CHAPTER 7

Infinite Domains

In Chapter 2, we managed to reduce the problem to a bounded region. This is not always possible. We shall address here a typical magnetostatics modelling, for which the computational domain is a priori the whole space. This will be an opportunity to introduce the technique of "finite elements and boundary elements in association", which is essential to the treatment of all "open space" problems, static or not, and will be applied to eddy-currents modelling in Chapter 8.

7.1 ANOTHER MODEL PROBLEM

Figure 7.1 describes the configuration we shall study: an electromagnet, with its load.

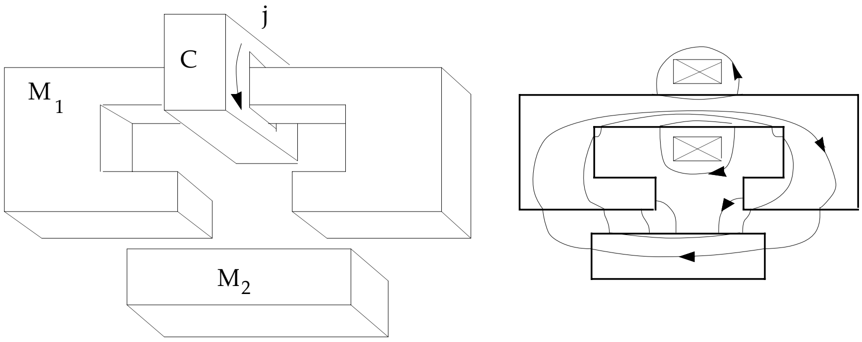


FIGURE 7.1. Left: Electromagnet, with its ferromagnetic core M_1, its coil C, and a load M_2. When a continuous current j is fed into C, the load is attracted upwards. Right: Typical flux lines, in a vertical cross-section.

The working principle of such devices is well known: A direct current creates a permanent magnetic field. Channeled by an almost closed magnetic circuit, the induction flux closes through a ferromagnetic piece (here M_2), to which the load is attached. Force lines (cf. Fig. 7.1, right) "tend to shorten", and the load is thus lifted towards the poles of the electromagnet, upwards in the present case, whatever the sense of the current.

This "shortening of field lines" is of course a very naive explanation, but a useful one all the same, and rigorous analysis does support it, as follows. Suppose for definiteness the electromagnet is fixed, the load M_2 being free to move vertically. Let u denote its position on the vertical axis (oriented upwards). For a given u, some field {h, b} settles, the magnetic coenergy[1] of which is $W_u(h) = \frac{1}{2}\int \mu\ |h|^2$. Thanks to the virtual work principle, one may prove that the lifting force upon M_2 is[2] $f = \partial_u W$. Now, let us verify that $W_u(h)$ *increases* with u (whence the sign of f), by the reasoning that follows.

First, μ is large in M. Let us take it as infinite. In that case, $h = 0$ in M. By Ampère's theorem, the circulation of h along a field line is equal to the intensity in the coil, call it I. If H is the magnitude of the field in the air gap, of width d, one thus has $Hd \sim I$, since only the part of the field line contained in the air gap contributes to the circulation. The air gap volume being proportional to d, the coenergy is thus proportional to $\mu_0 H^2\, d/2$, that is $\mu_0\, I^2/2d$, so it increases when d decreases. Hence the direction of the force: upwards, indeed.

This reasoning is quite useful, and moreover, it suggests how to pass from qualitative to quantitative statements: If one is able to compute the field with enough accuracy to plot W as a function of u, one will have the force as a function of u by simple differentiation[3]. So, if the mechanical characteristics of the system are known (masses, inertia tensors, restoring

[1]As already pointed out (Remark 2.6), one should carefully distinguish between a *function* of the field which, by its evaluation, yields the energy, and the numerical *result* of such an evaluation (the real number one refers to when one speaks of the energy stored in the field). Energy can be obtained in two different ways, by evaluating either the function $b \to V(b) = \frac{1}{2}\int \mu^{-1} |b|^2$ or the function $h \to W(h) = \frac{1}{2}\int \mu\ |h|^2$, and to tell them apart, one calls them *energy* and *coenergy function(al)s* respectively. Making this distinction is essential, because the results of both evaluations coincide in the linear case only (and otherwise the correct value of the energy is obtained by computing V, not W). In problems with motion, V and W are parameterized by the configuration of the system, here denoted by u.

[2]Force is also given by $f = -\partial_u V$, as can be seen by differentiating with respect to u the equality $V_u(b_u) + W_u(h_u) = \int b_u \cdot h_u$, which holds when b_u and h_u are the induction and the field that effectively settle in configuration u.

[3]A better, more sophisticated approach, is available [Co], by which $\partial_u W$ instead of W_u is obtained via a finite-element computation. It consists essentially of differentiating the elementary matrices with respect to u *before* proceeding to their assembly, and solving the linear system thus obtained.

forces, etc.), one may study its dynamics. This opens the way to a whole realm of applications: electromagnets, of course, but also electromagnetic "actuators" of various kinds (motors, linear or rotatory, launchers, etc.).

The underlying problem, in all such applications, is thus: Knowing the geometry of the system at hand, and the values of μ, compute the magnetic field, with its coenergy (or its energy) as a by-product. The principal difficulty then comes from the nonlinearity of the b–h relation (to say nothing of hysteresis).

Without really addressing this difficulty, let us only recall that nonlinear problems are generally solved by successive approximations, Newton–Raphson style, and thus imply the solution of a sequence of linear problems. In the present case, each of these problems assumes the form

(1) $\operatorname{rot} h = j,$ (2) $\operatorname{div} b = 0,$

(3) $b = \mu\, h\,,$

in all space, with $\mu = \mu_0$ outside M, and μ function of the position x (via the values of the field at this point at the previous iterations) inside M. This points to (1–3), the *magnetostatics model* in all space, as the basic problem.

Apart from this spatial variation of μ, and the explicit presence of the source-term j, the problem is quite alike the "Bath cube" one of Chapter 2. The only really new element is the non-boundedness of the domain, which will allow us to concentrate on that.

