Appendix D

Measure theory and Lebesgue integration

"As far as the laws of mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality."
Albert Einstein (1879-1955)

In this appendix, we will briefly recall the main concepts and results about the measure theory and Lebesgue integration. Then, we will introduce the L^p spaces that prove especially useful in the analysis of the solutions of partial differential equations.

Contents

D.1 Measure theory

Measure theory initially was proposed to provide an analysis of and generalize notions such as length, area and volume (not strictly related to physical sizes) of subsets of Euclidean spaces. The approach to measure and integration is axomatic, *i.e.* a measure is any function μ defined on subsets which satisfy a cetain list of properties. In this respect, measure theory is a branch of real analysis which investigates, among other concepts, measurable functions and integrals.

D.1.1 Definitions and properties

Definition D.1 *A collection S of subsets of a set X is said to be a* topology *in X is S has the following three properties:*

(i) $\emptyset \in S$ and $X \in S$,

(ii) *if* $V_i \in S$ *for* $i = 1, \ldots, n$ *then,* $V_1 \cap V_2 \cap \cdots \cap V_n \in S$,

(iii) *if* $\{V_\alpha\}$ *is an arbitrary collection of members of* S *(finite, countable or not), then* $\bigcup_\alpha V_\alpha in S$.

if S is a topology in X, then X is called a *topological space* and the members of S are called the *open sets* in X.

Definition D.2 *A collection* F *of subsets of a set* X *is said to be a* σ*-algebra in* X *if* F *has the following three properties:*

(i) $X \in F$,

(ii) *if* $A \in F$, *then* $A^c \in F$, *where* A^c *is the complement of* A *relatie to* X,

(iii) *if* $A = \bigcup_{n=1}^{\infty} A_n$ *and if* $A_n \in F$ *for all* n, *then* $A \in F$.

If F is a σ-algebra in X, then the pair (X, F) is called a *measurable space* . If X is a measurable space, Y is a topological space and f is a mapping of X into Y, then f is said to be *measurable* provided that $f^{-1}(V)$ is a measurable set in X for every open set V in Y.

Definition D.3 *A* measure *is a function defined on a* σ*-algebra* F *over a set* X *and taking values in the interval* $[0, \infty[$ *such that the following properties are satisfied:*

(i) *the emptyset has measure zero,* $\mu(\emptyset) = 0$;

(ii) *countable additivity: if* (E_i) *is a countable sequence of pairwise disjoint sets in* F, *then*

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)\,.$$

The triple (X, F, μ) *is then called a* measure space *and the members of* F *are called* measurable sets.

For measure spaces that are also topological spaces, various compatibility conditions can be placed for the measure and the topology.

Theorem D.1 *Let* f *and* g *be real measurable functions on a measurable space* X, *let* Φ *be a continuous mapping of the plane into a topological space* Y *and define* $h(x) = \Phi(u(x), v(x))$ *for* $x \in X$. *Then,* $h : X \to Y$ *is measurable.*

Proposition D.1 *If* E *is a measurable set in* X *and if*

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

then, χ_E *is a measurable function called the* characteristic function *of the set* E.

Definition D.4 *Let* (a_n) *be a sequence in* $[-\infty, \infty]$ *and put*

$$b_k = \sup\{a_k, a_{k+1}, a_{k+2}, \ldots\} \quad k = 1, 2, 3, \ldots$$

and $\beta = \inf\{b_1, b_2, b_3, \ldots\}$. *We call* β *the* upper limit *of* (a_n) *and write*

$$\beta = \limsup_{n\to\infty} a_n\,.$$

The *lower limit* is defined analogously by interchanging sup and inf in the previous definition. Moreover, we have

$$\liminf_{n\to\infty} a_n = -\limsup_{n\to\infty}(-a_n)\,,$$

and, if (a_n) converges, then

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \lim_{n\to} a_n\,.$$

Supose (f_n) is a sequence of real functions on a set X. Then $\sup_n f_n$ and $\limsup_{n\to\infty} f_n$ are the functions defined on X by:

$$(\sup_n f_n)(x) = \sup_n(f_n(x))\,, \qquad (\limsup_{n\to\infty} f_n)(x) = \limsup_{n\to\infty}(f_n(x))\,.$$

Moreover, if $f(x) = \lim_{n\to\infty} f_n(x)$ the limit being assumed to exist at every $x \in X$, then f is called the *pointwise limit* of the sequence (f_n).

Theorem D.2 *If $f_n : X \to \mathbb{R}$ is measurable, for $n = 1, 2, \ldots$ and*

$$g = \sup_{n\geq 1} f_n\,, \qquad h = \limsup_{n\to\infty} f_n\,,$$

then g and h are both measurable.

