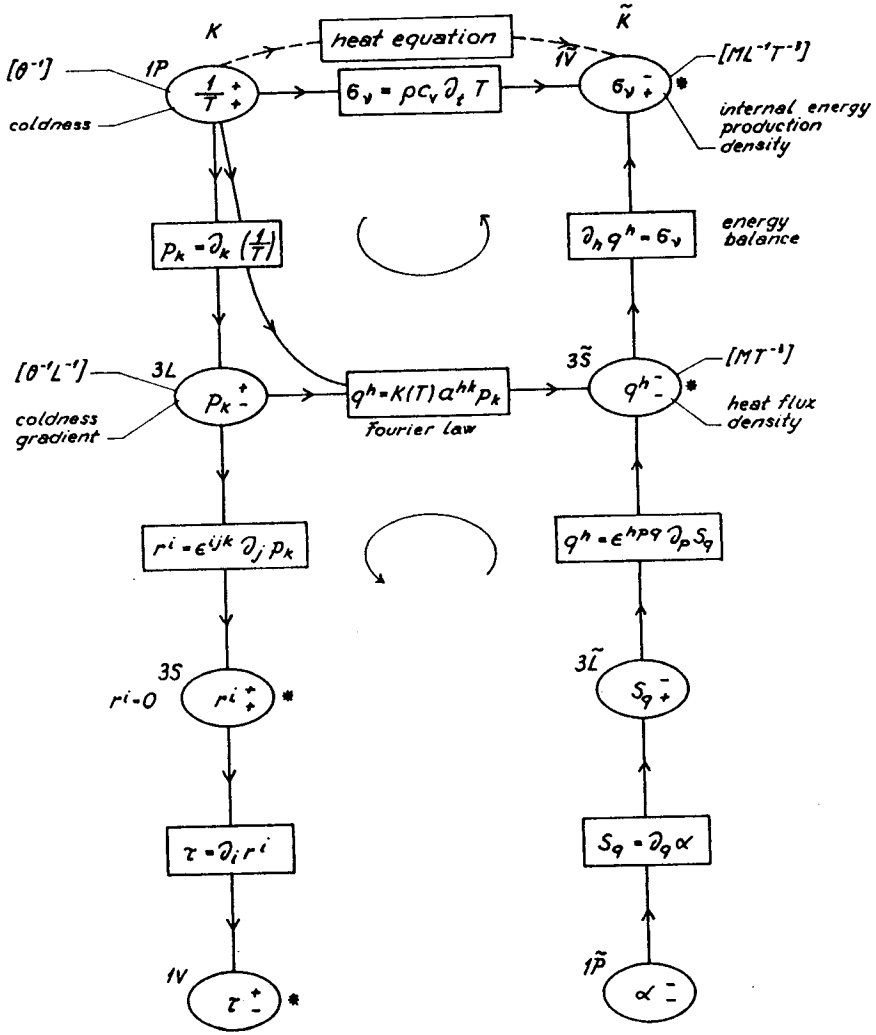
$\delta S = F_k \delta X^k$

thermostatics of a monoatomic ideal gas

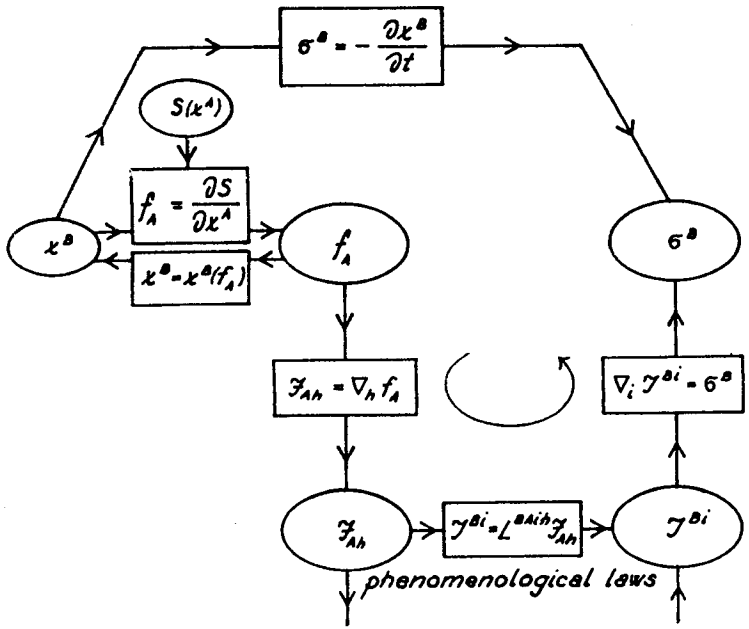


thermal conduction

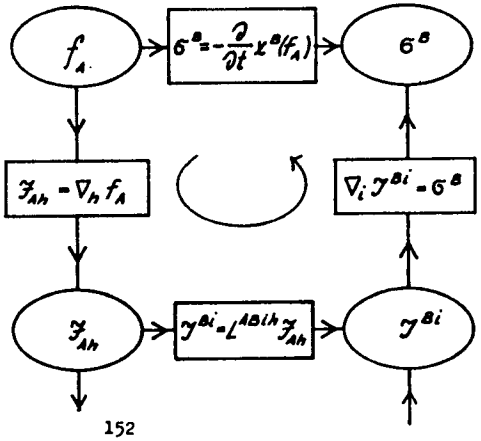
the time t is considered as a parameter



irreversible thermodynamics

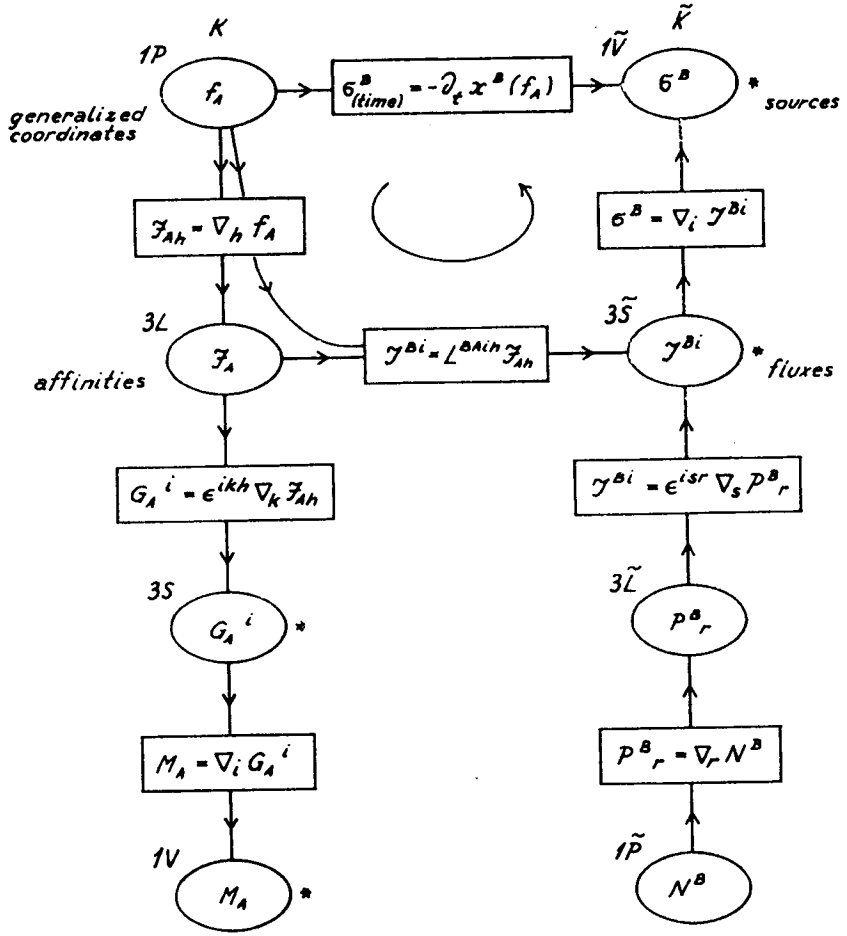


equivalent scheme



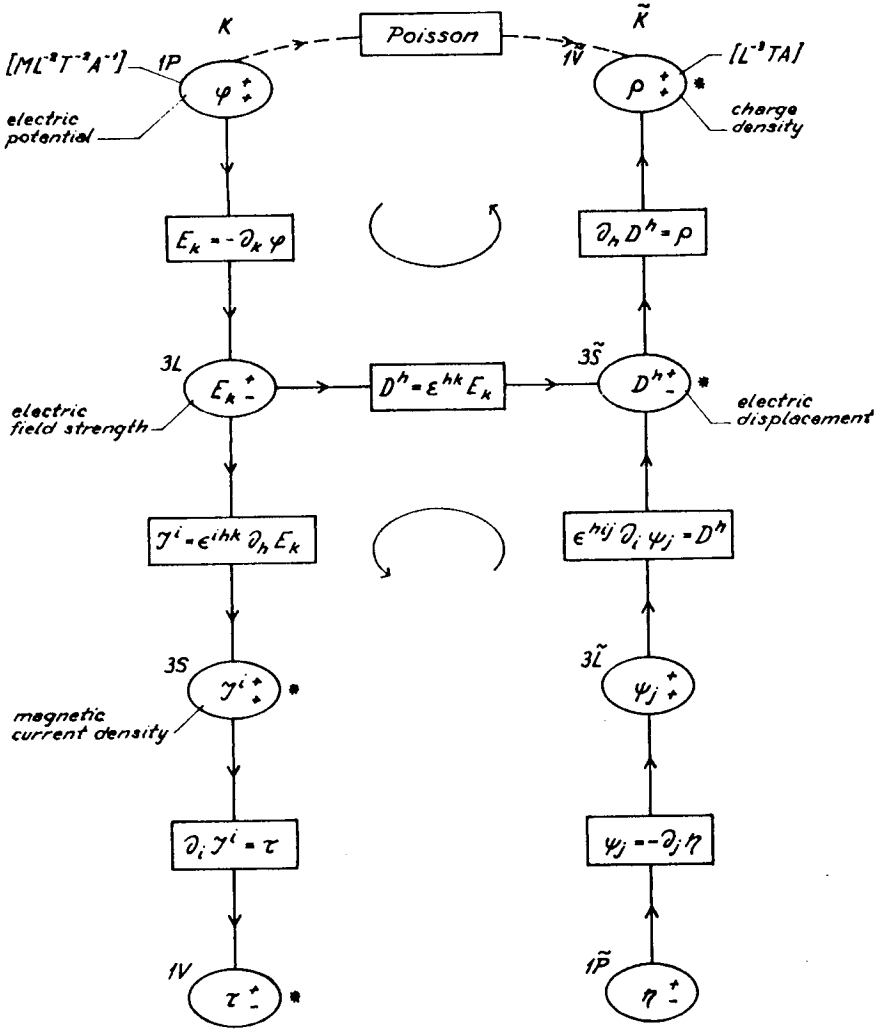
irreversible thermodynamics

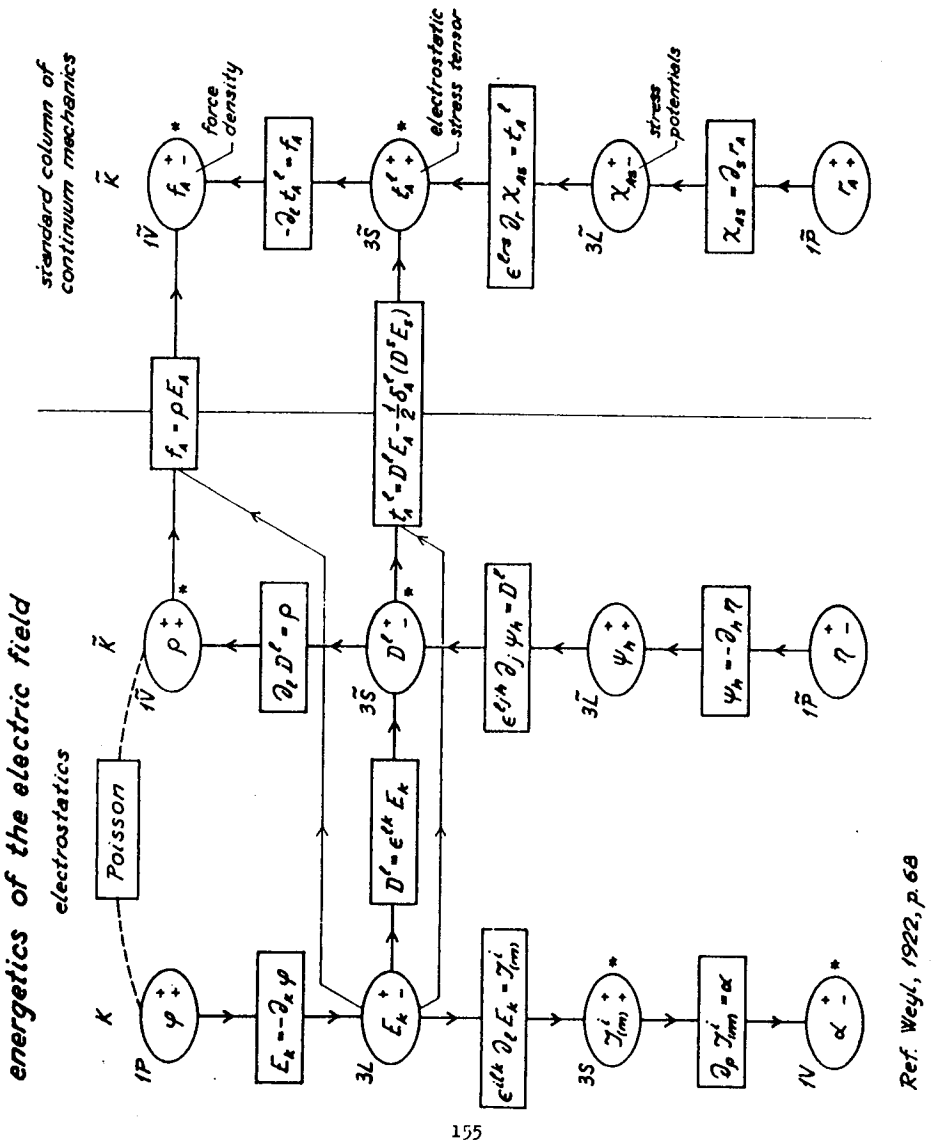
(the time t is considered as a parameter)



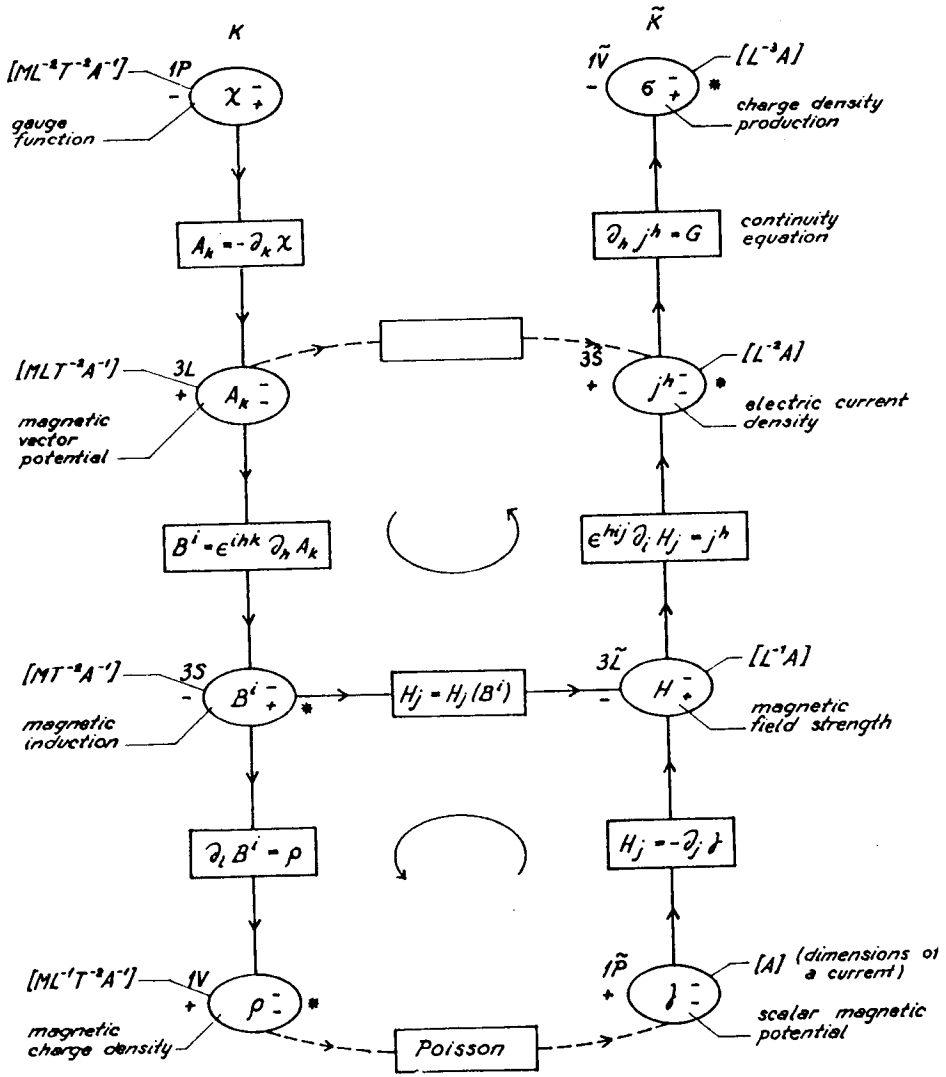
Ref. Boccara, 1968, p.111

electrostatics

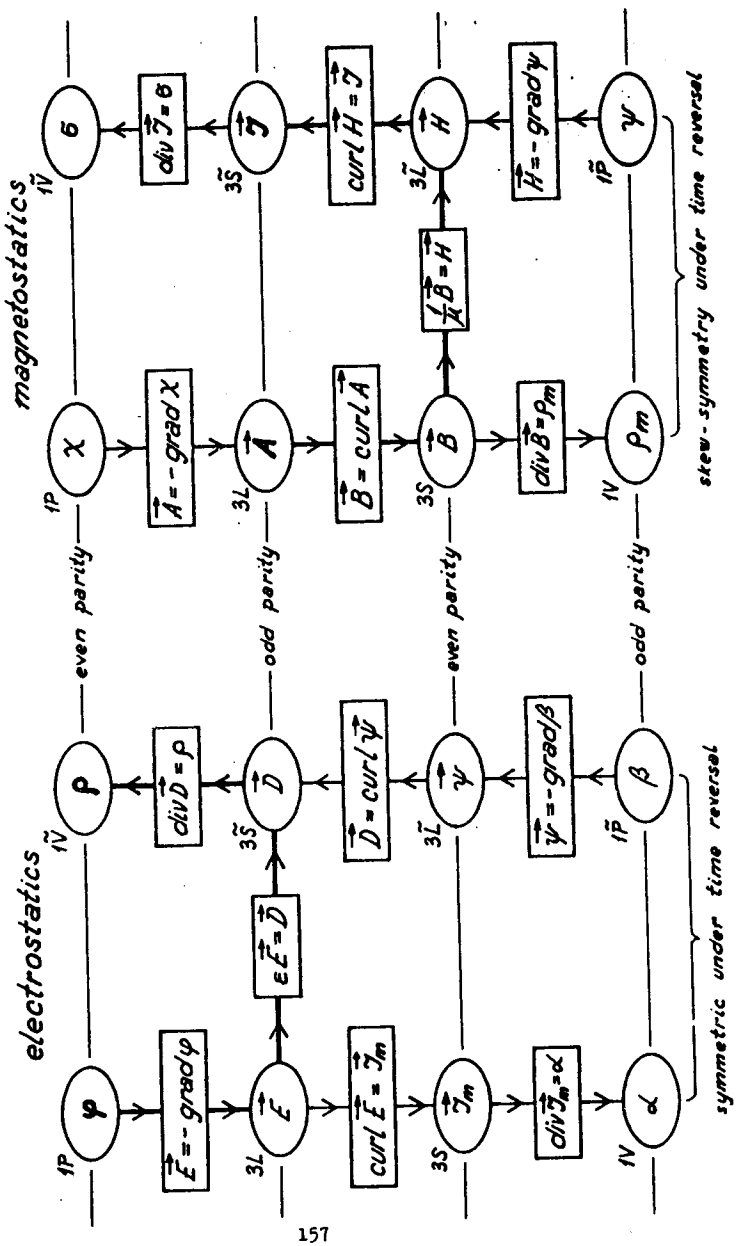




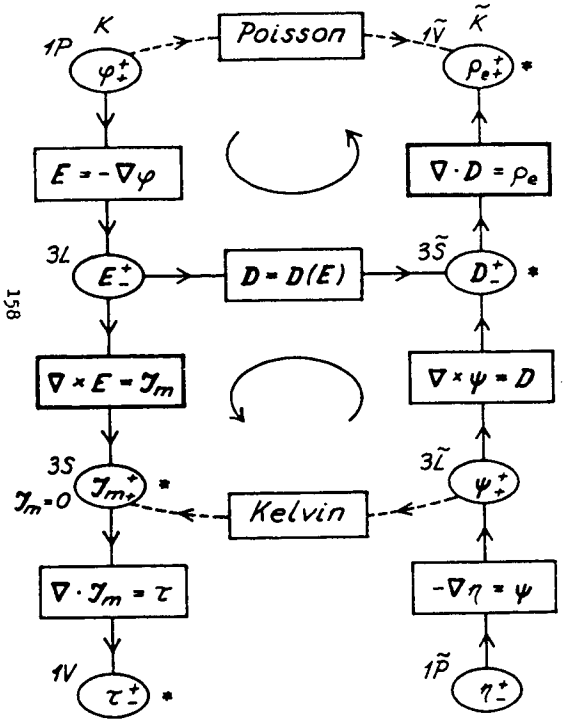
magnetostatics



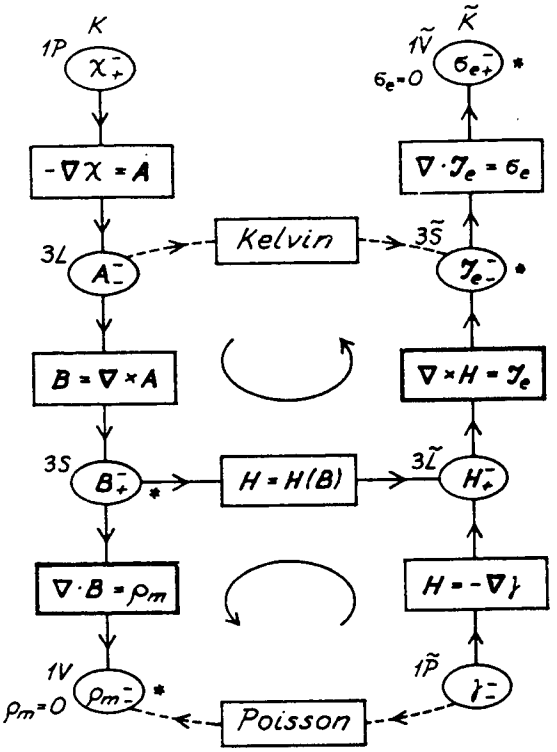
behaviour under space reflection and time reversal



electrostatics



magnetostatics



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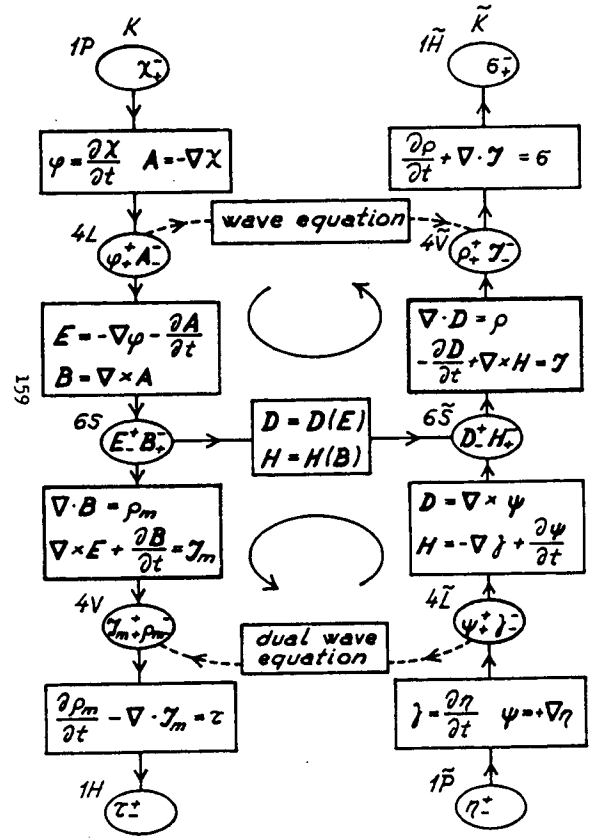
electromagnetic field