7.2 FORMULATION

According to the functional point of view, we look for a precise formulation of (1–3): *find* h *and* b *in* . . . *such that* . . . , etc. The first item on the agenda is thus to delimit the field of investigation: exactly in which functional space are we looking for b and h ? Once these spaces of "admissible fields" are identified, one realizes that Problem (1–3) has a *variational* formulation, which means it can be expressed in the form *find* h *and* b *which minimize*, etc. All that is left to do is then to replace spaces of admissible fields by "large enough" *finite* dimensional subspaces in order to obtain approximate formulations open to algebraic treatment on a computer. Let us thus try to find this "variational framework", as one says, in which problem (1–3) makes sense.

7.2.1 Functional spaces

First, set $\mathbb{H} = \mathbb{L}^2_{rot}(E_3)$ and $\mathbb{B} = \mathbb{L}^2_{div}(E_3)$, and put irrotational and solenoidal fields apart:

(4) $$\mathbb{H}^0 = \{h \in \mathbb{H} : \text{ rot } h = 0\}, \quad \mathbb{B}^0 = \{b \in \mathbb{B} : \text{ div } b = 0\}.$$

Then,

Proposition 7.1. *Subspaces* $\mathbb{H}^0$ *and* $\mathbb{B}^0$ *are ortho-complementary in* $\mathbb{L}^2(E_3)$:

(5) $$\mathbb{L}^2(E_3) = \mathbb{H}^0 \oplus \mathbb{B}^0.$$

In other words, any vector u *of* $\mathbb{L}^2(E_3)$ *can be written, in a unique way, as* $u = h + b$, *with* $\text{rot } h = 0$, $\text{div } b = 0$, *and* $\int_{E_3} h \cdot b = 0$.

Proof. First, let u be a smooth field with bounded support, form div u and rot u, and set

$$\varphi(x) = -\frac{1}{4\pi}\int_{E_3} \frac{(\text{div } u)(y)}{|x-y|}\, dy, \quad a(x) = \frac{1}{4\pi}\int_{E_3} \frac{(\text{rot } u)(y)}{|x-y|}\, dy.$$

Then, differentiating under the integral signs, and applying the formula $\text{rot rot} = \text{grad div} - \Delta$, one obtains that

(6) $$u = \text{grad } \varphi + \text{rot } a,$$

which is called the *Helmholtz*[4] *decomposition* of u, a standard result. (Beware, neither φ nor a has bounded support.) Setting $h = \text{grad } \varphi$ and $b = \text{rot } a$, one sees that $\int_{E_3} b \cdot h = 0$, as announced. If instead of considering a single field u we look at functional spaces wholesale, (6) is equivalent to

(6') $$\mathbb{C}_0^\infty(E_3) = -\text{ grad}(\text{div}(\mathbb{C}_0^\infty(E_3))) \oplus \text{rot}(\text{rot}(\mathbb{C}_0^\infty(E_3))),$$

which we can write $\mathbb{C}_0^\infty(E_3) = \mathcal{H}^0 \oplus \mathcal{B}^0$, where $\mathcal{H}^0$ and $\mathcal{B}^0$ are subspaces —not *closed* subspaces—of $\mathbb{L}^2(E_3)$ composed of smooth curl-free and divergence-free fields, respectively. Now call $\mathbb{H}^0$ and $\mathbb{B}^0$ the *closures* in $\mathbb{L}^2(E_3)$ of $\mathcal{H}^0$ and $\mathcal{B}^0$. For each pair $h \in \mathbb{H}^0$ and $b \in \mathbb{B}^0$, we thus have sequences of smooth fields $\{h_n \in \mathcal{H}^0 : n \in \mathbb{N}\}$ and $\{b_n \in \mathcal{B}^0 : n \in \mathbb{N}\}$ which converge towards h and b, while satisfying $\text{rot } h_n = 0$, $\text{div } b_n = 0$, and $\int b_n \cdot h_n = 0$. All these properties "pass to the limit" by continuity. For instance (to do it only once), $\text{div } b_n = 0$ means $\int b_n \cdot \text{grad } \varphi = 0$ for all φ in $C_0^\infty(E_3)$, hence, by continuity of the scalar product, $\int b \cdot \text{grad } \varphi = 0$ for all these test functions, which is weak solenoidality, and which we are writing

[4]Due to Stokes (1849), actually, according to [Hu], p. 147.

div $b = 0$ since we decided to adopt the "weak" interpretation for differential operators in Chapter 5. Last step: Since $\mathbb{C}_0^\infty(E_3)$ is dense in $\mathbb{L}^2(E_3)$ by construction of the latter, $\mathbb{H}^0 \oplus \mathbb{B}^0$ fills out all $\mathbb{L}^2(E_3)$, hence (5). ◊

But can one go further and have (6) for all, not only smooth, square-integrable fields? Yes, if one accepts having φ and a in the right functional spaces, those obtained by completion. Take $h \in \mathbb{H}^0$. By the foregoing, there is a sequence $\varphi_n \in C_0^\infty(E_3)$ such that grad φ_n converges to h. Thus, the sequence $\{\varphi_n\}$ is Cauchy with respect to the norm $\|\varphi\| = [\int_{E_3} |\text{grad}\ \varphi|^2]^{1/2}$. Now *complete* $C_0^\infty(E_3)$ with respect to this norm: There comes a complete space, which we shall denote Φ, an extension-by-continuity of grad (called, again,[5] the *weak* gradient), and we do have $h = \text{grad}\ \varphi$, with $\varphi \in \Phi$.

A sleight of hand? Yes, in a sense, since elements of Φ are abstract objects (equivalence classes of Cauchy sequences of functions), but not really, because one can identify this space Φ with a subspace[6] of a functional space, the Sobolev space $L^6(E_3)$. This is an immediate consequence of the inequality $\int_{E_3} |\varphi|^6 \leq C \int_{E_3} |\text{grad}\ \varphi|^2$, a proof of which (not simple) can be found in [Br], p. 162. So as regards irrotational fields, we have $\mathbb{H}^0 = \text{grad}\ \Phi$, where Φ is a well-defined functional space. This is the *Beppo Levi space* alluded to in Chapter 3, Note 5.

The representation (6) of a field u, now with $\varphi \in \Phi$ and $a \in A$, extends the Helmholtz decomposition, just as the Poincaré lemma was extended in Chapter 5. (Note that a is not unique, but φ is, because $\|\varphi\|$ is a norm, since there are no constants in $C_0^\infty(E_3)$.)

