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The reason for analogies between physical theories

E. Tonti

Istituto di Matematica del Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy
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An investigation on the reason for analogies between physical theories is carried out. The reason is found in the fact that in every physical theory there are basic physical quantities which are referred to geometrical and chronometrical elements. The analysis is pursued further using the rudiments of algebraic topology, i.e. the notions of 'chain' and that of 'coboundary'. A rational classification scheme for the physical variables and equations of whatever physical theory is then constructed. Some examples of these schemes are given.

Introduction

It is a common experience that there are different physical theories which exhibit formal similarities, i.e. well known analogies. Analogies exist between physical theories even though they may differ profoundly in physical content.

Essentially an analogy consists of a one-to-one construction between the physical quantities and the equations of a theory and those of another theory. This is usually described giving a correspondence table (see *Table 1*).

It may happen that the physical variables which correspond to one another are of the same mathematical nature, i.e. both scalars, both vectors, both tensors, etc. In this case the equations connecting two corresponding physical variables are not only similar but are the *same* equations.

This is the case for analogies between the stationary thermal conduction, the electric conduction, the electrostatic field and the hydrodynamics (stationary, irrotational flow), which are all described by Laplace equations. This leads to the use of harmonic functions and, for plane problems, to the use of the functions of one complex variable. These analogies are largely employed as a device in obtaining the solution of a problem using the known solution of the corresponding problem of another theory.

But there are analogies between physical theories in which the one-to-one construction is between two physical variables that are not of the same mathematical nature. Thus one may be a scalar the other a vector, one may be a vector and the other a tensor, etc. The analogy between the theory of dislocations in solid mechanics and magnetism is of this kind. It happens that these analogies cannot be used for solution purposes because the corresponding equations are not the same. Nevertheless, they can be used, and are used, as a means of transferring mathematical methods and numerical techniques from one theory to another.

In the study of a physical theory we are consciously inspired by the experience made in other physical theories. This is typical of all sciences but is especially valid for physics.

It is apparent that if we had a greater understanding of the origin of analogies between different physical theories we could make systematic use of them. This would permit us to transfer information, mathematical techniques, numerical experiences, etc. from one theory to another. Moreover, one can construct a unique mathematical model for many physical theories in which the common features are exhibited once and for all, the main properties are investigated with a great economy of mental labour, time and money. This is apart from the intellectual pleasure of the synthesis.

All these facts lead us to raise the question: what is the origin of the analogies?

In this paper we attempt to answer this question. Moreover, we show how one may construct a classification scheme for the physical quantities and equations of every physical theory. This classification scheme has led in a natural way to the construction of a mathematical model valid for a large class of physical theories[1,2].

Origin of the analogies

At the foundation of every physical theory we introduce some measurable physical quantities from which, in subsequent stages, other physical quantities are derived. It is at this initial stage that one makes the important observation contained in the following statement:

In every physical theory there are basic physical quantities that are naturally referred to the most simple geometrical and chronometrical elements such as points, lines, surfaces, volumes, time instants and time intervals and combinations of them (A).

Table 1
Comparison among some physical theories

Physical theory	Equations of structure	Constitutive equations	Equations of structure
electrostatic field	$E_k = -\partial_k \varphi$	$D^h = \epsilon^{hk} E_k$	$\partial_h D^h = \rho$
magnetostatic field	$B^h = \varepsilon^{hkl} \partial_k A_l$	$B^h = \mu^{hk} H_k$	$\varepsilon^{mlk} \partial_l H_k = j^m$
electromagnetic field	$\begin{cases} E_k = -\partial_k \varphi - \partial_t A_k \\ B^h = \varepsilon^{hkl} \partial_k A_l \end{cases}$	$\begin{cases} D^h = \epsilon^{hk} E_k \\ B^h = \mu^{hk} H_k \end{cases}$	$\begin{cases} \partial_h D^h = \rho \\ -\partial_t D^m + \epsilon^{mlk} \partial_l H_k = j^m \end{cases}$
perfect fluid motion	$\begin{cases} \alpha = -\partial_t \varphi \\ v_k = \partial_k \varphi \end{cases}$	$\begin{cases} \rho = f(d) \\ p^h = \rho(\alpha) a^{hk} v_k \end{cases}$	$\partial_t \rho + \partial_h p^h = 0$
thermal field	$p_k = \partial_k \dfrac{1}{T}$	$\begin{cases} q^h = k(T) a^{hk} p_k \\ \sigma = \rho c_v \partial_t T \end{cases}$	$\partial_h q^h = \sigma$
irreversible thermodynamics	$\mathcal{F}_{Ak} = \nabla_k f_A$	$\mathcal{J}^{Bh} = L^{BhAk} \mathcal{F}_{Ak}$	$\nabla_h \mathcal{J}^{Bh} = \sigma^B$
analytical dynamics	$v^A = \dfrac{D}{Dt} q^A$	$\begin{cases} p_B = \gamma_{BA} v^A \\ Q_B = Q_B(q) \end{cases}$	$\dfrac{D}{Dt} p_B = Q_B$
longitudinal vibrations	$\begin{cases} v = \partial_t S \\ \epsilon = \partial_x S \end{cases}$	$\begin{cases} p = \rho v \\ \sigma = E\epsilon \end{cases}$	$\partial_t p - \partial_x \sigma = f$
elastostatics (small displacement)	$\gamma_h^A = \nabla_h u^A$	$p_B^k = A_{BA}^{kh} \gamma_h^A$	$-\nabla_k p_B^k = f_B$
elastodynamics (small displacement)	$\begin{cases} \gamma_h^A = \nabla_h u^A \\ v^A = \partial_t u^A \end{cases}$	$\begin{cases} p_B^k = A_{BA}^{kh} \gamma^{Ah} \\ p_B = \rho a_{BA} v^A \end{cases}$	$\partial_t p_B - \nabla_h p_B^k = f_B$
Schrödinger field	$u_k = \partial_k \psi$	$V^h = \dfrac{\hbar^2}{2m} a^{hk} u_k$	$\partial_h v^h = \sigma$
meson field	$u_\alpha = \partial_\alpha \psi$	$\begin{cases} q^B = \hbar^2 g^{\beta\alpha} u_\alpha \\ \sigma = -(m_0 c)^2 \psi \end{cases}$	$\partial_\beta q^\beta = \sigma$

Symbols and names are explained in the corresponding Tables at the end of the paper. $h, k, l, i, \ldots = 1, 2, 3$; $\alpha, \beta, \gamma, \ldots = 0, 1, 2, 3$; $x^0 = ct$; ∇_k = covariant derivative; D = absolute differential; $Dt = dt$; a_{hk} = metric tensor of the space; $g_{\alpha\beta}$ = metric tensor of space–time with signature $(+ - - -)$; $\partial_t = \dfrac{\partial}{\partial t}$, $\partial_k = \dfrac{\partial}{\partial_x k}$; ϵ^{hkl} is the Ricci tensor $(a)^{1/2}, 0, -(a)^{1/2}$.

Let us cite some examples: we consider mass or electric charge contained in a *volume* at a given *time instant*, of the probability of finding a particle in a *volume* at a given *time instant*. We consider the electric potential at a *point* at a given *time instant*, of the displacement of a *point* of a material continuum during a *time interval*. We consider the position vector of a particle at a given *time instant*, of the impulse given to the particle during a *time interval*. We consider the electric flux through a *surface*, of the energy and momentum flux through a *surface* during a *time interval*, of the internal energy and entropy production in a *volume* during a *time interval*.

We emphasize the fact that the physical quantities of different physical theories which are associated with the same geometrical entity may be different mathematically. Hence with a surface one may associate respectively:

magnetostatics: ⟶⟶ magnetic flux ⟶⟶ scalar

continuum mechanics: ⟶ contact force ⟶⟶ vector

mechanics of polar continua: ⟶⟶ { contact force and contact couple ⟶ { motor or complex vector }

As a natural consequence of the statement A there is the following statement:

In every physical theory there are basic physical laws which state that a physical quantity referred to a p*-dimensional manifold* ω *such as lines, surfaces, volumes, time intervals, etc. is equal to a physical quantity referred to its boundary* $\partial\omega$ (B).

Typical laws of this kind are those expressed by *balance equations*, in particular *continuity equations, equilibrium equations, equations of motion, circuital equations*, and

compatibility equations; the equations that give the *general solution* of one of the preceding equations; the *equations defining the gradients;* etc.

Examples of balance laws are: in magnetostatics the statement that the sum of the magnetic fluxes through the boundary of a volume vanishes; in continuum mechanics the law of equilibrium that states that the sum of the forces acting on the boundary of a volume and of the external forces acting on that volume vanishes. The principle of conservation of energy states that the outgoing energy flux through the boundary of a volume during a time interval plus the energy stored in the volume in the same time interval is equal to the energy produced in the volume in the time interval considered. This law can be restated, with reference to space–time, by saying that the outgoing energy flux through the three-dimensional boundary of a four-dimensional region is equal to the energy produced inside it.

As an example of circuital law we mention the Ampère circuital law: it states that the magnetomotive force along the boundary of a surface is equal to the current flowing through the surface. Often circuital equations arise as compatibility conditions of gradient-like equations. So in fluid dynamics the condition that the circulation of the velocity vector along a closed line vanishes implies that the velocity vector is the gradient of a scalar function (velocity potential). In the thermodynamic configuration space the statement of the vanishing of the circuital of the vector field, whose components are the 'intensive' variables, along a closed line (Maxwell reciprocity relations) amounts to the statement of the existence of entropy.

Perhaps this connection between physics and geometry has been well known for a long time. But it seems that it has not been realized how far one can go in the understanding of the formal structure of physical theories by exploring the consequence of this remarkable connection.

One of the consequences of this connection is the possibility of rationally investigating the analogies between two physical theories according to the following criterion: to every physical quantity of one theory there corresponds that physical quantity of the other theory which is referred to the same geometrical entity. Comparing the perfect fluid motion with the magnetic field one may see that the analogy of the vorticity vector $\vec{w}$ is the magnetic induction $\vec{B}$ because both are associated with a surface. The velocity vector $\vec{v}$ in fluid dynamics corresponds to the magnetic vector potential $\vec{A}$ in magnetism because both are referred to lines. Such analogies are easily detected by comparing the classification schemes of the various physical theories exhibited in the Tables at the end of this paper.