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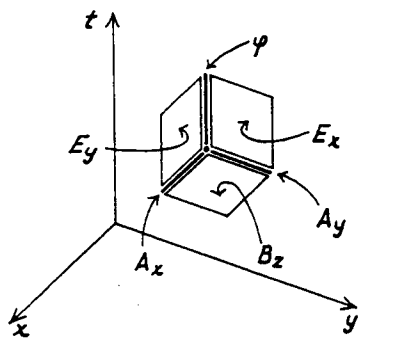
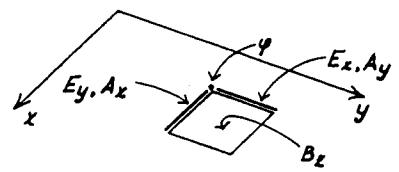
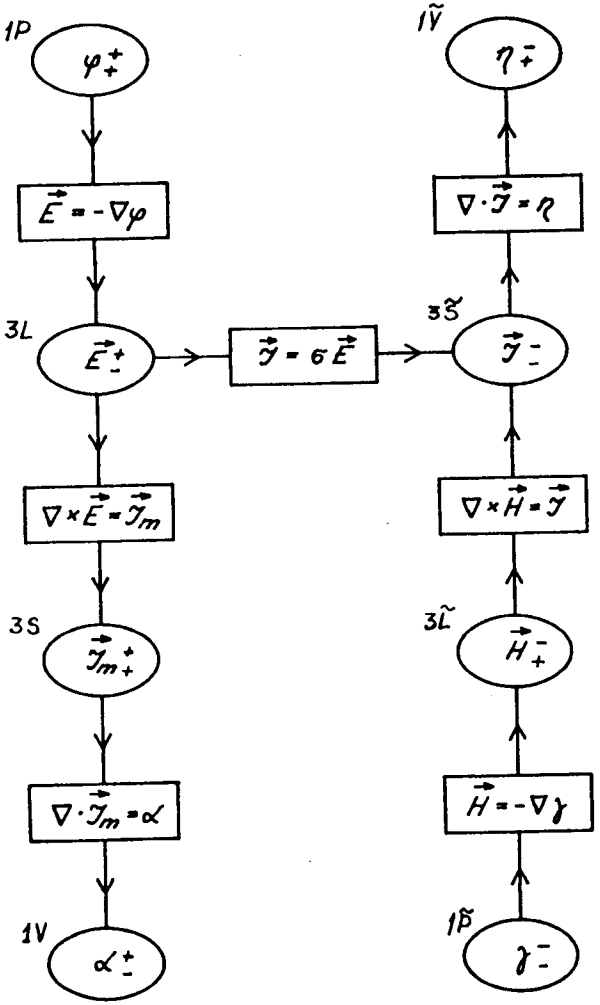
a three-dimensional space-time and the corresponding projection on the two dimensional space





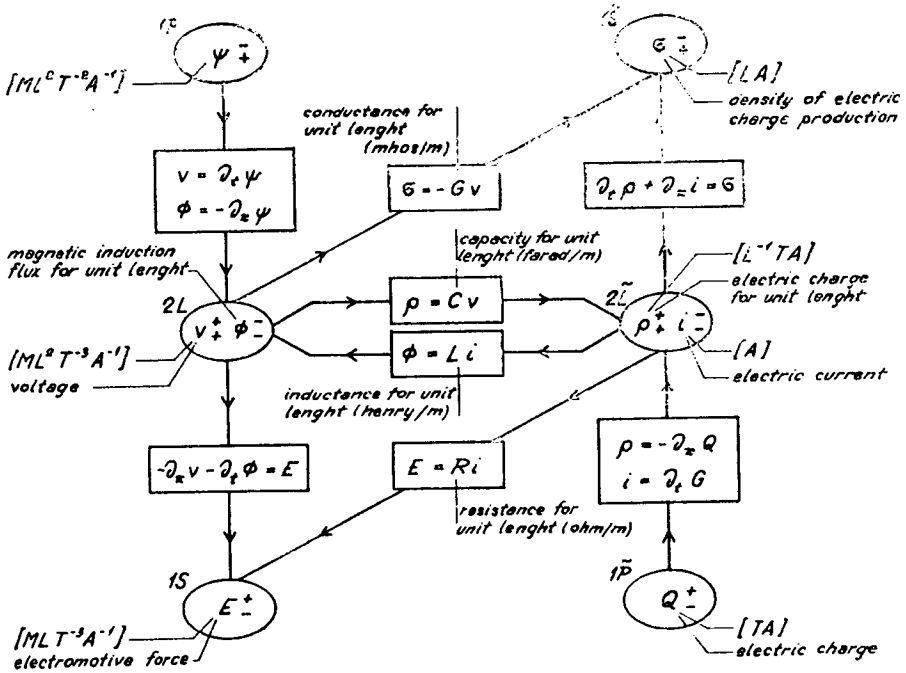
Abraham Joffe Electrodynamics

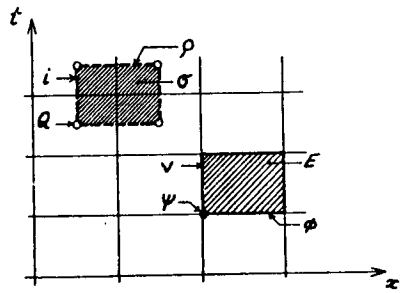
electric conduction

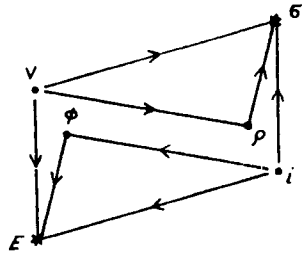


telegrapher's equation

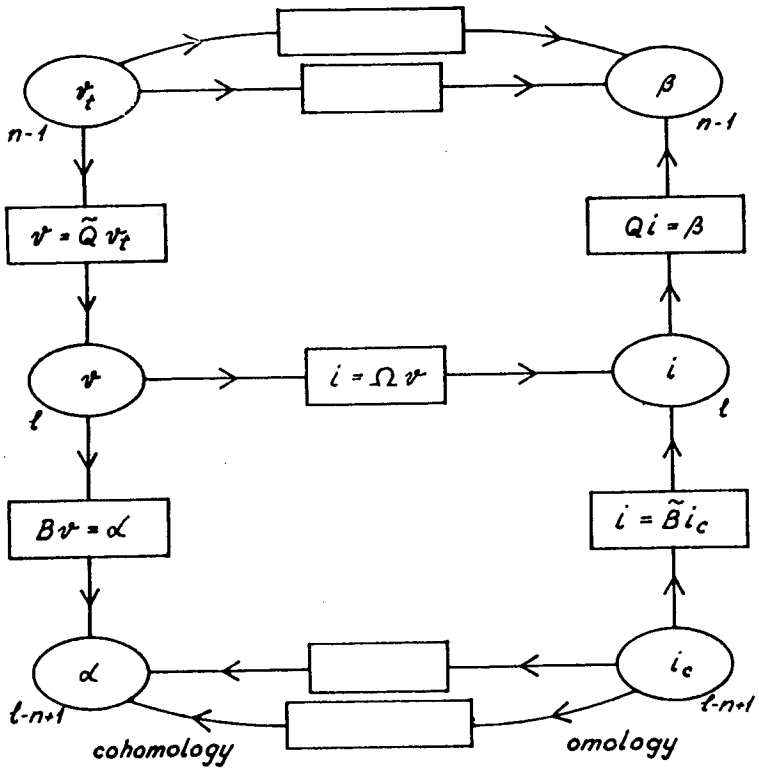
(variables t, x)







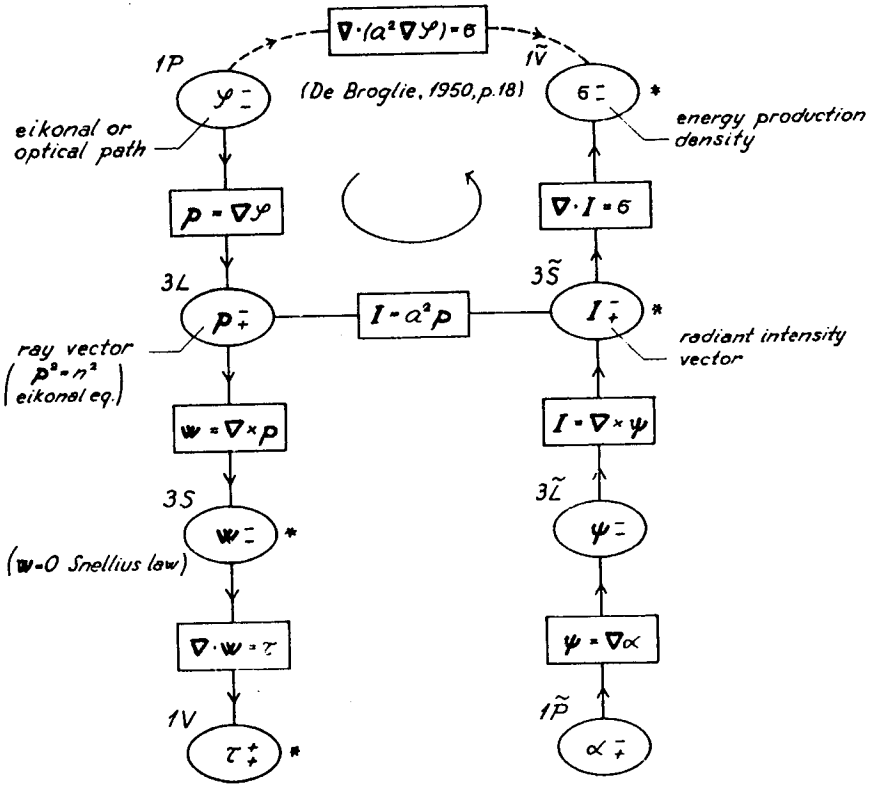
network theory



i_c = chord currents
i = edge currents
v_t = branch voltages
v = edge voltage

$$\Omega = \begin{bmatrix} G & 0 & 0 \\ 0 & C\frac{d}{dt} & 0 \\ 0 & 0 & \frac{1}{L}\int_0^t \ldots dt \end{bmatrix} \quad \text{(in the linear case)}$$

geometrical optics



Ref: Born-Wolf, 1970; De Broglie, 1950

geometrical optics

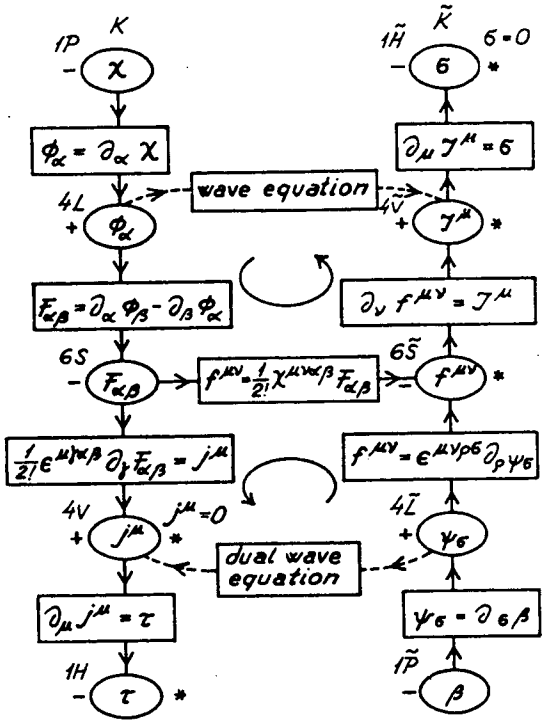
variables

particle dynamics in a conservative field: trajectory equation

variables

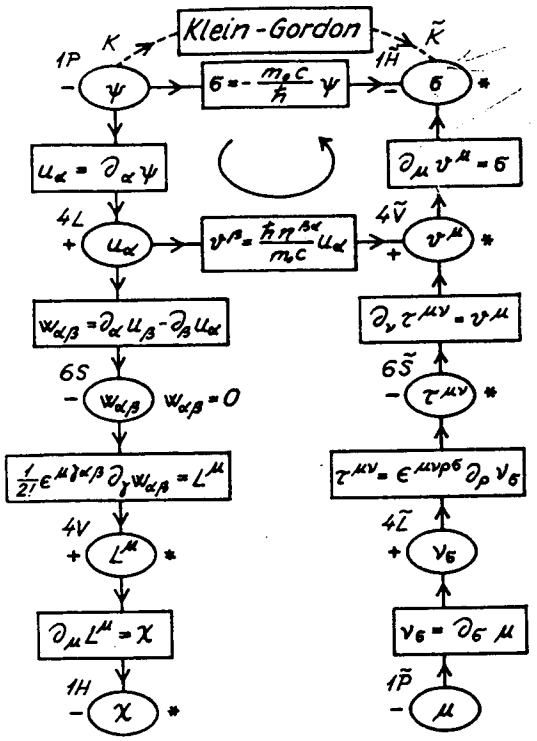
Maxwell equations

Ref. Post, 1962



Klein-Gordon equation

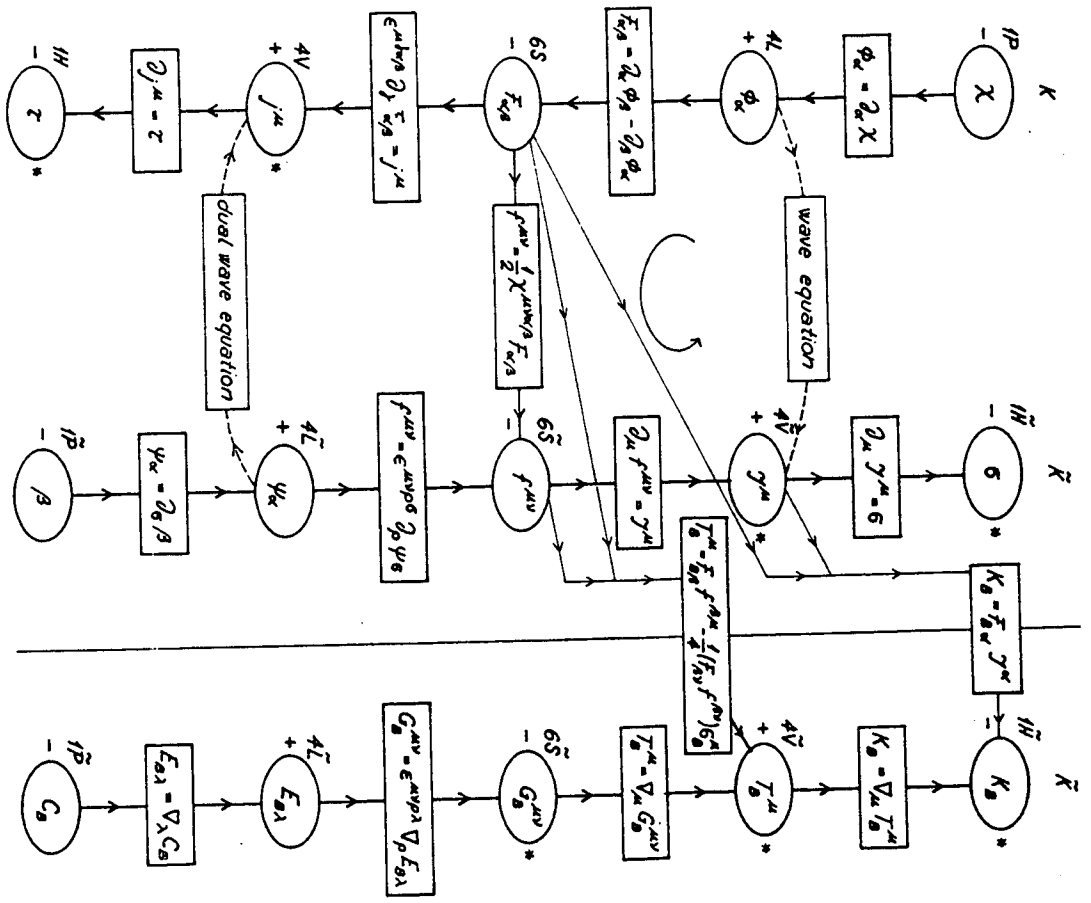
Ref. Corson, 1955



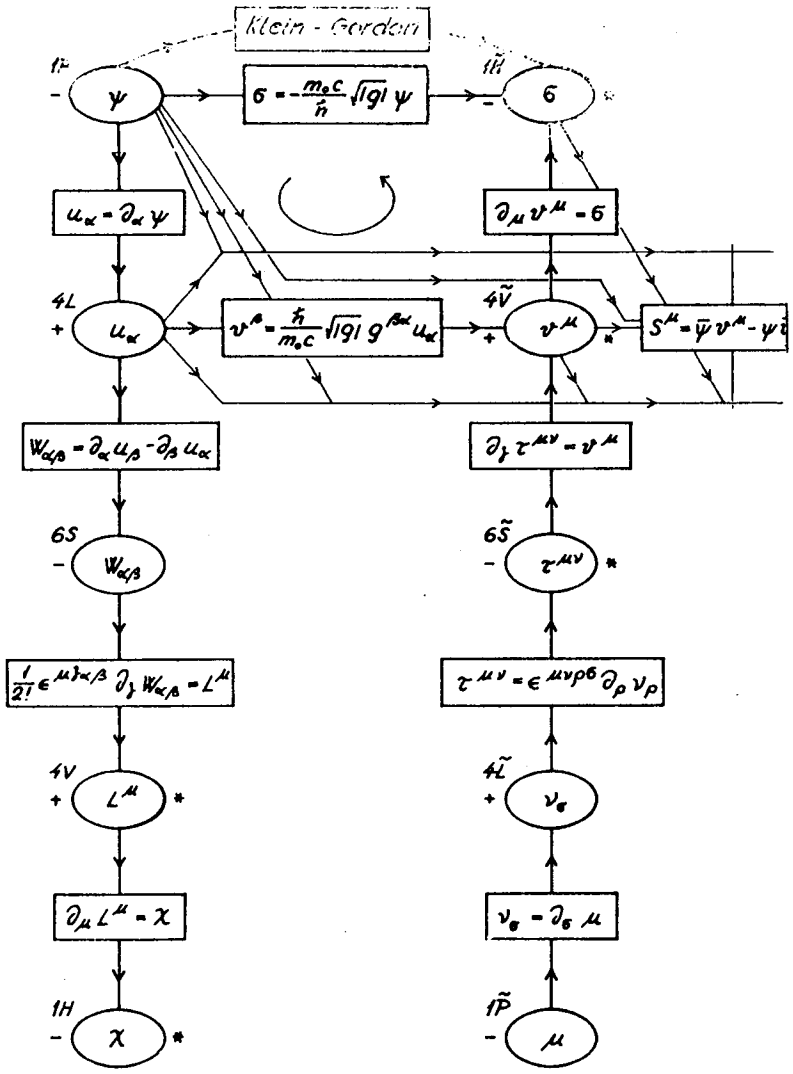
PT → VT

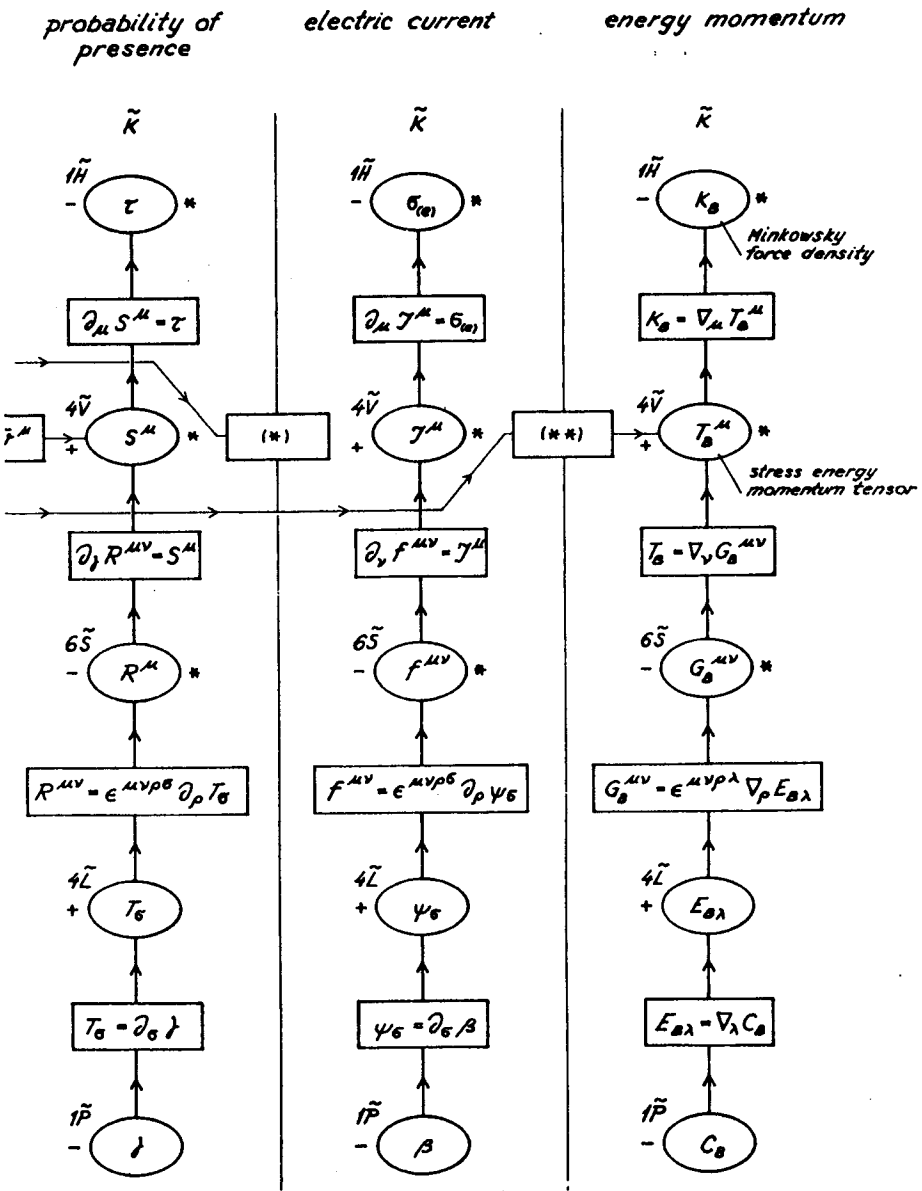
energetics of the electromagnetic field

(space - time formulation)