Let now $j \in \mathbb{L}^2(E_3)$ be given, with bounded support, and div $j = 0$. The field $h^j = \text{rot}\ a^j$, where a^j is given by the integral

$$a^j(x) = \frac{1}{4\pi} \int_{E_3} \frac{j(y)}{|x-y|}\ dy,$$

is in[7] $\mathbb{L}^2_{\text{rot}}(E_3)$ and satisfies $\text{rot}\ h^j = j$. Let us set $\mathbb{H}^j = h^j + \mathbb{H}^0$.

[5]In spite of its domain being larger than the closure of the strong gradient in $L^2(E_3) \times \mathbb{L}^2(E_3)$. No Poincaré inequality is available in the present case; hence the two methods of extension of the differential operators examined in Subsection 5.1.2 are no longer equivalent.

[6]There is a more precise characterization of Φ as the space of functions φ such that $\text{grad}\ \varphi \in \mathbb{L}^2(E_3)$ and $\int (1 + |x|^2)^{-1/2} |\varphi(x)|^2\ dx < \infty$.

[7]This is not totally obvious, and j having a bounded support (or at least, a support of finite volume) plays a decisive role there. (Show that $\text{rot rot}\ a^j = j$, then compute $\int \text{rot}\ a^j \cdot \text{rot}\ a^j = \int \text{rot rot}\ a^j \cdot a^j$, etc.)

7.2.2 Variational formulations

Now we have all the ingredients required to make the problem "well posed":

Proposition 7.2. *Let there be given* $j \in \mathbb{L}^2(E_3)$, *with bounded support and* $\operatorname{div} j = 0$, *and a function* μ *such that* $\mu_1 \geq \mu(x) \geq \mu_0$ *a.e. in* E_3. *The problem* find $h \in \mathbb{H}^j$ and $b \in \mathbb{B}^0$ such that[8]

$$(7) \qquad \int_{E_3} \mu^{-1} |b - \mu h|^2 \leq \int_{E_3} \mu^{-1} |b' - \mu h'|^2 \quad \forall\, h' \in \mathbb{H}^j, \ \forall\, b' \in \mathbb{B}^0,$$

has a unique solution, which satisfies (1–3).

Proof. If $h' \in \mathbb{H}^j$, then $h' - h^j \in \mathbb{H}^0$. Thus, $\int_{E_3} h' \cdot b' = \int_{E_3} h^j \cdot b'$ for each pair $\{h', b'\} \in \mathbb{H}^j \times \mathbb{B}^0$, after (5), and therefore,

$$\int_{E_3} \mu^{-1} |b' - \mu h'|^2 = \int_{E_3} \mu^{-1} |b'|^2 + \int_{E_3} \mu |h'|^2 - 2\int_{E_3} h^j \cdot b',$$

so that Problem (7) is equivalent to the following pair of *independent* optimization problems, taken together:

$$(8) \qquad \textit{find } h \in \mathbb{H}^j \textit{ such that } \ W(h) \leq W(h') \quad \forall\, h' \in \mathbb{H}^j,$$

where $W(h) = \frac{1}{2} \int_{E_3} \mu |h|^2$ is the magnetic coenergy introduced earlier, and

find $b \in \mathbb{B}^0$ *such that*

$$(9) \qquad V(b) - \int_{E_3} h^j \cdot b \leq V(b') - \int_{E_3} h^j \cdot b' \quad \forall\, b' \in \mathbb{B}^0,$$

where $V(b) = \frac{1}{2} \int_{E_3} \mu^{-1} |b|^2$ is the magnetic energy. These functionals being continuous on $\mathbb{L}^2(E_3)$, with coercive[9] quadratic parts, both (8) and (9) have a unique solution. The Euler equations of (8) and (9) being

$$(10) \qquad \textit{find } h \in \mathbb{H}^j \textit{ such that } \int_{E_3} \mu\, h \cdot h' = 0 \quad \forall\, h' \in \mathbb{H}^0,$$

$$(11) \qquad \textit{find } b \in \mathbb{B}^0 \textit{ such that } \int_{E_3} \mu^{-1} b \cdot b' = \int_{E_3} h^j \cdot b' \quad \forall\, b' \in \mathbb{B}^0,$$

the pair $\{h, b\}$ thus found satisfies $\int_{E_3} \mu |h|^2 = \int_{E_3} \mu\, h \cdot h^j$ and $\int_{E_3} \mu^{-1} |b|^2 =$

[8]The quantity on the right-hand side of (7) is again the "error in constitutive law" of Chapter 6. It measures the failure of the pair $\{h', b'\}$ to satisfy the behavior law $b' = \mu h'$, and to minimize it over $\mathbb{H}^j \times \mathbb{B}^0$ amounts to looking, among the pairs which obey other equations (here, $\operatorname{rot} h = j$ and $\operatorname{div} b = 0$), for the one that, by minimizing the error (and actually, cancelling it), best obeys (and actually, exactly obeys) the constitutive law.

[9]This is said (cf. A.4.3) of a quadratic functional $u \to (Au, u)$ over a Hilbert space U for which exists $\alpha > 0$ such that $(Au, u) \geq \alpha\, ||u||^2$ for all u. By the Lax–Milgram lemma, the equation $Au = f$ has then a unique solution, which is the minimizer of the functional $u \to \frac{1}{2}(Au, u) - (f, u)$.

$\int_{E_3} h^j \cdot b$ (set $h' = h - h^j$ and $b' = b$), whence

$$\int_{E_3} \mu^{-1} |b - \mu h|^2 = \int_{E_3} \mu^{-1} |b|^2 + \int_{E_3} \mu |h|^2 - 2\int_{E_3} h^j \cdot b = 0,$$

and therefore $b = \mu h$. ◊

Problems (10) and (11) are of the now-familiar kind of "constrained linear problems", and it is natural to try to solve them via "unconstrained" representations of the affine spaces $\mathbb{H}^j$ and $\mathbb{B}^0$. Since, as we saw earlier,

(12) $\quad \mathbb{H}^j = h^j + \text{grad}\,\Phi, \qquad \mathbb{B}^0 = \text{rot}\,A,$

(8) and (9) amount to

(13) $\quad$ *find* $\varphi \in \Phi$ *such that* $W(\varphi) \le W(\varphi') \quad \forall\, \varphi' \in \Phi,$

where $W(\varphi) = \frac{1}{2} \int_{E_3} \mu\, |h^j + \text{grad}\,\varphi|^2$ (magnetic coenergy again, but now considered as a function of φ, as signaled by the slight notational variation), and

