CHAPTER 2

Lebesgue Integration

With a basic knowledge of the Lebesgue measure theory, we now proceed to establish the Lebesgue integration theory.

In this chapter, unless otherwise stated, all sets considered will be assumed to be measurable.

We begin with simple functions.

1. Simple functions vanishing outside a set of finite measure

Recall that the characteristic function $\mathcal{X}_A$ of any set A is defined by

$$\mathcal{X}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

A function $\varphi\colon E \to \mathbb{R}$ is said to be *simple* if there exists $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and $E_1, E_2, \ldots, E_n \subseteq E$ such that $\varphi = \sum_{i=1}^n a_i \mathcal{X}_{E_i}$. Note that here the E_i's are implicitly assumed to be measurable, so a simple function shall always be measurable. We have another characterization of simple functions:

Proposition 2.1. *A function $\varphi\colon E \to \mathbb{R}$ is simple if and only if it takes only finitely many distinct values $a_1, a_2, \ldots, a_n$ and $\varphi^{-1}\{ a_i \}$ is a measurable set for all $i = 1, 2, \ldots, n$.*

Proof. Exercise. □

With the above proposition we see that every simple function φ can be written uniquely in the form

$$\varphi = \sum_{i=1}^n a_i \mathcal{X}_{E_i}$$

where the a_i's are all non-zero and distinct, and the E_i's are disjoint. (Simply take $E_i = \varphi^{-1}\{ a_i \}$ for $i = 1, 2, \ldots, n$ where $a_1, a_2, \ldots, a_n$ are all the distinct values of φ.) We say this is the *canonical representation* of φ.

We adopt the following notation:

Notation. A function $f\colon E \to \mathbb{R}$ is said to *vanish outside a set of finite measure* if there exists a set A with $m(A) < \infty$ such that f vanishes outside A, i.e.

$$f = 0 \qquad \text{on } E \setminus A$$

or equivalently $f(x) = 0$ for all $x \in E \setminus A$. We denote the set of all simple functions defined on E which vanish outside a set of finite measure by $S_0(E)$. Note that it forms a vector space.

We are now ready for the definition of the Lebesgue integral of such functions.

Definition. For any $\varphi \in S_0(E)$ and any $A \subseteq E$, we define the Lebesgue integral of φ over A by

$$\int_A \varphi = \sum_{i=1}^{n} a_i m(E_i \cap A)$$

where $\varphi = \sum_{i=1}^{n} a_i \mathcal{X}_{E_i}$ is the canonical representation of φ. (From now on we shall adopt the convention that $0 \cdot \infty = 0$. We need this convention here because it may happen that one a_i is 0 while the corresponding $E_i \cap A$ has infinite measure. Also note that here A is implicitly assumed to be measurable so $m(E_i \cap A)$ makes sense. We shall never integrate over non-measurable sets.)

It follows readily from the above definition that

$$\int_A \varphi = \int_E \varphi \mathcal{X}_A$$

for any $\varphi \in S_0(E)$ and for any $A \subseteq E$.

We now establish some major properties of this integral (with monotonicity and linearity being probably the most important ones). We begin with the following lemma.

Lemma 2.2. *Suppose $\varphi = \sum_{i=1}^{n} a_i \mathcal{X}_{E_i} \in S_0(E)$ where the E_i's are disjoint. Then for any $A \subseteq E$,*

$$\int_A \varphi = \sum_{i=1}^{n} a_i m(E_i \cap A)$$

holds even if the a_i's are not necessarily distinct.

Proof. If $\varphi = \sum_{j=1}^{m} b_j \mathcal{X}_{B_j}$ is the canonical representation of φ, we have

$$B_j = \bigcup_{\{\, i\colon a_i = b_j \,\}} E_i \tag{1}$$

for $j = 1, 2, \ldots, m$ and

$$\{\, 1, 2, \ldots, n \,\} = \bigcup_{j=1}^{m} \{\, i\colon a_i = b_j \,\}, \tag{2}$$

where both unions are disjoint unions. Hence for any $A \subseteq E$, we have

$$\begin{aligned}
\int_A \varphi &= \sum_{j=1}^{m} b_j m\left(B_j \cap A\right) && \text{(by definition of the integral)} \\
&= \sum_{j=1}^{m} b_j m\left(\bigcup_{\{\,i:\ a_i=b_j\,\}} (E_i \cap A)\right) && \text{(by (1))} \\
&= \sum_{j=1}^{m} b_j \sum_{\{\,i:\ a_i=b_j\,\}} m(E_i \cap A) && \text{(by finite additivity of } m\text{)} \\
&= \sum_{j=1}^{m} \sum_{\{\,i:\ a_i=b_j\,\}} a_i m(E_i \cap A) \\
&= \sum_{i=1}^{n} a_i m(E_i \cap A) && \text{(by (2))}
\end{aligned}$$

This completes our proof. □

Proposition 2.3. *(Properties of the Lebesgue integral) Suppose $\varphi, \psi \in S_0(E)$. Then for any $A \subseteq E$,*

(a) $\int_A (\varphi + \psi) = \int_A \varphi + \int_A \psi$. *(Note that $\varphi + \psi \in S_0(E)$ too by the vector space structure of $S_0(E)$.)*

(b) $\int_A \alpha\varphi = \alpha \int_A \varphi$ *for all* $\alpha \in \mathbb{R}$. *(Note $\alpha\varphi \in S_0(E)$ again.)*

(c) If $\varphi \le \psi$ a.e. on A then $\int_A \varphi \le \int_A \psi$.