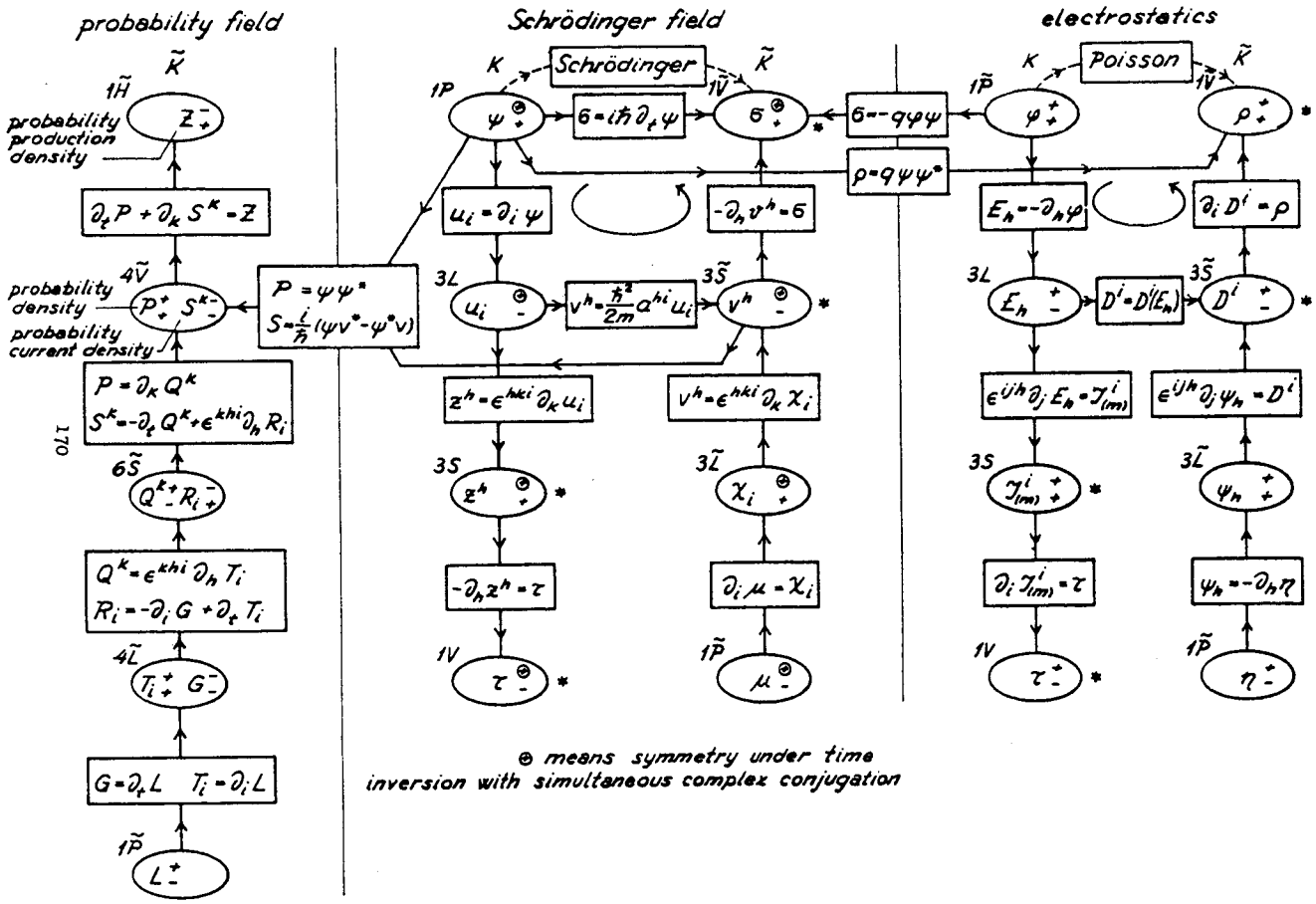


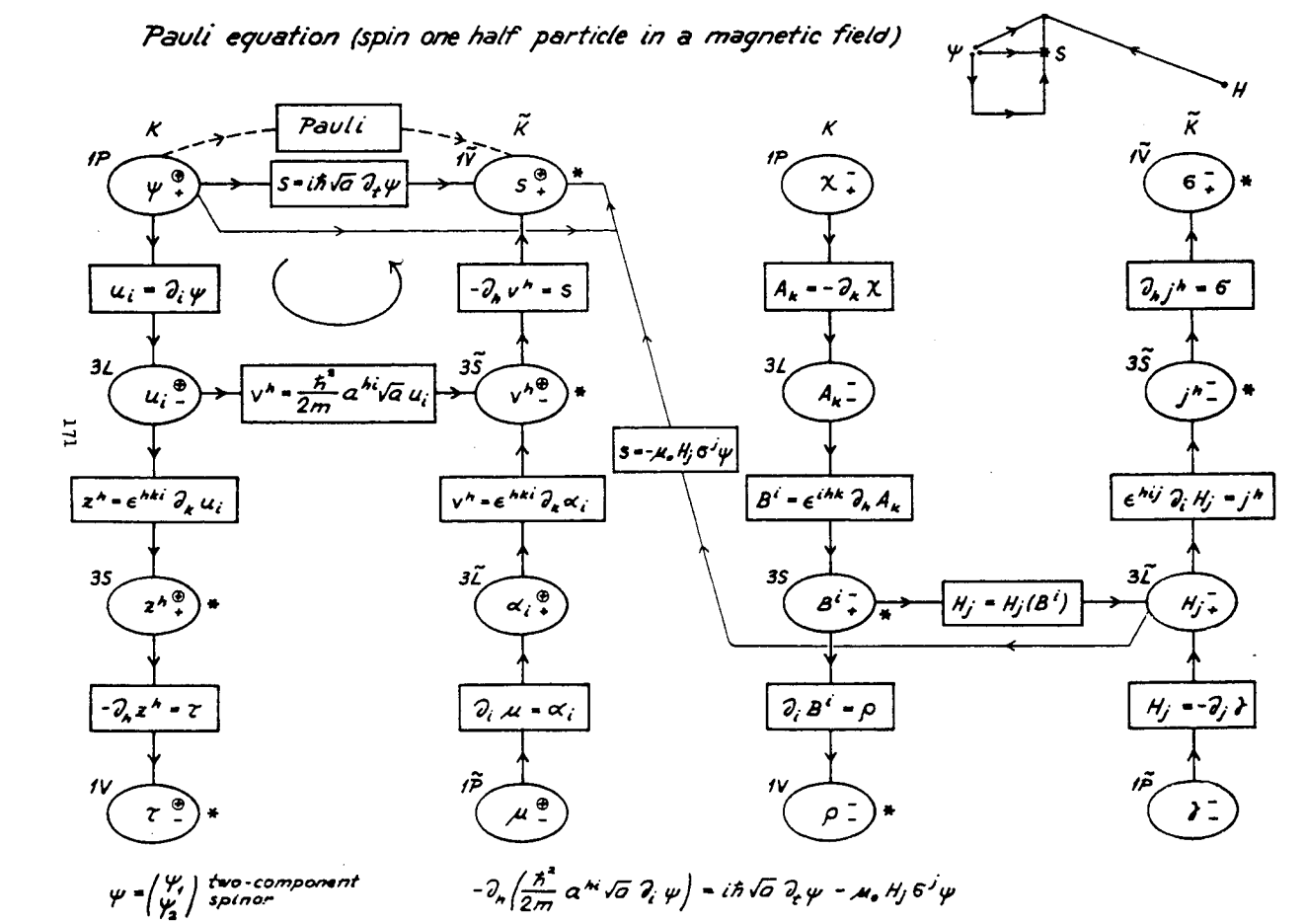
Klein - Gordon equation (meson field)



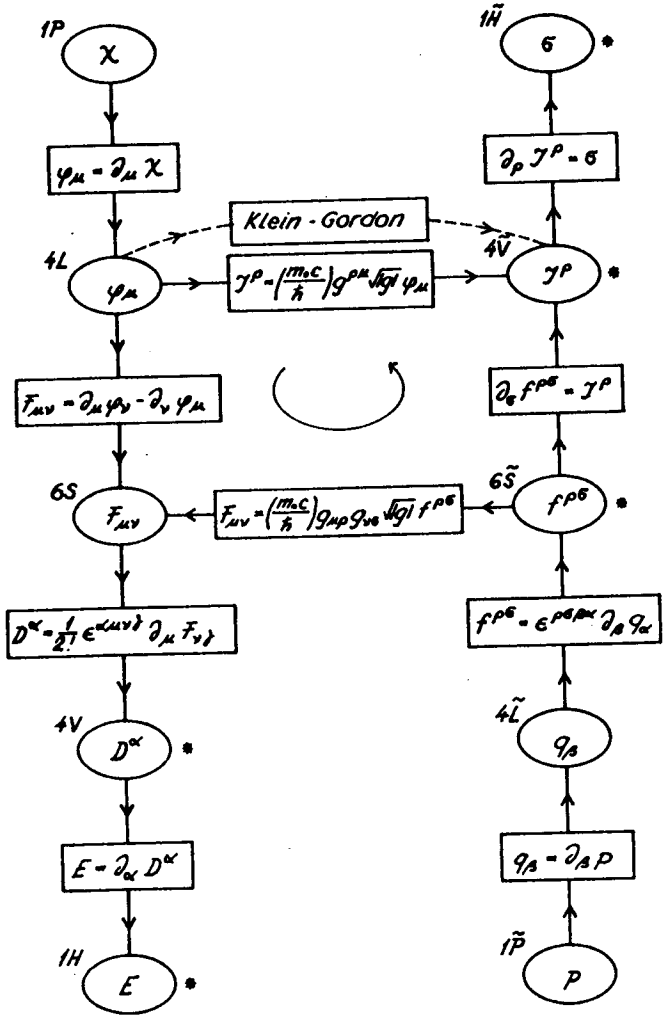


(*) $J^\mu = e(\bar{\psi} u^\mu - \psi \bar{u}^\mu)$ (**) $T_\beta^\mu = (\bar{v}_\beta u^\mu + u_\beta \bar{v}^\mu) - \delta_\beta^\mu (\bar{v}_\alpha u^\alpha + \bar{6}\psi)$


Schrödinger field interacting with a scalar field: its link with probability
probability field
Schrödinger field
electrostatics
Schrödinger
Poisson
probability production density
probability density
probability current density
$\partial_t P + \partial_k S^k = Z$
$P = \partial_K Q^K$
$S^k = -\partial_t Q^K + \epsilon^{khi}\partial_h R_i$
$Q^K = \epsilon^{khi}\partial_h T_i$
$R_i = -\partial_i G + \partial_t T_i$
$G = \partial_t L \quad T_i = \partial_i L$
$P = \psi\psi^*$
$S = \frac{i}{\hbar}(\psi v^* - \psi^* v)$
$\sigma = -i\hbar\partial_t\psi$
$\sigma = -q\varphi\psi$
$\rho = q\psi\psi^*$
$u_i = \partial_i\psi$
$-\partial_h v^h = \sigma$
$v^h = \frac{\hbar^2}{2m}a^{hi}u_i$
$z^h = \epsilon^{hki}\partial_k u_i$
$v^h = \epsilon^{hki}\partial_k\chi_i$
$-\partial_h z^h = \tau$
$\partial_i\mu = \chi_i$
$E_h = -\partial_h\varphi$
$\partial_i D^i = \rho$
$D^i = D^i(E_h)$
$\epsilon^{ijh}\partial_j E_h = \mathcal{T}^i_{(im)}$
$\epsilon^{ijh}\partial_j\psi_h = D^i$
$\partial_i\mathcal{T}^i_{(im)} = \tau$
$\psi_h = -\partial_h\eta$
170
⊛ means symmetry under time inversion with simultaneous complex conjugation

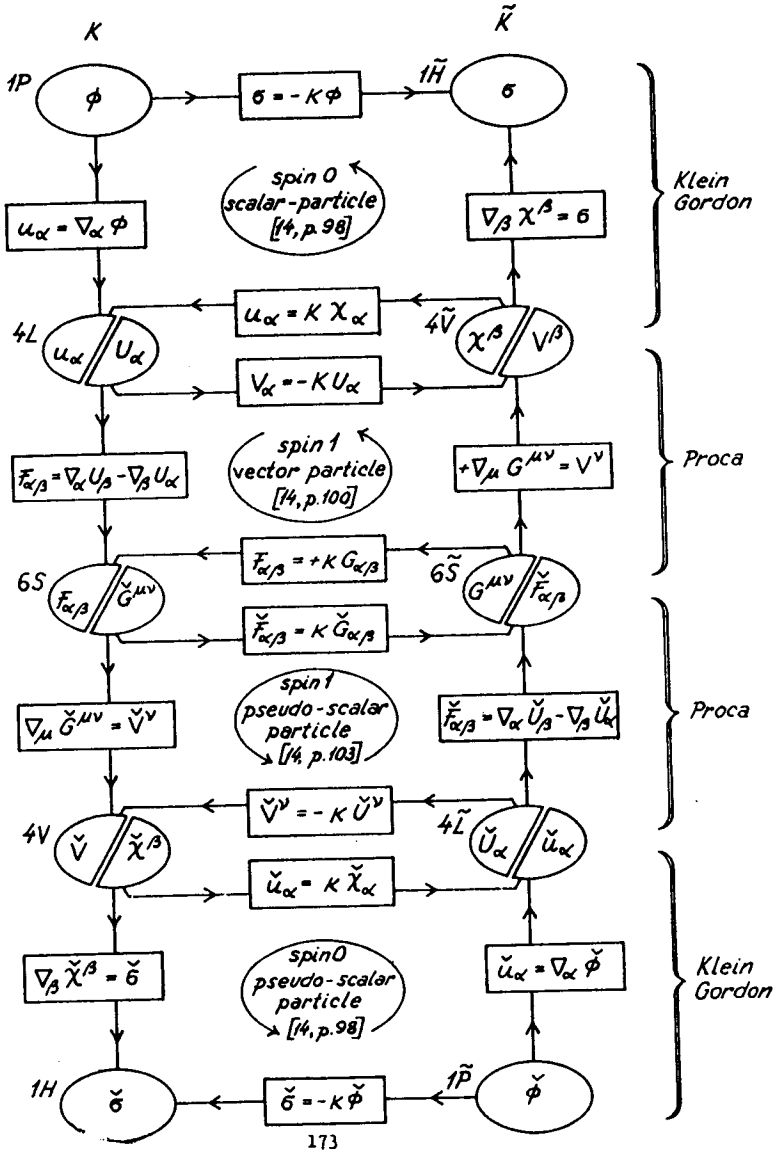

Pauli equation (spin one half particle in a magnetic field)
Pauli
$s = i\hbar\sqrt{a}\,\partial_t\psi$
$u_i = \partial_i\psi$
$-\partial_h v^h = s$
$v^h = \frac{\hbar^2}{2m}a^{hi}\sqrt{a}\,u_i$
$z^h = \epsilon^{hki}\partial_k u_i$
$v^h = \epsilon^{hki}\partial_k\alpha_i$
$-\partial_h z^h = \tau$
$\partial_i\mu = \alpha_i$
$s = -\mu_0 H_j\sigma^j\psi$
$A_k = -\partial_k\chi$
$B^i = \epsilon^{ihk}\partial_h A_k$
$\partial_i B^i = \rho$
$H_j = H_j(B^i)$
$\partial_h j^h = 6$
$\epsilon^{hij}\partial_i H_j = j^h$
$H_j = -\partial_j\lambda$
171
$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ two-component spinor
$-\partial_h\left(\frac{\hbar^2}{2m}a^{hi}\sqrt{a}\,\partial_i\psi\right) = i\hbar\sqrt{a}\,\partial_t\psi - \mu_0 H_j\sigma^j\psi$

the Proca field



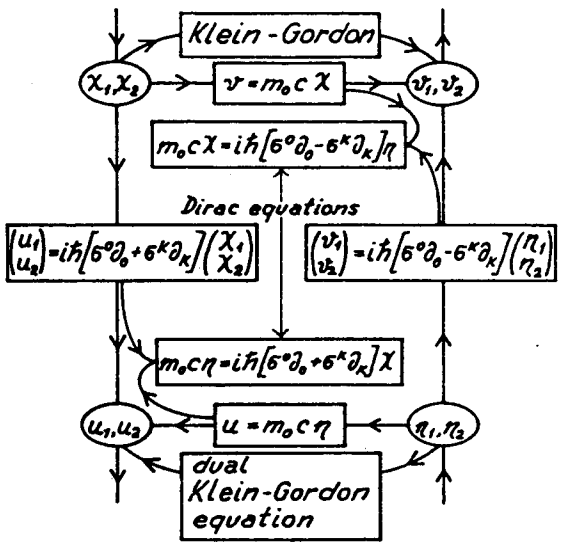
: the standard schemes for spin 0 and spin 1 particles, scalar and pseudo-scalar, vector and pseudo-vector particles

(the four schemes have been superimposed)



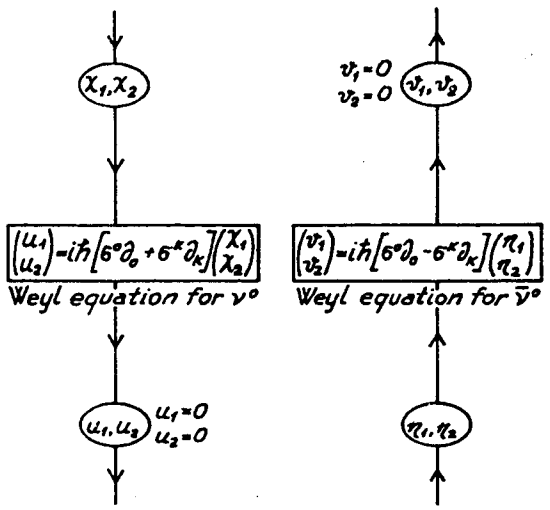
electron field (Dirac equations)

$n=4$



neutrino field (Weyl equation)

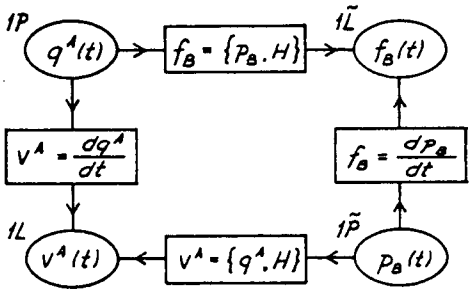
$n=4$



the quantization process

classical particle mechanics

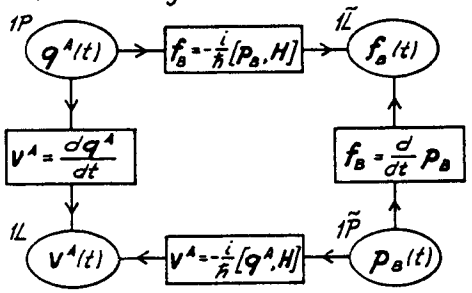
Ref. Ter Haar, 1961, p. 104



$\{q^A, p_B\} = \delta^A_B \quad \{q^A, q^B\} = 0 \quad \{p_A, p_B\} = 0$

quantum particle mechanics (Heisemberg formulation)

Ref. Born-Jordan, 1967

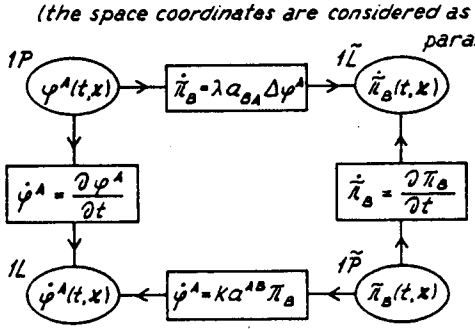


$[q^A, p_B] = i\hbar\delta^A_B I \quad [q^A, q^B] = 0 \quad [p_A, p_B] = 0$

classical field

(the space coordinates are considered as parameters)

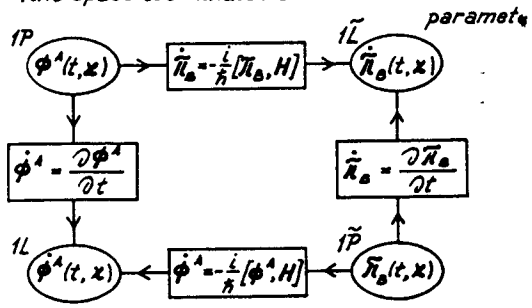
Ref. Goldstein, 1957, p. 362



quantized field

(the space coordinates are considered as parameters)

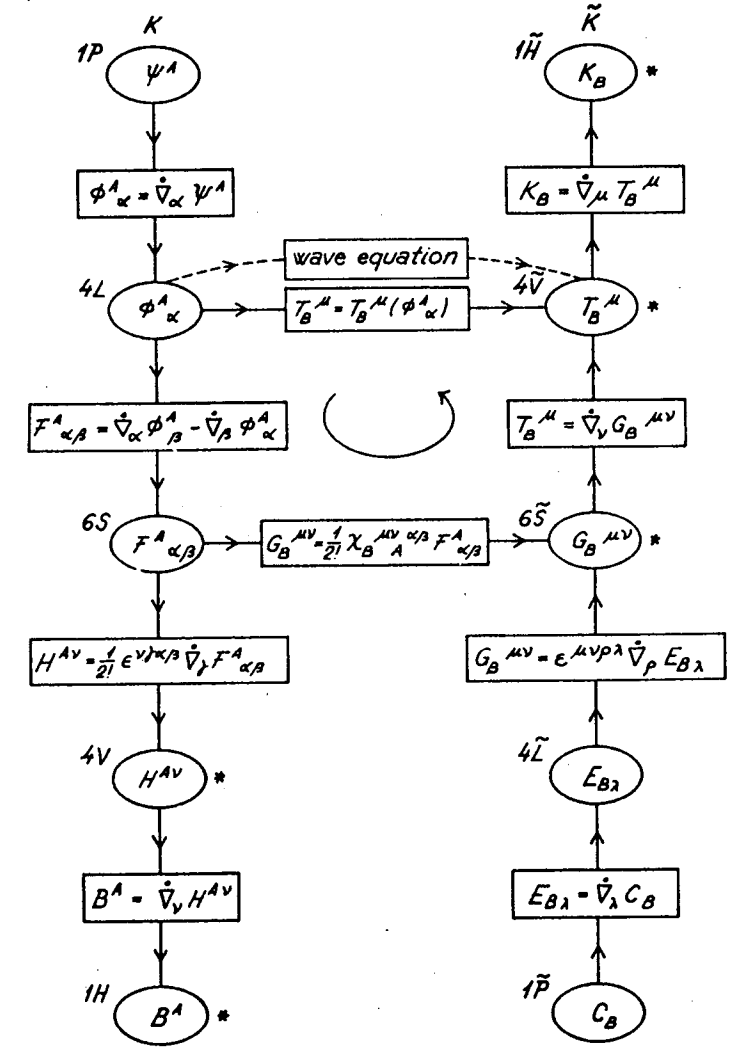
Ref. Kurşunoğlu, 1962, p. 310



$[\phi^A(t,x), \pi_B(t,x')] = i\hbar\delta^A_B\delta(x-x')$

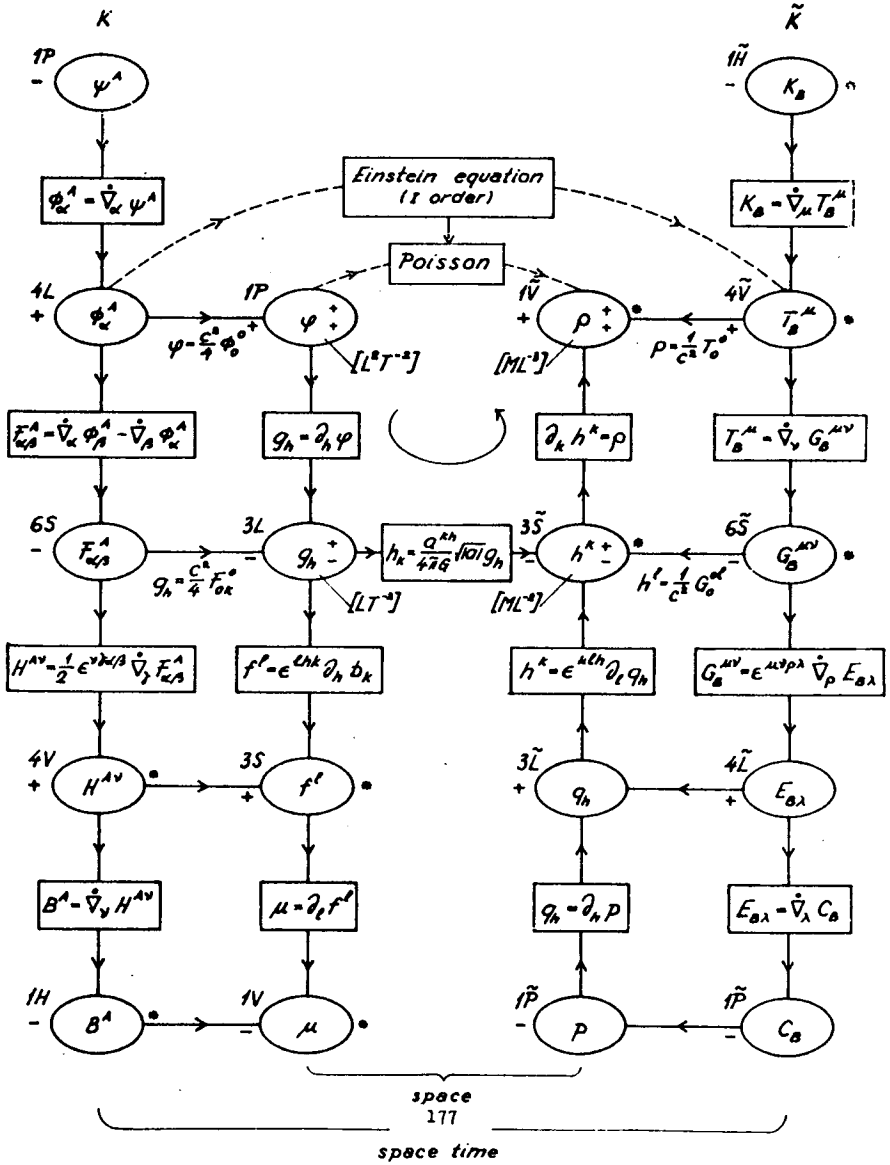
$[\phi^A(t,x), \phi^B(t,x')] = 0 \qquad [\pi_A(t,x), \pi_B(t,x')] = 0$