Corollary D.1 *The limit of very pointwise convergent sequence of complex measurable functions is measurable. If f and g are measurable then so are* $\max(f, g)$ *and* $\min(f, g)$*. In particular, this is true of the functions*

$$f^+ = \max(f, 0) \quad \textit{and} \quad f^- = -\min(f, 0)\,,$$

respectively called the positive*and* negative parts *of f.*

We have $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

Proposition D.2 *If $f = g - h$, $g \geq 0$ and $h \geq 0$ then $f^+ \leq g$ and $f^- \leq h$.*

D.1.2 Completeness

A measurable set X is called a *null set* if $\mu(X) = 0$. By extension, a subset of a null set is called a *negligible* set. A negligible set need not be measurable, but every measurable negligible set is automatically a null set. A measure is called *complete* if every negligible set is measurable.

A measure can be extended to a complete one by considering the σ-algebra of subsets Y which differ by a negligible set from a measurable set X, that is, such that the *symmetric difference* of X and Y is contained in a null set. One defines $\mu(Y)$ to equal $\mu(X)$.

D.1.3 Non-measureable sets

D.2 Riemann integration

Suppose that a function f is bounded on the interval $[a, b]$, where $a, b \in \mathbb{R}$ and $a < b$ and consider a *dissection* $\Delta : a = x_0 < x_1 < \cdots < x_n = b$ of $[a, b]$. Then,

Definition D.5 (Riemann sum) *The* lower Riemann sum *of f(x) corresponding to the dissection* Δ *is defined as the following sum:*

$$s(f,\Delta) = \sum_{j=1}^{n} (x_j - x_{j-1} \inf_{x\in[x_{j-1},x_j]} f(x)$$

anf the upper Riemann sum of $f(x)$ *corresponding to the dissection* Δ *is given by the sum:*

$$S(f,\Delta) = \sum_{j=1}^{n} (x_j - x_{j-1} \sup_{x\in[x_{j-1},x_j]} f(x)\,.$$

Theorem D.3 *Suppose that a function* $f : \mathbb{R} \to \mathbb{R}$ *is bounded on* $[a,b]$*, where* $a < b$*,* $a,b \in \mathbb{R}$ *and that* Δ *and* Δ' *are two dissections of* $[a,b]$ *such that* $\Delta' \subseteq \Delta$*. Then,*

$$s(f,\delta') \leq s(f,\Delta) \qquad \textit{and} \qquad S(f,\Delta) \leq S(f,\Delta')\,.$$

If Δ'' *is another dissection of* $[a,b]$ *then,*

$$s(f,\Delta') \leq S(f,\Delta'')\,.$$

Definition D.6 *For all dissections* Δ *of* $[a,b]$*, the real number* $I^-(f,a,b) = \sup_{\Delta} s(f,\Delta)$ *is called the* lower integral *of* $f(x)$ *over* $[a,b]$ *and the real number* $I^+(f,a,b) = \inf_{\Delta} S(f,\Delta)$ *is called the* upper integral *of* $f(x)$ *over* $[a,b]$*.*

Theorem D.4 *Suppose that a function* f *is bounded on the interval* $[a,b]$*, where* $a < b$ *and* $a,b \in \mathbb{R}$*. Then*

$$I^-(f,a,b) \leq I^+(f,a,b)\,.$$

Definition D.7 *Suppose that a function* f *is bounded on the interval* $[a,b]$*, where* $a < b$ *and* $a,b \in \mathbb{R}$ *and suppose that* $I^-(f,a,b) = I^+(f,a,b)$*. Then, the function* f *is said to be* Riemann integrable *over* $[a,b]$ *and we write*

$$\int_a^b f(x)\,dx = I^-(f,a,b) = I^+(f,a,b)\,.$$

Lemma D.1 *Suppose that a function* f *is bounded on the interval* $[a,b]$*, where* $a < b$ *and* $a,b \in \mathbb{R}$*. The following two statements are equivalent:*

(i) f *is Riemann integrable over* $[a,b]$

(ii) given any $\varepsilon > 0$*, there exists a dissection* Δ *of* $[a,b]$ *such that*

$$S(f,\Delta) - s(f,\Delta) < \varepsilon\,.$$

We have several additional properties of Riemann integrals.

Lemma D.2 *Suppose that* f,g *are Riemann integrable over* $[a,b]$*, where* $a < b$ *and* $a,b \in \mathbb{R}$*. Then*

(i) $f+g$ *is Riemann integrable over* $[a,b]$ *and* $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$*,*

(ii) for every $c \in \mathbb{R}$*,* cf *is Riemann integrable over* $[a,b]$ *and* $\int_a^b cf(x)\,dx = c\int_A^b f(x)\,dx$*,*

(iii) if $f(x) \geq 0$ *for every* $x \in [a, b]$ *then* $\int_a^b f(x)\,dx \geq 0$,

(iv) if $f(x) \geq g(x)$ *for every* $x \in [a, b]$ *then* $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

Proposition D.3 *Suppose that* f *is Riemann integrable over* $[a, b]$ *where* $a, b \in \mathbb{R}$ *and* $a < b$. *Then, for every real number* $c \in [a, b]$, f *is Riemann integrable over* $[a, c]$ *and Riemann integrable over* $[c, b]$. *Moreover, we can write:*

$$\int_A^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\,.$$

Similarly, we have the following result.