Preliminary classification of physical variables

Configuration-type variables

Since we propose to study the link between physical variables and basic geometrical elements we need a preliminary examination of the main physical variables and equations of a physical theory. This is a difficult task because of the great difference in terminology of different physical theories: whatever name we choose to denote a typical entity of a theory we are sure to be exposed to criticism. Since we must start somewhere, we decided to use as far as possible the terminology of mechanics, both analytical mechanics and continuum mechanics. The reason is that many physical theories are modelled on mechanics, among them field theories.

One of the basic notions of mechanics is that of 'configuration' of a mechanical system. In discrete mechanical systems the configuration is described by a set of n generalized coordinates, universally denoted by q^k. In continuum statics the configuration is described giving the displacement vector of every point of the continuum: this is almost universally denoted by $\vec{u}$. The coordinates q^k and the vector $\vec{u}$ will be called 'configuration variables' of the respective mechanical systems. In field theories the analogies of the mechanical configuration variables are the field functions (also called field variables). The variables that describe the configuration of a system or of a field will be called *configuration variables.*

So the configuration variable of an electromagnetic field is the vector potential A_μ; the configuration variable of the gravitational field in the relativistic description is the metric tensor $g_{\mu\nu}$. The configuration variables of a thermodynamical system may be the intensive variables. In quantum mechanics the configuration variable of a quantum–mechanical system is the state vector ψ that gives the probability amplitude.

In a physical theory, once we have chosen the configuration variables and the independent variables, we may consider those variables that are linked to the configuration variables by means of operations of sum and difference, of total or partial derivatives and total or partial integrals with respect to the independent variables – without the intervention of physical constants, material parameters, coupling constants, phenomenological coefficients and other parameters linked to the geometry or the physics of the system. An exception is made for the speed of light *in vacuo.* These variables, including the configuration variables themselves, will be called *configuration-type variables.*

In continuum mechanics typical configuration-type variables are the geometrical and kinematical variables such as the displacement, strain, velocity, deformation gradient, angular velocity, vorticity, rate of deformation, etc.

Source-type variables

Another basic entity of mechanics is the force concept. In continuum statics the body force is considered as a source of the change of the configuration of the system. In particle mechanics the force is the source of the motion of the particle. The notion of 'force' is replaced in other physical theories by that of source of a field. So the electric charges are the source of the electrostatic field and the electric currents are the source of the magnetic field. The variables that describe the source of a field or of a phenomenon will be called *source variables.*

Let us consider those variables that are linked to the source variables by means of operations of sum and difference, of total or partial derivative and total or partial integrals with respect to the independent variables – without the intervention of physical constants, material parameters, coupling coefficients, phenomenological constants or any other parameter linked to the geometry or the physics of the system, with the exception of the speed of light *in vacuo.* All these variables, including the source variables themselves, will be called *source-type variables.*

In continuum mechanics typical source-type variables are static and dynamic variables such as body force, body couple, momentum, angular momentum, stress, stress functions, stream functions, etc.

The choice of a set of configuration variables and a set of source variables in a physical theory is not unique: reasons of convenience may lead to prefer one choice to another.

Typical pairs of configuration and source variables are: the Lagrangian coordinates q^k and the generalized forces Q_k of analytical mechanics; the electrostatic potential and the electric charge density in electrostatics; the displacement vector and the body force vector in continuum mechanics; the metric tensor $g_{\mu\nu}$ and the stress energy–momentum tensor $T^{\mu\nu}$ in the relativistic gravitational theory; and the state vector ψ and the interaction term σ in quantum mechanics.

However, other choices are possible: thus in analytical mechanics one may choose the momenta p_k and the generalized velocities v^k as configuration and source variables respectively. In continuum mechanics one may choose the stress potentials as configuration variables and the dislocation tensor as a source variable. In magnetostatics one may choose the pair vector potential (A_μ) – current density $(\mathfrak{J}^\mu)$ or the pair scalar potential (χ) – monopole charge density $(\rho_{(m)})$ as configuration and source variables respectively.

In the general field theory, a theory which essentially rests upon formal properties common to many field theories, the configuration variables are the field functions, usually denoted by ψ_a. Usually no explicit mention is made of the source variables that, in the case of interacting fields are implicitly included in the interaction Lagrangian.

In the mathematical nature of the configuration and source variables we find the largest variety of types: they may be real or complex numbers, vectors, tensors, quaternions, motors, multivectors, Clifford numbers, matrices, spinors, operators, etc.

Independent variables

In continuous systems and fields the configuration and source variables depend on some independent variables such as the time and space coordinates. When the physical system under study exhibits some symmetry (plane symmetry, spherical symmetry, etc.) one may use a lower number of variables. In other cases one may treat one or more variables as parameters, typically time. One may consider those independent variables as coordinates of a point of a space.

When we say that in every theory there are some physical quantities that are naturally referred to the geometrical elements of a space we refer to the space of independent variables. Since the choice of independent variables is not unique it follows that the same physical quantity may be referred to different geometrical elements in different spaces. Hence the electric potential φ will be referred to points of the three-dimensional physical space or to time lines of the four-dimensional space–time.

Other physical variables

In every physical theory one encounters other physical variables that are defined as functions of the configuration-type and source-type variables. So in particle mechanics one defines the potential energy V and the Lagrangian of a free particle respectively as:

$$V \triangleq -\int_0^{\vec{r}} \vec{f}(\vec{r}) \cdot \mathrm{d}\vec{r}; \qquad L_p \triangleq \int_0^{\vec{v}} \vec{p}(\vec{v}) \cdot \mathrm{d}\vec{v} \tag{1}$$

They depend on the source-type variables $\vec{f}, \vec{p}$ and the configuration-type variables $\vec{r}, \vec{v}$. Once the constitutive equations are given one may evaluate these physical variables. So when:

$$\vec{p} = m\vec{v} \quad \text{or} \quad \vec{p} = m_0 \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \vec{v} \tag{2}$$

we obtain:

$$L_p = \frac{1}{2} m v^2 \quad \text{and} \quad L_p = -m_0 c^2 \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} \tag{3}$$

which are the typical expressions for the Lagrangian of a free particle of classical and relativistic mechanics respectively. In electrostatics and in magnetostatics one introduces the variables:

$$w_{(e)} \triangleq \int_0^{\vec{D}} \vec{E}(\vec{D}) \cdot \mathrm{d}\vec{D}; \qquad w_{(m)} \triangleq \int_0^{\vec{B}} \vec{H}(\vec{B}) \cdot \mathrm{d}\vec{B} \tag{4}$$

which are defined as functions of the source and configuration-type variables of the respective theories. Once we specify the constitutive equations, we obtain these new variables as a function of the configuration-type variables only. Another example of physical variables of the electrostatic field is the Maxwell tensor giving the Maxwellian stresses:

$$t^h_{\cdot k} = D^h E_k - \frac{1}{2}(D^s E_s)\delta^h_k \tag{5}$$

This physical variable is a function of the two variables $\vec{E}$ and $\vec{D}$. Once we specify the phenomenological equation $\vec{D} = \vec{D}(\vec{E})$ the stress tensor becomes a function of the configuration-type variable only.

Equations of structure

The equations that link the various configuration-type variables of a single physical theory do not contain physical constants, nor material parameters, as we have said. These equations contain algebraic sums, time or space derivatives, time or space integrals, and their linear combinations. Typical are the equations defining the gradients, those arising from circuital laws and compatibility equations. Of this kind are the equations:

$$\vec{B} = \operatorname{curl} \vec{A}; \qquad \vec{v} \triangleq \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}; \qquad \epsilon^h_{\cdot k} \triangleq \nabla_k u^h \tag{6}$$

Also the equations that link the various source-type variables of a theory are of this kind. Typical are the balance equations and the conservation laws, for example the equations:

$$-\nabla_k t_h{}^k = f_h; \qquad \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \vec{f}; \qquad \iint_{\partial\omega} \vec{B} \cdot \vec{n}\, \mathrm{d}S = 0 \tag{7}$$

Equations of this kind do not have a general name in the physical literature. Since in the following we shall deal mainly with this kind of equation we use, tentatively, a single name and call them equations of structure, or structural equations.

We shall show that the equations of structure have a geometrical origin, more properly described using the elementary notions of algebraic topology. Such notions are that of 'chain' and of 'coboundary' explained later. There it will be shown that the 'structural equations' are

realizations of a single linear process, that of forming the 'coboundary' of a 'chain'. This will give a reason for the linearity of the structural equations.

Phenomenological equations

Configuration- and source-type variables of the same physical theory are linked by *constitutive* equations, also called material equations or equations of state. They include physical constants, geometrical and material parameters.

The interaction between two phenomena is described by giving the link between the variables of one theory and the variables of the other. These are the interaction equations that contain coupling coefficients.

Both the constitutive equations and the interaction equations are of the phenomenological kind and are usually called phenomenological equations.

While the structural equations are always linear, the phenomenological equations are generally non-linear. Moreover, they can be differential or integrodifferential, as in the theory of hereditary response (hereditary materials, optical dispersion).

Other equations

The equations of structure and the phenomenological equations do not exhaust, of course, the equations arising in a physical theory. But they are the building blocks of all equations of a physical theory. Hence the Poisson equation:

$$\operatorname{div}[\varepsilon(-\operatorname{grad}\varphi)] = \rho \qquad (8)$$

arises as a mixing of two equations of structure with one phenomenological equation, i.e.

$$\operatorname{div}\vec{D} = \rho; \qquad \vec{D} = \varepsilon\vec{E}; \qquad \vec{E} = -\operatorname{grad}\varphi \quad (9)$$

In general the wave equations and the field equations are obtained by combining phenomenological and structural equations.

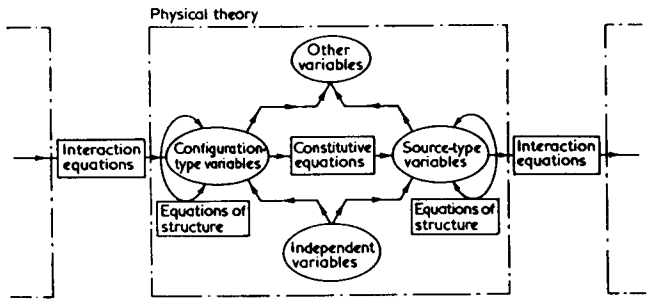
All the foregoing can be summarized in the block diagram of *Table 2*.