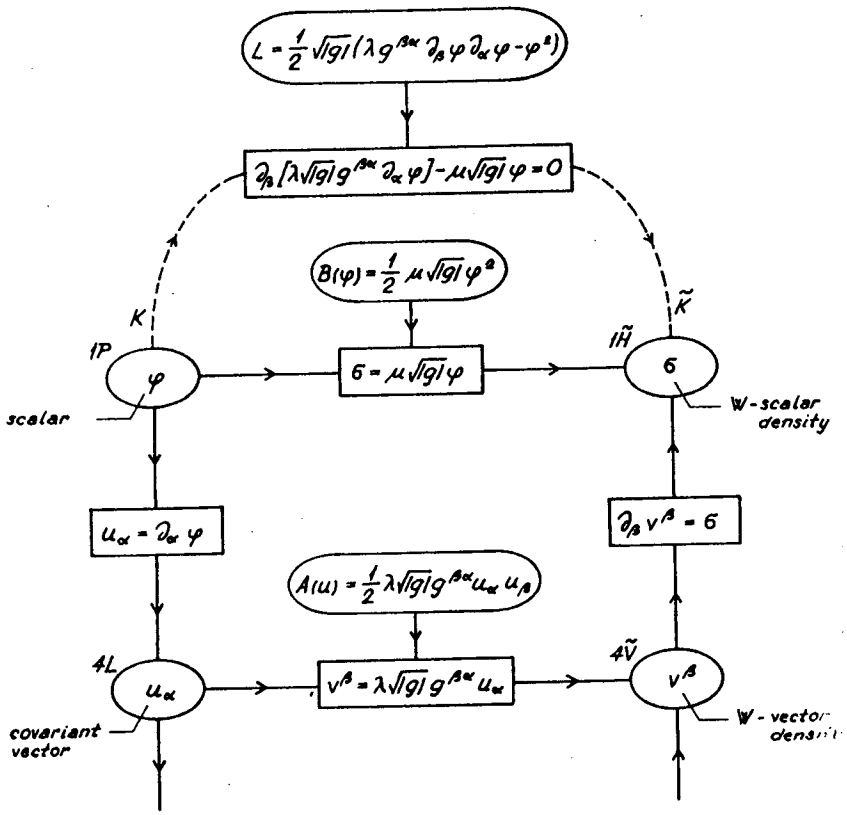
relativistic gravitation theory
(first order approximation)



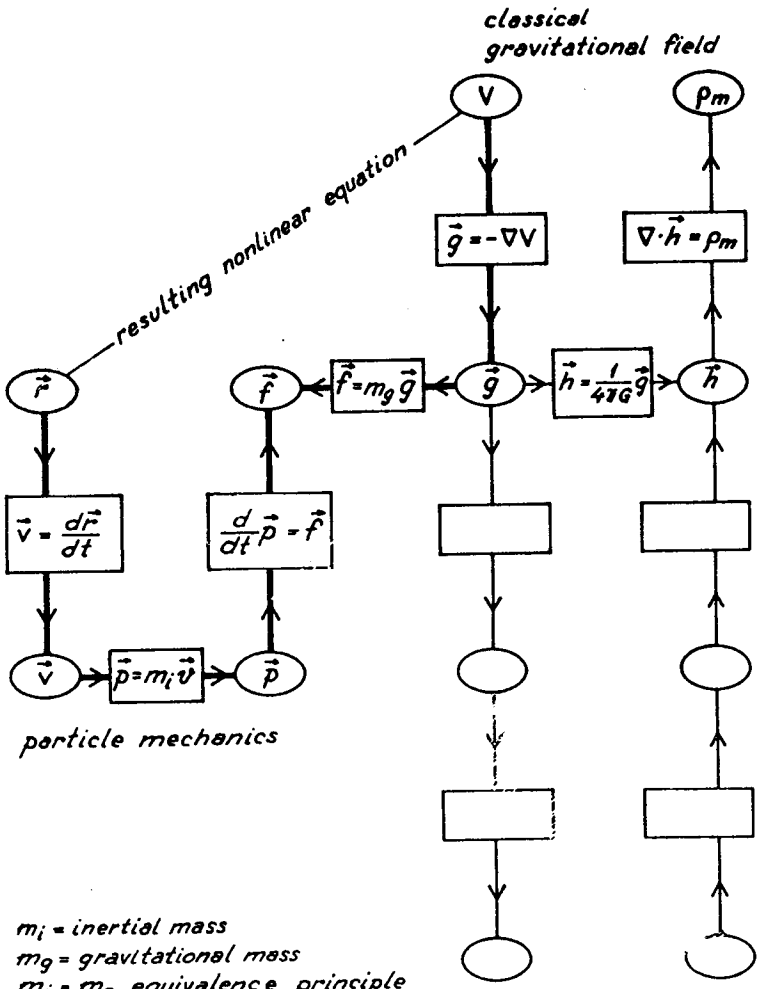
Ref. (Tonti, 1974)

gravitational theory
the passage from the relativistic to the classical theory



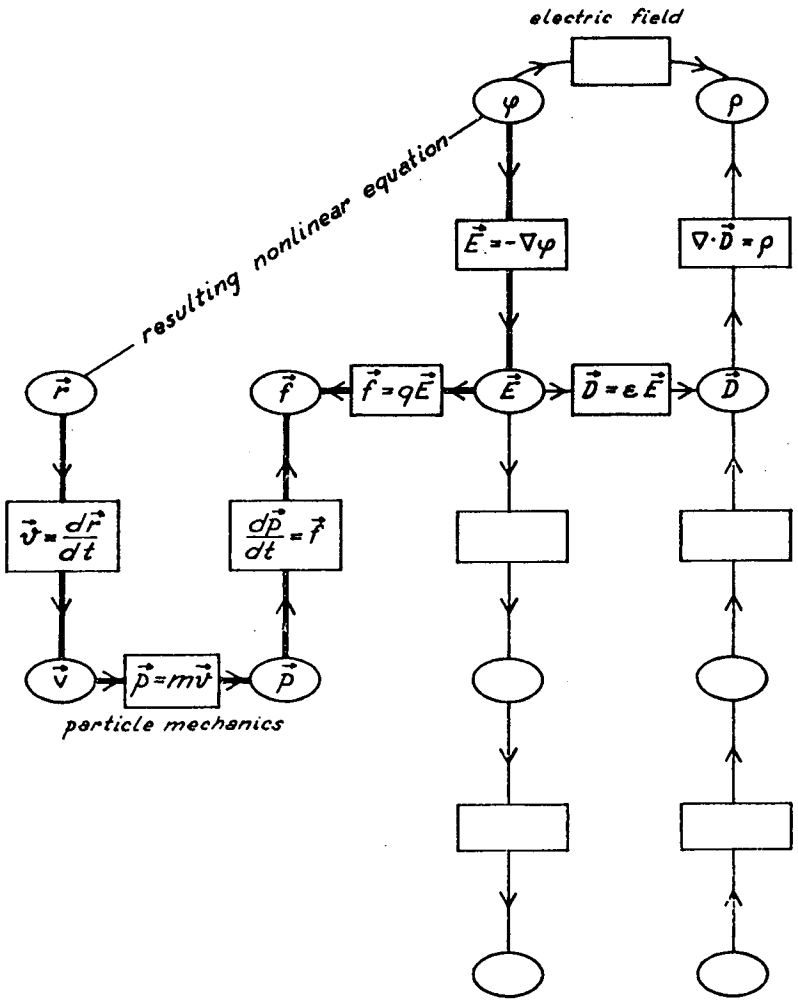


interactions: *motion of a particle in a gravitational field*

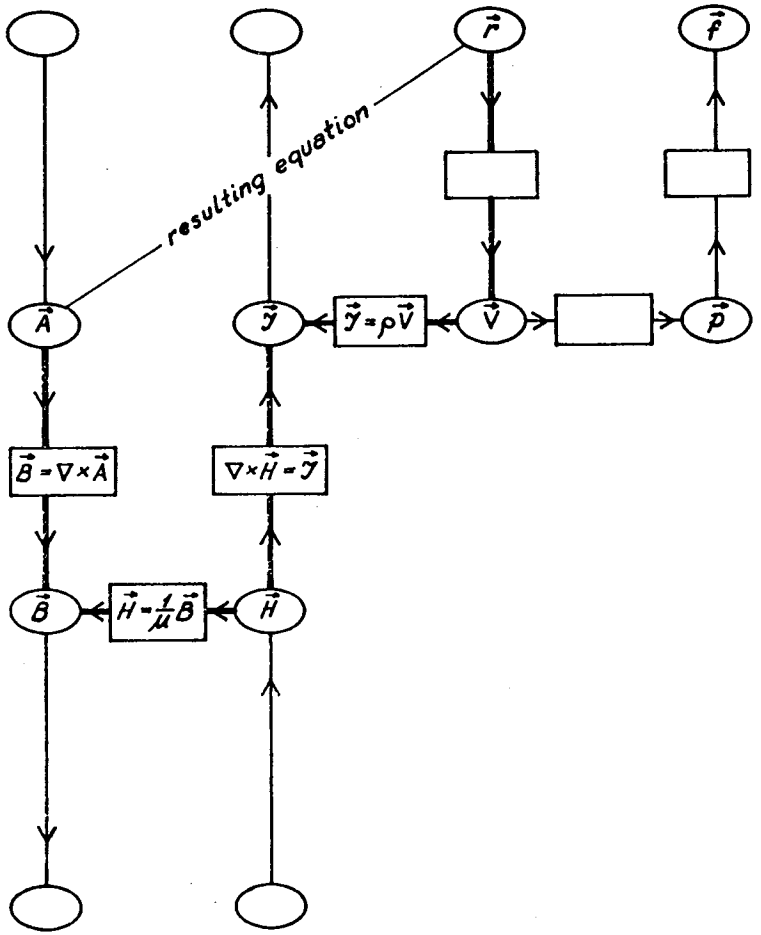


m_i = inertial mass
m_g = gravitational mass
$m_i = m_g$ equivalence principle

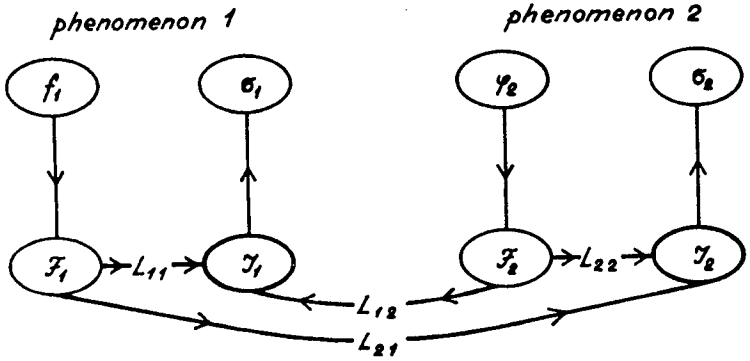
interaction: *motion of a charged particle in an electrostatic field*



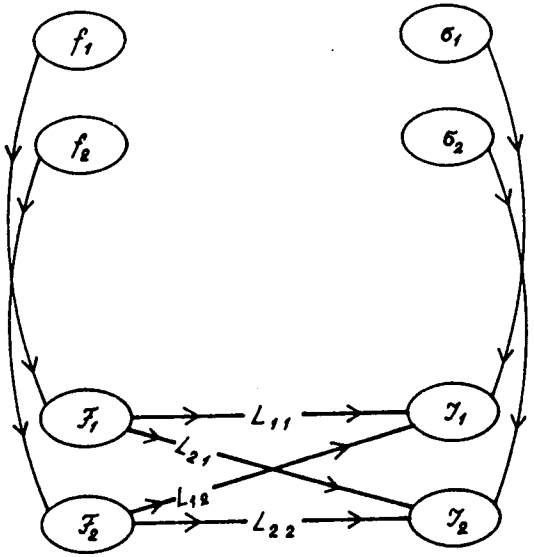
magnetic field generated from moving charges



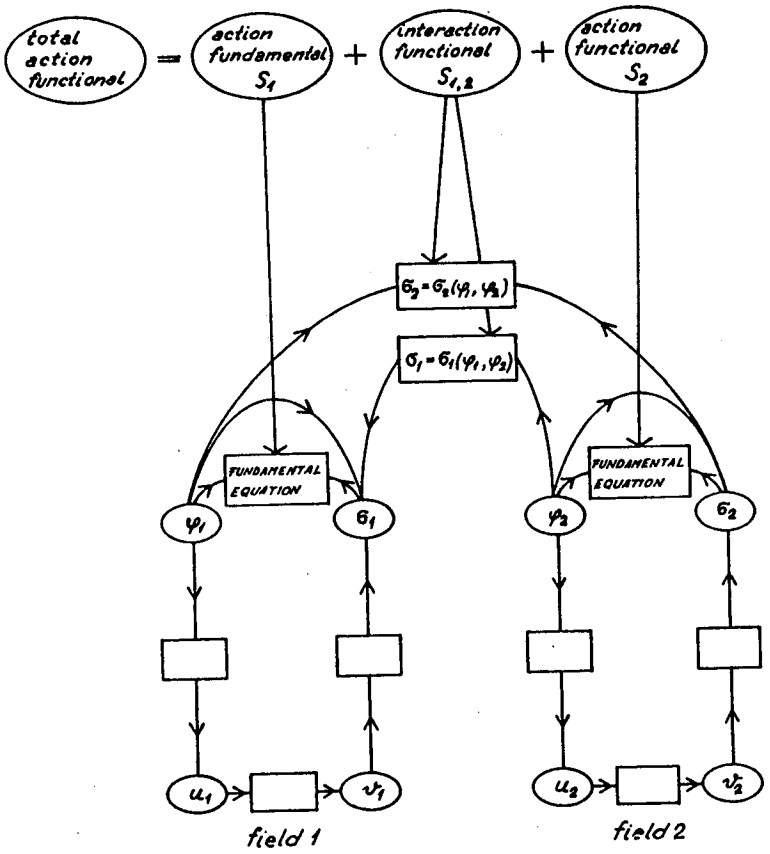
interaction between two phenomena:
the typical description of irreversible thermodynamics



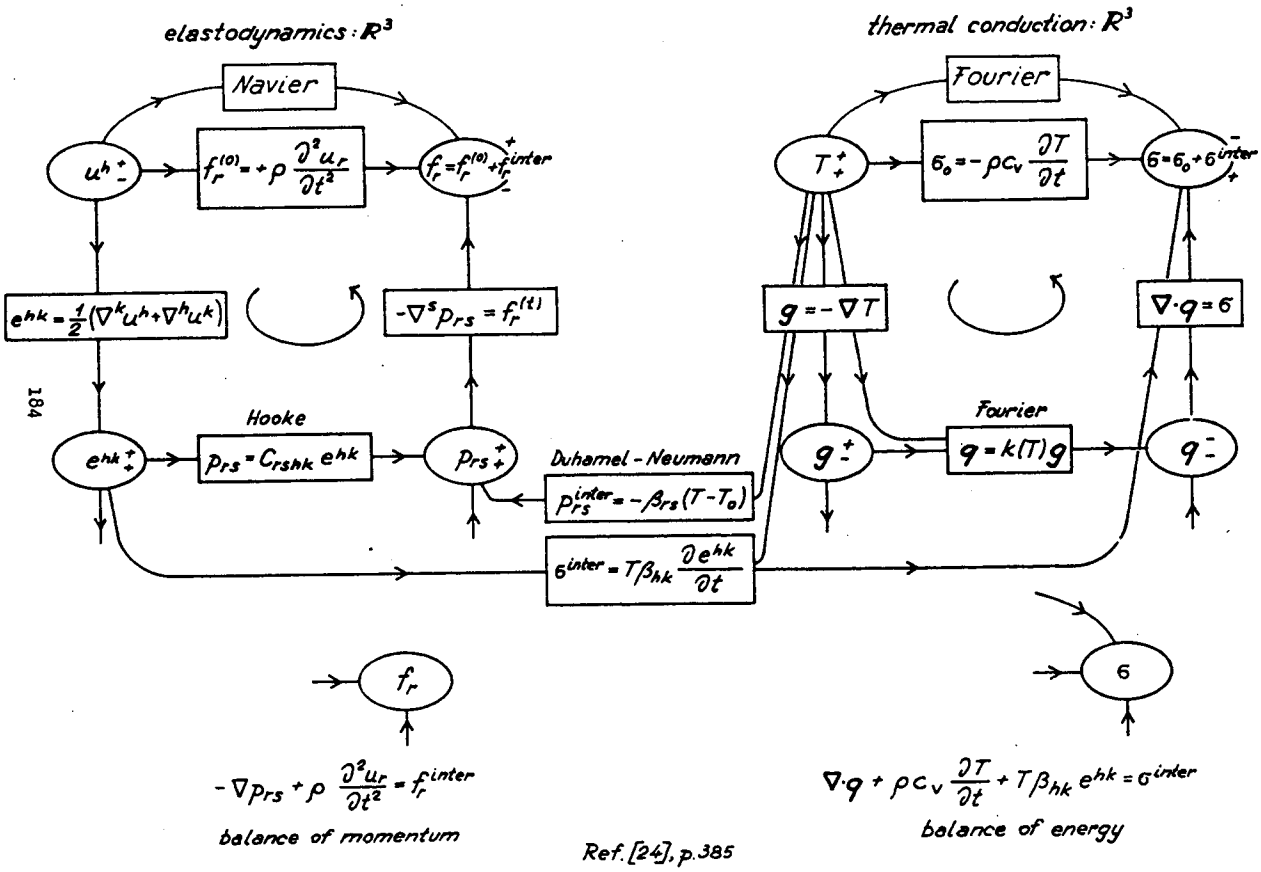
interactions 1 & 2



interactions between two fields:
the typical description of theoretical physics

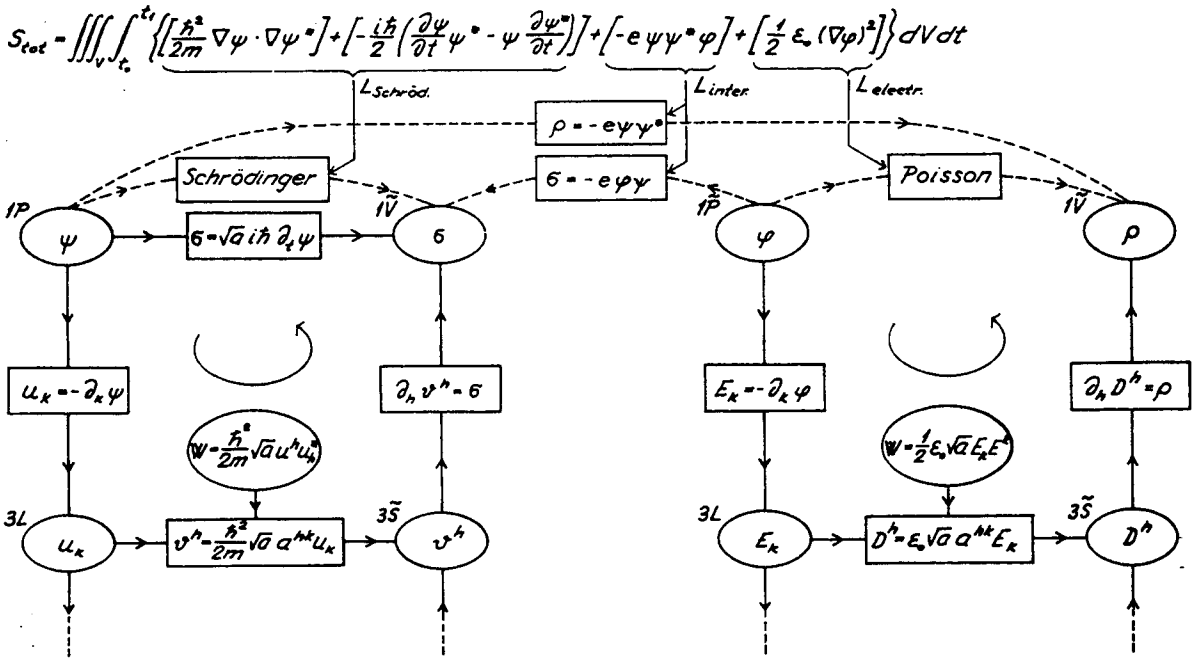


thermo-elasticity

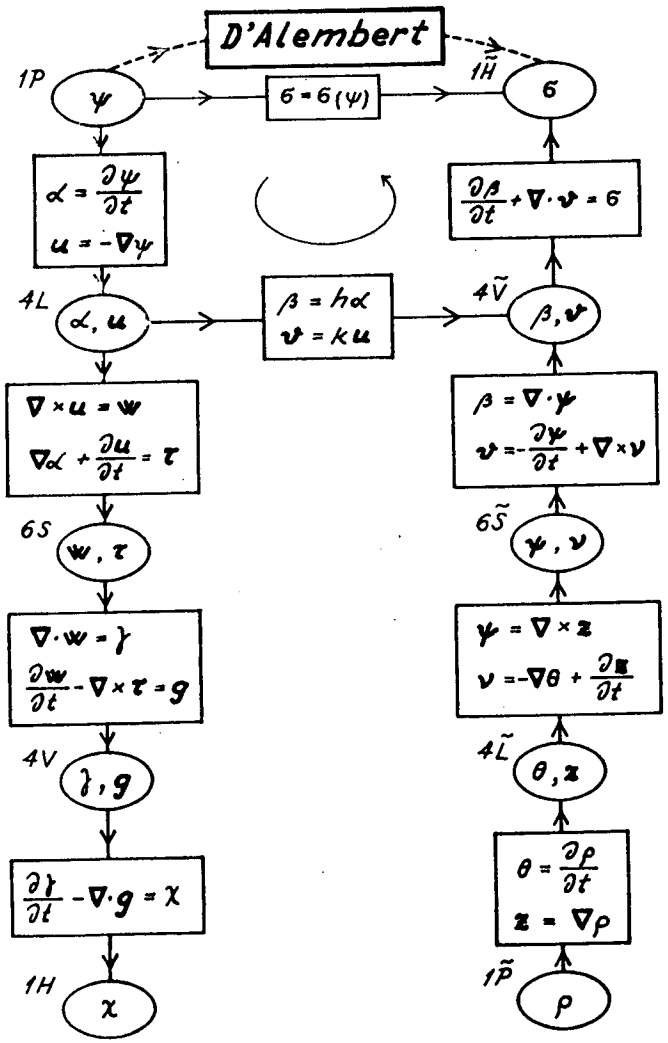


Ref. [24], p. 385

quantum dynamics of a particle in an electric field

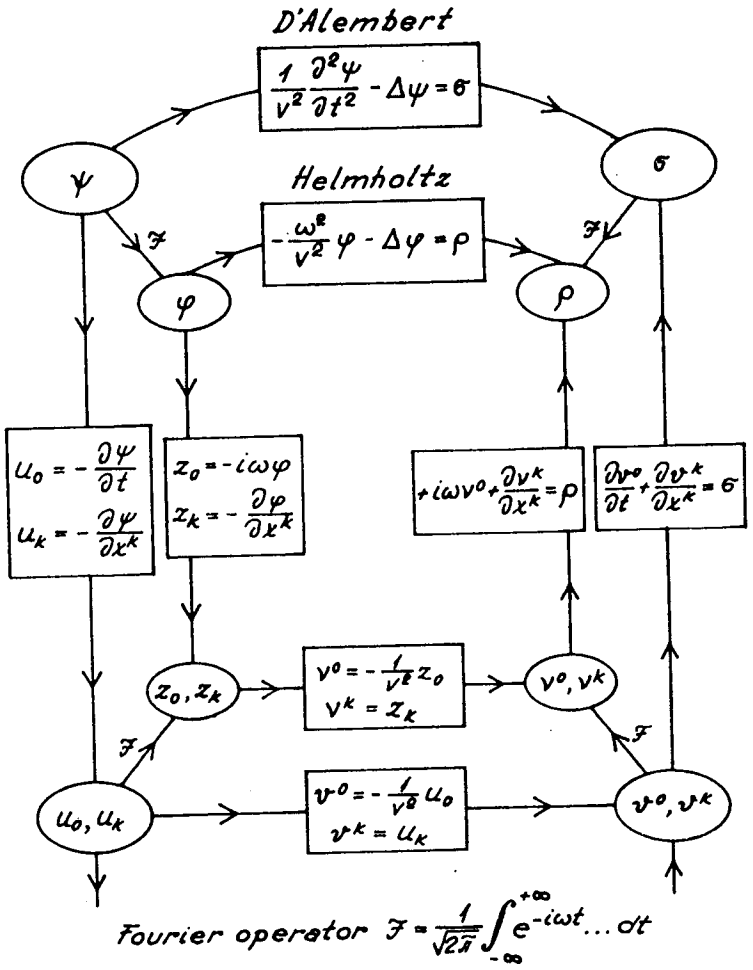


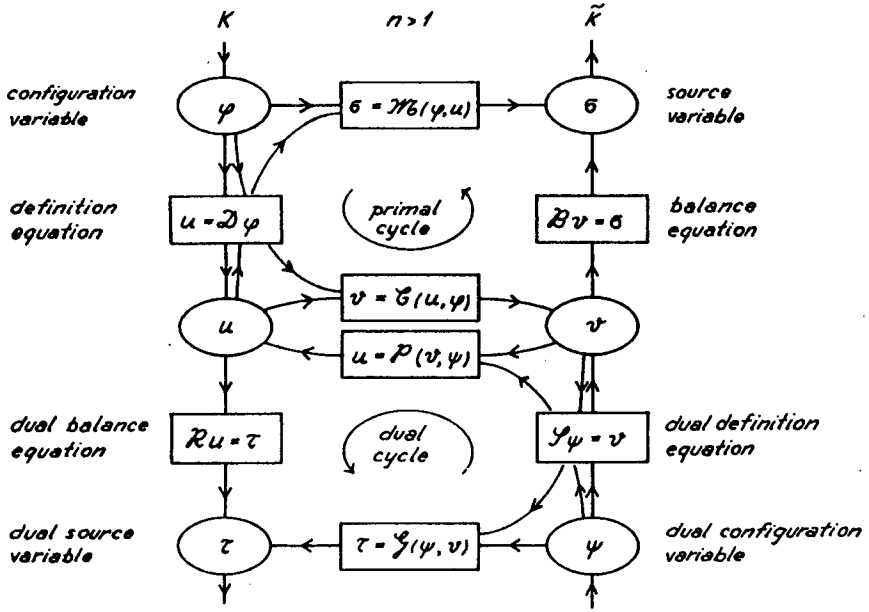
scalar wave equation

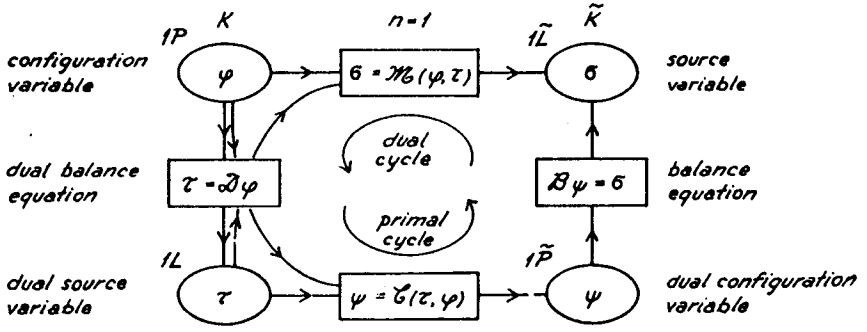


scalar wave equation
(D'Alembert)