(d) If $\varphi = \psi$ a.e. on A then $\int_A \varphi = \int_A \psi$.

(e) If $\varphi \ge 0$ a.e. on A and $\int_A \varphi = 0$, then $\varphi = 0$ a.e. on A.

(f) $\left|\int_A \varphi\right| \le \int_A |\varphi|$. *(Note $|\varphi| \in S_0(E)$ too. Why?)*

Remark. (a) and (b) are known as the *linearity* property of the integral, while (c) is known as the *monotonicity* property. Furthermore, Lemma 2.2 is now seen to hold by the linearity of the integral even without the disjointness assumption on the E_i's.

Proof. (a) Let $\varphi = \sum_{i=1}^{n} a_i \mathcal{X}_{A_i}$ and $\psi = \sum_{j=1}^{m} b_j \mathcal{X}_{B_j}$ be canonical representations of φ and ψ respectively. Then noting that $\mathcal{X}_{A_i} = \sum_{j=1}^{m} \mathcal{X}_{A_i \cap B_j}$ for all i and $\mathcal{X}_{B_j} =$

$\sum_{i=1}^n \mathcal{X}_{A_i \cap B_j}$ for all j we see that

$$\varphi = \sum_{i=1}^n a_i \mathcal{X}_{A_i} = \sum_{i=1}^n \sum_{j=1}^m a_i \mathcal{X}_{A_i \cap B_j}$$

$$\psi = \sum_{j=1}^m b_j \mathcal{X}_{B_j} = \sum_{i=1}^n \sum_{j=1}^m b_j \mathcal{X}_{A_i \cap B_j}$$

Consequently

$$\varphi + \psi = \sum_{i=1}^n \sum_{j=1}^m (a_i + b_j) \mathcal{X}_{A_i \cap B_j}.$$

But the $A_i \cap B_j$'s are disjoint. So by Lemma 2.2 we have

$$\int_A \varphi = \sum_{i=1}^n \sum_{j=1}^m a_i m(A_i \cap B_j \cap A)$$

$$\int_A \psi = \sum_{i=1}^n \sum_{j=1}^m b_j m(A_i \cap B_j \cap A)$$

and

$$\int_A (\varphi + \psi) = \sum_{i=1}^n \sum_{j=1}^m (a_i + b_j) m(A_i \cap B_j \cap A).$$

Hence $\int_A (\varphi + \psi) = \int_A \varphi + \int_A \psi$.

(b) If $\alpha = 0$ the result is trivial; if not, then let $\varphi = \sum_{i=1}^n a_i \mathcal{X}_{A_i}$ be the canonical representation of φ. We see that $\alpha\varphi = \sum_{i=1}^n \alpha a_i \mathcal{X}_{A_i}$ is the canonical representation of $\alpha\varphi$ and hence the result follows.

(c) Since $\int_A \varphi - \int_A \psi = \int_A (\varphi - \psi)$ by linearity, it suffices to show $\int_A \phi \geq 0$ whenever $\phi \geq 0$ a.e. on A. This is easy, since if $a_1, a_2, \ldots, a_n$ are the distinct values of ϕ, then

$$\int_A \phi = \sum_{\{ i \colon a_i < 0 \}} a_i m(\phi^{-1}\{ a_i \} \cap A) + \sum_{\{ i \colon a_i \geq 0 \}} a_i m(\phi^{-1}\{ a_i \} \cap A) \quad \geq \sum_{\{ i \colon a_i < 0 \}} a_i \cdot 0 = 0$$

where the inequality follows from the fact that $m(\phi^{-1}\{ a_i \cap A \}) = 0$ for all $a_i < 0$.

(d) This is immediate from (c).

(e) Since it is given that $\varphi \geq 0$ a.e. on A, it suffices to show $m(\{ x : \varphi(x) > 0 \} \cap A) = 0$. Suppose not, then there exists $a > 0$ such that $m(\{ x : \varphi(x) = a \} \cap A) > 0$ so $\int_A \varphi \geq a \cdot m(\{ x : \varphi(x) = a \} \cap A) > 0$. This leads to a contradiction.

(f) This follows directly from monotonicity since $-|\varphi| \leq \varphi \leq |\varphi|$.

□

EXERCISE 2.1. Show that if $A, B \subseteq E$, $A \cap B = \emptyset$ and $\varphi \in S_0(E)$, then $\int_{A \cup B} \varphi = \int_A \varphi + \int_B \varphi$.

EXERCISE 2.2. Show that if $\varphi \in S_0(E)$ vanishes outside F, then $\int_A \varphi = \int_{A\cap F} \varphi$ for any $A \subseteq E$.

EXERCISE 2.3. Show that if $A \subseteq B \subseteq E$ and $0 \leq \varphi \in S_0(E)$, then $\int_A \varphi \leq \int_B \varphi$.