Proposition D.4 *Suppose that* $a, b, c \in \mathbb{R}$ *and that* $a < c < b$. *Suppose further that* f *is Riemann integrable over* $[a, c]$ *and is Riemann integrable over* $[c, b]$. *Then,* f *is Riemann integrable over* $[a, b]$ *and*

$$\int_a^b f(x)dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\,.$$

Theorem D.5 *Suppose that* f *is Riemann integrable over* $[a, b]$, *where* $a, b \in \mathbb{R}$ *and* $a < b$. *Suppose that* $f(x) = g(x)$ *for every* $x \in [a, b]$, *except possibly at* $x = x_0$ *Then* g *is Riemann integrable over* $[a, b]$ *and*

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx\,.$$

D.3 Lebesgue integration

Suppose we are considering integrating functions like $\chi(x)$, the *characteristic function* of the set $S = \{x \in \mathbb{Q}\} \subset \mathbb{R}$ (*i.e.* $\chi_S(x) = 1$ if $x \in S$ and $\chi_S(x) = 0$ if $x \notin S$). Or suppose we are considering real-valued measurements x of a phenomenon and wondering what is the probability for x to be a rational number. From the prababilistic theory, we know that if the measurements are distributed normally with a *mean* of μ and a *standard deviation* of σ, then the probability is given by:

$$Pr[x \in \mathbb{Q}] = \int_S \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)\,dx\,, \tag{D.1}$$

and the Riemann integral is useless to evaluate this integral.

D.3.1 Sets of measure zero

The study of Lebesgue integral depends on the notion of zero measure sets in $\mathbb{R}$.

Definition D.8 *A set* $S \subseteq \mathbb{R}$ *is said to have* measure zero *if, for every* $\varepsilon > 0$ *there exiss a countable family* $\mathcal{F}$ *of intervals* I *such that*

$$S \subseteq \bigcup_{I \in \mathcal{F}} I \quad \textit{and} \quad \sum_{I \in \mathcal{F}} \mu(I) < \varepsilon\,,$$

where, for every $I \in \mathcal{F}$, $\mu(I)$ *denotes the length of the interval* I.

This definition states that the set S can be covered by a countable union of open intervals of arbitrarily small total length.

Proposition D.5 *Every countable set in* $\mathbb{R}$ *has measure zero. Furthermore, a outable union of sets of measure zero in* $\mathbb{R}$ *has measure zero.*

D.3.2 Compact sets

Definition D.9 *A set $S \subseteq \mathbb{R}$ is said to be* compact *if and only if, for every family $\mathcal{F}$ of open intervals I such that*

$$S \subseteq \bigcup_{I \in \mathcal{F}} I$$

there exists a finite subfamily $\mathcal{F}_0 \subseteq \mathcal{F}$ such that

$$S \subseteq \bigcup_{I \in \mathcal{F}_0} I\,.$$

This definition means that every open covering of S can be achieved by a subcovering.

Theorem D.6 (Heine-Borel) *Suppose that $F \subseteq \mathbb{R}$ is bounded and closed. Then, F is* compact*.*

D.3.3 Lebesgue integral

Definition D.10 *A function is* simple *if its range is a finite set.*

A simple function φ in $\mathbb{R}$ has always the following representation:

$$\varphi = \sum_{k=1}^{n} a_k \chi_{E_k}\,,$$

where a_k are distinct values of φ and $E_k = \varphi^{-1}(\{a_k\})$. Conversely, any expression of this form, where a_k need not be distinct and E_k not necessarily $\varphi^{-1}(\{a_k\})$ also defines a simple function. In the remainder, we will consider that E_k be measurable and that they partition the set X.

Definition D.11 *Let (X, μ) be a measure space. The* Lebesgue integral *over X of a $\mathbb{R}^+$-valued simple function φ is defined as:*

$$\int_X \varphi\, d\mu = \int_X \sum_{k=1}^{n} a_k \chi_{E_k}\, d\mu = \sum_{k=1}^{n} a_k \mu(E_k)\,.$$

The quantity on the right represents tha sum of the areas below the graph of φ. If $\varphi = \sum_i a_i \chi_{A_i} = \sum_j b_j \chi_{B_j}$, where A_i and B_j partition X, then

$$\sum_i a_i \mu(A_i) = \sum_j \sum_i a_i \mu(A_i \cap B_j) = \sum_j \sum_i b_j \mu(A_i \cap B_j) = \sum_j b_j \mu(B_j)\,.$$

The second equality follows since the value of φ is $a_i = b_j$ on $A_i \cap B_j$, so $a_i = b_j$ whenever $A_i \cap B_j \neq \emptyset$. The integral is thus well-defined. Similarly, for two simple functions $\varphi \leq \psi$ then $\int_X \varphi\, d\mu \leq \int_X \psi\, d\mu$ (monotonicity of the integral).