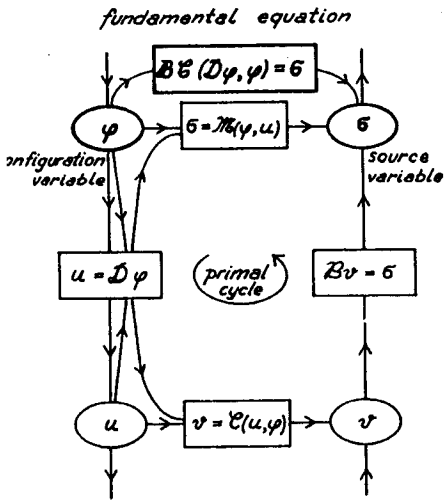
the passage from D'Alembert to Helmholtz equation

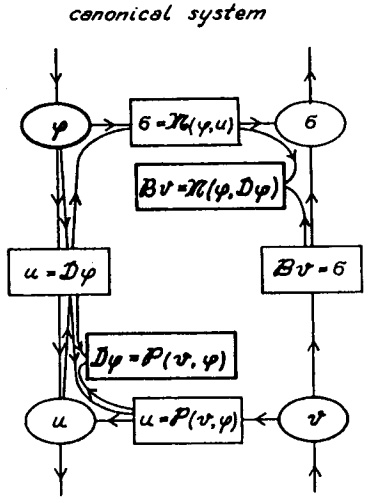


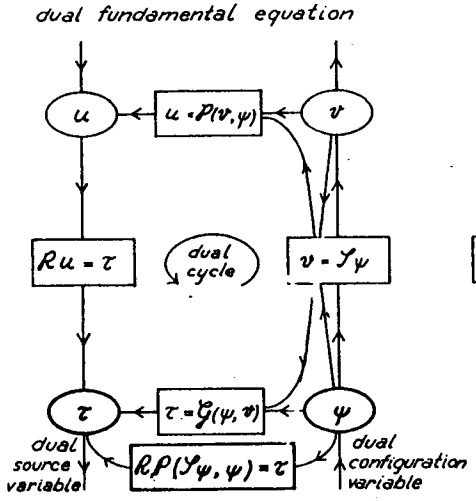


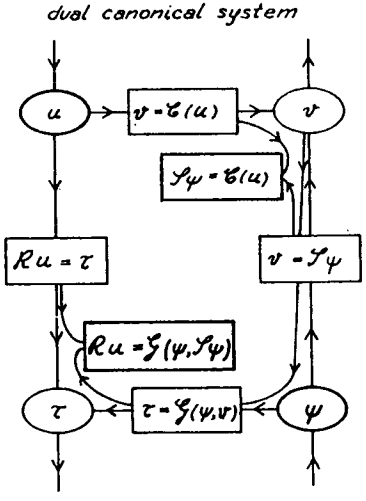


Some typical equations



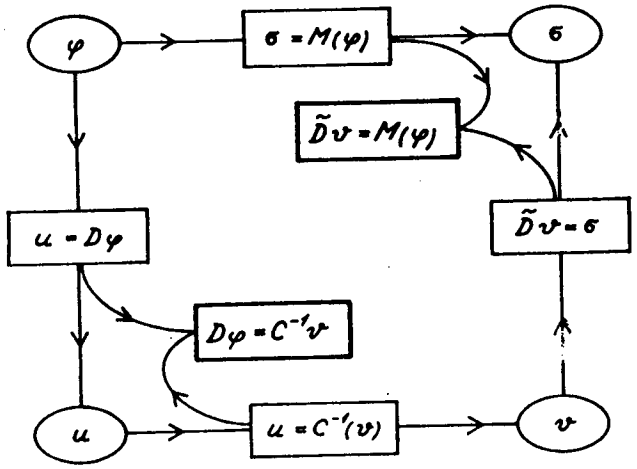






the canonical equations

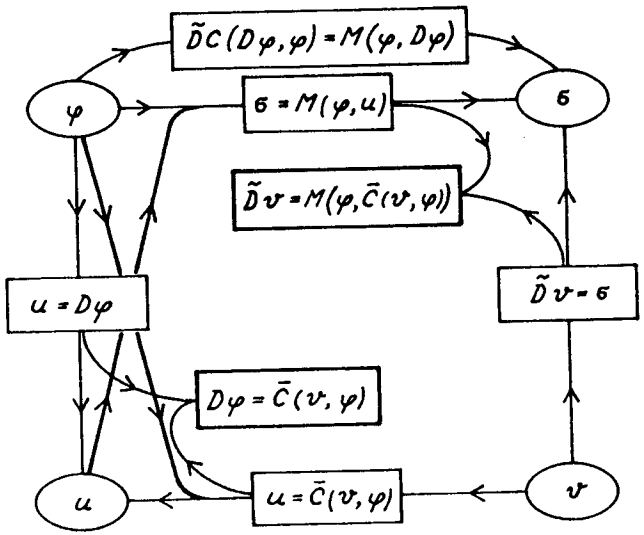
a) when the constitutive equations are not mixed, i.e. $v = C(u)$ $\sigma = M(\varphi)$



$$\tilde{D} v = M(\varphi)$$
$$D\varphi = C^{-1}(v)$$
$$\begin{pmatrix} 0 & \tilde{D} \\ D & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ v \end{pmatrix} = K(\varphi, v)$$
$$L\psi = K(\psi)$$

the canonical equations

b) when the constitutive equations are mixed, i.e. $v = C(u, \varphi)$ $\sigma = M(\varphi, u)$



$$\tilde{D} v = M(\varphi, \bar{C}(v, \varphi))$$
$$D\varphi = \bar{C}(v, \varphi)$$
$$\begin{pmatrix} 0 & \tilde{D} \\ D & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ v \end{pmatrix} = K(\varphi, v)$$
$$L\psi = K(\psi)$$

analytical mechanics:

$U(q)$ = force potential

$T^*(q,p)$ = dual kinetic energy

$D \overset{def}{=} \frac{d}{dt}$

bilinear forms:

$\langle Q,q \rangle \overset{def}{=} -\int_{t_o}^{t_1} Q_\kappa q^\kappa dt$

$\langle p,\nu \rangle \overset{def}{=} +\int_{t_o}^{t_1} p_\kappa \nu^\kappa dt$

$H(q,p) \overset{def}{=} T^*(q,p) + U(q)$

$L \overset{def}{=} \begin{pmatrix} 0 & \tilde{D} \\ D & 0 \end{pmatrix} =$

$= \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \frac{d}{dt} = E \frac{d}{dt}$

$$\delta\left[\int_{t_o}^{t_1} (p,q)\,dt - \int_{t_o}^{t_1} H(q,p)\,dt\right] = 0$$

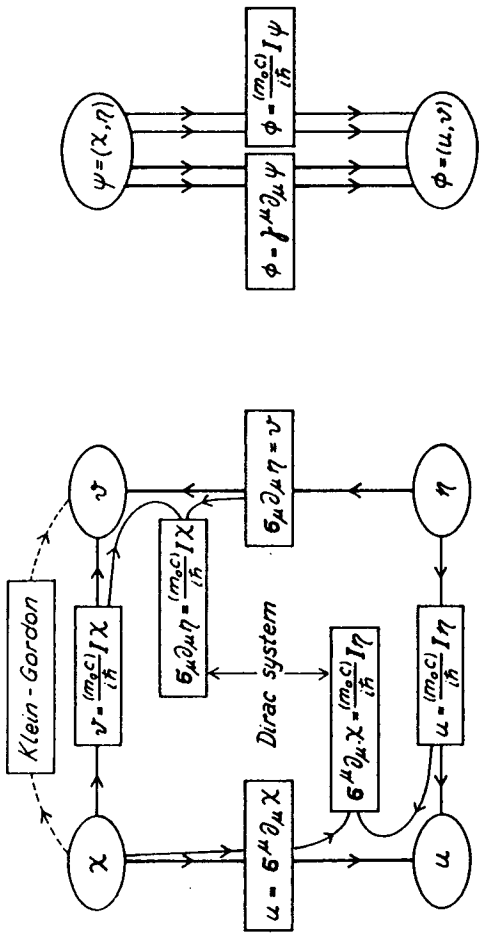
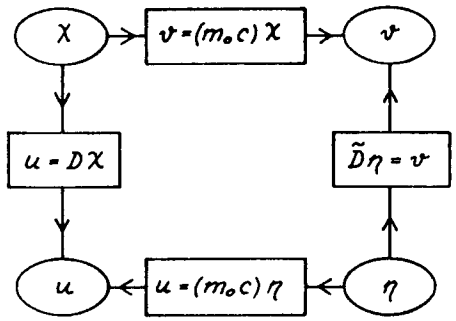


table 26: Dirac equation

Dirac field

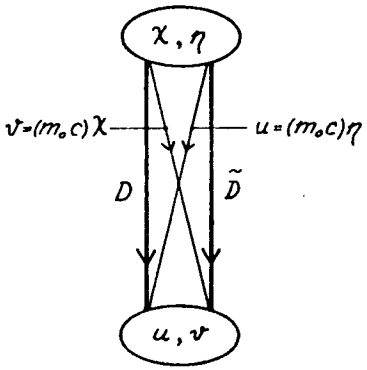


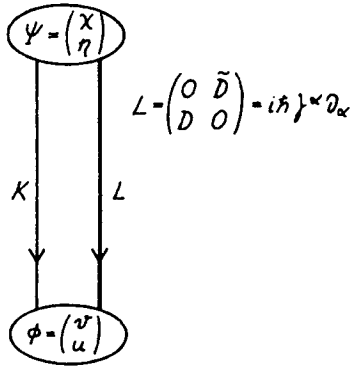
$$D = i\hbar(\sigma^0\partial_0 + \sigma^\kappa\partial_\kappa)$$

$$\tilde{D} = i\hbar(\sigma^0\partial_0 - \sigma^\kappa\partial_\kappa)$$

$$\langle v, \chi\rangle = \iiiint_\Omega (v_2\chi_1 - v_1\chi_2)\,d\Omega$$

$$\langle \eta, u\rangle = \iiiint_\Omega (\eta' u^2 - \eta^2 u')\,d\Omega$$





$$L = \begin{pmatrix} 0 & \tilde{D} \\ D & 0 \end{pmatrix} = i\hbar\,\gamma^\kappa\partial_\alpha$$

$$L\Psi = K\Psi$$

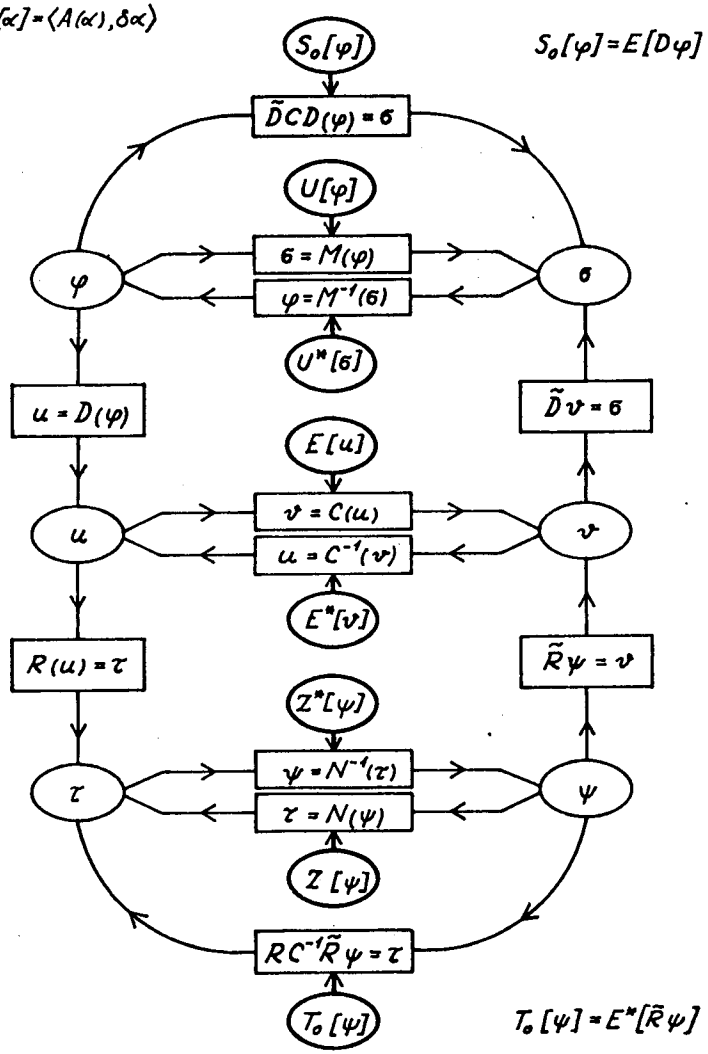
$$i\hbar\left[\left(\begin{array}{c|c} 0 & \sigma^0 \\ \hline \sigma^0 & 0 \end{array}\right)\partial_0 + \left(\begin{array}{c|c} 0 & -\sigma^\kappa \\ \hline \sigma^\kappa & 0 \end{array}\right)\partial_\kappa\right]\begin{pmatrix}\chi\\ \eta\end{pmatrix} = m_0 c\left(\begin{array}{c|c} \sigma^0 & 0 \\ \hline 0 & \sigma^0 \end{array}\right)\begin{pmatrix}\chi\\ \eta\end{pmatrix}$$

$$i\hbar\,\gamma^\kappa\partial_\alpha\Psi = m_0 c\,I\Psi$$

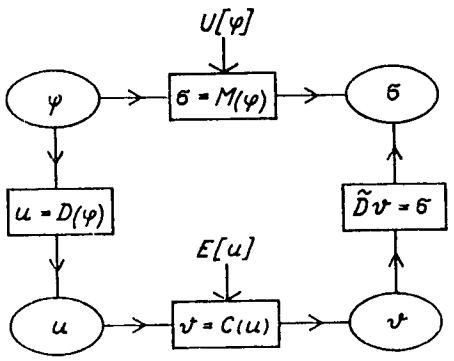
the potentials

the potential of an operator $A(\alpha)$ is defined as a functional $F[\alpha]$ such that

$\delta E[\alpha] = \langle A(\alpha), \delta\alpha \rangle$

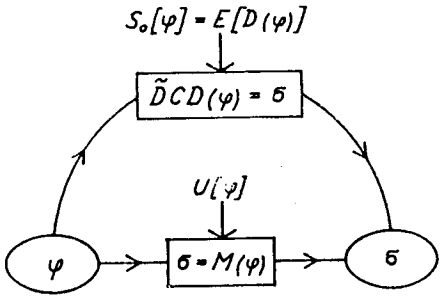


the birth of a variational statement

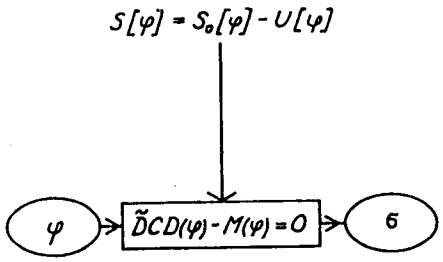


$\delta U[\varphi] = \langle M(\varphi), \delta\varphi \rangle$

$\delta E[u] = \langle C(u), \delta u \rangle$



$\delta S_0[\varphi] = \langle \tilde{D}CD(\varphi), \delta\varphi \rangle$



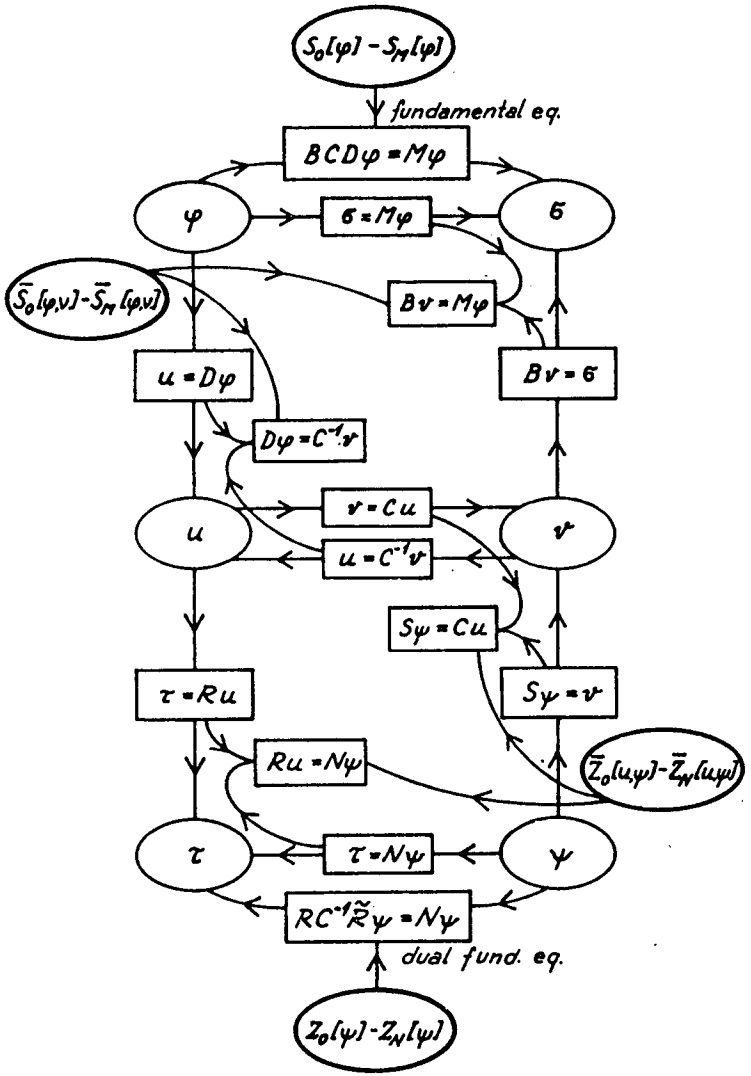
$\delta S[\varphi] =$

$= \delta(S_0[\varphi] - U[\varphi]) =$

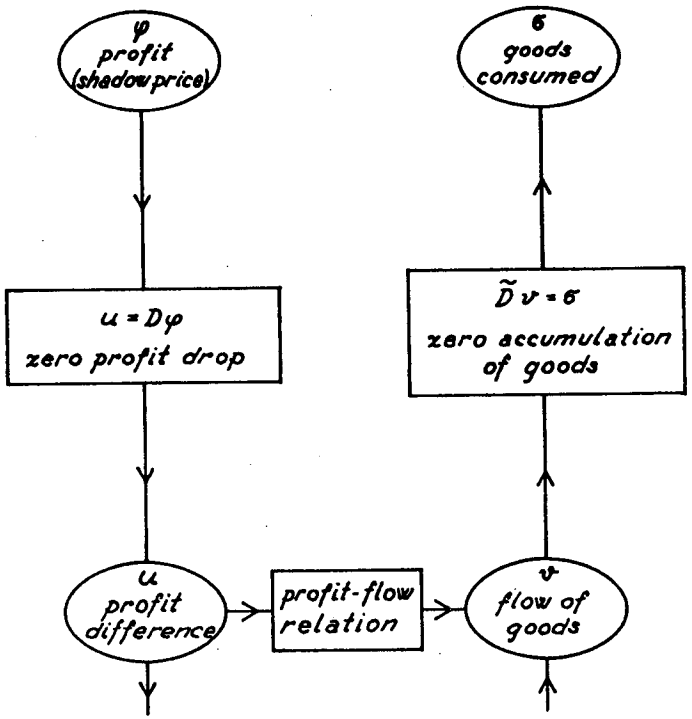
$= \langle \tilde{D}CD(\varphi) - M(\varphi), \delta\varphi \rangle = 0$

this is a variational statement

the four main variational statements

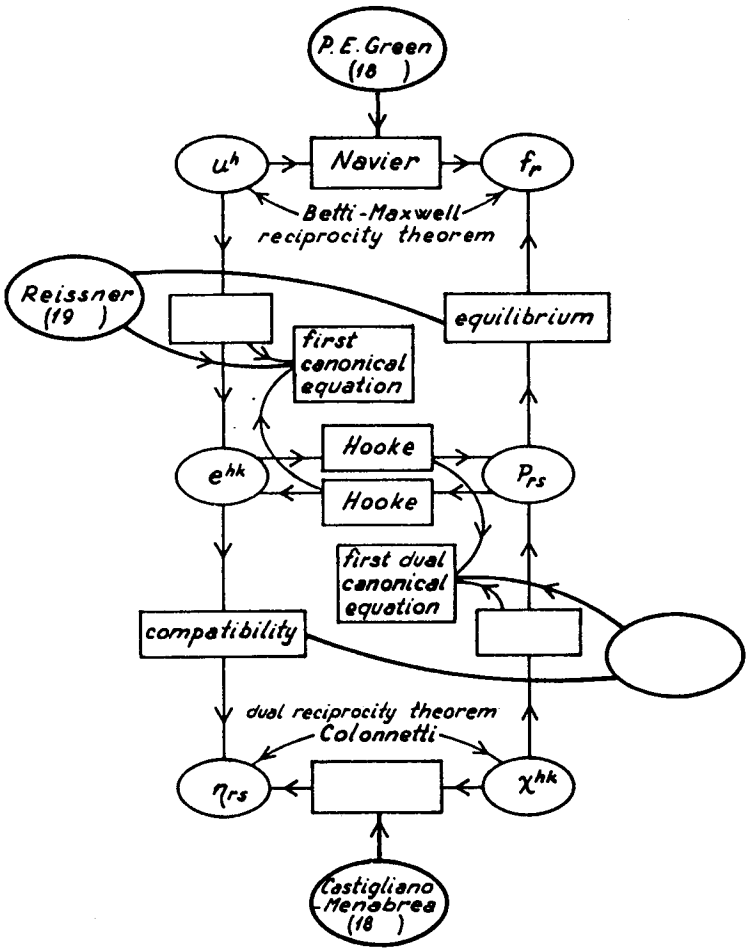


an extraphysical example: transportation economics



elastostatics (small displacement theory)

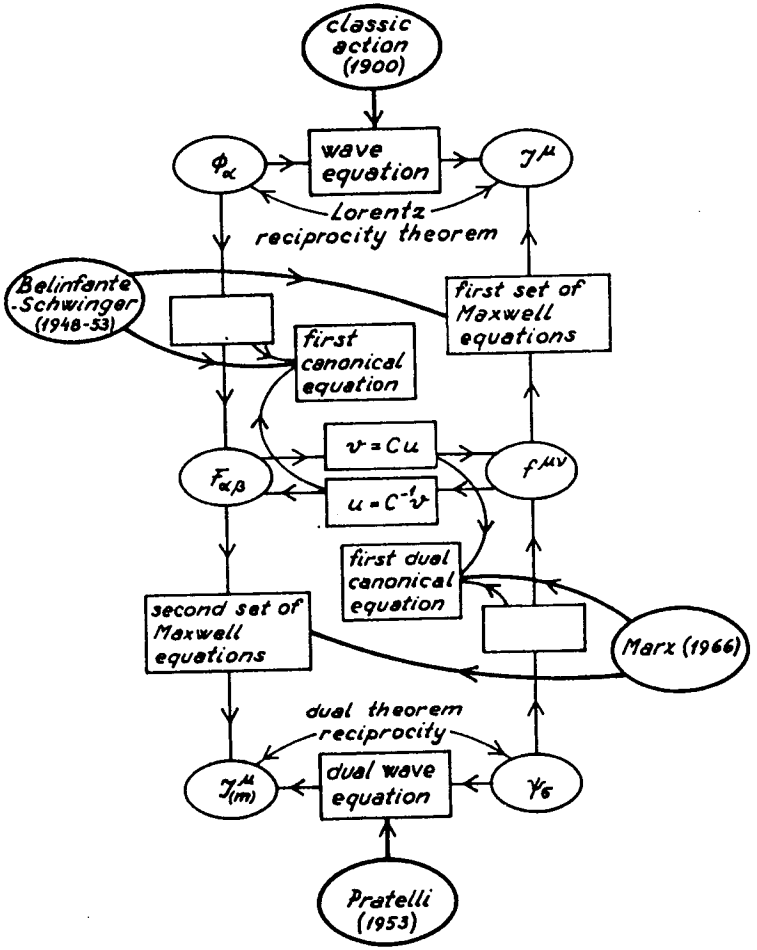
variational statements and reciprocity theorems