2. Bounded measurable functions vanishing outside a set of finite measure

Resembling the construction of the Riemann integral (using simple functions in place of step functions), we define the upper and lower Lebesgue integrals.

DEFINITION. Let $f\colon E \to \mathbb{R}$ be a bounded function which vanish outside a set of finite measure. For any $A \subseteq E$, we define the *upper integral* and the *lower integral* of f on A by

$$\overline{\int_A} f = \inf\left\{ \int_A \psi \colon f \leq \psi \text{ on } A, \psi \in S_0(E) \right\}$$
$$\underline{\int_A} f = \sup\left\{ \int_A \varphi \colon f \geq \varphi \text{ on } A, \varphi \in S_0(E) \right\}$$

If the two values agree we denote the common value by $\int_A f$. (Again the set A is implicitly assumed to be measurable so that $\int_A \psi$ and $\int_A \varphi$ make sense.)

Note that both the infimum and the supremum in the definitions of the upper and lower integrals exist because f is bounded and vanishes outside a set of finite measure. (This is why f has to be a bounded function here.) It is evident that for the functions we investigated in Section 1 (namely simple functions vanishing outside a set of finite measure) have their upper and lower integrals both equal to their integral as defined in the last section. In other words, if $\varphi \in S_0(E)$ then $\overline{\int}_A \varphi = \underline{\int}_A \varphi = \int_A \varphi$, where the last integral is as defined in the last section. It is also clear that $-\infty < \underline{\int}_A f \leq \overline{\int}_A f < \infty$ whenever they are defined; we investigate when $\underline{\int}_A f = \overline{\int}_A f$.

PROPOSITION 2.4. *Let f be as in the above definition. Then $\underline{\int}_A f = \overline{\int}_A f$ for all $A \subseteq E$ if and only if f is measurable.*

PROOF. ($\Leftarrow$) Let f be a bounded measurable function defined on E which vanishes outside F with $F \subseteq E$ and $m(F) < \infty$. Then for each positive integer n there are simple functions $\varphi_n, \psi_n \in S_0(E)$ vanishing outside F such that $\varphi_n \leq f \leq \psi_n$ and $0 \leq \psi_n - \varphi_n \leq$

$1/n$ on E (Why?). Hence for any $A \subseteq E$, we have

$$
\begin{aligned}
0 &\le \overline{\int}_A f - \underline{\int}_A f && \text{(subtraction makes sense since both integrals are finite)}\\
&\le \int_A \psi_n - \int_A \varphi_n && \text{(by definition of } \overline{\int}_A f \text{ and } \underline{\int}_A f)\\
&= \int_A (\psi_n - \varphi_n) && \text{(by linearity of Section 1)}\\
&= \int_{A\cap F} (\psi_n - \varphi_n) && \text{(by Exercise 2.2, } \varphi_n = \psi_n = 0 \text{ outside } F)\\
&\le \int_F (\psi_n - \varphi_n) && \text{(by Exercise 2.3, } \psi_n - \varphi_n \ge 0 \text{ on } F \text{ and } A\cap F \subseteq F)\\
&\le m(F)/n && \text{(by monotonicity of Section 1, } \psi_n - \varphi_n \le 1/n \text{ on } F)
\end{aligned}
$$

for all n. Letting $n \to \infty$ we have $\underline{\int}_A f = \overline{\int}_A f$. (Note here we used the fact that $m(F) < \infty$.)

($\Rightarrow$) Suppose $\overline{\int}_A f = \underline{\int}_A f$ for any $A \subseteq E$. Then $\overline{\int}_E f = \underline{\int}_E f$. Denote the common value by L. Then for all positive integers n there exists $\varphi_n, \psi_n \in S_0(E)$ such that $\varphi_n \le f \le \psi_n$ on E and $L - 1/n \le \int_E \varphi_n \le \int_E \psi_n \le L + 1/n$. (Note here we used the fact that $L \in \mathbb{R}$, a fact we have observed before.) Let $\varphi = \sup_n \varphi_n$ and $\psi = \inf_n \psi_n$. We shall show $\varphi = \psi$ a.e. on E. (Then the desired conclusion follows since then $\varphi \le f \le \psi$ on E implies that $\varphi = f = \psi$ a.e. on E and hence f is measurable.) To show that $\varphi = \psi$ a.e. on E, let $\triangle = \{\, x \in E \colon \varphi(x) \neq \psi(x) \,\}$ and $\triangle_i = \{\, x \in E \colon \psi(x) - \varphi(x) > 1/i \,\}$. Then $\triangle = \bigcup_{i=1}^{\infty} \triangle_i$. We wish to show $m(\triangle) = 0$, which will be true if we can show $m(\triangle_i)=0$ for all i. Now for any i and n, since $\psi_n - \varphi_n \ge \psi - \varphi \ge 1/i$ on $\triangle_i$, we have

$$
\begin{aligned}
\frac{1}{i} m(\triangle_i) &= \int_{\triangle_i} \frac{1}{i} && \text{(by definition of the integral)}\\
&\le \int_{\triangle_i} (\psi_n - \varphi_n) && \text{(by monotonicity in Section 1)}\\
&\le \int_E (\psi_n - \varphi_n) && \text{(by Exercise 2.3, } \psi_n - \varphi_n \ge 0 \text{ on } E \text{ and } \triangle_i \subseteq E)\\
&\le \int_E \psi_n - \int_E \varphi_n && \text{(by linearity of Section 1)}\\
&\le 2/n && \text{(by our choice of } \varphi_n, \psi_n).
\end{aligned}
$$

Letting $n \to \infty$ we have $m(\triangle_i) = 0$ for all i, completing our proof. □

Notation. We shall denote the set of all (real-valued) bounded measurable functions defined on E which vanishes outside a set of finite measure by $B_0(E)$.