(14) $\quad$ *find* $a \in A$ *such that* $V(a) - \int_{E_3} j \cdot a \le V(a') - \int_{E_3} j \cdot a' \quad \forall\, a' \in A,$

where $V(a) = \frac{1}{2} \int_{E_3} \mu^{-1} |\text{rot}\,a|^2$ (magnetic energy, as a function of a). The Euler equations of (13) and (14) are

(15) $\quad$ *find* $\varphi \in \Phi$ *such that* $\int_{E_3} \mu\,(h^j + \text{grad}\,\varphi) \cdot \text{grad}\,\varphi' = 0 \quad \forall\, \varphi' \in \Phi,$

(16) $\quad$ *find* $a \in A$ *such that* $\int_{E_3} \mu^{-1}\,\text{rot}\,a \cdot \text{rot}\,a' = \int_{E_3} j \cdot a' \quad \forall\, a' \in A.$

Being assured of existence and uniqueness for h and b solutions of (8) and (9), we know that (15) has a unique solution φ (the only $\varphi \in \Phi$ such that $h^j + \text{grad}\,\varphi = h$) and that (16) has a family of solutions all of which verify $\text{rot}\,a = b$. One might as well, of course, study (15) and (16) ab initio: The mapping $\varphi' \to \int_{E_3} \mu\, h^j \cdot \text{grad}\,\varphi'$ is continuous on Φ (since h^j is $\mathbb{L}^2$, and μ is bounded), therefore (15) has a unique solution, by the Lax–Milgram lemma. (For (16), it's less direct, because one must apply the lemma to the *quotient* of A by the kernel of rot.)

7.3 DISCRETIZATION

Whatever the selected formulation, the problem concerns the whole space, which cannot be meshed with a *finite* number of bounded elements. There are essentially three ways to deal with this difficulty. I shall be very brief and allusive about the first two, but this should not imply that they are less important.

7.3.1 First method: "artificial boundary, at a distance"

Start from a mesh $\{\mathcal{N}, \mathcal{E}, \mathcal{F}, \mathcal{T}\}$ of a bounded region of space $\hat{D}$, bounded by surface $\hat{S}$. Denote by $\mathcal{N}(\hat{S})$, etc., as before, the sets of nodes, etc., contained in $\hat{S}$. Then consider a smooth injective mapping u, called a *placement* of $\hat{D}$ in E_3, built in such a way that the image $D = u(\hat{D})$, bounded by $S = u(\hat{S})$, cover the region of interest as well as "enough" space around it (Fig. 7.2). Define "u-adapted" Whitney elements by setting ${}^u w_n(x) = w_n(u^{-1}(x))$ for node n, then ${}^u w_e(x) = {}^u w_m(x)\ \nabla\ {}^u w_n(x) - {}^u w_n(x)\ \nabla\, {}^u w_m(x)$ for edge $e = \{m, n\}$, etc.[10] Then set

$$W^0{}_m = \{\varphi : \varphi = \textstyle\sum_{n \in \mathcal{N} - \mathcal{N}(\hat{S})} \boldsymbol{\varphi}_n\, {}^u w_n\},$$

$$W^1{}_m = \{h : h = \textstyle\sum_{e \in \mathcal{E} - \mathcal{E}(\hat{S})} \mathbf{h}_e\, {}^u w_e\},$$

etc., where the $\boldsymbol{\varphi}_n$, $\mathbf{h}_e$, etc., assume real values. (The notation m now refers to both the mesh and its placement.)

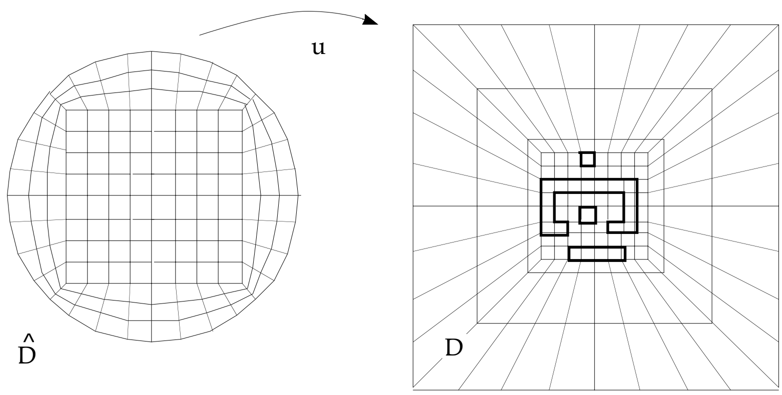


FIGURE 7.2. Illustrating the idea of "placement" of a reference mesh, for a problem similar to the one of Fig. 7.1. What we see is actually a 3D "macro-mesh", the "bricks" of which, once placed in order to accommodate the material interfaces, will be subdivided as required.

[10]This is the precise definition of finite elements on "curved tetrahedra", informally introduced in Chapter 3. In all generality, ${}^u f$ defined by ${}^u f(x) = f(u(x))$ is the *push-forward* of f by u, and f is the *pull-back* of ${}^u f$. So ${}^u w_n$ comes from w_n by push-forward.

With only tiny variations, we may carry on with the former notational system: $\boldsymbol{\varphi}$, $\mathbf{h}$, etc., are the vectors of degrees of freedom, $\boldsymbol{\Phi} = \mathbb{R}^{\mathcal{N} - \mathcal{N}(\hat{S})}$, $\mathbf{A} = \mathbb{R}^{\mathcal{E} - \mathcal{E}(\hat{S})}$, etc., are the spaces they generate (isomorphic to W^0_m, W^1_m, etc.), and the incidence matrices $\mathbf{G}$, etc., are such that $\mathbf{h} = \mathbf{G}\boldsymbol{\varphi}$ if $h = \text{grad}\ \varphi$, etc. Observe that $W^0_m \subset \Phi$ and $W^1_m \subset A$, thanks to our having set to zero the degrees of freedom of surface nodes and edges. The intersections $\Phi_m = W^0_m \cap \Phi$ and $A_m = W^1_m \cap A$ are thus Galerkin approximation subspaces for Φ and A. The approximations of (15) and (16) determined by m and u are then

(17) *find* $\varphi_m \in \Phi_m$ *such that* $\int_D \mu\, (h^j + \text{grad}\ \varphi_m) \cdot \text{grad}\ \varphi' = 0 \quad \forall\ \varphi' \in \Phi_m$,

(18) *find* $a_m \in A_m$ *such that* $\int_D \mu^{-1}\, \text{rot}\ a_m \cdot \text{rot}\ a' = \int_D j \cdot a' \quad \forall\ a' \in A_m$.