Chains and the coboundary process in physics

Cell complex

In order to give a rational classification of the basic geometrical and chronometrical elements we shall consider a region Ω of a space $\mathbf{R}^n$. This may be the three-dimensional space ($n = 3$); the one-dimensional time axis ($n = 1$); the four dimensional space–time ($n = 4$) or any other space whose points are the n-ples of coordinates formed by the independent variables used in a physical theory. To fix our ideas we consider a region of $\mathbf{R}^3$. To display the geometrical elements of $\mathbf{R}^3$ it is expedient to subdivide the region Ω into three-dimensional cells whose faces are formed by the coordinate surfaces of a coordinate system x^1, x^2, x^3. Every cell is composed of vertices, edges and faces that, in turn may be considered as zero, one, two-dimensional cells respectively. Then we have four geometrical elements that are called 0-cells, 1-cells, 2-cells, 3-cells respectively.

Table 2 Preliminary classification of physical variables of a physical theory



In a time axis a cell complex is formed by 0-cells (time instants) and 1-cells (time intervals). In space–time the 0-cells are the events. Since spaces with $n \leqslant 4$ are the most commonly used we shall use special symbols for the cells of the various dimensions.

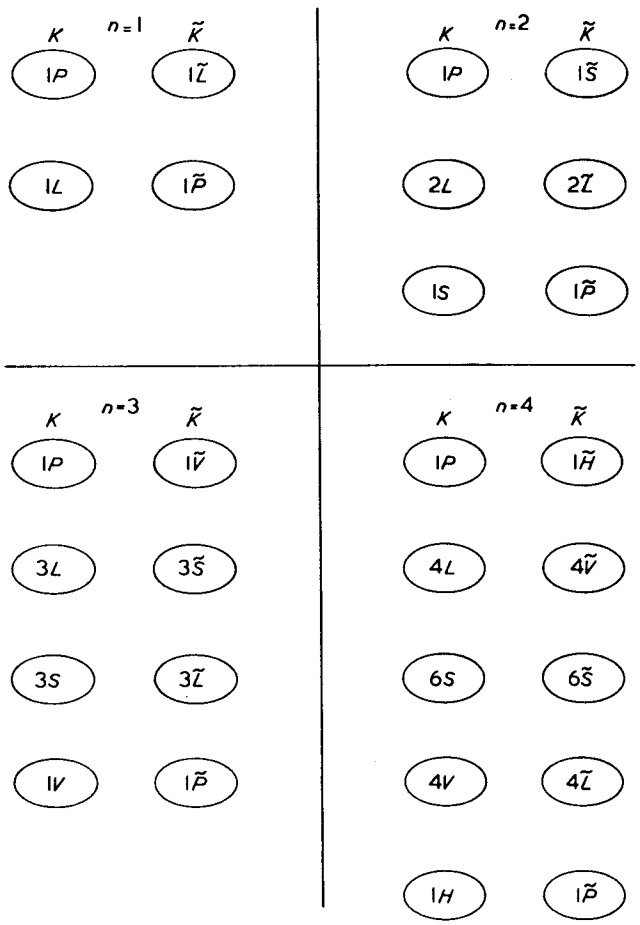
0-cell: point, time-instant, event: P
1-cell: line segment, time interval: L
2-cell: surface segment, line segment x time interval: S
3-cell: volume, surface segment x time interval: V
4-cell: hypervolume, volume x time interval: H
For every point of $\mathbf{R}^3$ they pass:

$\binom{3}{1}$ = 3 coordinate lines;

$\binom{3}{2}$ = 3 coordinate surfaces

as is shown in *Figure 1c*. This permits the p-cells to be grouped into families: all p-cells that lie on a coordinate manifold, e.g. x^1x^2 belong to the same family. Then we have 3 families of 1-cells and 3 families of 2-cells; these will be denoted by the symbols $3L$ and $3S$. In this way we obtain a rational classification of the geometrical elements of $\mathbf{R}^n$ that is quoted in the left column of *Table 3*. The set of all the cells of the various orders is called *cell complex* and will be denoted by K.

Table 3 Classification scheme of the basic geometrical elements of a cell complex K and its dual $\tilde{K}$ in $\mathbf{R}^n$



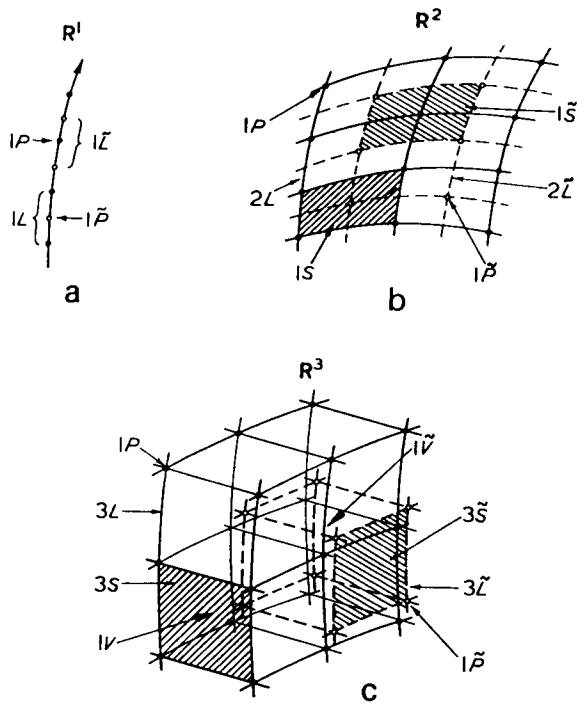


Figure 1 Cell complex for (a) $\mathbf{R}^1$, (b) $\mathbf{R}^2$ and (c) $\mathbf{R}^3$

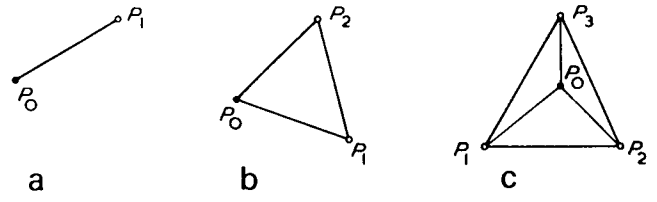


Figure 2 Simplex of (a) $\mathbf{R}^1$, (b) $\mathbf{R}^2$ and (c) $\mathbf{R}^3$

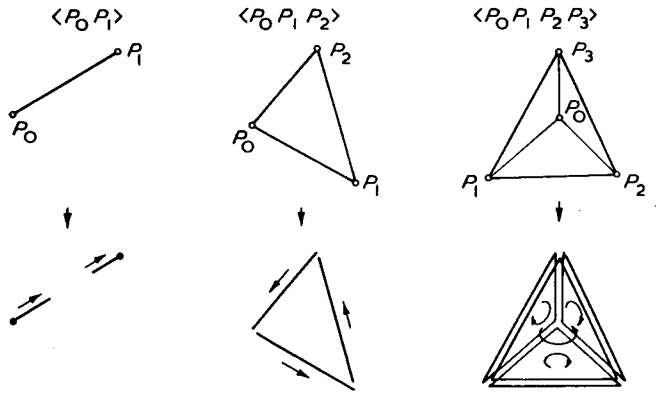


Figure 3 Induced inner orientation

Dual cell complex

For physical theories it will be useful to consider the centres of gravity of the 3-cells and to consider them as vertices of another cell complex, as shown in *Figure 1.* This will be called the dual cell complex and we denote it by $\tilde{K}$. A characteristic of the dual cell complex is the fact that for every p-cell of K there corresponds a $(n-p)$ cell of $\tilde{K}$ and *vice versa:* this is shown in the right column of *Table 3* which gives a classification scheme of the geometrical elements of a space $\mathbf{R}^n$, for $n = 1,2,3,4$. Cell complexes are one of the typical subjects of algebraic topology, founded by Poincaré in 1895.

Inner orientation of a p-cell

The triangle is the simplest polygon, the tetrahedron is the simplest polyhedron; they are called the simplexes of the spaces of two and three dimensions respectively (*Figure 2*). The number of vertices of a simplex is one more dimension of the corresponding space. It is then customary to denote the vertices of a simplex of $\mathbf{R}^n$ with the notation:

$$\langle P_0, P_1, \ldots P_n \rangle \tag{10}$$

in which the index zero is included.

Let us consider a given arrangement of the vertices, say for a tetrahedron, the arrangement $\langle P_2, P_3, P_0, P_1 \rangle$. All other arrangements can be grouped in two classes: those that differ for an even or an odd number of permutations from the initial one. We say that the given arrangement and all those of the same class define an inner orientation of the simplex. The arrangement of the other class defines the opposite orientation. In this way one sees the combinatorial character of the notion of orientation; it is easy to realize that the inner orientation defined in this way reduces to the intuitive notion of orientation (the way to go along the perimeter of a triangle). This can be done using the notion of induced orientation.

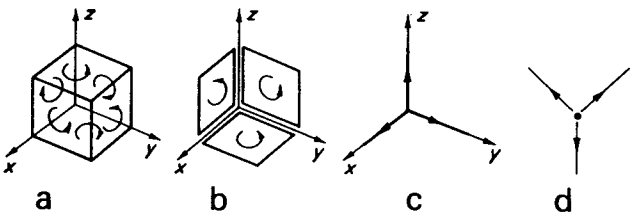


Figure 4 Inner orientation of the p-cells in $\mathbf{R}^3$. (a) Oriented 3-cells; (b) oriented 2-cells; (c) oriented 1-cell; (d) oriented 0-cell

Induced inner orientation

The inner orientation of a simplex induces in a natural way an inner orientation on its faces according to the rule[3]:

$$(-1)^k \langle P_0, P_1, \ldots \hat{P}_k, \ldots P_n \rangle \tag{11}$$

where the circumflex upon P_k means that this vertex must be omitted. When this rule is applied to a 1-simplex, 2-simplex, 3-simplex respectively it gives the orientation indicated in *Figure 3.*

Since the set of all faces of a simplex forms its boundary we may say that the orientation of a simplex implies the orientation of its boundary.

We now show that the converse is also true. Let us observe that the orientation of two adjacent faces is such that the orientations induced on their common element are opposite. This property is known as the Möbius law of edges[4]. If we orient one face of a simplex and then propagate the orientation according to this law, one obtains a compatible orientation of the whole boundary of the simplex, and this is equivalent to the orientation of the simplex itself, i.e. to the choice of an order of its vertices. Then to orient a simplex we may give a compatible orientation to its faces.

To give an inner orientation to a p-cell we may proceed as follows: we divide the p-cell into p-simplexes, then we

orient one simplex and propagate the orientation to the adjacent simplexes using the Möbius law of edges. It is easily seen that this process is equivalent to the choice of an orientation of one face of the p-cell and to the propagation of the orientation with the edge law. Then to orient all p-cells of the same family, i.e. lying on the same coordinate manifold, we need only fix an orientation of a simplex lying on that coordinate manifold.