So from now on for $f \in B_0(E)$, we have

$$\int_A f = \inf\left\{ \int_A \psi \colon f \leq \psi \in S_0(E) \right\} = \sup\left\{ \int_A \varphi \colon f \geq \varphi \in S_0(E) \right\}$$

for any $A \subseteq E$.

Note also that $B_0(E)$ is a vector lattice, by which we mean it is a vector space partially ordered by $\leq$ (such that $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in E$) and every two elements of it (say $f, g \in B_0(E)$) have a least upper bound in it (namely $f \vee g \in B_0(E)$). (Why is it a least upper bound?)

We have the following nice proposition concerning the relationship between the Riemann and the Lebesgue integrals.

PROPOSITION 2.5. *If $f \colon [a, b] \to \mathbb{R}$ is Riemann integrable on the closed and bounded interval $[a, b]$, then $f \in B_0([a, b])$ and*

$$(\mathcal{R}) \int_a^b f = (\mathcal{L}) \int_{[a,b]} f, \tag{3}$$

where the $(\mathcal{R})$ and $(\mathcal{L})$ represents Riemann integral and Lebesgue integral respectively.

PROOF. Since step functions defined on closed and bounded interval $[a, b]$ are simple and have the same Lebesgue and Riemann integral over $[a, b]$ (why?), we see from the definitions

$$\begin{aligned}
(\mathcal{R})\underline{\int_a^b} f &= \sup\left\{ \int_a^b \varphi \colon f \geq \varphi \text{ step on } [a, b] \right\} \\
(\mathcal{L})\underline{\int_{[a,b]}} f &= \sup\left\{ \int_{[a,b]} \varphi \colon f \geq \varphi \text{ simple on } [a, b] \right\} \\
(\mathcal{L})\overline{\int_{[a,b]}} f &= \inf\left\{ \int_{[a,b]} \psi \colon f \leq \psi \text{ simple on } [a, b] \right\} \\
(\mathcal{R})\overline{\int_a^b} f &= \inf\left\{ \int_a^b \psi \colon f \leq \psi \text{ step on } [a, b] \right\}
\end{aligned}$$

that

$$(\mathcal{R})\underline{\int_a^b} f \leq (\mathcal{L})\underline{\int_{[a,b]}} f \leq (\mathcal{L})\overline{\int_{[a,b]}} f \leq (\mathcal{R})\overline{\int_a^b} f \tag{4}$$

whenever the four quantities exist. Now if f is Riemann integrable over $[a, b]$, then f is bounded on $[a, b]$. Since $[a, b]$ is of finite measure, we see that all four quantities in (4) exist. In that case $(\mathcal{R})\underline{\int_a^b} f = (\mathcal{R})\overline{\int_a^b} f$ as well so all four quantities in (4) are equal, which implies that f is measurable (so $f \in B_0([a, b])$) and (3) holds. $\square$

Proposition 2.6. *(Properties of the Lebesgue integral) Suppose* $f, g \in B_0(E)$*. Then* $f+g, \alpha f, |f| \in B_0(E)$*, and for any* $A \subseteq E$*, we have*

(a) $\int_A (f+g) = \int_A f + \int_A g$.

(b) $\int_A \alpha f = \alpha \int_A f$ *for all* $\alpha \in \mathbb{R}$.

(c) $\int_A f = \int_E f \mathcal{X}_A$.

(d) If $B \subseteq A$ *then* $\int_A f = \int_B f + \int_{A\setminus B} f$.

(e) If $B \subseteq A$ *and* $f \geq 0$ *a.e. on* A *then* $\int_B f \leq \int_A f$.

(f) If $f \leq g$ *a.e. on* A *then* $\int_A f \leq \int_A g$.

(g) If $f = g$ *a.e. on* A *then* $\int_A f = \int_A g$.

(h) If $f \geq 0$ *a.e. on* A *and* $\int_A f = 0$*, then* $f = 0$ *a.e. on* A.

(i) $\left|\int_A f\right| \leq \int_A |f|$.

Proof. We prove only (h); the others are easy and left as an exercise.