In order to set these linear systems in standard form, let us redefine the "mass matrices" of Chapter 5 as follows (p is the dimension of the simplices):

$$(\mathbf{M}_p(\alpha))_{s\,s'} = \int_D \alpha\ {}^u w_s \cdot {}^u w_{s'} \quad \text{if } p = 1 \text{ or } 2,$$

$$= \int_D \alpha\ {}^u w_s\ {}^u w_{s'} \quad \text{if } p = 0 \text{ or } 3,$$

the indices s and s' being restricted to the sets of "internal" simplices (those not in $\hat{S}$). Then (17) and (18) can be rewritten as

(19) $\mathbf{G}^t \mathbf{M}_1(\mu)\,(\mathbf{G}\boldsymbol{\varphi} + \mathbf{h}^j) = 0,$ (20) $\mathbf{R}^t \mathbf{M}_2(\mu^{-1}) \mathbf{R}\ \mathbf{a} = \mathbf{j},$

where vectors $\mathbf{h}^j$ and $\mathbf{j}$ are defined by $\mathbf{h}^j_e = \int_e \tau \cdot h^j$ and $\mathbf{j}_f = \int_f n \cdot j$.

7.3.2 Second method: "infinite elements"

This method makes use of a more sophisticated placement u, but otherwise coincides with the first one. The difference is that here u maps $\hat{D}$ onto the *whole* space E_3, all points of the boundary $\hat{S}$ being sent to infinity. The elements of the mesh which are immediately under $\hat{S}$ are then sent onto regions of infinite volume, called infinite elements. See, e.g., [BM] for a construction of such a mapping. It is often convenient, in this respect, to use a geometric inversion with respect to some point [IM, LS].

The literature on infinite elements is huge. For a bibliography and a comparative study, from a practical viewpoint, cf. C. Emson's contribution (in English . . .) to [B &].

7.3.3 Third method: "finite elements and integral method in association"

There, in contrast, D is made as small as possible, including the region of interest while still having a boundary of simple shape. Applying the first method would then amount to neglecting the field outside D, which is not acceptable. However, the far field is not of primary interest by itself: All that matters is its contribution to the energy or the coenergy, as the following informal approach will suggest.

Let D, bounded by surface S, be such that $\mu \equiv \mu_0$ outside D. Then[11]

$$\inf\{W(h^j + \operatorname{grad}\varphi) : \varphi \in \Phi\}$$
$$= \inf\{\tfrac{1}{2}\int_D \mu \, |h^j + \operatorname{grad}\varphi|^2 + W_{ext}(j, \varphi_{|S}) : \varphi \in \Phi\},$$

where, by way of definition,

$$W_{ext}(j, \varphi_S) = \inf\{\tfrac{1}{2}\int_{E_3 - D} \mu_0 \, |h^j + \operatorname{grad}\varphi|^2 : \varphi \in \Phi,\ \varphi_{|S} = \varphi_S\}. \tag{21}$$

This "exterior coenergy" term only depends on the *boundary* values of the potential φ. So, in order to be able to solve Problem (15) by meshing region D only, it would be sufficient to know some approximation of $W_{ext}(j, \varphi_S)$ as a function of nodal values of φ on S (which are *not* set to 0 here, in contrast with the first method). For the same reason, having an approximation of the functional $V_{ext}(a) - \int_{E_3 - D} j \cdot a$ (where $V_{ext}(a) = \frac{1}{2}\int_{E_3 - D} \mu_0^{-1} |\operatorname{rot} a|^2$) in terms of the circulations of a along edges of S would allow one to solve (16) without having to mesh the outer region.

Now let φ be the potential (outside D) that minimizes (21). One has $\varphi_{|S} = \varphi_S$ and

$$\int_{E_3 - D} \mu_0 \, (h^j + \operatorname{grad}\varphi) \cdot \operatorname{grad}\varphi' = 0 \quad \forall\, \varphi' \in \Phi \text{ such that } \varphi'_{|S} = 0.$$

So $\operatorname{div}(h^j + \operatorname{grad}\varphi) = 0$ outside D. Since $\operatorname{div} h^j = 0$, by construction, $\Delta\varphi = 0$ outside D, thus φ is solution to the "exterior Dirichlet problem":

$$\Delta\varphi = 0 \text{ outside D}, \quad \varphi_{|S} = \varphi_S.$$

On the other hand, thanks to the integration by parts formula,

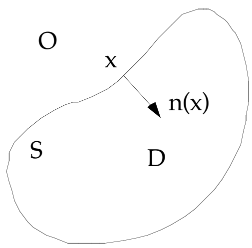
$$W_{ext}(j, \varphi_S) = \tfrac{1}{2}\int_{E_3 - D} \mu_0 \, |h^j + \operatorname{grad}\varphi|^2 = \tfrac{1}{2}\,\mu_0 \int_S n \cdot (h^j + \operatorname{grad}\varphi)\ \varphi_S$$
$$= \tfrac{1}{2}\,\mu_0 \int_S (n \cdot h^j + P\varphi_S)\ \varphi_S,$$

[11]For a while, we need to distinguish $\varphi_{|S}$, the restriction of φ to S, and φ_S, which will denote some function defined on S.

where the normal n is oriented towards D (beware!) and P the operator that maps φ_S to the normal derivative $n \cdot \text{grad}\ \varphi$. So the problem would be solved if P was known, or at least if one could compute an approximation of $\int_S P\ \varphi_S\ \varphi_S$ in terms of the DoF $\boldsymbol{\varphi}_n$, where n spans $\mathcal{N}(S)$. We shall therefore look for a matrix approximation of this "Dirichlet-to-Neumann" operator (also called "Poincaré–Steklov" operator [AL], or else "capacity" [DL], because of its interpretation in electrostatics). This operator is a quite delicate object to handle, and we'll have to spend some time on its precise definition and its properties. The reader who feels the foregoing overview was enough (despite, or perhaps thanks to, many abuses) may skip what follows and proceed to Subsection 7.4.4.