Ref Tonti E., Variational principles in elastostatics, Meccanica No 1, vol II p. 1, (1967)

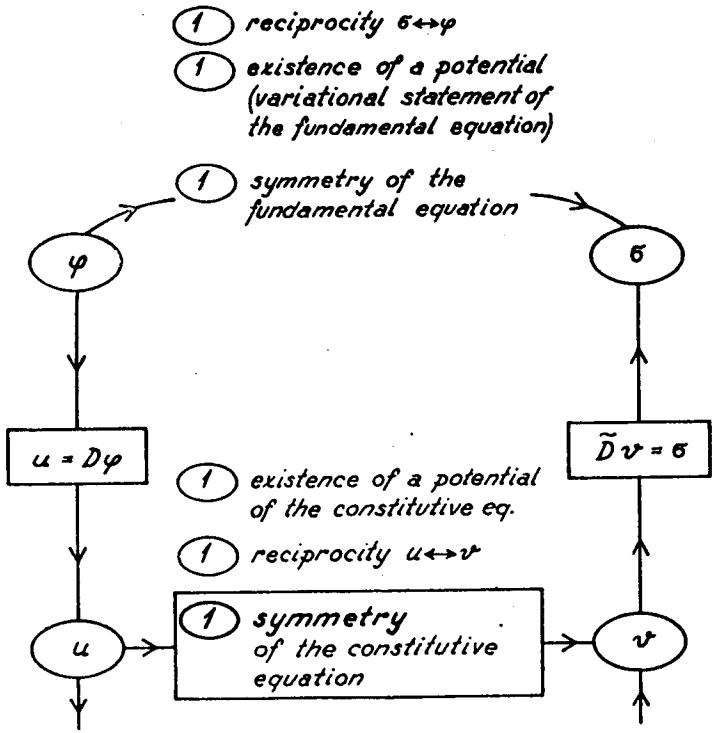
electromagnetism

variational statements and reciprocity theorems

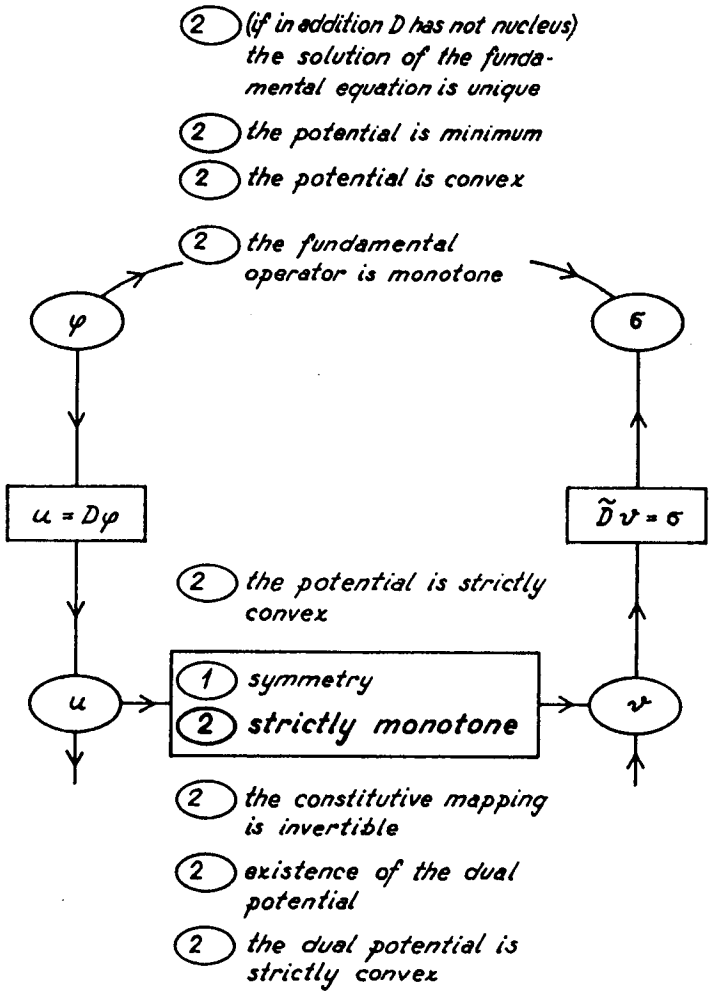


Ref. Tonti E., Variational principles in electromagnetism. Rend. Ist. Lomb. p. 845, (1968)

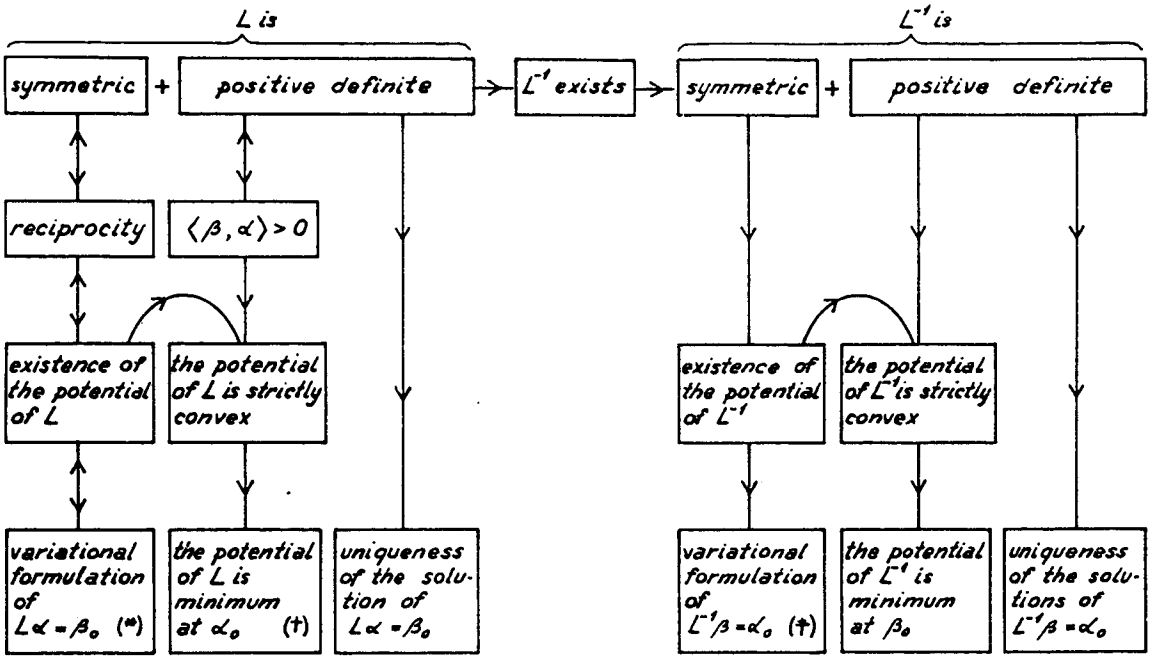
properties that follow from the symmetry of the constitutive operator



properties that follows from the strictly monotonic character of the constitutive operator

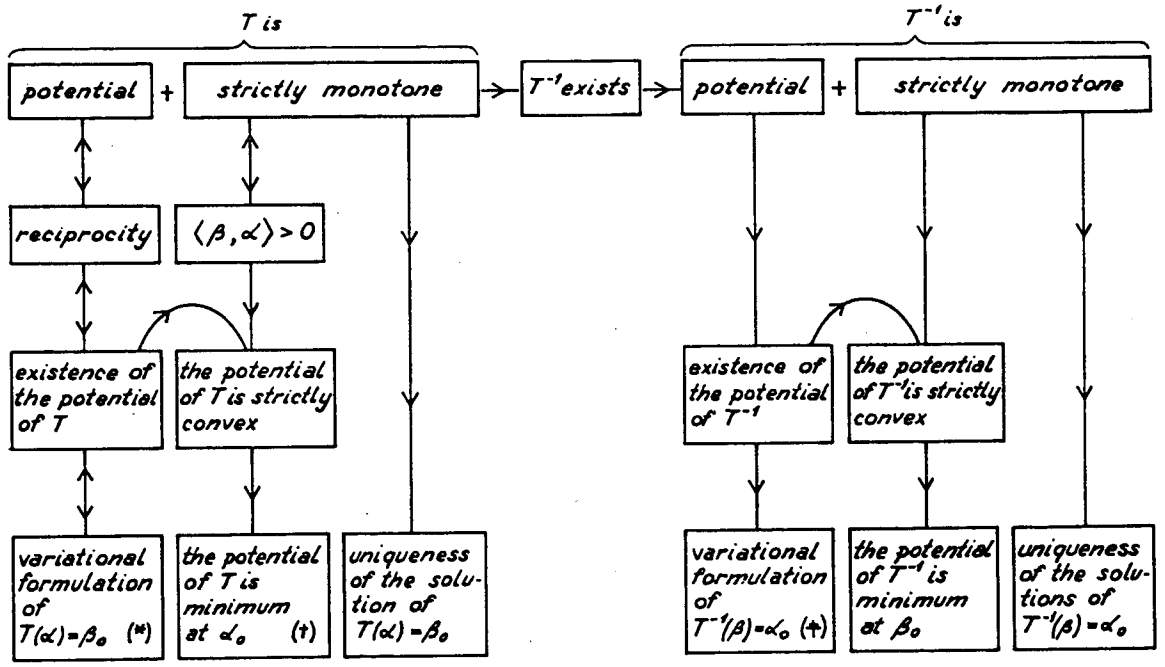


properties of a symmetric and positive-definite linear operator L



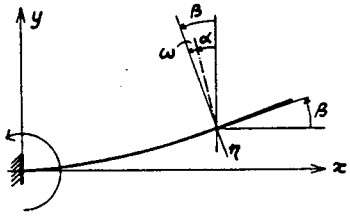
(*) $\beta_0 \in R(L)$ (†) α_0 is the solution of $L\alpha = \beta_0$ (†) $\alpha_0 \in R(L^{-1})$

properties of a potential and strictly monotone nonlinear operator T

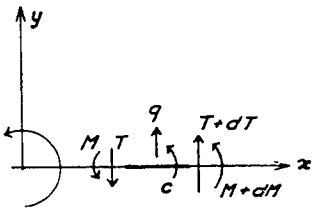


(*) $\beta_0 \in R(T)$ (†) α_0 is the solution of $T(\alpha) = \beta_0$ (†) $\alpha_0 \in R(T^{-1})$

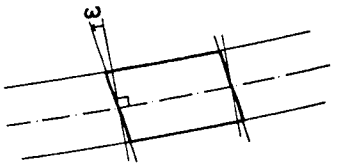
table **bending of beams**

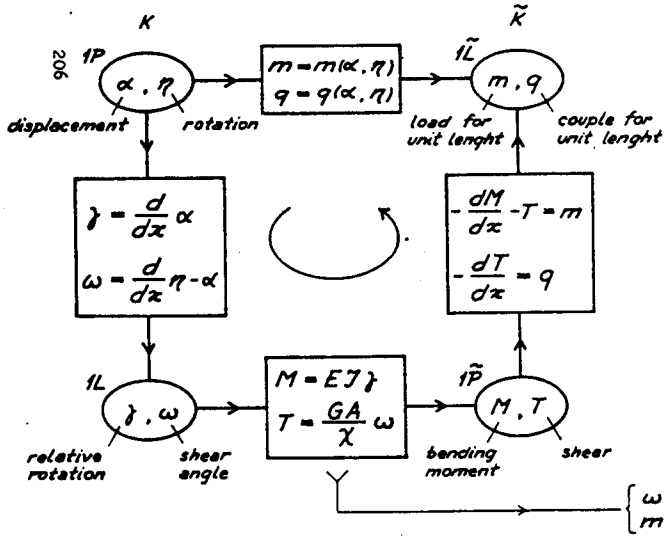


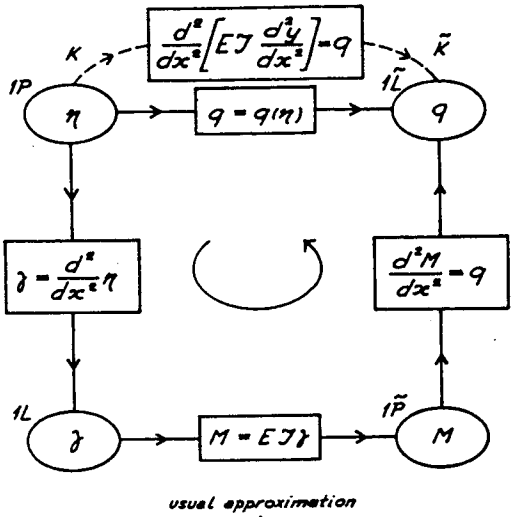
kinematical variables



statical variables







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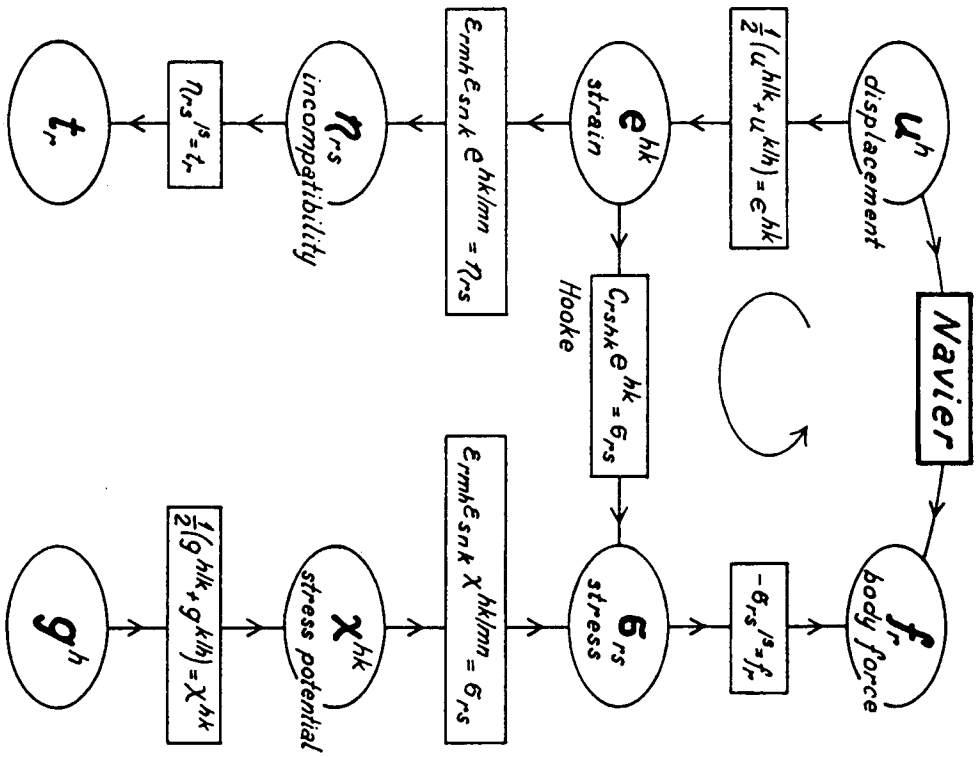

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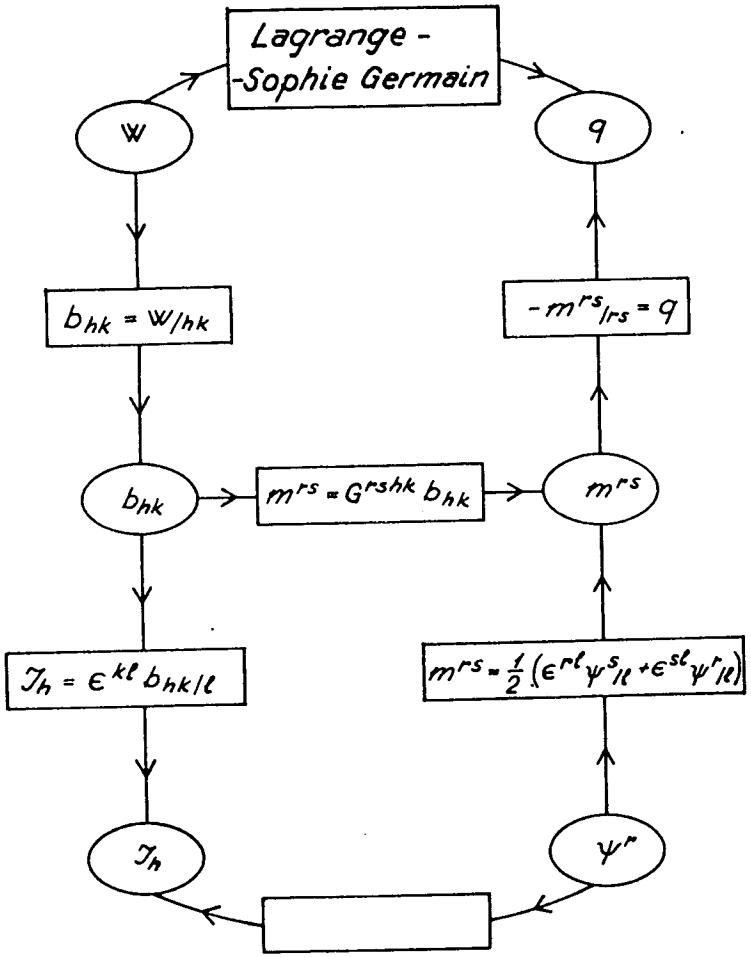
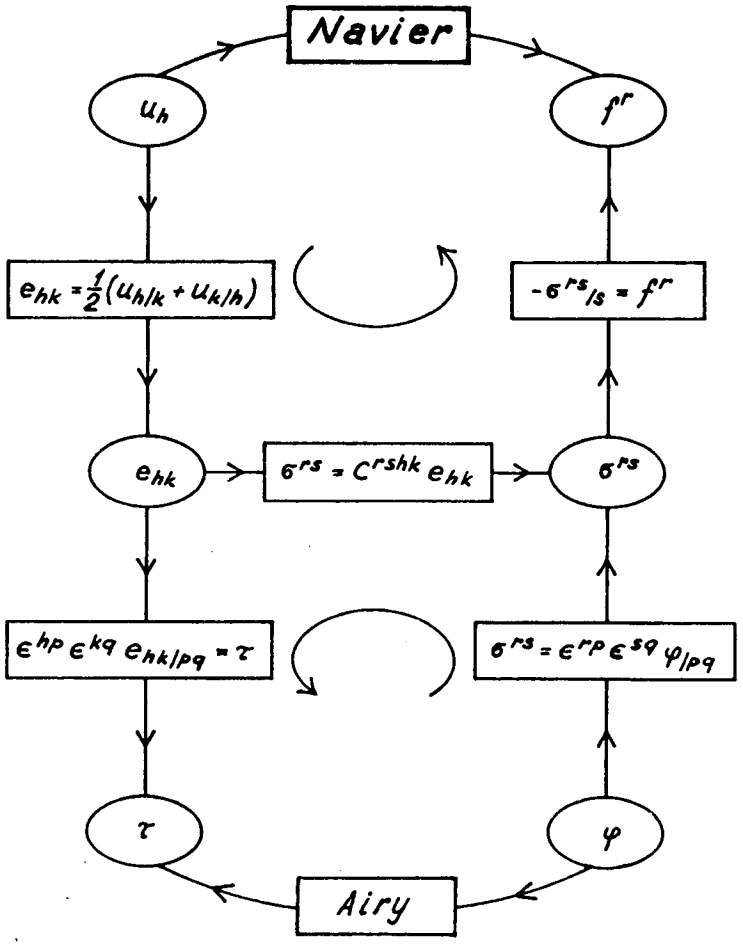
thin plates theory
(small deflections)

variables x^1, x^2



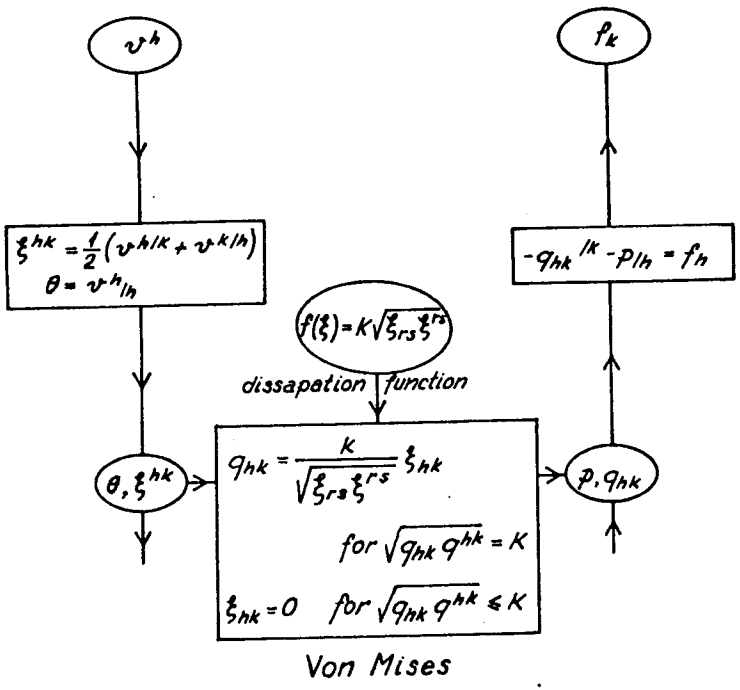
plane elasticity

variables x^1, x^2



plasticity

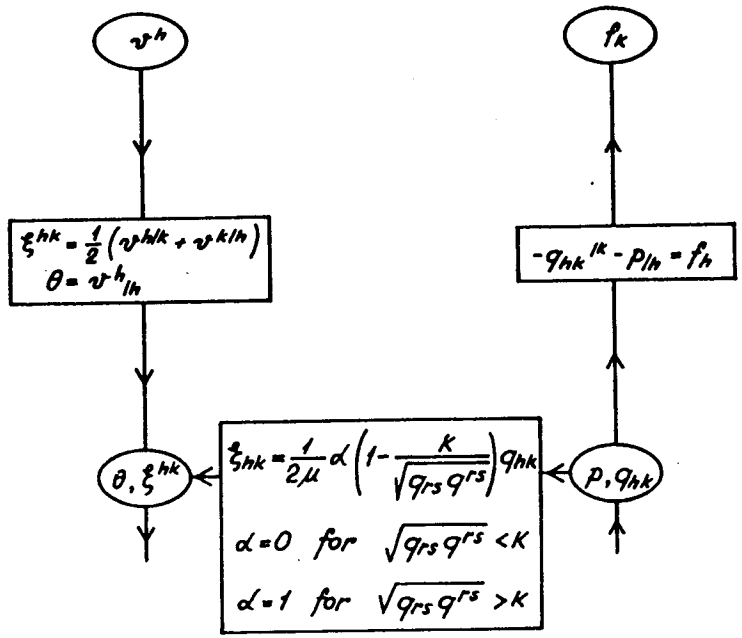
perfectly plastic material



Von Mises

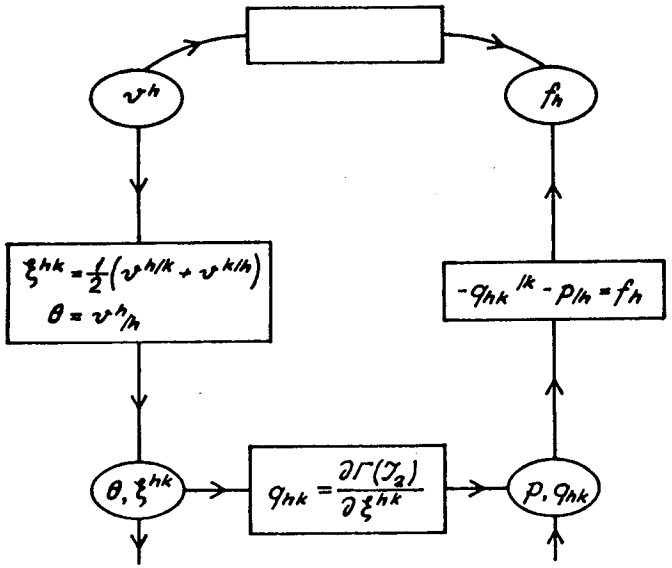
plasticity

visco-plastic material



viscous fluid motion

(incompressible, viscous, stationary flow)