(h) For each positive integer n let $A_n = \{\, x \in A \colon f(x) \geq 1/n \,\}$. Then

$$\begin{aligned} 0 = \int_A f &\geq \int_{A_n} f && \text{(by (e))} \\ &\geq \int_{A_n} \frac{1}{n} && \text{(by (f))} \\ &= \frac{1}{n} m(A_n) && \text{(by definition of the integral)} \\ &\geq 0 \end{aligned}$$

so $m(A_n) = 0$. Since this holds for all n, we see from $f^{-1}(0,\infty) \cap A = \cup_{n=1}^{\infty} A_n$ that $0 \leq m(f^{-1}(0,\infty) \cap A) \leq \sum_{n=1}^{\infty} m(A_n) = 0$ so $m(f^{-1}(0,\infty) \cap A) = 0$. Together with $f \geq 0$ a.e. on A, we see that $f = 0$ a.e. on A. □

We end this section with the important *Bounded Convergence Theorem.*

Theorem 2.7. *(Bounded Convergence Theorem) Suppose* $m(E) < \infty$*, and* $\{ f_n \}$ *is a sequence of measurable functions defined and uniformly bounded on* E *by some constant* $M > 0$*, i.e.*

$$|f_n| \leq M \qquad \text{for all } n \text{ on } E.$$

If $\{ f_n \}$ *converges to a function* f *(pointwisely) a.e. on* E*, then* f *is also bounded measurable on* E*,* $\lim_{n\to\infty} \int_E f_n$ *exists (in* $\mathbb{R}$*) and is given by*

$$\lim_{n\to\infty} \int_E f_n = \int_E f \tag{5}$$

PROOF. Under the given assumptions it is clear that f, being the pointwise limit of $\{ f_n \}$ a.e. on E, is bounded (by M) and measurable on E. We wish to show $\lim_{n\to\infty} \int_E f_n$ exists and (5) holds. The result is trivial if $m(E) = 0$. So assume $m(E) > 0$ and let $\varepsilon > 0$ be given. Then for each natural number i let

$$E_i = \{ x \in E \colon |f_j(x) - f(x)| \geq \varepsilon/2m(E) \text{ for some } j \geq i \}.$$

Then $\{ E_i \}$ is a decreasing sequence of sets with $m(E_1) \leq m(E) < \infty$. So

$$m(E_i) \downarrow m\left(\bigcap_{i=1}^{\infty} E_i\right) = 0,$$

the last equality follows from the fact that

$$m\left(\bigcap_{i=1}^{\infty} E_i\right) \leq m\left(\{ x \in E \colon f_n(x) \nrightarrow f(x) \}\right) = 0.$$

Choose N large enough such that $m(E_N) < \varepsilon/4M$ and let $A = E_N$. Then $|f_n - f| < \varepsilon/2m(E)$ everywhere on $E \setminus A$ for all $n \geq N$, and hence whenever $n \geq N$ we have

$$\begin{aligned}
\left| \int_E f_n - \int_E f \right| &\leq \int_E |f_n - f| && \text{(by linearity and (i))} \\
&= \int_{E\setminus A} |f_n - f| + \int_A |f_n - f| && \text{(by (e))} \\
&\leq \int_{E\setminus A} \frac{\varepsilon}{2m(E)} + \int_A 2M && \text{(by our choice of } N \text{ and that } n \geq N) \\
&= \frac{\varepsilon m(E \setminus A)}{2m(E)} + 2Mm(A) \\
&\leq \frac{\varepsilon}{2} + 2M\frac{\varepsilon}{4M} \\
&= \varepsilon.
\end{aligned}$$

Hence $\lim_{n\to\infty} \int_E f_n$ exists (in $\mathbb{R}$) and (5) holds.

(Alternatively when $\varepsilon > 0$ is given, by Littlewood's 3rd Principle we can choose a subset A of E with $m(A) < \varepsilon/4M$ such that $\{ f_n \}$ converges uniformly to f on $E \setminus A$. Then choose N large enough such that $|f_n - f| < \varepsilon/2m(E)$ everywhere on $E \setminus A$ for all $n \geq N$, we see that whenever $n \geq N$, we have (as in the above)

$$\left| \int_E f_n - \int_E f \right| < \varepsilon.$$

Hence $\lim_{n\to\infty} \int_E f_n$ exists (in $\mathbb{R}$) and (5) holds.) □

REMARK. Note that the first argument is just an adaptation of the proof of Littlewood's 3rd Principal to the present situation.

EXERCISE 2.4. Find an example to show that the assumption $m(E) < \infty$ cannot be dropped in the Bounded Convergence Theorem.

EXERCISE 2.5. Prove or disprove the following: Let E be of finite or infinite measure. If $\{ f_n \}$ is a sequence of uniformly bounded measurable functions on E which vanishes outside a set of finite measure and converges pointwisely to $f \in B_0(E)$ a.e. on E, then $\lim_{n\to\infty} \int_E f_n = \int_E f$. (Compare with the statement of the Bounded Convergence Theorem.)

3. Integration of non-negative measurable functions

We integrate non-negative measurable functions through approximation by bounded measurable functions vanishing outside a set of finite measure, which we studied in the last section.

DEFINITION. For a non-negative measurable function $f\colon E \to [0,\infty]$ (where E is a set which may be of finite or infinite measure), we define

$$\int_A f = \sup\left\{ \int_A \varphi \colon \varphi \le f \text{ on } A, \varphi \in B_0(E) \right\}$$

for any $A \subseteq E$.