7.4 THE "DIRICHLET-TO-NEUMANN" MAP

So let D be a regular bounded domain, inside a closed surface S. We shall denote by O (for "outside") the complement of $D \cup S$, which is also the interior of $E_3 - D$. Domains D and O have S as common boundary, and the field of normals to S is taken as outgoing from O (not from D).



7.4.1 The functional space of "traces"

The theory, unfortunately, is more demanding than anything we have done up to now, and the time has come to introduce something which could be evaded till this point: *traces* of functions in L^2_{grad}, and the Sobolev space of traces, $H^{1/2}(S)$.

Smooth functions φ over D have restrictions $\varphi_{|S}$ to S, which are piecewise smooth functions. Let us denote by γ the mapping $\varphi \to \varphi_{|S}$. We shall base our approach on the following lemma:

Lemma 7.1. *There exists a constant* C(D), *depending only on* D, *such that, for all functions* φ *smooth over* D,

(22) $$\int_S |\gamma\ \varphi|^2 \le C(D)\ [\int_D |\varphi|^2 + \int_D |\text{grad}\ \varphi|^2].$$

This is technical, but not overly difficult if one accepts cutting a few corners, and Exer. 7.6 will suggest an approach to this result. Our purpose is to extend to $L^2_{\text{grad}}(D)$ this operator γ, by using the prolongation principle of A.4.1.

Pick some $\varphi \in L^2_{grad}(D)$. By definition of the latter space, there exists a Cauchy sequence of smooth functions $\varphi_n \in C_0^\infty(E_3)$ which, once restricted to D, converge towards φ in the sense of $L^2_{grad}(D)$, and thus $\{\varphi_n\}$ and $\{\text{grad } \varphi_n\}$ are Cauchy sequences in $L^2(D)$ and $\mathbb{L}^2(D)$, respectively. Then, by an immediate corollary of (22), $\{\gamma\ \varphi_n\}$ also is a Cauchy sequence and therefore converges towards a limit in $L^2(S)$, which we define as $\gamma\ \varphi$ and call the *trace* of φ. By (22), γ is a continuous linear map from $L^2_{grad}(D)$ into $L^2(S)$.

Its image, however, has no reason to be all of $L^2(S)$, and constitutes only a dense[12] subspace, which we shall call T(S). Let us provide T(S) with the so-called *quotient norm*, as follows. Pick some φ_S in T(S). There is, by definition of T(S), at least one function φ the trace of which is φ_S, so the pre-image of φ_S is a non-empty affine space, that we may denote $\Phi_D(\varphi_S)$. Now, let us set

$$[|\varphi_S|] = \inf\{\, [\int_D |\varphi|^2 + \int_D |\text{grad } \varphi|^2]^{1/2} : \varphi \in \Phi_D(\varphi_S)\,\}. \tag{23}$$

This defines a *norm* $[|\ |]$ on T(S), with respect to which γ keeps being continuous, since $[|\gamma\ \varphi|] = [|\varphi_S|] \leq |||\varphi|||$, where $|||\ |||$ denotes the $L^2_{grad}(D)$-norm. Now,

Definition 7.1. *The normed space* $\{T(S), [|\ |]\}$ *is denoted* $H^{1/2}(S)$.

Why this name, that we shall explain, but let's just accept this notation for the moment. More importantly,

Proposition 7.3. $H^{1/2}(S)$ *is a Hilbert space.*

Proof. Note that $\Phi_D(\varphi_S)$ is closed in $L^2_{grad}(D)$, as the pre-image of φ_S by the continuous map γ. Since $L^2_{grad}(D)$ is complete by definition, the infimum in (23) is reached at a (unique) point φ, which is the projection of 0 on $\Phi_D(\varphi_S)$. Let us call $\pi(\varphi_S)$ this special element, and remark that $\pi\,\gamma\,\varphi = \varphi$ and $\gamma\,\pi\,\varphi_S = \varphi_S$ (operator π is called a "lifting" from S to D). Now let us set $[(\varphi_S, \psi_S)] = \int_D \pi\varphi_S\, \pi\psi_S + \int_D \text{grad } \pi\varphi_S \cdot \text{grad } \pi\psi_S$, thus defining a scalar product on T(S). Since the norm associated with this scalar product is precisely $[|\ |]$, the norm of $H^{1/2}(S)$, the latter is pre-Hilbertian, and since its Cauchy sequences lift to Cauchy sequences of $L^2_{grad}(D)$, which is complete, they converge, so $H^{1/2}(S)$ is complete, and thus a Hilbert space. ◊

Note that, after (22), $||\varphi_S|| \leq [|\varphi_S|]$, where $||\ ||$ denotes the $L^2(S)$-norm. Therefore, sequences which converge for $[|\ |]$ also converge for $||\ ||$, and the identity mapping $\varphi_S \to \varphi_S$ is continuous from $H^{1/2}(S)$ into $L^2(S)$. One says that the new norm $[|\ |]$ is *stronger*[13] than $||\ ||$, and that there is

[12]Because it contains the restrictions of smooth functions, which are dense in $L^2(S)$.

"topological inclusion" of $H^{1/2}(S)$ into $L^2(S)$, not only the mere "algebraic" inclusion of a set (here $T(S)$) into another set (here $L^2(S)$). Note that the norm couldn't be made any stronger (authorize fewer converging sequences) without breaching the continuity of γ : It's the *strongest* norm with respect to which γ stays continuous.

Now why this name, $H^{1/2}$? This notation pertains to the theory of Sobolev spaces [Ad, Br, Yo]: On any "measured differentiable manifold" X, which S is, there exists a whole family of functional spaces, denoted $H^s(X)$, which includes $L^2(X)$ for $s = 0$, and the one I have been calling here $L^2_{grad}(X)$, for the sake of notational consistency, for $s = 1$. They form a kind of ladder, H^s being topologically included in H^t, for $t < s$. It happens that our trace space is midway from $L^2(S) \equiv H^0(S)$ to $H^1(S)$ in this hierarchy, hence its name. Giving sense to "midway" and proving the point is not easy, but not important either, since we *know,* having provided a definition, what $H^{1/2}(S)$ is. So let's just accept the name as a dedicated symbol.