By definition the inner orientation of a point (0-cell) means that incoming lines are considered positive (or negative) (see *Figure 4*).

Outer orientation of a p-cell

If we consider the dual cell complex $\tilde{K}$ we may use the same criterion to give an inner orientation to all its cells of the various dimensions. Since every p-cell of the primary cell complex K is crossed by a $(n-p)$ cell of $\tilde{K}$ we may consider the inner orientation of this $(n-p)$ cell as defining a new kind of orientation of the corresponding p-cell (*Figure 5*).

Precisely we call the *outer orientation* of a p-cell the orientation of its dual $(n-p)$ cell[5].

The outer orientation for a 3-cell amounts to the choice of the outward or inward normals, while the outer

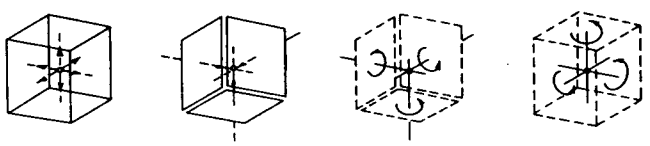


Figure 5 Outer orientation of the p-cells of $\mathbf{R}^3$

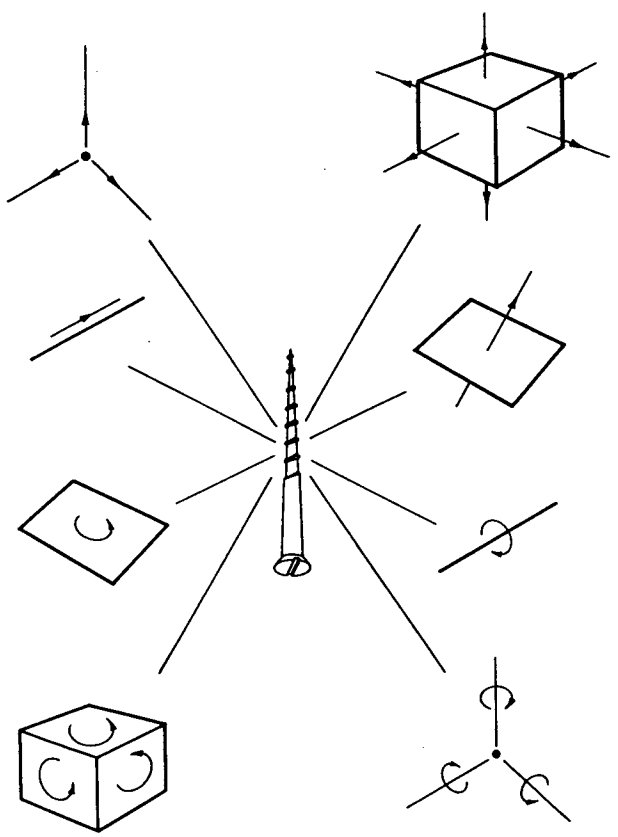


Figure 6 Illustration of screw to show link between inner and outer orientation in three-dimensional spaces

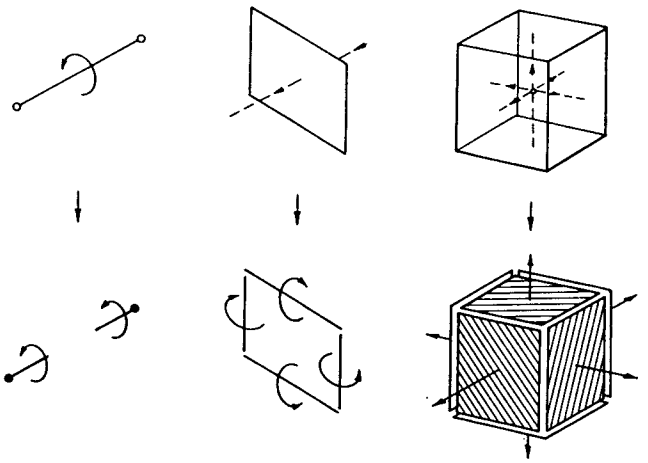


Figure 7 Induced outer orientation

orientation of a point is defined fixing a sense of rotation around the lines that arrive or leave the point.

The link between inner and outer orientation in three-dimensional spaces may be established by means of a screw (see *Figure 6*).

Induced outer orientation

The outer orientation of a p-cell induces in a natural way an outer orientation on its faces. This is described in *Figure 7*.

Incidence numbers

Let us consider an oriented cell complex, i.e. a cell complex in which all p-cells have been equipped with an inner or outer orientation, and this for $p = 0, 1, \ldots n$.

If we consider a p-cell we may induce an orientation on all its faces; for some faces the induced orientation will agree with the orientation previously assigned to the face, for others it will not. If we number all p-cells of the cell complex according to whatever criterion and we number also all $(p-1)$ cells, we may consider the incidence of the rth p-cell with the sth $(p-1)$ cell. If the sth $(p-1)$ cell is not a face of the rth p-cell we assign to the pair (r, s) the number 0. If the sth $(p-1)$ cell is a face of the p-cell we assign to the pair (r, s) the number $+1$ or -1 according to whether the induced orientation is compatible or not with the orientation of the $(p-1)$ cell. Then we are led to introduce the incidence numbers between the p-cells and the $(p-1)$ cells denoted by:

$$e_{rs}^{(p-1,p)} = \begin{cases} +1 \\ 0 \\ -1 \end{cases} \tag{12}$$

These incidence numbers may be considered as entries of a matrix called the incidence matrix and denoted:

$$E^{(p,p-1)}$$

So in $\mathbf{R}^3$ we have three incidence matrices:

$$E^{(3,2)} \qquad E^{(2,1)} \qquad E^{(1,0)} \tag{13}$$

These matrices reflect the connectivity of the various cells and then are of combinatorial nature.

Chains on a cell complex

One of the basic notions of the homology theory is that of 'chain'. Let us number with an arbitrary criterion all

p-cells of a cell complex and give an orientation to every p-cell. If we associate with every p-cell a mathematical element like a number, a vector, a tensor, a matrix, etc. we have defined a p-chain. A p-chain is then a mapping between the oriented p-cell of a cell complex and the elements of a set S. For later purposes we shall suppose that this set has at least the structure of an additive commutative group, i.e. that an operation of sum is defined on every pair of its elements u and v, and that $u + v = v + u$.

Chains in physical theories

It happens that in physics we use p-chains without having explicitly recognized it. So writing an equilibrium equation we use a small volume element, usually a parallelepiped in a rectangular coordinate system. This volume element is just a 3-cell of an underlying cell complex that we usually do not explicitly mention. When we say that every face of the parallelepiped is associated with the force transmitted through it, we say that every 2-cell of the underlying cell complex is associated with a vector, i.e. an element of a space $\mathbf{S} = \mathbf{R}^3$. To define the contact force transmitted through a 2-cell it is necessary to define an outer orientation of the 2-cell because we must select one of the two opposite forces transmitted through it. Then the distribution of the force on the 2-cell of a cell complex is described by assigning with every oriented 2-cell of the cell complex a vector. This amounts to saying that we have constructed a 2-chain whose 'coefficients' are elements of the same linear space $\mathbf{S} = \mathbf{R}^3$. Let us denote the force transmitted through the sth 2-cell by $\vec{f}_s$; the corresponding 2-chain will be denoted by:

$$f^{(2)} = (\vec{f}_1, \vec{f}_2, \ldots \vec{f}_{\alpha_2}) \tag{14}$$

where α_2 is the number of 2-cells of the cell complex.

Let us consider as a second example a magnetic field in a finite region Ω of the space. If we cover this region with a cell complex K then with every oriented 2-cell of K is associated the amount of magnetic flux transmitted through it. If ϕ_s is the magnetic flux transmitted across the sth oriented 2-cell the whole distribution of magnetic fluxes is described by the 2-chain:

$$\phi^{(2)} = (\phi_1, \phi_2, \ldots \phi_{\alpha_2}) \tag{15}$$

In this case the coefficients of the 2-chain i.e. the amounts ϕ_s are numbers and then it is $\mathbf{S} = \mathbf{R}$.

As a third example let us consider a Cosserat continuum: with every oriented 2-cell is associated not only a force but also a couple. If $\vec{f}_s$ denotes the force transmitted through the sth oriented 2-cell and $\vec{\mu}_s$ the corresponding couple we may consider the vector with complex components:

$$\vec{m}_s = \vec{f}_s + i\vec{\mu}_s \tag{16}$$

where i denotes the imaginary unit. Then we can say that in a polar continuum with every oriented 2-cell is associated a complex vector. We then obtain a 2-chain:

$$m^{(2)} = (\vec{m}_1, \vec{m}_2, \ldots \vec{m}_{\alpha_2}) \tag{17}$$

whose coefficients are elements of the space $\mathbf{S} = \mathbf{C}^3$.

In these three examples we have 2-chains with different space coefficients.

We now pass to consider examples of 0-chains. Let us consider the displacement vector of continuum mechanics: with every 0-cell of a cell complex K we may associate the displacement vector $\vec{u}$. Then the displacement field is described by a 0-chain:

$$u^{(0)} = (\vec{u}_1, \vec{u}_2, \ldots \vec{u}_{\alpha_0}) \qquad \vec{u}_s \in \mathbf{R}^3 \tag{18}$$

where α_0 is the number of 0-cells of K.

Another example of 0-chain is that of the velocity potential in the fluid dynamics of an irrotational, perfect stationary fluid motion. Let $\varphi(x, y, z)$ denote the velocity potential, and φ_s its value at the sth 0-cell. The field of the velocity potential is then described by the 0-chain:

$$\varphi^{(0)} = (\varphi_1, \varphi_2, \ldots \varphi_{\alpha_0}) \qquad \varphi_s \in \mathbf{R} \tag{19}$$

Let us give now some examples of 3-chains. First let us consider a continuum body that occupies a region Ω of the space in which a cell complex has been build up. With every 3-cell may be associated the mass contained inside it. Then the mass distribution is described by the 3-chain:

$$m^{(3)} = (m_1, m_2, \ldots m_{\alpha_3}) \tag{20}$$

We may also consider the external force acting on every 3-cell. Let $\vec{F}_s$ be the external force acting on the sth 3-cell. The external force distribution is then described by the 3-chain:

$$F^{(3)} = (\vec{F}_1, \vec{F}_2, \ldots \vec{F}_{\alpha_3}) \qquad \vec{F}_s \in \mathbf{R}^3 \tag{21}$$

Up to now we have considered cell complexes in the three-dimensional space because we have considered physical quantities that depend on the three space coordinates. If we consider physical quantities that depend on a set of n independent variables we may consider the space $\mathbf{R}^n$ and build up a cell complex on it. If we consider a theory whose physical quantities depend from the space and time variables, x, y, z, t, the space is $\mathbf{R}^4$. But we may consider the time t as a parameter and the three space coordinates as independent variables and then the space is $\mathbf{R}^t$. On the contrary we may consider the three space co-ordinates as parameters and the time t as an independent variable; in this case the representative space is the real axis $\mathbf{R}$.