Note that for non-negative bounded measurable functions vanishing outside a set of finite measure, this definition agrees with the old one. Also note that we allow the functions to take infinite value here.

We verify the monotonicity and linearity of such integrals.

PROPOSITION 2.8. *Suppose $f, g\colon E \to [0,\infty]$ are non-negative measurable and $A \subseteq E$.*

(a) If $f \le g$ a.e. on A then $\int_A f \le \int_A g$.

(b) For $\alpha > 0$, $f + g$ and αf are non-negative measurable functions too and

$$\int_A (f+g) = \int_A f + \int_A g$$

$$\int_A \alpha f = \alpha \int_A f.$$

PROOF. (a) This is clearly true, for if $\varphi \in B_0(E)$ and $\varphi \le f$ on A, then $\varphi \le g$ on A so $\int_A \varphi \le \int_A g$ by definition of $\int_A g$. Taking supremum over all such φ's, we get $\int_A f \le \int_A g$.

(b) The assertion on $\int_A \alpha f$ can be proved using supremum arguments similar to that in (a) by noting that for $\alpha > 0$ and $\varphi \in B_0(E)$, $\varphi/\alpha \le f$ on A whenever $\varphi \le \alpha f$ on A, and $\alpha\varphi \le \alpha f$ on A whenever $\varphi \le f$ on A.

To verify $\int_A (f+g) = \int_A f + \int_A g$, note that if $\varphi, \psi \in B_0(E)$ and $\varphi \leq f$, $\psi \leq g$ on A, then $\varphi + \psi \in B_0(E)$ and $\varphi + \psi \leq f + g$ on A so

$$\begin{aligned}\int_A (f+g) &\geq \int_A (\varphi + \psi) && \text{(by definition of } \int_A (f+g)\text{)}\\ &= \int_A \varphi + \int_A \psi && \text{(by linearity of the last section);}\end{aligned}$$

take supremum over all such φ's and ψ's we have $\int_A (f+g) \geq \int_A f + \int_A g$. For the opposite inequality, note that if $\phi \in B_0(E)$ with $\phi \leq f + g$ on A, then write $\varphi = \min\{\phi, f\}$ and $\psi = \phi - \varphi$ we see that $\varphi, \psi \in B_0(E)$ (note (i) $-M \leq \varphi \leq \phi \leq M$ if $|\phi| \leq M$ so φ is bounded on E; (ii) $\psi = \phi - \varphi$ is bounded on E because both ϕ and φ are; (iii) measurability of φ, ψ is clear; and (iv) from $\varphi = \min\{\phi, f\}$ and $\psi = \max\{0, \phi - f\}$ we see that $\varphi, \psi = 0$ whenever $\phi = 0$ so φ, ψ vanishes outside a set of finite measure). Further, we have $\varphi \leq f$, $\psi \leq g$ on A. Hence

$$\begin{aligned}\int_A \phi &= \int_A \varphi + \int_A \psi && \text{(by linearity in the last section)}\\ &\leq \int_A f + \int_A g && \text{(by definition of } \int_A f \text{ and } \int_A g\text{)}\end{aligned}$$

Taking supremum over all such ϕ's we get $\int_A (f+g) \leq \int_A f + \int_A g$ and we are done.

□

THEOREM 2.9. *(Fatou's Lemma) Suppose $\{ f_n \}$ is a sequence of non-negative measurable functions defined on E and $\{ f_n \}$ converges (pointwisely) to a non-negative function f a.e. on E. Then*

$$\int_E f \leq \liminf_{n\to\infty} \int_E f_n.$$

PROOF. Let $h \in B_0(E)$ and $h \leq f$ on E. Then there exists $A \subseteq E$ with $m(A) < \infty$ such that $h = 0$ outside A. Let $h_n = \min\{f_n, h\}$ on A, we have h_n is uniformly bounded and measurable on A: in fact if $|h| \leq M$ on E, then $h_n = \min\{f_n, h\} \geq \min\{0, h\} \geq -M$ and $h_n = \min\{f_n, h\} \leq h \leq M$ so $|h_n| \leq M$ on A. Further, with the observation that $\min\{a, b\} = (a + b - |a - b|)/2$ for all real a, b we have

$$h_n = \frac{f_n + h - |f_n - h|}{2} \to \frac{f + h - |f - h|}{2} = \min\{f, h\} = h$$

on A. Since $m(A) < \infty$, we can conclude by Bounded Convergence Theorem that $\int_A h = \lim_{n\to\infty} \int_A h_n$. So assuming $h_n = 0$ on $E \setminus A$, we have

$$\int_E h = \int_A h = \lim_{n\to\infty} \int_A h_n = \lim_{n\to\infty} \int_E h_n \leq \liminf_{n\to\infty} \int_E f_n$$

where the first equality follows from $h = 0$ on $E \setminus A$ and the last inequality holds because $h_n \leq f_n$ on E for all n. Taking supremum over all such h's, we get the desired inequality.