Remark 7.1. Now that we have a linear continuous map γ from $L^2_{grad}(D)$ to $H^{1/2}(S)$, which is by construction surjective, but certainly not injective, since functions supported inside D map to 0, a natural question arises: What is $\ker(\gamma)$? This closed subspace, which is traditionally denoted $H^1_0(D)$, obviously contains $C_0^\infty(D)$. Less obviously, and we'll admit this result [LM], $\ker(\gamma) \equiv H^1_0(D)$ is the closure of $C_0^\infty(D)$ in $L^2_{grad}(D)$. ◊

Exercise 7.1. Remember that $C_0^\infty(D)$ is dense in $L^2(D)$, which is its completion. How can the closure of $C_0^\infty(D)$ then be *smaller* than $L^2_{grad}(D)$, which is already smaller than $L^2(D)$?

7.4.2 The interior Dirichlet-to-Neumann map

Next step, let's have a closer look at the lifting π. Finding the minimizer in (23) is a variational problem, which has an associated Euler equation. The latter is *find* $\varphi \in \Phi_D(\varphi_S)$ *such that*

(24) $$\int_D \varphi\, \varphi' + \int_D \operatorname{grad} \varphi \cdot \operatorname{grad} \varphi' = 0 \quad \forall\, \varphi' \in \ker(\gamma),$$

since $\ker(\gamma)$ is the vector subspace parallel to $\Phi_D(\varphi_S)$. This implies, by specializing to smooth test functions,

$$\int_D \varphi\, \varphi' + \int_D \operatorname{grad} \varphi \cdot \operatorname{grad} \varphi' = 0 \quad \forall\, \varphi' \in C_0^\infty(D),$$

[13] Because it is "more demanding", letting fewer sequences converge, having more closed sets or open sets (all these things are equivalent).

which means that $\text{div}(\text{grad}\ \varphi) = \varphi$ in the weak sense. So (24) is the weak formulation of a boundary value problem, *find* φ *such that*

(25) $$-\Delta\varphi + \varphi = 0, \quad \gamma\ \varphi = \varphi_S,$$

which gives a nice[14] interpretation of the lifting: π maps the Dirichlet data φ_S to the solution of (25).

Now in case this solution is smooth over[15] D, let us denote by P the "Dirichlet-to-Neumann" linear map $\varphi_S \to \partial_n\varphi$. The divergence integration by parts formula gives

(26) $$\int_D \varphi\, \psi + \int_D \text{grad}\ \varphi \cdot \text{grad}\ \psi = \int_S \partial_n\varphi\, \psi = \int_S \mathrm{P}\varphi_S\, \psi$$

for any smooth function ψ, and hence an *explicit* formula for the scalar product in $H^{1/2}(S)$,

(27) $$[(\varphi_S\,,\ \psi_S)] = \int_S \mathrm{P}\varphi_S\, \psi_S\,,$$

when φ_S is smooth enough for P to make sense. Here, $\mathrm{P}\varphi_S$ is a function, defined on S. But from the point of view which we have so often adopted, a function is known by its effect on test functions, so we may identify $\mathrm{P}\varphi_S$ and the linear map $\psi_S \to \int_S \mathrm{P}\ \varphi_S\psi_S$. The latter map, in turn, being continuous on $H^{1/2}(S)$, constitutes an element of the space *dual* to $H^{1/2}(S)$, which is denoted by $H^{-1/2}(S)$ (again the question "why $-1/2$?" arises and will be answered, to some extent, in a moment). Now, (27) shows that the map $\varphi_S \to (\psi_S \to [(\varphi_S\,,\ \psi_S)])$, which sends $\varphi_S \in H^{1/2}(S)$ to an element of $H^{-1/2}(S)$, is an extension, a prolongation of the operator P. Quite naturally, we denote by P this extended map, and call it the *Dirichlet-to-Neumann operator*, even though the image $\mathrm{P}\varphi_S$ may fail to be a function.

All these identifications suggest some notational conventions. It is customary to denote $<f, v>$ the scalar which results from applying an element f of the dual space V' to an element v of V. (In case of ambiguities, the more precise notation $<f, v>_{V', V}$ may help.) We then have

(28) $$[(\varphi_S, \psi_S)] = <\mathrm{P}\varphi_S, \psi_S> = \int_S \mathrm{P}\varphi_S\, \psi_S = \int_S \partial_n(\pi\varphi_S)\, \psi_S$$

[14]And useful: This is the paradigm of a classical approach to boundary-value problems by Hilbertian methods [LM].

[15]Smoothness inside D is assured by a variant of this Weyl lemma we mentioned in Chapter 2. But smoothness *over* D, in the sense of Chapter 2, Subsection 2.2.1, is not warranted, even if S and φ_S are piecewise smooth.

when the latter terms make sense. So we shall abuse the notation and use whichever form is most convenient.

Note that $\sup\{|\langle P\varphi_S, \psi_S\rangle| / [|\psi_S|] : \psi_S \in H^{1/2}(S)\} = [|\varphi_S|]$ after (28). Thus, P is isometric. Moreover, its restriction to $L^2(S)$ is self-adjoint (since $\int_S P\varphi_S\, \psi_S = \int_S \varphi_S\, P\psi_S$) and positive definite (for $\int_S P\varphi_S\, \varphi_S \geq 0$, with equality for $\varphi_S = 0$ only).

As for the terminology, $H^{-1/2}(S)$ is of course one of the Sobolev spaces, which one proves is isomorphic to the dual of $H^{1/2}(S)$. It is made of distributions, not of functions, and contains $L^2(S)$. It may come as a surprise that $H^{1/2}(S)$ can be, like all Hilbert spaces, isomorphic with its dual $H^{-1/2}(S)$, and at the same time, can be identified with a subspace of it, via a continuous injection. But the latter is not *bi*-continuous (continuous in both directions), so there is no contradiction in that.

7.4.3 The exterior Dirichlet-to-Neumann map

For the exterior region, the approach is strictly the same, except for the basic functional space and for the absence of the term φ in the analogue of (25). We'll go much faster, directing attention only to the differences with respect to the previous case.