Theorem D.7 *For non-negative simple functions, the Lebesgue integral is linear.*

Definition D.12 *Let $f : X \to [0, +\infty]$ be measurable. Consider the set S_f if all measurable functions $0 \leq \varphi \leq f$. The integral of f over X is defined as:*

$$\int_X f\, d\mu = \sup_{\varphi \in S_f} \int_X \varphi\, d\mu\,.$$

The simple functions in S_f are supposed to approximate f as close as possible. The integral of f is obtained by computing the integrals of these approximations.

Theorem D.8 (Approximation theory) *Let $f : X \to [0, +\infty]$ be measurable. Then, there exists a sequence of non-negative functions $\{\varphi_n\} \nearrow f$, meaning φ_n are increasing pointwise and converging pointwise toward f. Moreover, if f is bounded, it is possible for the φ_n to converge toward f uniformly.*

Definition D.13 *If f is not necessarily non-negative, we define:*

$$\int_X f\, d\mu = \int_X f^+\, d\mu - \int_X f^-\, d\mu\,,$$

provided that the two integrals are not both ∞.

The functions f^+ and f^- are measurable and represent the positive and negative part of f, respectively:

$$f^+(x) = \max(+f(x), 0) \qquad f^-(x) = \max(-f(x), 0)\,.$$

Let A be a measurable subset of X, we can define:

$$\int_A d\, d\mu = \int_X f\chi_A\, d\mu\,,$$

to introduce the Lebesgue integral on subsets of X.

Proposition D.6 *A measurable function $f : X \to [0, \infty]$ vanishes almost everywhere if and only if $\int_X f = 0$.*

D.3.4 Convergence results

Theorem D.9 (Monotone convergence) *Let (X, μ) be a measure space. Let f_n be non-negative measurable functions increasing pointwise toward f. Then,*

$$\int_X f\, d\mu = \int_X \left(\lim_{n\to\infty} f_n\right) d\mu = \lim_{n\to\infty} \int_X f_n\, d\mu\,.$$

This theorem allows to prove linearity of the Lebesgue integral for non-simple functions. Given any two non-negative measurable functions f, g we know (Theorem D.8) that there are non-negative simple functions $\{\varphi_n\} \nearrow f$ and $\{\psi_n\} \nearrow g$. Then, $\{\varphi_n + \psi_n\} \nearrow f + g$ and so, since the integral is linear for simple functions:

$$\int f + g = \lim_{n\to\infty} \int \varphi_n + \psi_n = \lim_{n\to\infty} \int \varphi_n + \int \psi_n = \int f + \int g\,.$$

And, if f, g are not necessarily non-negative, then:

$$\begin{aligned}\int f + g &= \int (f^+ - f^-) + (g^+ - g^-) &&= \int (f^+ + g^+ - (f^- + g^-)) \\ &= \int (f^+ + g^+) - \int (f^- + g^-) &&= \int f^+ + \int g^+ - \left(\int (f^- + \int g^-\right) = \int f + \int g\,.\end{aligned}$$

Theorem D.10 (Beppo Levi) *Let $f_n : X \to [0, \infty]$ be measurable. Then:*

$$\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n .$$

Theorem D.11 (Fatou's lemma) *Let $f_n : X \to [0, \infty]$ be measurable. Then,*

$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n .$$

Definition D.14 *A function $f : X \to \mathbb{R}$ is called* integrable *if it is measurable and if $\int_X |f| < \infty$.*

It follows that f is integrable if and only if f^+ and f^- are both integrable. Moreover, $\int |f| < \infty$ implies that $|f| < \infty$ almost everywhere.

Theorem D.12 (Dominated convergence) *Let (X, μ) be a measure space. Let $f_n : X \to \mathbb{R}$ be a sequence of measurable functions converging pointwise toward f. Moreover, supose that there is an integrable function g such that $|f_n| \leq g$ for all n. Then f_n and f are also integrable and:*

$$\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0 .$$

It is sufficient to require that f_n converge to f pointwise almost everywhere, or that $|f_n|$ is bounded above by g almost everywhere. Using the triangle inequality, we conclude that

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu ,$$

and this corresponds to the common application of this theorem.

D.3.5 Generalization

Theorem D.13 (Beppo Levi) *Let (X, μ) be a measure space and $f_n : X \to \mathbb{R}$ be measurable functions with $\int \sum |f_n| = \sum \int |f_n| < \infty$. Then,*

$$\sum_{n=1}^{\infty} \int f_n = \int \sum_{n=1}^{\infty} f_n .$$

Proposition D.7 *Let $g : X \to [0, \infty]$ be measurable in the measure space (X, A, μ). Let*

$$\nu(E) = \int_E f \, d\mu , \quad E \in A .$$

Then, ν is a measure on (X, A) and for any measurable function f on X,

$$\int_X f \, d\nu = \int_X f g \, d\mu ,$$

and this result is usually written as: $d\nu = g d\mu$.