To do some examples let us consider the motion of a particle in space. The physical variables that are used in particle dynamics depend on the time t. Then we may consider a one-dimensional representative space, the time axis. On it we may construct a cell complex dividing the

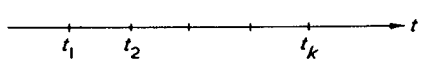


Figure 8 The continuum analogous of chains and coboundary by time intervals

time axis in time intervals (*Figure 8*). The radius vector $\vec{r}(t)$ is naturally referred to time instants (0-cells) and then its variation with time is described by a 0-chain:

$$r^{(0)} = (\vec{r}_1, \vec{r}_2, \ldots \vec{r}_{\alpha_0}) \qquad \vec{r}_k \in \mathbf{R}^3 \tag{22}$$

where $\vec{r}_k = \vec{r}(t_k)$. The displacement vector $\vec{s}$ is relative to a couple of time instants and then is naturally referred to the time intervals (1-cells). Then its variation in time is described by a 1-chain on the time axis:

$$s^{(1)} = (\vec{s}_1, \vec{s}_2, \ldots \vec{s}_{\alpha_1}) \qquad \vec{s}_k \in \mathbf{R}^3 \tag{23}$$

where $\vec{s}_k$ is the displacement of the particle relative to the kth 1-cell. The impulse $\vec{h}$ given to the particle is referred to

time intervals then its time distribution is expressed by a 1-chain:

$$h^{(1)} = (\vec{h}_1, \vec{h}_2, \ldots \vec{h}_{\alpha_1}) \qquad \vec{h}_k \in \mathbf{R}^3 \tag{24}$$

on the time axis etc.

Lastly we consider a thermal field, and assume as independent variables the three space coordinates x, y, z and the time t. Let K be a cell complex in the region Ω of $\mathbf{R}^4$ that forms the domain of the independent variables. If we consider the heat current density $\vec{q}$ transmitted through a surface during a time interval we see that this physical quantity is referred to the 3-cells of K. Then its space–time distribution is given by a 3-chain:

$$q^{(3)} = (\vec{q}_1, \vec{q}_2, \ldots \vec{q}_{\alpha_3}) \tag{25}$$

These few examples may be enough to state that those physical quantities of a physical theory that are naturally referred to the basic geometrical elements of some space give rise to chains. The coefficients of these chains are the amounts of the physical quantity associated with every cell.

As a matter of fact we can see that among the physical quantities of a physical theory there are some that are referred to the cells of a cell complex K and others that are naturally referred to the cells of the dual cell complex $\tilde{K}$. To work an example, if we consider the bending of a plate subjected to a continuous load when we discretize the plate, for example dividing it into rectangular elements, with every such element associated with the corresponding load. It is then natural to consider as unknown the vertical displacement of the barycentric point of every finite element. But these barycentric points are just the 0-cells of $\tilde{K}$ while the rectangular elements are the 2-cells of K. Then when the configuration variable (here the vertical displacement) is referred to the cells of K the corresponding source variable (here the load) is referred to the corresponding cells of $\tilde{K}$.

As a general rule if the geometrical and kinematical variables are referred to the p-cells of K the corresponding statical and dynamical variables are referred to the p-cells of $\tilde{K}$.

The coboundary process

In the homology theory of cell complexes there is one fundamental process that, starting with a p-chain leads to the construction of a $(p+1)$ chain: this is called the 'coboundary' of the given p-chain. The process is the following: let us consider a p-chain defined on a cell complex K. Taking an arbitrary p-cell we consider all $(p+1)$ cells of K that are incident on it: these are called the cofaces on the p-cell. The mathematical entity a associated with every p-cell is then transferred to every coface with the same sign or with the opposite sign depending whether the incidence number is $+1$ or -1. Since every $(p+1)$ cell is a common coface of several p-cells, to every $(p+1)$ cell there come as many values of the mathematical entity a as are the faces of the $(p+1)$ cell. We then sum up all the amounts a_k coming to every $(p+1)$ cell: this sum is associated with the $(p+1)$ cell. In this way, we have obtained a $(p+1)$ chain that is called the coboundary of the given p chain. This two-stage process is illustrated in *Figure 9*. If $b^{(p+1)} = (b_1, b_2, \ldots b_{\alpha_{p+1}})$ is the coboundary of the p-chain $a^{(p)} = (a_1, a_2, \ldots a_{\alpha_p})$ we may write:

$$b^{(p+1)} = \delta a^{(p)} \tag{26}$$

δ is called the coboundary operator, which is a linear operator that maps the p-chains into the $(p+1)$ chains.

Since a p-chain is essentially a vector of $\mathbf{R}^{\alpha_p}$ with coefficients in S the coboundary operator from $a^{(p)}$ to $b^{(p+1)}$ may be represented by a matrix $E^{(p+1,p)}$ that operates from $\mathbf{R}^{\alpha_p}$ and $\mathbf{R}^{\alpha_{p+1}}$. We may write:

$$b_r = \Sigma_s e_{rs}^{(p+1,p)} a_s \tag{27}$$

When the coboundary operator δ is applied in sequence twice, it gives rise to the null chain:

$$\delta(\delta a^{(p)}) \equiv \Theta^{(p+2)} \tag{28}$$

where $\Theta^{(p+2)}$ denotes the chain of rank $(p+2)$ whose coefficients are all vanishing. This property is known as the Poincaré lemma. A p-chain whose coboundary vanishes is

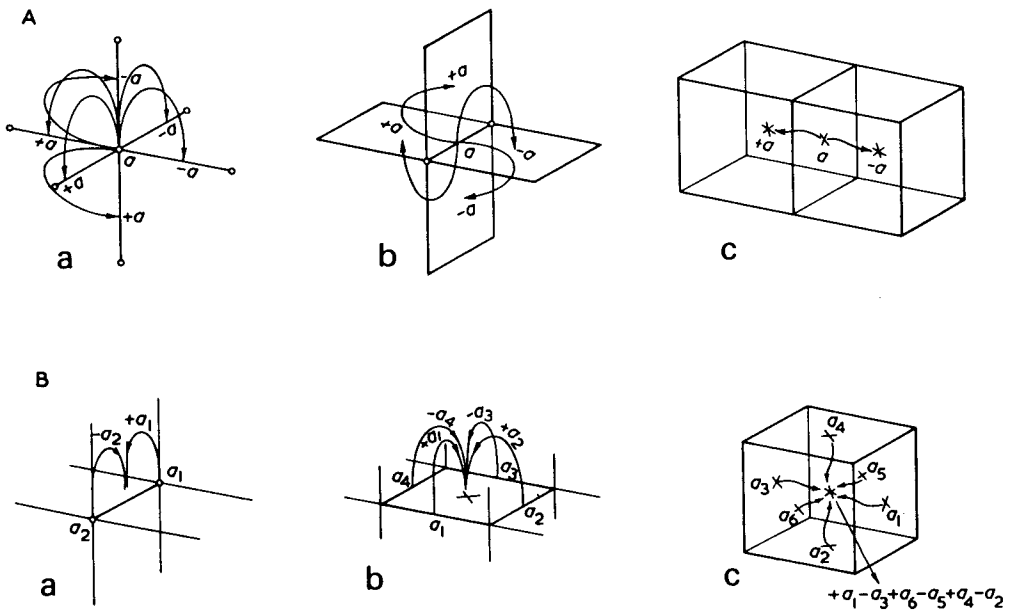
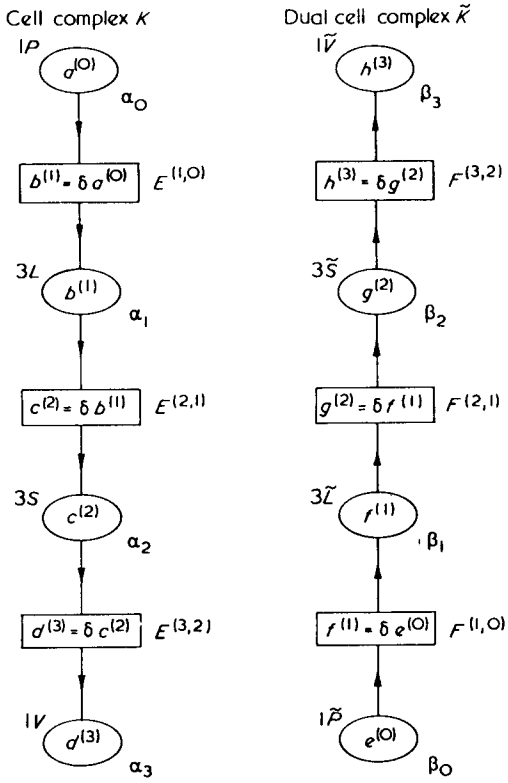


Figure 9 The process of forming the coboundary. A, Transferring the mathematical element to the cofaces of a p-cell; B, summing the mathematical elements assigned to every $(p+1)$ cell

Table 4 Classification scheme for the chains of K and $\tilde{K}$ in $\mathbf{R}^3$



called a cocycle. The Poincaré lemma states that the coboundary of a given p-chain is a cocycle.

One may also consider the p-chains on the dual cell complex $\tilde{K}$. We denote β_p the number of p-cells of $\tilde{K}$; $F^{(p+1,\,p)}$ the incidence matrices of $\tilde{K}$. *Table 4* gives a classification scheme of the chains of a cell complex K and its dual $\tilde{K}$ in a region $\Omega C \mathbf{R}^3$.

The coboundary process in physical theories

The physical interest of this process when applied to the chains formed by physical quantities comes from the fact that the equations of structure of every physical theory states that one chain is the coboundary of another. Let us give some examples.