□

Theorem 2.10. *(Monotone Convergence Theorem) If $\{ f_n \}$ is an increasing sequence of non-negative measurable functions defined on E (increasing in the sense that $f_n \le f_{n+1}$ for all n on E) and $f_n \to f$ a.e.on E, then*

$$\int_E f_n \uparrow \int_E f$$

by which it means $\{\int_E f_n\}$ is an increasing sequence with limit $\int_E f$.

In symbol,

$$0 \le f_n \uparrow f \ \text{a.e.on } E \Rightarrow \int_E f_n \uparrow \int_E f$$

Proof.

$$\int_E f \le \liminf_{n\to\infty} \int_E f_n \le \limsup_{n\to\infty} \int_E f_n \le \int_E f,$$

the first inequality follows from Fatou's Lemma, the last inequality follows from $f_n \le f$ on E for all n. Hence $\int_E f_n \uparrow \int_E f$. (That $\int_E f_n$ increases as n increases is immediate from monotonicity of such integrals.) □

Corollary 2.10.1. *(Extension of Fatou's lemma) If $\{ f_n \}$ is a sequence of non-negative measurable functions on E, then $\int_E \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int_E f_n$.*

The proof is easy and left as an exercise.

The following proposition is concerned with the *absolute continuity* of the integral. (The concept of absolute continuity is to be defined in Chapter 3.)

Proposition 2.11. *Suppose f is a non-negative measurable function defined on E such that $\int_E f < \infty$. Then for all $\varepsilon > 0$, there is a $\delta > 0$ such that*

$$\int_A f < \varepsilon$$

whenever $A \subseteq E$ with $m(A) < \delta$.

Proof. The result clearly holds if f is bounded on E. Suppose now f is not necessarily bounded, we see that $(f \wedge n) \uparrow f$ so by Monotone Convergence Theorem

$$\int_A f = \lim_{n\to\infty} \int_A (f \wedge n)$$

for all $A \subseteq E$. Note that by assumption $\int_E f < \infty$ so both sides of the equality above are finite. Hence if $\varepsilon > 0$ is given, then there is a N such that $\left|\int_A f - \int_A (f \wedge N)\right| < \varepsilon/2$. Take $\delta = \varepsilon/2N$, we see that

$$\int_A f \le \left|\int_A f - \int_A (f \wedge N)\right| + \int_A (f \wedge N) \le \varepsilon/2 + Nm(A) \le \varepsilon/2 + N\delta < \varepsilon$$

whenever $A \subseteq E$ with $m(A) < \delta$. So we are done. □

EXERCISE 2.6. For a non-negative measurable function f defined on E, show that $\int_A f = \int_E f\mathcal{X}_A$ for any $A \subseteq E$. Also show that $\int_A f \leq \int_B f$ if $A \subseteq B \subseteq E$.

EXERCISE 2.7. Show that if $A, B \subseteq E$ are disjoint and f is a non-negative measurable function defined on E, then $\int_{A\cup B} f = \int_A f + \int_B f$.

EXERCISE 2.8. Show that if f is a non-negative measurable function defined on E and $\int_E f = 0$, then $f = 0$ a.e. on E.

EXERCISE 2.9. Show that if f is a non-negative measurable function defined on E and $\int_E f < \infty$, then f is finite a.e.

4. Extended real-valued integrable functions

In the last section we integrated non-negative measurable functions, and in this section we wish to drop the non-negative requirement. Recall that it is a natural requirement that our integral be linear, and now we can integrate a general non-negative measurable function, so it is tempting to define the integral of a general (not necessarily non-negative) measurable function f to be $\int f^+ - \int f^-$ where $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$, since f^+, f^- are non-negative measurable and they sum up to f. But it turns out that we cannot always do that, because it may well happen that $\int f^+$ and $\int f^-$ are both infinite, in which case their difference would be meaningless. (Remember that $\infty - \infty$ is undefined.) So we need to restrict ourselves to a smaller class of functions than the collection of all measurable functions when we drop the non-negative requirement and come to the following definition.

DEFINITION. For $f \colon E \to [-\infty, \infty]$, denote $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$. Then f is said to be *integrable* if and only if both $\int_E f^+$ and $\int_E f^-$ are finite, in which case we define the integral of f by

$$\int_A f = \int_A f^+ - \int_A f^-$$

for any $A \subseteq E$.

NOTATION. We shall denote the class of all (extended real-valued) integrable functions defined on E by $\mathcal{L}(E)$.

Note that in the above definition, f^+ and f^- are both non-negative measurable, so for any set $A \subseteq E$, $\int_A f^+$ and $\int_A f^-$ are both defined according to Section 3. Furthermore, $\int_A f^+ \leq \int_E f^+ < \infty$ (by Exercise 2.6) and similarly $\int_A f^- < \infty$ so their difference makes sense now. Also note that for non-negative integrable functions this definition agrees with our old one.

We provide an alternative characterization of integrable functions.