Lemma D.3 (Change of variables) *Let X, Y be measure spaces and $g : X \to Y$, $f : Y \to \mathbb{R}$. Then,*

$$\int_X (f \circ g) \, d\mu = \int_Y f d\nu ,$$

where $\nu(B) = \mu(g^{-1}(B))$ is a measure defined for all measurable $B \subseteq Y$.

Applying these results together leads to:

Theorem D.14 (Differential change of variables in $\mathbb{R}^n$) *Let $g : X \to Y$ be a diffeomorphism of open sets in $\mathbb{R}^n$. If $A \subseteq X$ is measurable and $f : Y \to \mathbb{R}$ is measurable then,*

$$\int_{g(A)} f\,d\lambda = \int_A (f \circ g)\,d\mu = \int_A (f \circ g) \cdot |\det Dg|\,d\lambda\,.$$

And we have Lebesgue versions of results about the Riemann integral.

Theorem D.15 (First fundamental theorem of calculus) *Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be integrable with Lebesgue measure in $\mathbb{R}$. Then, the function*

$$F(x) = \int_a^x f(t)\,dt\,,$$

is continuous. Moreover, if f is continuous at x, then $F'(x) = f(x)$.

Theorem D.16 (Second fundamental theorem of calculus) *Suppose $f : [a, b] \to \mathbb{R}$ is measurable and bounded above and below. If $f = g'$ for some g, then*

$$\int_a^b f(x)\,dx = g(b) - g(a)\,.$$

In this result, we shall not assume as strong hypotheses that f' is continuous or even that it is Lebesgue integrable. If g exists, then it can also be computed as countable limit $\lim_{n\to\infty} n(g(x + 1/n) - g(x))$, thus showing that g' is measurable.

Theorem D.17 (Continuous dependence on integral parameter) *Let (X, μ) be a measure space, T be any metric space and $f : X \times T \to \mathbb{R}$ with $f(\cdot, t)$ being measurable for each $t \in T$. Consider the function*

$$F(t) = \int_{x\in X} f(x, t)\,.$$

Then, we have F continuous at $t_0 \in T$ if the following conditions are met:

(i) for each $x \in X$, $f(x, \cdot)$ is continuous at $t_0 \in I$,

(ii) there is an integrable function g such that $|f(x, t)| \leq g(x)$ for all $t \in T$.

Theorem D.18 (Differentiation under the integral sign) *Using the same notations, with T being an open real interval, we have*

$$F'(t) = \frac{d}{dt} \int_{x\in X} f(x, t) = \int_{x\in X} \frac{\partial}{\partial t} f(x, t)\,,$$

if the following conditions are satisfied:

(i) for each $x \in X$, $\dfrac{\partial}{\partial t} f(x, t)$ exists,

(ii) there is an integrable function g such that $\left|\dfrac{\partial}{\partial t} f(x, t)\right| \leq g(x)$, for all $t \in T$.

This result can be generalized to T being any open set in $\mathbb{R}^n$, taking partial derivatives.

D.4 L^p spaces

The L^p spaces, named after H. Lebesgue (1875-1941), are spaces of p-power integrable functions and form an important class of examples of Banach spaces.

D.4.1 Definitions

Let Ω be a domain in $\mathbb{R}^d$ and let $p \in [1,+\infty[$ be a positive real number. We denote $L^p(\Omega, d\mu)$ the space of all measurable functions u, defined on Ω, for which

$$\|f\|_{L^\infty} \stackrel{def}{=} \int_\Omega |f(x)|^p \, d\mu < \infty \,.$$

The elements of L^p are indeed equivalence classes of measurable functions satisfying the previous equality (two functions being equivalent if they coincide almost everywhere in Ω). If $p = \infty$, we denote by $L^p(\Omega, d\mu)$ the space of all measurable functions, defined on Ω, such that:

$$\|f\|_{L^\infty} \stackrel{def}{=} \sup\left\{\lambda \,/\, \mu\{x\,, |f(x)| > \lambda\} > 0\right\} \,.$$

We need to introduce the notion of exponent conjugate. If $p \in]1,\infty[$, $p' \stackrel{def}{=} \dfrac{p}{p-1}$, if $p = 1$ then $p' \stackrel{def}{=} +\infty$ and if $p = +\infty$, $p' \stackrel{def}{=} 1$. The exponents p and p' are said to be (Hölder) conjugates of each other and, under the convention that $1/\infty = 0$, we have

$$\frac{1}{p} + \frac{1}{p'} = 1 \,.$$

Theorem D.19 *For any value $p \in [1,\infty[$, the space $L^p(\Omega, d\mu)$, endowed with the norm $\|\cdot\|_{L^p}$, is a Banach space.*

It is clear that if $u \in L^p(\Omega)$ and $c \in \mathbb{C}$, then $cu \in L^p(\Omega)$. Moreover, if $u, v \in L^p(\Omega)$, then since

$$|u(x)+v(x)|^p \;\le\; (|u(x)|+|v(x)|)^p \;\le\; 2^p(|u(x)|^p + |v(x)|^p) \,,$$

$u + v \in L^p(\Omega)$, so $L^p(\Omega)$ is a vector space.