Equilibrium equations. Let us consider a material continuum and, in it, a three-dimensional region. The equilibrium condition asserts that the sum of the contact forces transmitted through the boundary of the region and the external force acting on the matter contained in the region vanishes. If we consider as usual a 3-cell and call $\vec{F}_s$ the exterior force acting on the sth cell, $\vec{f}_k$ the contact force acting on the kth 2-cell we may consider the two chains:

$$f^{(2)} = (\vec{f}_1, \vec{f}_2, \ldots \vec{f}_{\alpha_2}) \qquad F^{(3)} = (\vec{F}_1, \vec{F}_2, \ldots \vec{F}_{\alpha_3}) \quad (29)$$

With reference to *Figure 10a* the equilibrium equation can be written:

$$-\vec{F}_h = (+1)\vec{f}_j + (-1)\vec{f}_i + (+1)\vec{f}_l + (-1)\vec{f}_m + (+1)\vec{f}_q + (-1)\vec{f}_p \quad (30)$$

or

$$-\vec{F}_h = \Sigma_k\, e_{hk}^{(3,2)} \vec{f}_k \quad (31)$$

i.e.

$$-F^{(3)} = \delta f^{(2)} \quad (32)$$

Then an equilibrium equation states that a 3-chain is the coboundary of a 2-chain.

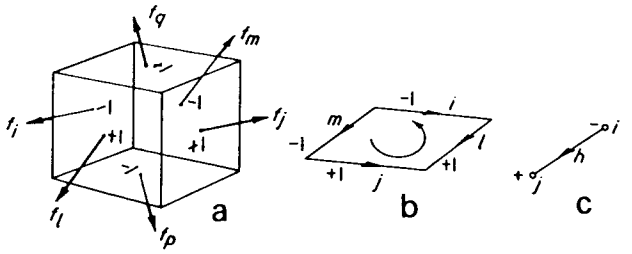


Figure 10

Circuital laws. Let us consider a small parallelogram as indicated in *Figure 10b* and a physical quantity g referred to the lines. A typical circuital law has the form:

$$G = (+1)g_j + (-1)g_i + (+1)g_e + (-1)g_m \quad (33)$$

where G is the amount of a physical quantity associated with the 2-cell. This law can be written:

$$G_h = \Sigma_k e_{hk}^{(2,1)} g_k \quad (34)$$

or

$$G^{(2)} = \delta g^{(1)} \quad (35)$$

where we have put

$$G^{(2)} = (G_1, G_2, \ldots G_{\alpha_2}) \qquad g^{(1)} = (g_1, g_2, \ldots g_{\alpha_1}) \quad (36)$$

Then a circuital law is a statement that a 2-chain is the coboundary of a 1-chain.

The gradients. Let us consider the process of forming the gradient of a scalar, vector or tensor field. The first act is that of forming the differences between the physical quantities φ_i and φ_j referred to two points P_i and P_j (see *Figure 10c*). This difference:

$$\phi_s = (+1)\varphi_j + (-1)\varphi_i \quad (37)$$

is associated with the line segment: we may write:

$$\phi_s = \Sigma_k e_{sh}^{(1,\,0)} \varphi_h \quad (38)$$

or

$$\phi^{(1)} = \delta \varphi^{(0)} \quad (39)$$

Then we see that the construction of the gradients is equivalent to the process of forming the coboundary of a 0-chain.

In an analogous way one can show that the compatibility equations, the circuital equations, the balance equations, the equations giving the general solution of another equation, the conservation laws are all realizations of a single statement: a $(p+1)$ chain is the coboundary of a p-chain.

Classification scheme of physical variables

As shown in many physical theories there are physical quantities that are associated with the basic geometrical elements of the space of the independent variables chosen. This association is completed by the fact that the sign of the physical quantity depends on the orientation (inner or outer) of the geometrical element. This permits the distribution of the physical quantity to be described by a chain. Now we have seen that there exists a typical linear process, the coboundary process, that allows us to obtain a chain from a chain of one dimension lower. In turn this process reflects a typical process used in physics to link physical quantities: the corresponding equations identify with the equations of structure (see above). If the configuration-type variables are associated with the cells of

a cell complex K, the corresponding source-type variables are naturally associated with the dual cell complex $\tilde{K}$.

These facts lead us towards a classification scheme for physical quantities of a physical theory analogous to the scheme of the chains of K and $\tilde{K}$ of *Table 4*.

Extensive physical quantities

Let us consider those physical quantities that are referred to extended geometrical manifolds of the space of the independent variables, e.g. lines, surfaces, volumes, hypervolumes, time intervals, etc.

By their definition they are additive to the parts of the corresponding geometrical manifolds; hence there is a good reason to call them extensive. The name 'extensive' is used here in a more general sense than in thermodynamics where it means additivity to the volume parts only.

These extensive physical quantities may be scalars, vectors, matrices, operators, etc. When there are more than scalars their components with respect to a base have one or more indices. So the displacement of a particle during a time interval is associated with an extensive chronometrical element and is an extensive physical quantity (extensive in time). Since it is a vector it can be represented by its components s^A, with $A = 1, 2, 3$. When the finite rotation of a rigid body with a fixed point is associated with the time intervals, it is an extensive physical quantity. It can be described by an orthogonal matrix whose entries may be denoted R^B_A, i.e. with two indices.

Examples of extensive physical quentities are magnetic flux, electromotive force, electric charge, energy, contact force, etc.

Forming of densities

From the extensive physical quantities one is led to introduce the corresponding specific quantities, e.g. the flux for unit area, the contact force for unit area, the magneto-motive force for unit length, the charge for unit volume etc. These specific quantities are the common densities. So we have the electric charge density, the mass density, the velocity (time density of the displacement), etc.

When the physical quantity is associated to lines or surfaces one must introduce a directional density or equivalently a vectorial or tensorial variable. So

$$\text{magnetomotive force } F_{(m)} = \int_L \vec{H} \cdot \mathrm{d}\vec{L}$$

$$\text{magnetic flux } \phi = \int\int_S \vec{B} \cdot \vec{n}\,\mathrm{d}S$$

$$\text{contact force } f_A = \int\int_S P^k_A n_k \mathrm{d}S$$

The vectors H_k and B_k and the stress tensor P^k_A are the densities of the corresponding extensive physical quantities. The index k refers to the geometrical element with which the extensive physical quantity is associated, and will be called geometrical index. On the contrary the index A, as in the contact force f_A, is referred to the mathematical nature of the extensive physical quantity and will be called intrinsic or internal index. The distinction of these two kinds of indices is of fundamental importance for the classification we are doing. *Table 5* shows some extensive physical variables.

Since every extensive physical quantity is associated with a corresponding density we may classify these densities. As a classification criterion we choose the following: we refer a density to the same geometrical object to which the corresponding extensive physical quantity is referred.

Table 5 Some extensive physical variables and their densities

Extensive physical variable	Defining relation	Corresponding density
potential difference, V	$V = \int_L E_k \mathrm{d}L^k$	electric field strength, E_k
displacement, s^A	$s^A = \int_0^\tau v^A \mathrm{d}t$	velocity, v^A
impulse, h_A	$h_A = \int_0^\tau f_A \mathrm{d}t$	force, f_A
magnetic flux, ϕ	$\phi = \int\int_S B^k n_k \mathrm{d}S$	magnetic induction, B^k
contact force, t_A	$t_A = \int\int_S \sigma^k_A n_k \mathrm{d}S$	stress tensor, σ^k_A
mass, m	$m = \int\int\int_V \rho \mathrm{d}V$	mass density, ρ
energy production, Σ	$\Sigma = \int\int\int_V \int_0^\tau \sigma \mathrm{d}V\mathrm{d}t$	energy production density, σ
fluid flow, Q	$Q = \int\int_S \int_0^\tau q^k n_k \mathrm{d}S\mathrm{d}t$	current density, q^k

Table 6 The continuum analogous of chains and coboundary

chains:		integral on the sth p-cell:
$a^{(0)} = (a_1, a_2, \ldots a_{\alpha_0})$	⟶	$a_s = a(P_s)$
$b^{(1)} = (b_1, b_2, \ldots b_{\alpha_1})$	⟶	$b_s = \int_{L_s} B_k(P)\mathrm{d}L^k$
$c^{(2)} = (c_1, c_2, \ldots c_{\alpha_2})$	⟶	$c_s = \int\int_{S_s} c^k(P) n_k \mathrm{d}S$
$d^{(3)} = (d_1, d_2, \ldots d_{\alpha_3})$	⟶	$d_s = \int\int\int_{V_s} D(P)\mathrm{d}V$
coboundary:		differential operators:
$b^{(1)} = \delta a^{(0)}$	⟶	$\vec{B}(P) = \mathrm{grad}\, a(P)$
$c^{(2)} = \delta b^{(1)}$	⟶	$\vec{C}(P) = \mathrm{rot}\, \vec{B}(P)$
$d^{(3)} = \delta c^{(2)}$	⟶	$D(P) = \mathrm{div}\, \vec{C}(P)$
when the coefficients of the chain are vectors:		
$t^{(2)} = (\vec{t}_1, \vec{t}_2, \ldots \vec{t}_{\alpha_2})$	⟶	$(t_A)_s = \int\int_{S_s} \sigma^k_A n_k \mathrm{d}S$
$F^{(3)} = (\vec{F}_1, \vec{F}_2, \ldots \vec{F}_{\alpha_3})$	⟶	$(F_A)_s = \int\int\int_{V_s} f_A \mathrm{d}V$
$F^{(3)} = \delta f^{(2)}$	⟶	$-F_A = \nabla_k \sigma^k_A$
Poincaré lemma:		differential identities:
$\delta(\delta a^{(0)}) \equiv 0$	⟶	$\mathrm{rot}\,(\mathrm{grad}\, a) \equiv 0$
$\delta(\delta b^{(1)}) \equiv 0$	⟶	$\mathrm{div}\,(\mathrm{rot}\, \vec{B}) \equiv 0$

The density field of a physical quantity is the continuum analogy of the corresponding chain. To the coboundary operator, that links two chains, it corresponds to a differential operator that links the corresponding densities, as is shown in *Table 6*. In this way one realizes that the commonest differential operators 'grad', 'rot', 'div' are the continuum analogous of the coboundary operator applied respectively to 0-chains, 1-chains and 2-chains.