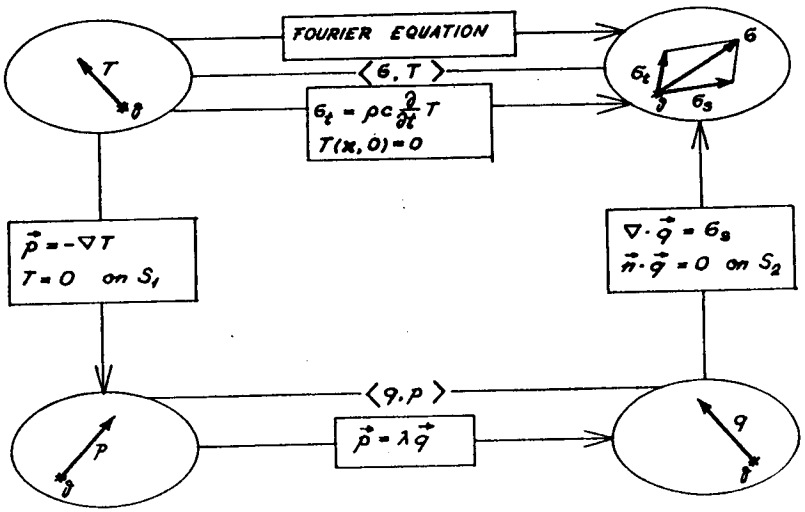


$$\mathcal{I}_2 = \xi_{rs}\,\xi^{rs}$$

newtonian flow $\Gamma = -\mu \mathcal{I}_2 \implies q_{hk} = -2\mu\,\xi_{hk}$

non newtonian flow $\Gamma = \Gamma(\mathcal{I}_2) \implies q_{hk} = \frac{\partial \Gamma}{\partial \mathcal{I}_2} 2\,\xi_{hk}$

heat conduction



configuration variable : temperature $T(x,t)$

source variable : heat production for unity volume and time $\sigma(x,t)$

variable of first kind : gradient of temperature $\vec{p}(x,t)$

variable of second kind : heat flux $\vec{q}(x,t)$

bilinear forms:

$$\langle \sigma, T \rangle \overset{def}{=\!=} \iiint_V \int_0^T \sigma(x, T-t)\, T(x,t)\, dV\, dt$$

$$\langle q, p \rangle \overset{def}{=\!=} \iiint_V \int_0^T \vec{q}(x, T-t) \cdot \vec{p}(x,t)\, dV\, dt$$

λ = *thermal conductivity,* ρ = *density,* c = *specific heat*

$$\underbrace{\nabla \cdot \left\{ \lambda \left[-\nabla T(x,t) \right] \right\}}_{\sigma_s} + \underbrace{\left[\rho c \frac{\partial}{\partial t} \right] T(x,t)}_{\sigma_t} = \sigma(x,t)$$

9. CLIFFORD ALGEBRA AND SPINORS

9.1 From Grassmann to Clifford algebra

Pursuing our exploration on the formal structure of physical theories we now examine the various kind of mathematical objects used in physics to describe physical quantities. We encounter real and complex numbers, vectors, skew-symmetric tensors, quaternions, complex vectors, spinors, and so on. We shall show that a large number of these mathematical objects are elements of an algebra, the Dirac algebra, that can be considered as a natural evolution of the Grassmann algebra. This is summarized in table(9.10)

Given an n-dimensional linear space U we have explained how to construct the exterior space ΛU i.e. the space of the aggregate of multivectors of different orders formed with the vectors of U the reason of the space ΛU is that of making close the space U with respect to the exterior product.

In this way the space ΛU becomes an algebra, the Grassmann or exterior algebra. Considering a dual space V and then of the exterior space ΛV one may introduce the scalar product of two multivectors of the same order and the inner products(left and right) of multivectors of different orders. Such products are defined between a multivector of ΛU and a multivector of ΛV.

These products have not meaning, up to now, for two multivectors of the same space, say ΛU. It becomes possible to introduce the scalar and inner products for two multivectors of the same space when we introduce a metric tensor defined as a linear and invertible mapping between ΛU and ΛV. This is what we shall show in the next sections. Once this will be done we have defined in ΛU essentially two kinds of products, the exterior and the inner products (the scalar product being a particular case of the inner one).

At this point Clifford has shown how it is possible to define an overall product between two elements of ΛU of which the

exterior and inner products are particular cases. For those who know the quaternion calculus this is the analogous of the quaternion product of which the cross and scalar products are particular cases (Brand, 1947, p. 411).

9.2 The metric tensor

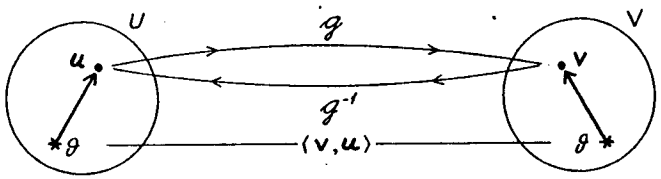
The theory of linear spaces in duality leads to the concept of scalar product between vectors of the two spaces. No scalar product has been introduced up to now between the elements of the same space. In order to do this it appears natural to define a one-to-one linear mapping g of the linear space U on the dual space V. In this way to every element u of U it corresponds an element

$$v = g(u) \qquad u \in U,\ v \in V \tag{9.2.1}$$

Now to realize this correspondence it is sufficient to prescribe the vectors of V that correspond to the base vectors e_h of U, i.e. to prescribe the numbers g_{hk} in the relation

$$g(e_h) = g_{hk}\, e^k. \tag{9.2.2}$$

The linear operator g is called the metric tensor and the numbers g_{hk} are its components pertinent to the base vectors e_h and to their dual e^k. Once we have prescribed the components g_{hk} we may obtain the relation



between the components of the vectors u and v that correspond one another by the relation

$$v = g(u) = g(u^h e_h) = u^h g(e_k) = u^h g_{hk}\, e^k \tag{9.2.3}$$

then

$$v_k = u^h g_{hk} \tag{9.2.4}$$

In this proof we have used the linearity of the operator g. Since the u^h are controvariant and the v_k are covariant it follows that g_{hk} are covariant for a change of the basis.

The metric tensor g is said symmetric if

$$\langle g(u'), u'' \rangle = \langle g(u''), u' \rangle \tag{9.2.5}$$

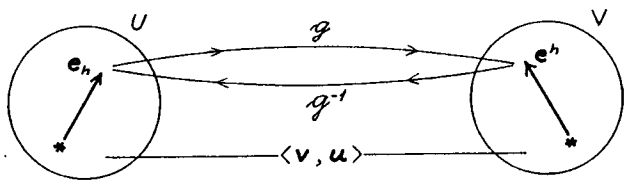
with $u', u'' \in U$. Remembering that $\langle v, u \rangle = v^h u_h$ the symmetry condition implies

$$g_{hk} = g_{kh} \tag{9.2.6}$$

We emphasize the fact that the metric tensor, that is here conceived as a messenger between the two spaces U and V, does not necessarily transform the base vectors e_h into the dual base vectors e^k. This is a very particular case that correspond to the metric tensor g such that

$$g(e_h) = e^h \tag{9.2.7}$$

The corresponding geometry is called the euclidean geometry

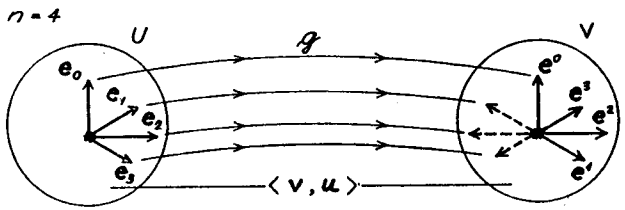


euclidean geometry

To deal with relativity one need a four-dimensional space with metric tensor

$$g(e_0) = e^0 \qquad (9.2.8)$$

$$g(e_k) = -e^k \qquad (k = 1,2,3)$$



minkowskian geometry

the corresponding components are

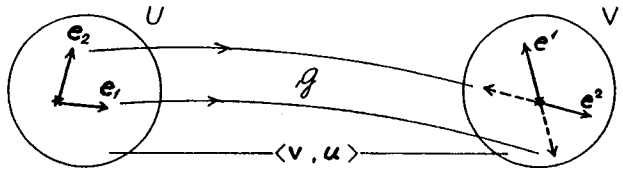
$$\begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad (9.2.9)$$

and the corresponding geometry is the minkowskian geometry. To define the quaternions, as we shall see later, we may use a two dimensional space and a metric tensor

$$g(e_h) = -e^h \qquad (h = 1,2) \qquad (9.2.10)$$

with components

$$n = 2 \qquad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (9.2.11)$$

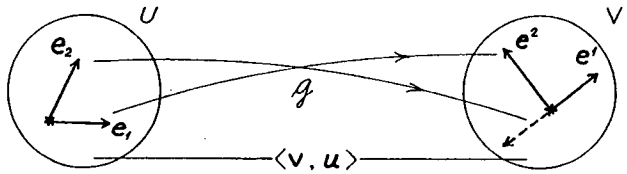


In a bidimensional space (or more in general in an even-dimensional space) the metric tensor

$$g(e_1) = e^2 \qquad g(e_2) = -e^1 \qquad (9.2.12)$$

gives rise to the symplectic geometry (Artin, 1957). The components of the metric tensor are

$$n = 2 \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad (9.2.13)$$



symplectic geometry

Symplectic geometry is used in analytical dynamics (Abraham, 1967, p. 87) and in the theory of spinors (Rastall, 1973). In the last case the metric tensor is called the fundamental metric spinor (Corson, 1955 p. 15).

The inverse mapping $\mathcal{g}^{-1}$ is described by the set of numbers

$$\mathcal{g}^{-1}(e^h) = g^{hk} e_k \tag{9.2.14}$$

that, on account of the property $g^{-1}g = I$ gives

$$e^h = \mathcal{g}\mathcal{g}^{-1}(e^h) = \mathcal{g}(g^{hk}e_k) = g^{hk}\mathcal{g}(e_k) = g^{hk}g_{kj}e^j \tag{9.2.15}$$

and then

$$g^{hk}g_{kj} = \delta^h_j \tag{9.2.16}$$

One may easily show that the components g^{hk} are controvariant for a change of the basis of the U -space.

With the introduction of the metric tensor one need not to distinguish the two spaces U and V because every operation made on V can be transferred in a corresponding operation made on U .

The two vectors that are in one-to-one correspondence can be denoted with the same letter. In particular one can define the scalar product between two vectors of the <u>same</u> space U by means of the induced scalar product

$$(u,u') \overset{def}{=} \langle \mathcal{g}u, u' \rangle = g_{kh}u^h u'^k \tag{9.2.17}$$

The space U equipped with such scalar product is called a <u>scalar product space</u>. In particular

$$(e_m, e_n) = g_{mn} \tag{9.2.18}$$

If the metric tensor $\mathcal{g}$ is symmetric it follows

$$(u', u'') = (u'', u') \tag{9.2.19}$$

and moreover if it is positive definite it follows

$$(u,u) = \langle \mathcal{g}(u), u \rangle > 0 \tag{9.2.20}$$

In symplectic geometry the scalar product is skew-symmetric, i. e.

$$(u', u'') = -(u'', u') \tag{9.2.21}$$

that implies the usual property

$$(u,u) = 0 \tag{9.2.22}$$

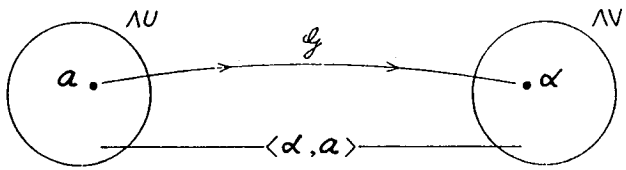
(Corson, 1955 p. 15)

The scalar product of two vectors $u, u' \in U$ can be extended to two simple p-vectors of $\Lambda^p U$ as *follows*

$$\begin{aligned}(u_1 \wedge u_2 \wedge \dots \wedge u_p,\ u'_1 \wedge u'_2 \wedge \dots \wedge u'_p)_p = \\ = \det \| \langle \mathcal{g}u_h, u'_k \rangle \| = \\ = \det \| (u_h, u'_k) \|\end{aligned} \tag{9.2.23}$$

(Greub, 1967, p. 106). We now come to the space ΛU . The mapping $\mathcal{g}$ between the vectors of U and those of V induces a mapping between the p-vectors of $\Lambda^p U$ and the p-vectors of $\Lambda^p V$.

In fact given a simple p-vector $p \in \Lambda^p U$ one can associate with it the simple p-vector $\pi \in \Lambda^p V$, whose vectors are those corresponding to the vectors that form p . In this way we *construct* a mapping between $\Lambda^p U$ and $\Lambda^p V$ and then between ΛU and ΛV (Witney, 1957, p. 46). Denoted $\mathcal{G}$ the mapping between these two linear spaces one can introduce an inner product on ΛU by means of the relation



$$\begin{aligned}(a', a) = \langle \mathcal{G}a', a \rangle = (s', s)_0 + (v', v)_1 + (b', b)_2 + \dots \\ = \frac{1}{0!}s's + \frac{1}{1!}v'_h v^h + \frac{1}{2!}b'_{hk}b^{hk} + \dots\end{aligned} \tag{9.2.24}$$

In this way the space ΛU becomes a scalar product space. One can define the left and right inner products: they are defined implicitly by the formulae

$$(x \lrcorner a, c) = (a, c \wedge x)$$
$$(y \wedge a, c) = (a, c \llcorner y) \qquad (9.2.25)$$

9.3 The Clifford product

Let us consider an exterior space ΛU equipped with a symmetric scalar product. One may introduce a new product between two elements $a, b \in \Lambda U$, as giving an element $c \in \Lambda U$. This is called the Clifford product of a and b and is denoted (Artin, 1957)

$$c = a \cdot b \qquad (9.3.1)$$

It is defined implicitly by means of the five axioms (Raševskii, 1957, p. 8)

1) distributive property

$$(a+b)\cdot c = a\cdot c + b\cdot c \qquad c\cdot(a+b) = c\cdot a + c\cdot b \qquad (9.3.2)$$

2) associative property

$$(a\cdot b)\cdot c = a\cdot(b\cdot c) \qquad (9.3.3)$$

3) if λ is a number (0-vector) considered as a particular aggregate:

$$\lambda\cdot a = a\cdot\lambda = \lambda a \qquad (9.3.4)$$

4) if u is a vector considered as a particular aggregate

$$u\cdot u = (u,u) \qquad (9.3.5)$$

5) if $u_1, u_2, \dots u_p$ are p vectors

$$u_1 \wedge u_2 \wedge \dots \wedge u_p = \frac{1}{p!}\,\delta^{hk\dots\ell}_{12\dots p}\, u_h \cdot u_k \cdot \dots \cdot u_\ell \qquad (9.3.6)$$

The linear space ΛU equipped with the Clifford product becomes an algebra called the Clifford algebra

9.4 Some properties

From these requirements some properties easily follows. So if u and v are two arbitrary vectors it will be

$$u \wedge v = \frac{1}{2!}(u\cdot v - v\cdot u) \qquad (9.4.1)$$

and

$$(u+v, u+v) = (u,u) + (u,v) + (v,u) + (v,v) \qquad (9.4.2)$$

From which, remembering the properties 4) and 1) and the symmetry of the scalar product one obtains

$$(u,v) = \frac{1}{2}(u\circ v + v\circ u) \tag{9.4.3}$$

In particular

$$\begin{aligned}(e_h, e_k) &= \tfrac{1}{2}(e_h\cdot e_k + e_k\cdot e_h) = g_{hk}\\ (e^r, e^s) &= \tfrac{1}{2}(e^r\cdot e^s + e^s\cdot e^r) = g^{rs}\end{aligned} \tag{9.4.4}$$

From eqs. (9.4.1) and (9.4.3) one obtains $(u,v\in U)$

$$u\cdot v = (u,v) + u\wedge v \tag{9.4.5}$$

Property 5) gives

$$\begin{aligned}\frac{1}{p!}p^{hk\cdots}e_h\wedge e_k\wedge\cdots &= \frac{1}{p!}p^{hk\cdots}\left(\frac{1}{p!}\delta^{rs\cdots}_{hk\cdots}e_r\cdot e_s\cdot\cdots\right) =\\ &= \frac{1}{p!}p^{rs\cdots}e_r\cdot e_s\cdot\cdots\end{aligned} \tag{9.4.6}$$

then a Clifford number may be written

$$c = s + v^k e_k + \frac{1}{2!}b^{hk}e_h\cdot e_k + \frac{1}{3!}t^{hkl}e_h\cdot e_k\cdot e_l + \cdots \tag{9.4.7}$$

The computations with Clifford algebra become much more simple if we have an orthogonal basis, i.e. one for which $g_{hk}=0$ for $h\neq k$. In this case eq. (9.4.4) gives

$$e_h\cdot e_k = -e_k\cdot e_h \qquad (h\neq k) \tag{9.4.8}$$

i.e. the Clifford product for orthogonal vectors is skew-symmetric. The orthogonal basis makes simpler the performing of the Clifford product: so

$$\begin{aligned}(e_1\wedge e_2)\cdot(e_1\wedge e_3) &= (e_1\cdot e_2)\cdot(e_1\cdot e_3) = -e_2\cdot(e_1\cdot e_1)\cdot e_3 =\\ &= -g_{11}e_2\cdot e_3 = -g_{12}\,e_2\wedge e_3\end{aligned} \tag{9.4.9}$$

Thus in the four dimensional space-time with the Minkowsky metric have

$$\begin{aligned}(e_0\wedge e_1)\cdot(e_1\wedge e_2\wedge e_3) &= e_0\cdot e_1\cdot e_1\cdot e_2\cdot e_3 = e_0\cdot(-1)\cdot e_2\cdot e_3 =\\ &= -e_0\wedge e_2\wedge e_3\end{aligned} \tag{9.4.10}$$

We are now at position to carry out the Clifford product of two Clifford numbers: we write the Clifford numbers as done in eq. (9.4.7) and then we apply the axioms 1) 2) 3).

Another interesting consequence of the Clifford product is the following: if $u_1, u_2, \ldots u_p$ is a set of p orthogonal vectors, from the condition 4) one find

$$(u_1+u_2+\cdots+u_p, u_1+u_2+\cdots+u_p) = (u_1+u_2+\cdots+u_p)\cdot(u_1+u_2+\cdots+u_p)$$

and then

$$(u_1)^2+(u_2)^2+\cdots+(u_p)^2 = (u_1+u_2+\cdots+u_p)^2 \tag{9.4.12}$$

In words: when the square of a vector is understood as the Clifford product of the vector by itself, then the sum of the square of orthogonal vectors is equal to the square of the sum of the vectors.

This property characterizes the Clifford algebra from a purely algebraic point of view and is often used to define the Clifford algebra (Bourbaky, 1959, p. 139).

Let us consider now those Clifford numbers that are formed by multivectors of even order: these are called even Clifford numbers.

$$\begin{aligned}n=2 \qquad & c = s + b^{12}e_1\cdot e_2\\ n=3 \qquad & c = s + b^{12}e_1\cdot e_2 + b^{23}e_2\cdot e_3 + b^{13}e_1\cdot e_3\end{aligned} \tag{9.4.13}$$

The interesting fact about such even Clifford numbers is that their product gives another even Clifford number, i.e. the even Clifford numbers form a sub-algebra of the whole Clifford algebra.

9.5 Clifford numbers of P^1

Let us consider a one-dimensional space P^1 with the scalar product

$$e_1 \cdot e_1 = (e_1, e_1) = -1 \tag{9.5.1}$$

This will be called pseudo-euclidean one-dimensional space. The Clifford numbers are of the form

$$c = s + v' e_1 \tag{9.5.2}$$

The product of two Clifford numbers c and $\bar{c}$ gives

$$c \cdot \bar{c} = (s + v' e_1) \cdot (\bar{s} + \bar{v}' e_1) = (s\bar{s} - v'\bar{v}') + (v'\bar{s} + \bar{v}' s) e_1 \tag{9.5.3}$$

If we consider the complex number

$$z = s + v' i \tag{9.5.4}$$

we see that the Clifford algebra of P^1 is isomorphic to the algebra of complex numbers. The base vector e_1 behaves as the unit i. Alternatively we may define a complex number as a Clifford number of P^1.

9.6 Clifford numbers of P^2

Let us consider a two-dimensional space P^2 with the pseudo-euclidean metric

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9.6.1}$$

The Clifford numbers are of the form

$$c = s + v' e_1 + v^2 e_2 + b'^2 e_1 \cdot e_2 \tag{9.6.2}$$

i.e. they depend from four parameters. The multiplication table of the algebra is

	1	e_1	e_2	$e_1 \cdot e_2$
1	1	e_1	e_2	$e_1 \cdot e_2$
e_1	e_1	-1	$e_1 \cdot e_2$	$-e_2$
e_2	e_2	$-e_1 \cdot e_2$	-1	e_1
$e_1 \cdot e_2$	$e_1 \cdot e_2$	e_2	$-e_1$	-1

This is identical to the multiplication table of the quaternions (Brand, 1947) where

$$e_1 \to i \qquad e_2 \to j \qquad e_1 \cdot e_2 \to k \tag{9.6.3}$$

Then the Clifford algebra of P^2 is isomorphic to the quaternion algebra. Alternatively one may define a quaternion as the Clifford number of P^2.

Now if we consider the even subalgebra of P^2 i.e. the numbers

$$c = s + b'^2 e_1 \cdot e_2 \tag{9.6.4}$$

their multiplication table is the same of that of complex numbers (Clifford numbers of P^1). Then we may say that the even subalgebra of P^2 is isomorphic with the algebra of P^1 (see table 9.10).

9.7 Clifford numbers of E^3.

A Clifford number of E^3 can be written in the form

$$c = s + v^k e_k + \frac{1}{2!} b^{hk} e_h \circ e_k + \frac{1}{3!} t^{hkl} e_h \circ e_k \circ e_l \qquad (9.7.1)$$

The last term reduces to

$$t^{123} e_1 \circ e_2 \circ e_3 = t^{123} \epsilon \qquad (9.7.2)$$

The 3-vector ϵ enjoys the property

$$\epsilon^2 = -1 \qquad (9.7.3)$$

as can easily be seen performing the products of the e_k.

Then ϵ behaves as the complex number i. Then we have

$$e_1 \circ e_2 = \epsilon \circ e_3 \qquad e_2 \circ e_3 = \epsilon \circ e_1 \qquad e_3 \circ e_1 = \epsilon \circ e_2 \qquad (9.7.4)$$

It follows that a Clifford number of E^3 can be written as follows

$$c = (s + t^{123}\epsilon) + (v^1 + b^{23}\epsilon) \circ e_1 + (v^2 - b^{13}\epsilon) \circ e_2 + (v^3 + b^{12}\epsilon) \circ e_3 \qquad (9.7.5)$$

On account of the property (9.7.3) the terms contained in the round brackets are formally complex numbers. If we denote them by $\phi^0, \phi^1, \phi^2, \phi^3$ we can write a Clifford number of E^3 in the form (Hestenes, 1966, p. 37) (Kahan, 1960, p. 101)

$$c = \phi^0 + \phi^1 \circ e_1 + \phi^2 \circ e_2 + \phi^3 \circ e_3 \qquad (9.7.6)$$

Then a Clifford number of E^3 is characterized by four formally complex numbers. The Clifford algebra of E^3 is also called Pauli algebra (Hestenes, 1966, p. 20) (Corson, 1955, p. 177).

Let us consider an even Clifford number of E^3 if we introduce the following notations

$$e_1 \circ e_2 = k \qquad e_1 \circ e_3 = -j \qquad e_2 \circ e_3 = i \qquad (9.7.7)$$

we can write it in the form

$$c = s + b^{23} i + b^{13} j + b^{12} k \qquad (9.7.8)$$

The Clifford product of two base bivectors gives

$$\begin{aligned} &i \circ i = -1 \qquad j \circ j = -1 \qquad k \circ k = -1 \\ &i \circ j = k = -j \circ i \qquad j \circ k = i = -k \circ j \qquad k \circ i = j = -i \circ k \end{aligned} \qquad (9.7.9)$$

These are the multiplication rules used to define quaternions. Then we conclude that the quaterions may be identified with the <u>even Clifford numbers of E^3</u> (Riesz, 1958) - (Hestenes, 1966).