Proposition D.8 (Hölder inequality) *Let (Ω,μ) be a measure space, f be a function of $L^p(\Omega, d\mu)$ and g be a function of $L^{p'}(\Omega, d\mu)$. Then, the product fg is in $L^1(\Omega, d\mu)$ and*

$$\int_\Omega |f(x)g(x)| \, d\mu(x) \;\le\; \|f\|_{L^p} \|g\|_{L^{p'}} \,.$$

For $1 < p, p' < \infty$, the inequality becomes equality if and only if $|f|^p$ is proportional to $|g|^{p'}$ almost everywhere. When $p = p' = 2$, we retrieve the Cauchy-Schwarz inequality.

Proposition D.9 (Minkowski's inequality) *If $1 \le p < \infty$, then for all $u, v \in L^p(\Omega)$*

$$\|u+v\|_p \;\le\; \|u\|_p + \|v\|_p \,.$$

Theorem D.20 (Jensen's inequality) *Let $\rho \in L^1(\Omega)$ be a non-negative function such that $\int_\Omega \rho(x)dx = 1$. Then, for every measurable function f such that $f\rho \in L^1(\Omega)$ and for every convex measurable function $\varphi : \mathbb{R} \to \mathbb{R}$, we have:*

$$\varphi\left(\int_\Omega f(x)\rho(x)\, dx\right) \le \int_\Omega \varphi(f(x))\rho(x)\, dx \,.$$

In particular, for $\varphi(x) = x^2$, we have:

$$\left(\int_\Omega f(x)\rho(x)\,dx\right)^2 \le \int_\Omega (f(x))^2\rho(x)\,dx\,.$$

Theorem D.21 (Fubini's theorem) *Suppose $F \in L^1(\Omega_1 \times \Omega_2)$. Then, for almost every $x \in \Omega_1$, $F(x,\cdot) \in L^1(\Omega_2)$ and for almost every $y \in \Omega_2$, $F(\cdot,y) \in L^1(\Omega_1)$. Furthermore, we have, by noting $d(x,y) = dx \otimes dy$:*

$$\int_{\Omega_1\times\Omega_2} F(x,y)d(x,y) = \int_{\Omega_1}\int_{\Omega_2} F(x,y)dydx = \int_{\Omega_2}\int_{\Omega_1} F(x,y)dxdy\,.$$

D.4.2 Properties of L^p spaces

Corollary D.2 *Let $p,q \in]1,\infty[$ be two real numbers such that $1/P + 1/q \le 1$. Then, the function $L^p \times L^q \to L^r$, $(f,g) \mapsto fg$ is a bilinear continuous map if $1/r = 1/p + 1/q$.*

Corollary D.3 *If Ω is a space of finite measure, then $L^p(\Omega, d\mu) \subset L^q(\Omega, d\mu)$ if $p \ge q$.*

Lemma D.4 *Given $(\Omega, d\mu)$ a measure space. Let f be a measurable function and $p \in [1,\infty[$. Then, if*

$$\sup_{\|g\|_{L^{p'}}\le 1} \int_\Omega |f(x)g(x)|\,d\mu(x) \;<\; +\infty\,,$$

then, f is in $L^p(\Omega)$ and we have

$$\|f\|_{L^p} = \sup_{\|g\|_{L^{p'}}\le 1} \left|\int_\Omega f(x)g(x)\,d\mu(x)\right|\,.$$

The next result provides a reverse form of Hölder and Minkwski inequalities for the case $0 < p < 1$. This result is used to show the uniform convexity of certain L^p spaces.

Theorem D.22 *Let $0 < p < 1$ so that the conjugate writes $p' = p((p-1) < 0$. Suppose $f \in L^p(\Omega)$ and*

$$0 \;<\; g \int_\Omega |g(x)|^p\,d\mu(x) < \infty\,.$$

Then,

$$\int_\Omega |f(x)g(x)|\,d\mu(x) \ge \left(\int_\Omega |f(x)|^p\,d\mu(x)\right)^{1/p} \left(\int_\Omega |g(x)|^{p'}\,d\mu(x)\right)^{1/p'}\,.$$

and we have also

$$\||u| + |v|\|_p \;\ge\; \|u\|_p + \|v\|_p\,.$$

Definition D.15 *A function f, measurable on Ω is said to be* essentialy bounded *on Ω if there exists a constant C such that $f(x) \le C$ almost everywhere in Ω. The greatest lower bound of C is called the* essential supremum *of $|f|$ on Ω and is denoted by* $\operatorname{ess\,sup}_{x\in\Omega} |f(x)|$.

We denote by $L^\infty(\Omega)$ the space of all functions f that are essentially bounded on Ω . It is then easy to verify that the functional $\|\cdot\|_\infty$ defined by

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x\in\Omega} |f(x)|\,,$$

is a norm on $L^\infty(\Omega)$.