Standard classification scheme

The content of *Table 6* can be ordered as shown in *Table 7*. This first column represents the chains of the various orders, the second and third column give the continuum analogous of the first one using an intrinsic notation and a tensorial notation respectively. In both

Table 7 Various forms of a standard column

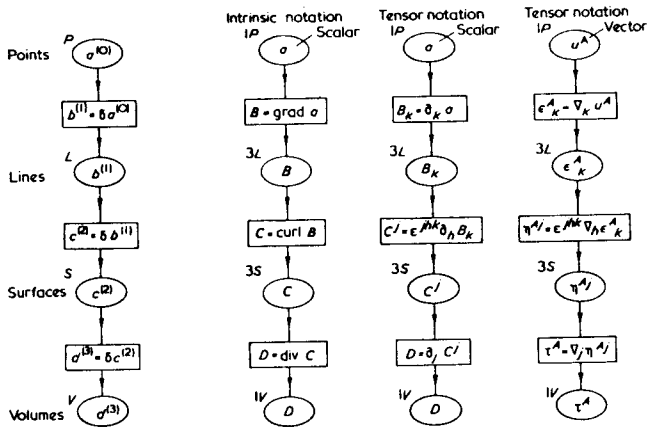


Table 8 Standard schemes for $n = 1, 2, 3$ and 4

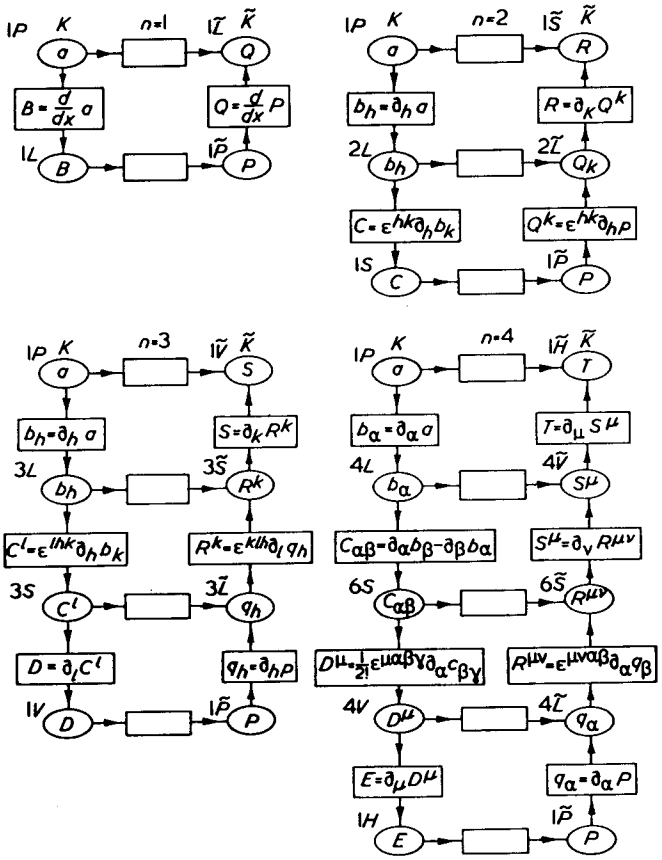
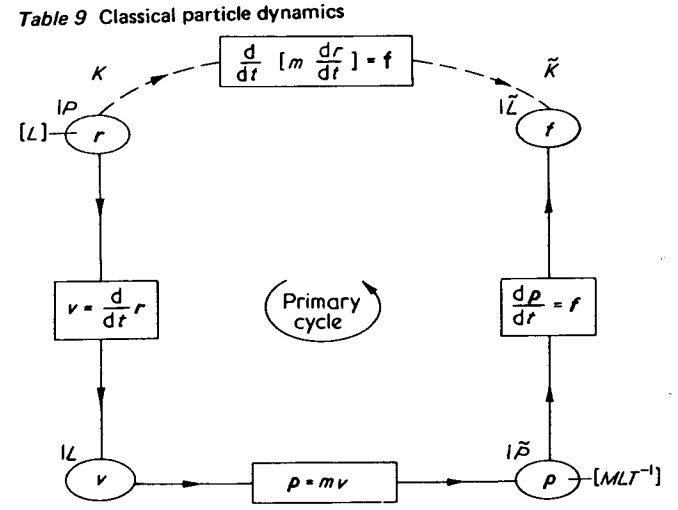


Table 9 Classical particle dynamics



columns the physical quantity associated with the cells has the nature of a scalar. This is reflected in the fact that the mathematical entity that appears in the first box is a scalar.

When the physical quantity associated with the cells has a vector nature, as is the case in continuum mechanics, we have the fourth column. The symbol ∇_k denotes the covariant derivative that is needed to take into account the space variation of the components u_A on account of the space variation of the base vectors. The last three columns will be called *standard columns.* A pair of standard columns, one relative to a cell complex K and the other to its dual will be called *standard scheme* $\tilde{K}$.

Table 8 gives the standard schemes for spaces of $n = 1, 2, 3, 4$ dimensions. In the rectangular boxes that link

Table 10 Relativistic particle dynamics

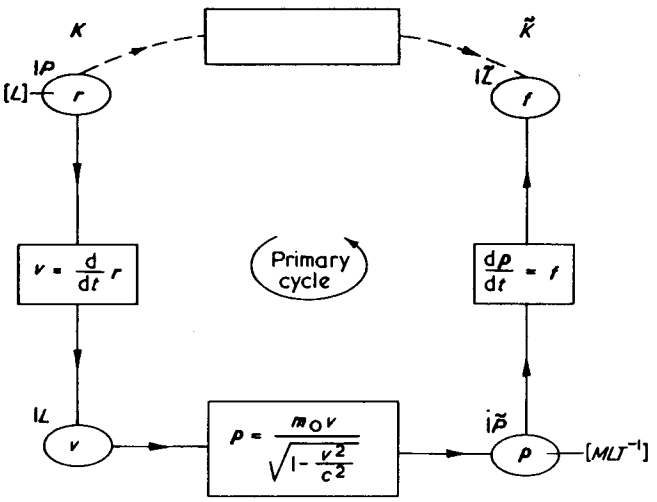


Table 11 Longitudinal vibrations of a rod with variables t and x

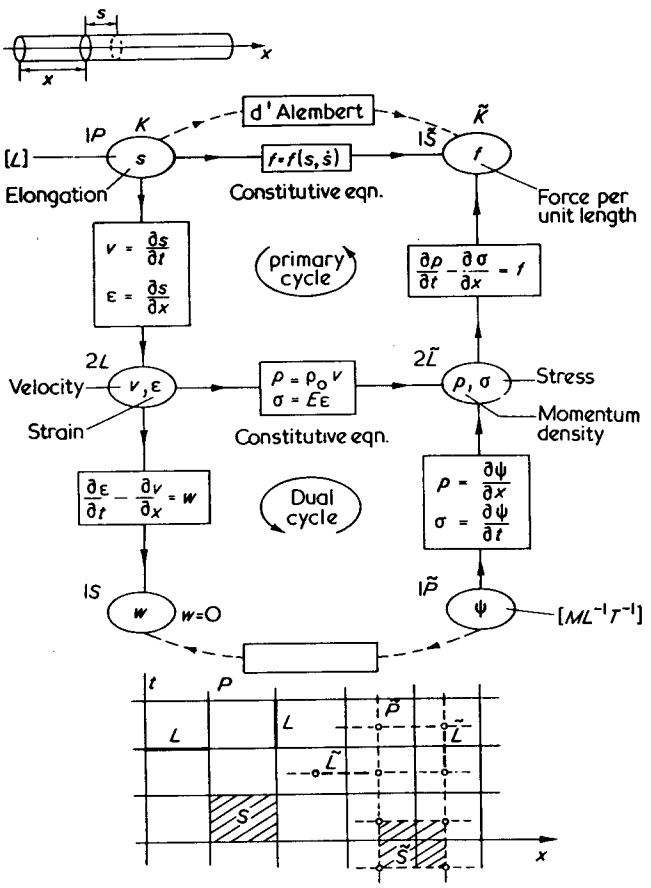


Table 12 Thermal conduction for variables x^1, x^2 and x^3 (time t is a parameter)

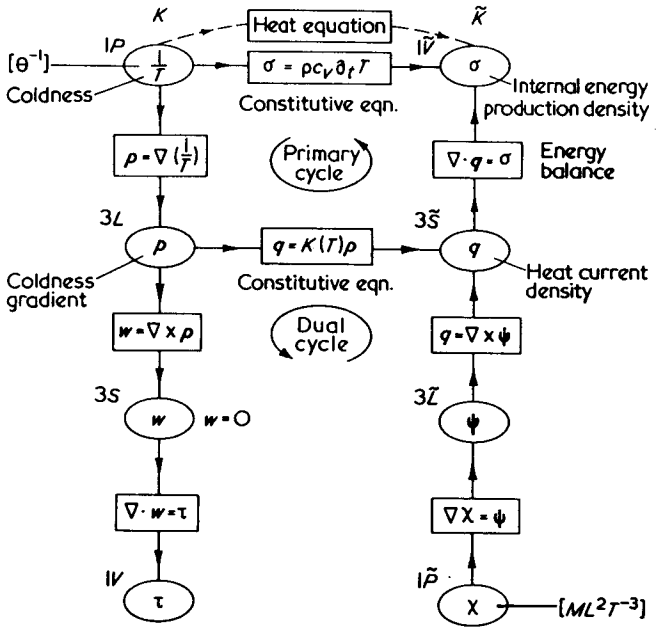


Table 13 Thermostatics of a monatomic ideal gas

the configuration-type variables (K) with the source-type variables ($\tilde{K}$) we must insert the constitutive relations.

With these rules we now construct the classification scheme of the physical quantities of some physical theories. Some of these schemes are shown in *Tables 9–20*. All symbols and names are those recommended by the International Union of Pure and Applied Physics, SUN Commission, 1965. The dimensions are those of the International System of Units (SI units). The schemes for other physical theories are contained in a previous paper[6]. In the same paper the analysis we have summarized here is given in much more detail.