A Clifford number of E^3 can also be written as follows

$$c = (s + t^{123}\epsilon) + (b^{12} - v^3\epsilon) \circ e_1 \circ e_2 + (b^{13} + v^2\epsilon) \circ e_1 \circ e_3 + (b^{23} + v^1\epsilon) \circ e_2 \circ e_3$$

If we use the three units i, j, k as indicated in eq. (9.7.7) we may write

$$c = \psi^0 + \psi^1 \circ i + \psi^2 \circ j + \psi^3 \circ k \qquad (9.7.11)$$

This is a <u>complex quaternion</u> also called <u>biquaternion</u>. Then the Clifford algebra of E^3 is isomorphic with the algebra of complex quaternions.

9.8 Rotations in E^3.

We now show how simply Clifford algebra permits to describe the composition of finite rotations in E^3.

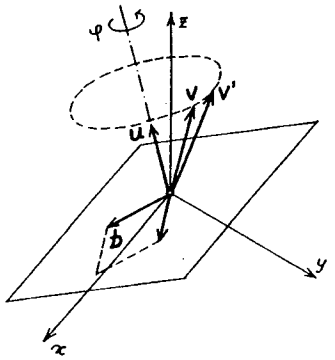
Let us consider a finite rotation of an angle φ around an axis, passing from the origin of the coordinates, described by the unit vector u. We shall conceive the finite rotation φ a ari-

sing from small rotations $\frac{\varphi}{n}$ around the same axis. An arbitrary vector v is rotated to a new vector v' given by the relation

$$v' = v + \frac{\varphi}{n} b \llcorner v \tag{9.8.1}$$

where b is the unit bivector that lies in the plane orthogonal to the axis: $b = u^{\perp}$.

This formula comes from



$$dv = \frac{\varphi}{n} u \times v = \left(\frac{\varphi}{n} u \wedge v\right)^{\perp} = \epsilon \llcorner \left(\frac{\varphi}{n} u \wedge v\right) = \\ = \frac{\varphi}{n} (\epsilon \llcorner u) \llcorner v = \frac{\varphi}{n} b \llcorner v \tag{9.8.2}$$

Now one may show that

$$b \llcorner v = \frac{1}{2}(v \circ b - b \circ v) \tag{9.8.3}$$

Then

$$v' = v + \frac{1}{2}\frac{\varphi}{n} v \circ b - \frac{1}{2}\frac{\varphi}{n} b \circ v = \\ = \left(1 - \frac{1}{n}\frac{\varphi}{2} b\right) \circ v \circ \left(1 + \frac{1}{n}\frac{\varphi}{2} b\right) \tag{9.8.4}$$

where we have neglected the square $\left(\frac{\varphi}{n}\right)^2$. If we now perform a second rotation of the same amount $\frac{\varphi}{n}$ around the same axis we obtain

$$v'' = \left(1 - \frac{1}{n}\frac{\varphi}{2} b\right) \circ v' \circ \left(1 + \frac{1}{n}\frac{\varphi}{2} b\right) = \left(1 - \frac{1}{n}\frac{\varphi}{2} b\right)^2 \circ v \circ \left(1 + \frac{1}{n}\frac{\varphi}{2} b\right)^2 \tag{9.8.5}$$

After n rotations we have

$$v^{(n)} = \left(1 - \frac{1}{n}\frac{\varphi}{2} b\right)^n \circ v \circ \left(1 + \frac{1}{n}\frac{\varphi}{2} b\right)^n \tag{9.8.6}$$

the limit for $n \to \infty$ gives

$$\bar{v} = e^{-\frac{\varphi}{2} b} \circ v \circ e^{\frac{\varphi}{2} b} \tag{9.8.7}$$

This shows that a finite rotation is properly described by a bivector. The composition of two rotations of amounts φ and ψ around two different axes whose unit bivectors are b and β, is given by

$$\bar{v} = e^{-\frac{\psi}{2}\beta} \circ e^{-\frac{\varphi}{2} b} \circ v \circ e^{\frac{\varphi}{2} b} \circ e^{\frac{\psi}{2}\beta} \tag{9.8.8}$$

It can be shown that

$$e^{\frac{\varphi}{2} b} = \cos\left(\frac{\varphi}{2}\right) + b \operatorname{sen}\left(\frac{\varphi}{2}\right) \tag{9.8.9}$$

This is an aggregate of a scalar and a bivector and then is a quaternion (with unit norm). Using eqs. (9.7.9) one may show that the product of two exponentials is another exponential but that the argument is not the sum of the two arguments according with the noncommutative character of the composition of rotations.

9.9 Clifford numbers of space-time

A Clifford number of M^4 may be written in the form

$$c = s + v^{\alpha} e_{\alpha} + \frac{1}{2!} b^{\alpha\beta} e_{\alpha} \circ e_{\beta} + \frac{1}{3!} t^{\alpha\beta\gamma} e_{\alpha} \circ e_{\beta} \circ e_{\gamma} + \\ + \frac{1}{4!} f^{\alpha\beta\gamma\rho} e_{\alpha} \circ e_{\beta} \circ e_{\gamma} \circ e_{\rho} \tag{9.9.1}$$

the last sum reduces to the single term

$$f^{0123}\epsilon \tag{9.9.2}$$

If we choose in M^4 a minkowskian metric with signature +- - - referring to an orthonormal base e_0, e_1, e_2, e_3 we have

$$\epsilon^2 = (e_0 \circ e_1 \circ e_2 \circ e_3)^2 = -1 \tag{9.9.3}$$

as is readly verified performing the Clifford product of the e_α.

Then the four-vector ϵ of an orthonormal base in a Minkowsky space behaves as the immaginary unit i.

The four-vector e defined by eq. in an orthonormal basis becomes

$$e = e^0 \circ e^1 \circ e^2 \circ e^3 = e_0 \circ (-e_1) \circ (-e_2) \circ (-e_3) = -\epsilon \tag{9.9.4}$$

and then

$$e^2 = (-\epsilon^2) = -1 \tag{9.9.5}$$

Let us consider an <u>even</u> Clifford number of M^4 : it can be written in the form

$$c = s + \frac{1}{2!} b^{0k} e_0 \circ e_k + \frac{1}{2!} b^{hk} e_h \circ e_k + t^{0123} e_0 \circ e_1 \circ e_2 \circ e_3 \tag{9.9.6}$$

It has eight components, as many as a Clifford number of E^3. If we perform the substitution

$$\begin{aligned} &e_0 \circ e_k \to e'_k \\ &e_h \circ e_k \to e'_h \circ e'_k \\ &e_0 \circ e_1 \circ e_2 \circ e_3 \to e'_1 \circ e'_2 \circ e'_3 \end{aligned} \tag{9.9.7}$$

one may see that the multiplication table of the base multivectors of the even Clifford numbers of M^4 is the same that the multiplication table of the base multivectors of the Clifford algebra of E^3.

Then the even subalgebra of the Clifford algebra of M^4 is isomorphic with the Clifford algebra of E^3. This isomorphism is shown in table (9.10).

9.10 Ideals

Clifford algebra has two salient features: the product of two Clifford numbers is not commutative and among its elements there are some which do not admit inverse.

The noninvertible elements of an algebra play a great role in the structure of the algebra. Firts of all if r is an element that has not right inverse then for every $c \in C$ the elements

$$r' = r \circ c \tag{9.10.1}$$

have not right inverse, as can be easily shown. This mean that from one noninvertible element one may obtain other noninvertible elements. The set R of elements rc is called <u>right invariant subalgebra</u>, or, more briefly <u>right ideal</u>. The element r is called the <u>generator</u> of the right ideal R.

Analogous considerations can be done starting with an element that has not a left inverse: we obtain in this way a <u>left ideal</u>.

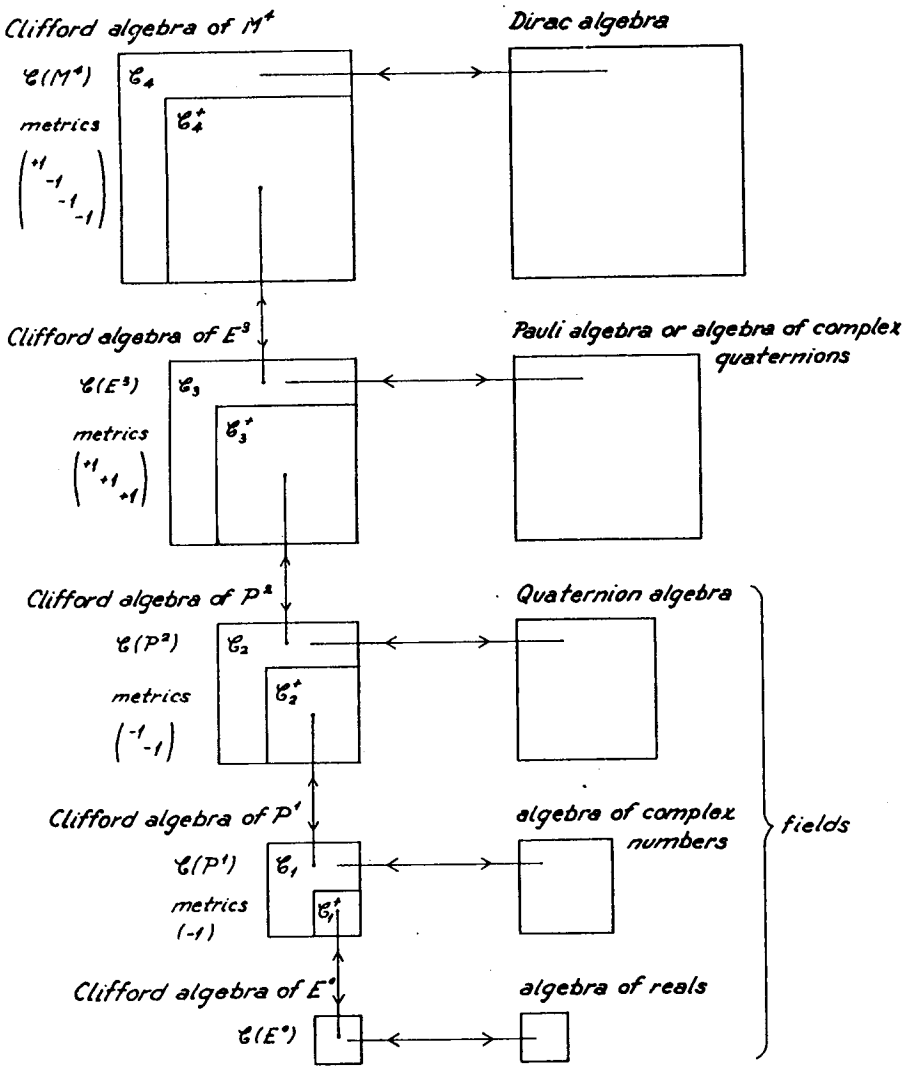
These concepts are not peculiar of the Clifford algebra but are common to all associative algebras. In particular in matrix algebra they become transparent. So let us consider the 2 x 2 matrices with real or complex elements. A noninvertible matrix is, for ex.

$$r = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} \tag{9.10.2}$$

and all the matrices

$$\begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a + \beta c & \alpha b + \beta d \\ 0 & 0 \end{pmatrix} \tag{9.10.3}$$

9.10 *Clifford algebras and their isomorphic algebras*



are of the same nature. Then they form a right ideal.

9.11 Idempotents

A particular kind of elements is that of the elements q that enjoy the property

$$q^2 = q \tag{9.11.1}$$

these are called idempotents. In particular the unit 1 is an idempotent. Excluding this particular case it is easily seen that an idempotent is noninvertible. In the algebra of 3 x 3 matrices examples of idempotents are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{9.11;2}$$

If q is idempotent also (1-q) is idempotent:

$$(1-q)^2 = 1-2q+q^2 = 1-2q+q = 1-q \tag{9.11.3}$$

Two idempotents q_1 and q_2 such that

$$q_1 \cdot q_2 = 0 \qquad q_2 \cdot q_1 = 0 \tag{9.11.4}$$

are said indipendent. (Weyl, 1950, p. 292). The idempotents q and $1-q$ are indipendent:

$$(1-q)\cdot q = 0 \qquad q\cdot(1-q) = 0 \tag{9.11.5}$$

The main use of idempotents is to generate left or right ideals.

The right ideal R generated by an idempotent q i.e. the set of elements

$$r = q \cdot c \tag{9.11.6}$$

enjoys the property

$$(1-q)\cdot r = 0 \tag{9.11.7}$$

The equations can be considered as a kind of compatibility condition for eq.(9.11.6).

The corresponding idempotent $(1-q)$ generates another right ideal

$$r' = (1-q)\cdot c \tag{9.11.8}$$

from which

$$q\cdot l = 0 \tag{9.11.9}$$

Since every Clifford number c can be written in the form

$$c = q\cdot c + (1-q)\cdot c \tag{9.11.10}$$

we see that

$$c = r + r' \tag{9.11.11}$$

i.e. a Clifford number can be decomposed in the sum of two elements that belong to two right ideals. On account of the property

$$q\cdot r = q\cdot(q\cdot c) = q^2\cdot c = q\cdot c = r \tag{9.11.12}$$

the idempotent behaves as the left unit element in the right ideal generated by it. An idempotent is said <u>primitive</u> (Weyl, 1950, p.293) or <u>irreducible</u> if it cannot be decomposed in the sum of two idempotents q_1 and q_2 i.e. if

$$q = q_1 + q_2 \tag{9.11.13}$$

implies that q_1 or q_2 vanishes. If q is primitive the ideal generated by it is <u>minimal</u>, i.e. does not contain other proper ideals.

The subspace formed by the ideal is said <u>irreducible</u>.

9.12 Matrix representation of the Clifford algebra

In the order to find the representation of an algebra a general procedure is the following. Let us choose a basis $E_1, E_2, \ldots E_p$ where p is the dimension of the algebra. The products $E_k\cdot c$ are again element of the algebra and therefore must be expressible linearly in terms of the E_k

$$\begin{cases} E_1\cdot a = \alpha_{11}E_1 + \alpha_{12}E_2 + \cdots + \alpha_{1p}E_p \\ E_2\cdot a = \alpha_{21}E_1 + \alpha_{22}E_2 + \cdots + \alpha_{2p}E_p \\ \cdots = \cdots \\ E_k\cdot a = \alpha_{p1}E + \alpha_{p2}E_2 + \cdots + \alpha_{pp}E_p \end{cases} \tag{9.12.1}$$

As can be easily shown the matrix $|\alpha_{hk}|$ gives a representation of the algebra that is called the <u>regular representation</u>.

To prove this, let us denote with a and b two elements of an associative algebra. If we put

$$\begin{aligned} E_h\cdot a &= \alpha_h{}^k E_k & E_h\cdot b &= \beta_h{}^k E_k \\ c &= a\cdot b & E_h\cdot c &= \gamma_h{}^k E_k \end{aligned} \tag{9.12.2}$$

it will be

$$E_h\cdot c = E_h(a\cdot b) = (E_h\cdot a)\cdot b = (\alpha_h{}^k E_k)\cdot b = \alpha_h{}^k\beta_k{}^\ell E_\ell \tag{9.12.3}$$

then

$$\gamma_h{}^\ell = \alpha_h{}^k\beta_k{}^\ell \tag{9.12.4}$$

i.e. to the product of the two elements a and b there corresponds a matrix that is the product of the matrices that represent a and b respectively.

To give an example, we consider the matrix representation of the algebra of complex numbers.

A base is formed by $E_1 = 1$, $E_2 = i$.

Since

$$\begin{cases} 1\cdot(x+iy) = 1x + iy \\ i\cdot(x+iy) = -y + ix \end{cases} \tag{9.12.5}$$

the matrix

$$\hat{z} = \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \qquad (9.12.6)$$

gives the regular representation of the algebra of complex numbers.

To find the matrix representation of the Clifford algebra of P^2 we choose as a basis $E_1 = 1$, $E_2 = e_1$, $E_3 = e_2$, $E_4 = e_1 \cdot e_2$. Then

$$\begin{cases} 1 \circ (a + be_1 + ce_2 + de_1 \cdot e_2) = +a + be_1 + ce_2 + de_1 \cdot e_2 \\ e_1 \circ (a + be_1 + ce_2 + de_1 \cdot e_2) = -b + ae_1 - de_2 + ce_1 \cdot e_2 \\ e_2 \circ (a + be_1 + ce_2 + de_1 \cdot e_2) = -c1 + de_1 + ae_2 - be_1 \cdot e_2 \\ (e_1 \cdot e_2) \circ (a + be_1 + ce_2 + de_1 \cdot e_2) = -d1 - ce_1 + be_2 + ae_1 \cdot e_2 \end{cases}$$

The matrix

$$\begin{pmatrix} +a & +b & +c & +d \\ -b & +a & -d & +c \\ -c & +d & +a & -b \\ -d & -c & +b & +a \end{pmatrix} \qquad (9.12.8)$$

gives the regular representation of the Clifford algebra of P^2 or, on account of the isomorphism, of the quaternion algebra. This matrix representation was found for the first time by Caley.

Proceding in analogous way one find the regular representation of every algebra as the Clifford algebra of E^3 and M^4.

Since the Clifford algebras all contain the unit element those representations are all faithful. The order of the matrices so obtained is equal to that of the algebra.

There may exist other representations of the algebra that are of lower degree: these are obtainad considering a right ideal

of the algebra. If we apply to a right ideal of an algebra the same procedure indicated above using a base in the ideal we find another representation of the algebra that is of lower degree.

In the Clifford algebra of E^3 to generate a right ideal we take the idempotent

$$q = \frac{1 + e_3}{2} \qquad (9.12.9)$$

The elements of the right ideal generated by it are (a,b,c,d, are complex numbers)

$$r = \frac{1 + e_3}{2}(a + be_1 + ce_2 + de_3) = \qquad (9.12.10)$$

$$= \frac{a + d}{2} \quad \frac{b - ic}{2} e_1 + \frac{c + ib}{2} e_2 + \frac{d + a}{2} e_3$$

or

$$r = A\left[\frac{1 + e_3}{2}\right] + B\left[\frac{(1 + e_3) \cdot e_1}{2}\right] \qquad (9.12.11)$$

One may choose as base vectors of the right ideal

$$E_1 = \frac{1 + e_3}{2} \qquad E_2 = \frac{(1 + e_3) \cdot e_1}{2} \qquad (9.12.12)$$

and then

$$\left[\frac{1 + e_3}{2}\right] \circ (P + Qe_1 + Re_2 + Se_3) = (P + S)\left[\frac{1 + e_3}{2}\right] + (Q - iR)\left[\frac{(1 + e_3) \cdot e_1}{2}\right]$$

$$\left[\frac{(1 + e_3) \cdot e_1}{2}\right] \circ (P + Qe_1 + Re_2 + Se_3) = (Q + iR)\left[\frac{1 + e_3}{2}\right] + (P - S)\left[\frac{(1 + e_3) \cdot e_1}{2}\right]$$

The matrix

$$C = \begin{pmatrix} P + S & Q - iR \\ Q + iR & P - S \end{pmatrix} \qquad (9.12.14)$$

gives a representation of the Clifford algebra of E^3. This representation is of degree two on the complex number field while that obtained with the regular representation is of degree eight on the real field. In particular the base vectors are represented by the matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9.12.15}$$

respectively, that are the Pauli matrices used in the context of the nonrelativistic quantum mechanics of spinning particles.
If we put

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{9.12.16}$$

the matrix C can be written

$$C = \sigma_\alpha \phi^\alpha. \tag{9.12.17}$$

To obtain the reduced representation of the Clifford algebra of space-time we start observing that the element

$$q = \frac{1 + e_3 \cdot e_0}{2} \tag{9.12.18}$$

is an idempotent. Then the elements (a,b,c ...p = complex numbers)

$$r = \left[\frac{1+e_3\cdot e_0}{2}\right]\cdot(a + be_0 + ce_1 + de_2 + he_3 + me_0\cdot e_1 + ne_0\cdot e_2 + pe_0\cdot e_3)$$

form a right ideal. Performing the product one may see that these elements have the form

$$r = A\left(\frac{1+e_3\cdot e_0}{2}\right) + B\left(\frac{e_1\cdot(e_0+e_3)}{2}\right) + C\left(\frac{e_0+e_3}{2}\right) + D\left(\frac{1+e_3\cdot e_0}{2}\right)\cdot e_1$$

where A,B,C,D are four complex numbers. A basis of the right ideal is then

$$E_1 = \frac{1+e_3\cdot e_0}{2} \qquad E_2 = \frac{e_0\cdot(e_0+e_3)}{2}$$
$$E_3 = \frac{e_0+e_3}{2} \qquad E_4 = \frac{(1+e_3\cdot e_0)\cdot e_1}{2} \tag{9.12.21}$$

whose multiplication table is

	E_1	E_2	E_3	E_4
E_1	E_1	E_2	E_3	E_4
E_2	0	0	0	0
E_3	0	0	0	0
E_4	E_4	$-E_3$	$+E_2$	$-E_1$

with these base elements one finds the representation of the Clifford algebra of the space-time: in particular the matrix that represents e_0 is

$$\begin{cases} E_1\cdot e_0 = 0E_1 + 0E_2 + 1E_3 + 0E_4 \\ E_2\cdot e_0 = 0E_1 + 0E_2 + 0E_3 + 1E_4 \\ E_3\cdot e_0 = 1E_1 + 0E_2 + 0E_3 + 0E_4 \\ E_4\cdot e_0 = 0E_1 + 1E_2 + 0E_3 + 0E_4 \end{cases} \tag{9.12.22}$$

in this way one discovers the matrix

$$\gamma_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix} \tag{9.12.23}$$

In analogous way one finds

$$e_k \longrightarrow \gamma_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \tag{9.12.24}$$

where the σ_k are the Pauli matrices. The matrices γ_μ that represent the base vectors e_μ are the well-known Dirac matrices used in the relativistic quantum mechanics of spin 1/2 particles.

The matrices associated with a simple multivector are the following: if $p = u_1 \wedge u_2 \wedge \cdots \wedge u_p$ one account of the relation (9.3.6) the corresponding matrix is (Cartan, 1966, p. 83).

$$p = \frac{1}{p!}\delta^{hki\cdots}_{123\cdots}\gamma_h\gamma_k\gamma_i\cdots \qquad (9.12.25)$$

In particular to the multivectors ϵ and e there correspond the matrices

$$\begin{aligned} \epsilon &\longrightarrow \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 \\ e &\longrightarrow \gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 \end{aligned} \qquad (9.12.26)$$

The right and left supplementary of a base vector e_μ are represented by the matrices

$$\begin{aligned} (e_\mu)^\perp = \epsilon \circ e_\mu &\longrightarrow \gamma_5\gamma_\mu \\ {}^\perp(e_\mu) = e_\mu \circ e &\longrightarrow \gamma_\mu\gamma^5 \end{aligned} \qquad (9.12.27)$$

Proceeding in this way one may recover many formulae used in the relativistic theory of elementary particles, with gain in the geometrical insight (Hestenes, 1966).

9.13 Spinors

The matrices that give an irreducible representation of the Clifford algebra have complex entries: they may be conceived as operators on a complex vector space, called the spin space. The elements of the spin space are called spinors.

So the 2 x 2 complex Pauli matrices σ_k may be conceived as operators on a two-dimensional vector space: its vectors with two complex components are called two-component spinors. The 4 x 4 complex Dirac matrices γ_μ may be considered as operators on a four-dimensional vector space whose elements are called four-component spinors.

Then spinor calculus appears as a natural consequence of the Clifford algebra.

9.14 The immaginary unit

The unit pseudoscalar ϵ for spaces of one, two, three and four dimensions satisfies the following relations

$$\begin{aligned} &P^1 \text{ (pseudoeuclidean metric)} && \epsilon^2 = -1 \\ &P^2 \text{ (pseudoeuclidean metric)} && \epsilon^2 = -1 \\ &E^2 \text{ (euclidean metric)} && \epsilon^2 = -1 \\ &E^3 \text{ (euclidean metric)} && \epsilon^2 = -1 \\ &M^4 \text{ (pseudoeuclidean metric)} && \epsilon^2 = -1 \end{aligned} \qquad (9.14.1)$$

In particular we notice that this is true in space-time on account of the signature $+---$; in an euclidean space-time would be

$$\epsilon^2 = +1 \qquad (9.14.2)$$

Then for there spaces the n-vector ϵ behaves as the immaginary unit i. This fact may suggest that the apparence of the immaginary unit in physical theories arises to take into account directional properties. So, as is well known, the immaginary unit i in electricity is linked to the rotation of $\frac{\pi}{2}$ of the axes in E^2.

Moreover the property of the Pauli matrices

$$\sigma_x\sigma_y = i\sigma_z \qquad (9.14.3)$$

states that the supplementary of the vector e_z is the bivector $e_x \wedge e_y$ because relation (9.14.3) is equivalent to

$$e_x \wedge e_y = \epsilon \llcorner e_z \qquad (9.14.4)$$

or that is the same

$$e_x \times e_y = e_z \qquad (9.14.5)$$

These are examples that show that the colourless symbol i often hides the geometrical content of a rotation.

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