D.4.3 Density of continuous functions in L^p spaces

At first, we shall introduce new functional spaces, the L^p_{loc} spaces.

Definition D.16 *We denote $L^p_{loc}(\Omega)$ the space of all p-power ($1 \leq p \leq \infty$) locally integrable functions on Ω. It is the set of functions f such that, for any compact subset K of Ω*

$$f \in L^p(K, d\mu) .$$

The Hölder inequality implies that $q \geq p \Rightarrow L^q(K, d\mu) \subset L^p(K, d\mu)$ thus yielding the following result.

Proposition D.10 *If $p < p'$ then, $L^{p'}_{loc}(\Omega)$ is included in the space L^p_{loc}.*

Definition D.17 *Let f be a function of $L^1_{loc}(\Omega)$. The* support *of f, denoted* Supp f *is the remainder of the largest open subset U such that $f_{|U} \equiv 0$.*

In other words, a function $f \in L^1_{loc}(\Omega)$ is said to have finite support if $f(x) = 0$ for all but finitely many x.

Proposition D.11 *Let f be a function of $L^1_{loc}(\Omega)$. The support of f is the set of points $x \in \Omega$ for which every open neighborhood $N(x)$ of x has positive measure:*

$$\mathrm{Supp}\, f = \{x \in \Omega\, , / x \in \textit{some open } N_x\, , \mu(N_x) > 0\, .$$

We introduce the following fundamental result about the density of compact support functions.

Theorem D.23 *Given $p \in [1, \infty[$, the space $C_c(\Omega)$ of all continuous functions with compact support in Ω is dense in the space $L^p(\Omega, d\mu)$.*

Finally, we have a useful imbedding result for L^p spaces over domains with finite volume.

Theorem D.24 *Suppose $vol\Omega = \displaystyle\int_\Omega \mathbf{1} dx < \infty$ and $1 \leq p, q \leq \infty$. If $f \in L^q(\Omega)$, then $f \in L^p(\Omega)$ and*

$$\|f\|_p \,\leq\, (vol\Omega)^{(1/p)-(1/q)} \|f\|_q\, .$$

Hence, $L^q(\Omega) \to L^p(\Omega)$. If $f \in L^\infty(\Omega)$, then

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty\, .$$

Finally, if $f \in L^p(\Omega)$ for $1 \leq p < \infty$ and if there is a constant C such that for all such p

$$\|f\|_p \,\leq\, C\, ,$$

then $f \in L^\infty(\Omega)$ and

$$\|f\|_\infty \,\leq\, C\, .$$

Corollary D.4 *$L^p(\Omega) \subset L^1_{loc}(\Omega)$ for $1 \leq p \leq \infty$ and any domain Ω.*

D.4.4 Completeness of L^p spaces

We give a sequence of important results about L^p spaces.

Proposition D.12 *(i) $L^p(\Omega)$ is a Banach space if $1 \le p \le \infty$.*

(ii) If $1 \le p \le \infty$, a Cauchy sequence in $L^p(\Omega)$ has a subsequence converging pointwise almost everywhere on Ω.

(iii) $L^2(\Omega)$ is a Hilbert space with respect to the inner product

$$\langle f, g\rangle = \int_\Omega f(x)\,\overline{g(x)}\,dx\,.$$

and Hölder inequality for $L^2(\Omega)$ is simply the Schwarz inequality:

$$|\langle f, g\rangle| \;\le\; \|f\|_{L^2}\,\|g\|_{L^2}\,.$$

(iv) The space $L^1(\Omega, d\mu) \cap L^\infty(\Omega, d\mu) \cap L^p(\Omega, d\mu)$ is dense in the space $L^p(\Omega, d\mu)$.

Corollary D.5 *The space $C_0(\Omega)$ is dense in $L^p(\Omega)$ if $1 \le p < \infty$.*
The space $L^p(\Omega)$ is separable if $1 \le p < \infty$.

Remark D.1 *$C(\Omega)$, considered as a closed subspace of $L^\infty(\Omega)$ is not dense in that space. The same consideration applies to $C_0(\Omega)$ and $C_0^\infty(\Omega)$; this leads to conclude that $L^\infty(\Omega)$ is not separable.*

Proposition D.13 *$C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$ if $1 \le p < \infty$.*

D.4.5 The uniform convexity of L^p spaces

Definition D.18 *The norm on any normed space X is called* uniformly convex *if for every number ε satisfying $0 < \varepsilon \le 2$, there exists a number $\delta(\varepsilon) > 0$ such that if $x, y \in X$ satisfy $\|x\|_X = \|y\|_X = 1$ and $\|x - y\|_X \ge \varepsilon$ then*

$$\|(x + y)/2\|_X \le 1 - \delta(\varepsilon)\,.$$

For $1 < p < \infty$, the space $L^p(\Omega)$ is uniformly convex, its norm $\|\cdot\|_p$ satisfying the condition above. Clarkson showed this result via a set of inequalities for $L^p(\Omega)$ that generalizes the parallelogram law in $L^2(\Omega)$.