Table 14 Irreversible thermodynamics (time t is considered as a parameter)

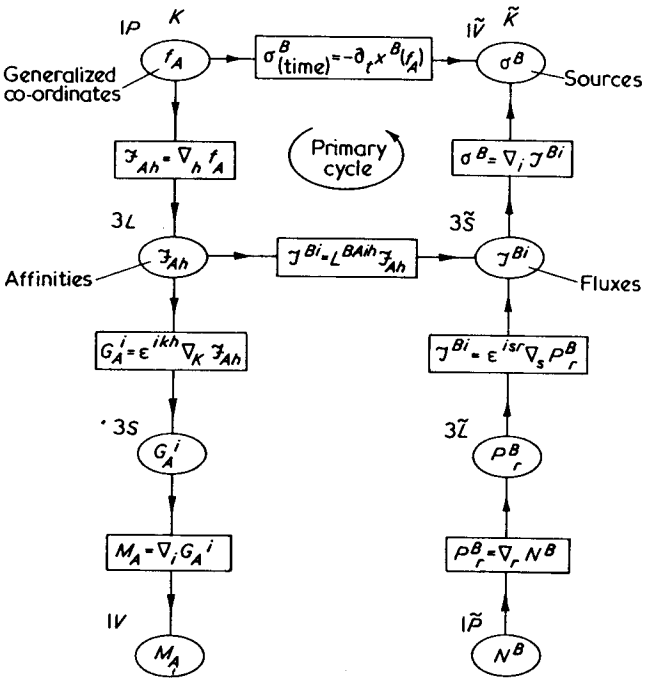
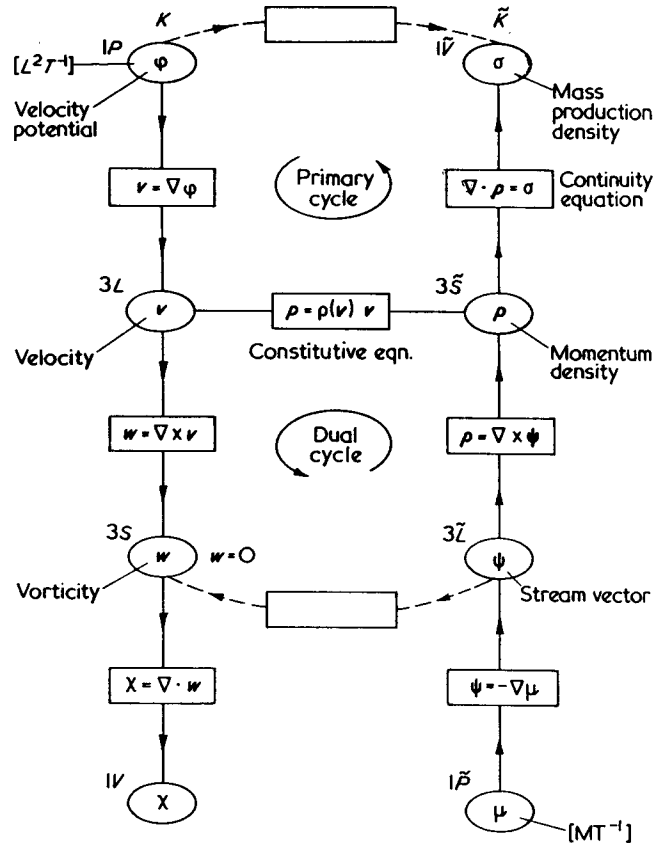


Table 15 Perfect fluid motion of barotropic, unrotational, stationary flow for variables x^1, x^2 and x^3



These classification schemes make evident the analogies between different physical theories. So the analogy among thermal conduction, fluid dynamics, electrostatics and magnetostatics is evident by a comparison of the corresponding schemes.

In order to utilize these schemes for solution purposes one must investigate the mathematical properties exhibited by the operators that appear in the structure equations and

Table 16 Elastostatics

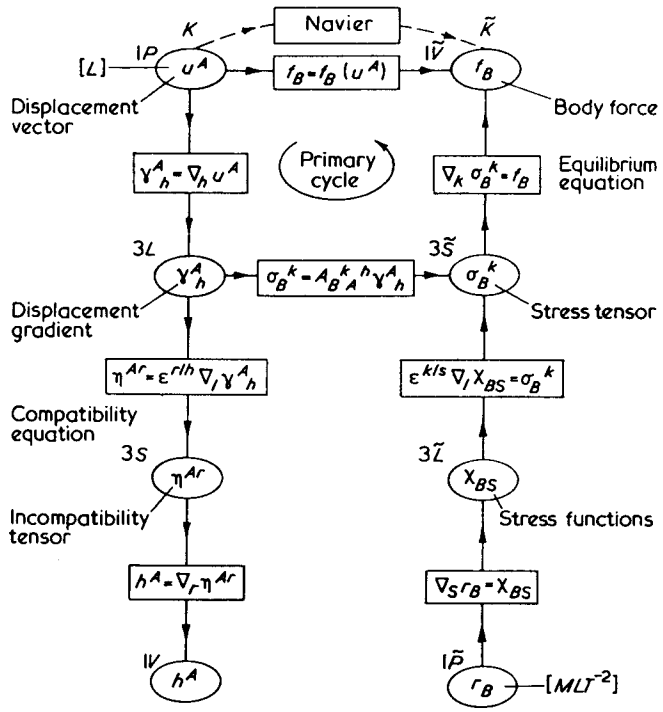


Table 17 Fluid dynamics

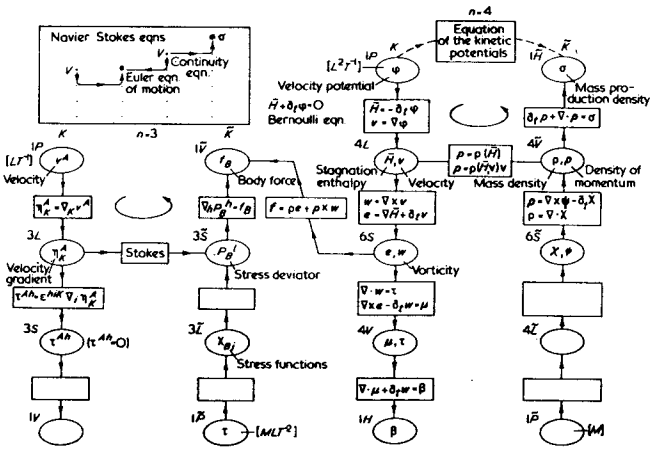
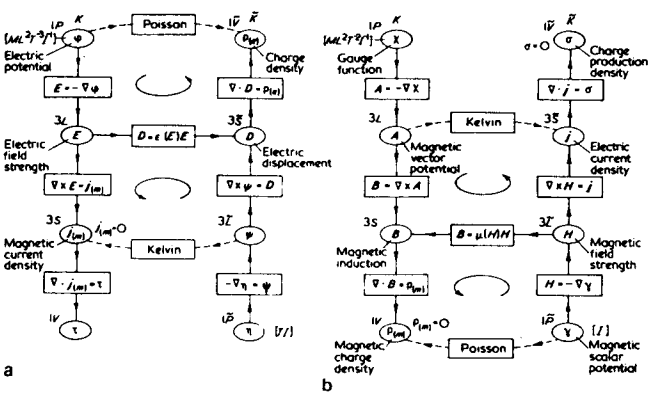


Table 18 (a) Electrostatics and (b) magnetostatics



in the constitutive equations. Such investigation has been performed and the main mathematical properties have been elucidated[1]. This has led to the construction of a mathematical model[2].

Acknowledgement

This work has been sponsored by the Italian CNR.

Table 19 Electromagnetic field with variables *t*, *x*, *y* and *z*

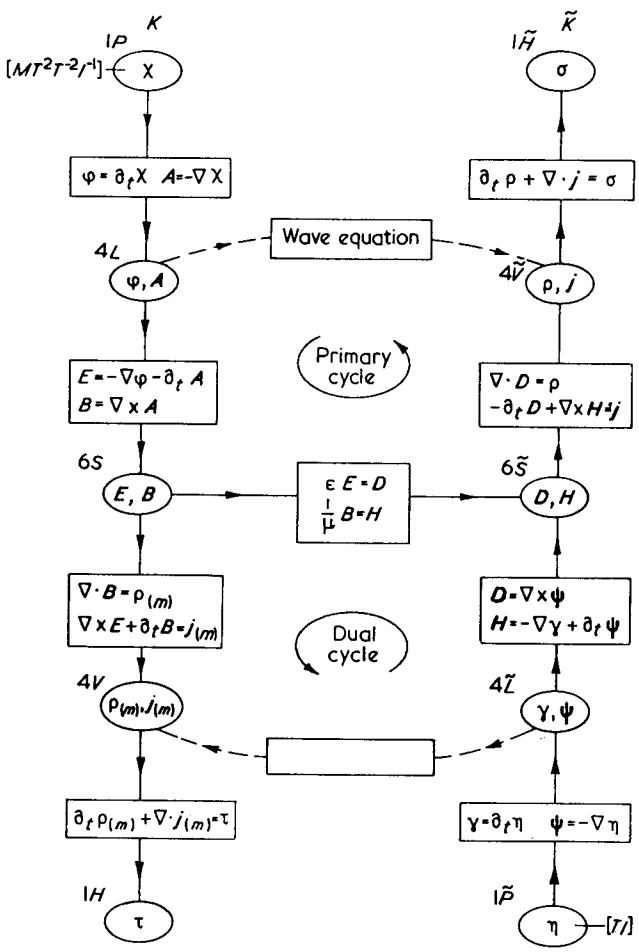
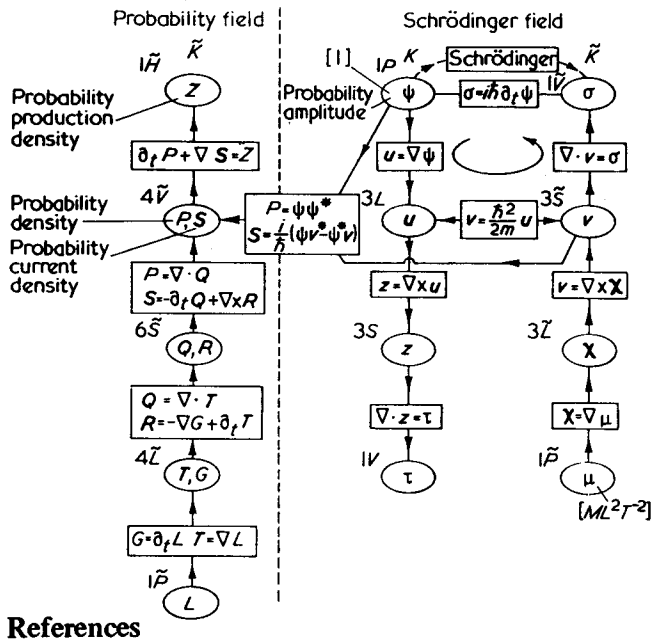


Table 20 Schrödinger field



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2 Tonti, E. *Rend. Accad. Lincei* 1972, 7, 175, 350
3 Franz, W. 'Algebraic Topology', Ungar Publ. Co., 1968
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6 Tonti, E. 'On the formal structure of physical theories', CNR. 1975 and references cited therein

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