PROPOSITION 2.12. *A measurable function f defined on E is integrable if and only if $\int_E |f| < \infty$.*

PROOF. Easy! Just note that $|f| = f^+ + f^-$. □

We proceed to investigate the structure of $\mathcal{L}(E)$. We want to say it is a vector lattice. But we have to be careful here: Given $f, g \in \mathcal{L}(E)$ it may well happen that $f(x) = +\infty$ and $g(x) = -\infty$ for some $x \in E$ and then $f + g$ cannot be defined by $f(x) + g(x)$ at that x. Luckily there cannot be too many such x's, in the sense that the set of all such x's is of measure zero. In fact every integrable function is finite a.e., a result which the reader should prove from Exercise 2.9. We know that the values of a function on a set of measure zero are not important as far as integration is concerned. (This was observed as in the case of bounded measurable functions vanishing outside a set of finite measure; the reader should verify this for the case of general integrable functions as well.) So that eliminates our previous worries: more precisely, let us agree from now on two functions $f, g\colon E \to [-\infty, \infty]$ are said to be equal (write $f = g$) if and only if they take the same values a.e.on E, and $f + g$ shall mean a function whose value at x is equal to $f(x) + g(x)$ for a.e.$x \in E$. Also say $f \leq g$ if and only if $f(x) \leq g(x)$ for a.e.$x \in E$. Then we have the following proposition.

PROPOSITION 2.13. *$\mathcal{L}(E)$ forms a vector lattice (partially ordered by $\leq$).*

PROOF. If $f, g \in \mathcal{L}(E)$, then $\int_E |f + g| \leq \int_E |f| + \int_E |g| < \infty$ (we are using linearity and monotonicity in Section 3 here) and hence $f + g \in \mathcal{L}(E)$ (the measurability of $f + g$ is previously known). The rest of the proposition is trivial. □

With the vector lattice structure of $\mathcal{L}(E)$ it is natural to ask whether the integral is linear and monotone or not. We expect it to be true; we verify it below.

PROPOSITION 2.14. *For any $f, g \in \mathcal{L}(E)$ and $A \subseteq E$, we have $\int_A (f+g) = \int_A f + \int_A g$ and $\int_A \alpha f = \alpha \int_A f$. Furthermore, if $f \leq g$ a.e.on A then $\int_A f \leq \int_A g$.*

PROOF. The parts for monotonicity and $\int_A \alpha f = \alpha \int_A f$ are easy and left as an exercise. (Simply make use of the corresponding results in Section 3.)

So now let $f, g \in \mathcal{L}(E)$ and $A \subseteq E$ be given, and we prove $\int_A (f + g) = \int_A f + \int_A g$. By definition of the integral, the LHS is just $\int_A (f + g)^+ - \int_A (f + g)^-$ and the RHS is $\int_A f^+ - \int_A f^- + \int_A g^+ - \int_A g^-$, all terms being finite. So it suffices to show

$$\int_A (f + g)^+ + \int_A f^- + \int_A g^- = \int_A (f + g)^- + \int_A f^+ + \int_A g^+, \tag{6}$$

which will be true if we can show

$$(f + g)^+ + f^- + g^- = (f + g)^- + f^+ + g^+ \tag{7}$$

a.e. on A because we can then use linearity of Section 3 to conclude that (6) is true. But (7) is clearly true a.e., because $(f + g)^+ - (f + g)^- = f + g = f^+ - f^- + g^+ - g^-$ a.e., all terms being finite a.e. This completes our proof. □

Finally we prove the important *Generalized Lebesgue Dominated Convergence Theorem.*

THEOREM 2.15. *If* $\{f_n\}, \{g_n\}$ *are sequences of measurable functions defined on* E, $|f_n| \leq g_n$, $f = \lim_{n\to\infty} f_n$, $g = \liminf_{n\to\infty} g_n$ *and* $\lim_{n\to\infty} \int_E g_n = \int_E g < \infty$, *then* $\lim_{n\to\infty} \int_E f_n$ *exists and is equal to* $\int_E f$.

PROOF. Since $|f_n| \leq g_n$ implies $g_n \pm f_n$ are non-negative measurable, we see (by the Extension of Fatou's Lemma, Corollary 2.10.1) that

$$\int_E g + \int_E f = \int_E \liminf_{n\to\infty}(g_n + f_n) \leq \liminf_{n\to\infty} \int_E (g_n + f_n) = \int_E g + \liminf_{n\to\infty} \int_E f_n.$$

and similarly

$$\int_E g - \int_E f = \int_E \liminf_{n\to\infty}(g_n - f_n) \leq \liminf_{n\to\infty} \int_E (g_n - f_n) = \int_E g - \limsup_{n\to\infty} \int_E f_n.$$

So $\int_E f \leq \liminf_{n\to\infty} \int_E f_n \leq \limsup_{n\to\infty} \int_E f_n \leq \int_E f$ (note here we used the assumption that $\int_E g < \infty$) and the desired conclusion follows. □

COROLLARY 2.15.1. *(Lebsegue Dominated Convergence Theorem) Suppose a sequence of measurable functions* $\{f_n\}$ *defined on* E *converges pointwisely a.e. on* E *to* f. *If* $|f_n| \leq g$ *on* E *for some integrable function* g, *then* $\int_E f_n$ *converges to* $\int_E f$.

A final word of remark: The idea of this section extends readily to complex-valued functions, and the readers who are familar with general measure theory should find that the results in the whole chapter is valid on a general measure space without needing the slightest modification.