Lemma D.5 *(i) If $1 \le p < \infty$ and $a \ge 0$, $b \ge 0$, then*

$$(a + b)^p \;\le\; 2^{p-1}\,(a^p + b^p)\,.$$

(ii) If $0 < s < 1$, the function $f(x) = (1 - s^x)/x$ is a decreasing function of $x > 0$.

(iii) If $1 < p \le 2$ and $0 \le t \le 1$, then

$$\left|\frac{1+t}{2}\right|^{p'} + \left|\frac{1-t}{2}\right|^{p'} \;\le\; \left(\frac{1}{2} + \frac{1}{2}t^p\right)^{1/(p-1)}\,,$$

where $p' = p/(p - 1)$ is the exponent conjugate to p.

(iv) *Let* $z, w \in \mathbb{C}$*. If* $1 < p \leq 2$*, then*

$$\left|\frac{z+w}{2}\right|^{p'} + \left|\frac{z-w}{2}\right|^{p'} \leq \left(\frac{1}{2}|z|^p + \frac{1}{2}|w|^p\right)^{1/(p-1)},$$

and if $2 \leq p < \infty$*, then*

$$\left|\frac{z+w}{2}\right|^{p} + \left|\frac{z-w}{2}\right|^{p} \leq \frac{1}{2}|z|^p + \frac{1}{2}|w|^p,$$

Theorem D.25 (Clarkson's inequalites) *Let* $f, g \in L^p(\Omega)$*. For* $1 < p < \infty$ *let* $p' = p/(p-1)$*. If* $2 \leq p < \infty$*, then*

$$\left\|\frac{f+g}{2}\right\|_p^p + \left\|\frac{f-g}{2}\right\|_p^p \leq \frac{1}{2}\|f\|_p^p + \frac{1}{2}\|g\|_p^p,$$

$$\left\|\frac{f+g}{2}\right\|_p^{p'} + \left\|\frac{f-g}{2}\right\|_p^{p'} \geq \left(\frac{1}{2}\|f\|_p^p + \frac{1}{2}\|g\|_p^p\right)^{p'-1}.$$

If $1 < p \leq 2$*, then*

$$\left\|\frac{f+g}{2}\right\|_p^{p'} + \left\|\frac{f-g}{2}\right\|_p^{p'} \leq \left(\frac{1}{2}\|f\|_p^p + \frac{1}{2}\|g\|_p^p\right)^{p'-1},$$

$$\left\|\frac{f+g}{2}\right\|_p^p + \left\|\frac{f-g}{2}\right\|_p^p \geq \frac{1}{2}\|f\|_p^p + \frac{1}{2}\|g\|_p^p.$$

Corollary D.6 *If* $1 < p < \infty$*, the space* $L^p(\Omega)$ *is uniformly convex.*

D.4.6 The dual of $L^p(\Omega)$

The main result of this section is the following theorem that establishes that a linear functional can be represented on L^p spaces when $p \in \mathbb{R}$.

Definition D.19 *Let* $1 \leq p \leq \infty$ *and let* p' *denote the exponent conjugate to* p*. For each element* $g \in L^{p'}(\Omega)$ *we can define a linear functional* L_g *on* $L^p(\Omega)$ *via*

$$L_g(f) = \int_\Omega f(x)vg(x)d\mu(x), \qquad f \in L^p(\Omega).$$

By Hölder's inequality, $|L_g(f)| \leq \|f\|_p\|g\|_{p'}$, so that $L_g \in (L^p(\Omega))'$ and

$$\|L_g\|_{(L^p(\Omega))'} \leq \|g\|_{p'}.$$

Indeed, the equality must holds in the previous expression. Hence, the operator map g to L_g is an isometric isomorphism of $L^{p'}(\Omega)$ onto a subspace of $(L^p(\Omega))'$.

Theorem D.26 (Riesz representation theorem for L^p) *Let* $1 < p < \infty$ *and let* $L \in (L^p(\Omega))'$*. Then, there exists* $g \in L^{p'}(\Omega)$ *such that for all* $f in L^p(\Omega)$

$$L(f) = \int_\Omega f(x)g(x)d\mu(x).$$

Moreover, $\|g\|_{p'} = \|L\|_{(L^p(\Omega))'}$*. Thus,* $(l^p(\Omega))' \cong L^{p'}(\Omega)$*.*
Let $L \in (L^1(\Omega))'$*. Then, there exists* $g \in L^\infty(\Omega)$ *such that for all* $f \in L^1(\Omega)$

$$L(f) = \int_\Omega f(x)g(x)\,d\mu(x),$$

and $\|f\|_\infty = \|L\|_{(L^1(\Omega))'}\|$*. Thus,* $(L^1(\Omega))' \cong L^\infty(\Omega)$*.*