So let Φ_O be the space[16] of restrictions to $O \equiv E_3 - D$ of elements of Φ. Fitted with the norm $\varphi \to (\int_O |\text{grad}\, \varphi|^2)^{1/2}$ (which *is* a norm, since O is connected), it becomes a Hilbert space (larger than $L^2_{\text{grad}}(O)$, this time; it's one of these small but irreducible technical differences that force one to do the same work twice, as we are doing here). We denote by $\Phi_O(\varphi_S)$ the affine subspaces of Φ_O of the form $\{\varphi \in \Phi_O : \gamma\, \varphi = \varphi_S\}$, where φ_S is a given function belonging to $H^{1/2}(S)$. The subspaces $\Phi_O(\varphi_S)$ are not empty (there exists a function of $L^2_{\text{grad}}(E_3)$, thus of Φ_O, the trace of which is φ_S), and are closed by continuity of the trace mapping.

Proposition 7.4. *Let* $\varphi_S \in H^{1/2}(S)$ *be given. There exists* $\varphi \in \Phi_O(\varphi_S)$, *unique, such that*

(29) $$\int_O |\text{grad}\, \varphi|^2 \leq \int_O |\text{grad}\, \varphi'|^2 \quad \forall\, \varphi' \in \Phi_O(\varphi_S).$$

Proof. This specifies φ as the projection of the origin on the affine subspace $\Phi_O(\varphi_S)$, which is non-empty and closed, hence existence and uniqueness for φ. ◊

We note that the Euler equation of the variational problem (29) is

[16] It can be defined, alternatively, as the completion of $C_0^\infty(E_3)$ with respect to the norm $\varphi \to (\int_O |\text{grad}\, \varphi|^2)^{1/2}$.

(30) $\int_O \operatorname{grad} \varphi \cdot \operatorname{grad} \varphi' = 0 \quad \forall \varphi' \in \Phi_O(0)$.

Hence, taking smooth compactly supported test functions and integrating by parts, $\Delta\varphi = 0$ in O. The function φ is thus the *harmonic continuation* of φ_S to O, i.e., the solution of the "exterior Dirichlet problem":

$$\Delta\varphi = 0 \text{ in } O, \quad \varphi_{|S} = \varphi_S.$$

(A "condition at infinity" is implicitly provided by the inclusion $\varphi \in \Phi$: Although functions of Φ do not necessarily vanish at infinity (**Exercise 7.2:** Find such a freak), smooth functions of Φ do.) Consider now the normal derivative $\partial_n\varphi$ of φ on S, and denote by P the linear operator $\varphi \to \partial_n\varphi$. By the same arguments as before, P extends to an isometry from $H^{1/2}(S)$ to its dual $H^{-1/2}(S)$. Only the scalar product differs, and

(31) $\langle P\varphi_S, \psi_S\rangle = \int_S \partial_n\varphi\, \psi_S = \int_O \operatorname{grad} \varphi \cdot \operatorname{grad} \psi$,

where ψ is the harmonic continuation of ψ_S, hence self-adjointness and positivity. Note that both operators, exterior and interior, realize isometries between $H^{1/2}(S)$ and its dual, but for *different* norms on $H^{1/2}(S)$.

7.4.4 Integral representation

We now explain how the knowledge of P is equivalent to solving a particular integral equation on surface S. Subsection 7.4.5 will deal with the discretization of this equation. This will give us a linear system, to be solved as a prelude to solving the magnetostatics problem, to which we shall return in Section 7.5.

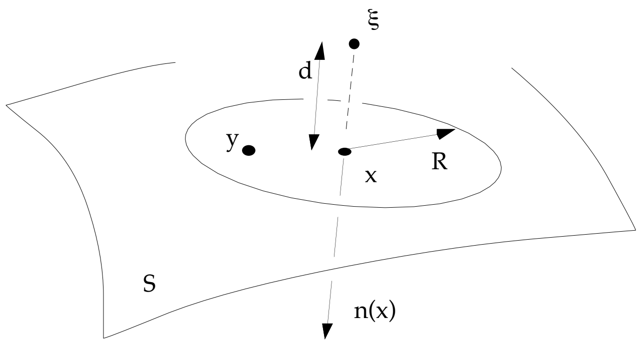


FIGURE 7.3. Notations. $S_R(x)$, or $S_R(\xi)$, is the disk of radius R drawn on S.

Let us introduce some notation: ξ will denote a point in space, and $x(\xi)$, or simply x, its projection on S (Fig. 7.3). If S is regular, the mapping $\xi \to x$ is well defined in some neighborhood of S. Let $d = |\xi - x|$ and $S_R(\xi) = \{y \in S : |y - x(\xi)| < R\}$. One has (this is trivial, but important):

(32) $$\int_{S_R(\xi)} |\xi - y|^{-1}\, dy \le C\, R,$$

where dy is the measure of areas on S, and C a constant depending on S but not on ξ. Last, we denote by $n(\xi)$ the vector at ξ parallel to $n(x)$. One thus obtains in the neighborhood of S a vector field, still denoted by n, which extends the field of normals. (Field lines of n are orthogonal to S.)

Remark 7.2. If S is only piecewise smooth, as we assumed from the beginning, the properties to be established below stay valid at all points of regularity of the surface (those in the neighborhood of which S has a tangent plane, and bounded principal curvatures). ◊

Now let q be a function defined on S, taken as smooth in a first approach (continuous is enough), and let us consider its potential φ:

(33) $$\varphi(x) = \frac{1}{4\pi}\int_S \frac{q(y)}{|x-y|}\, dy.$$

One may interpret q as an *auxiliary magnetic charge density* and φ as a magnetic potential (called "single layer potential"), from which derives a magnetic field $h = \text{grad}\, \varphi$. If $x \notin S$, the integral converges in an obvious way. But moreover, q being bounded, it also converges when $x \in S$ (study the contribution to the integral of a small disk centered at x, and invoke (32)). Finally, the function φ is continuous (thanks to the uniform bound (32), again), null at infinity, and harmonic outside S. Let us call K the operator $q \to \varphi_{|S}$.

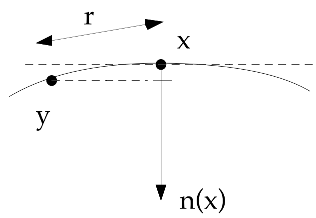
Next we study the field $h = \text{grad}\, \varphi$. By differentiation under the summation sign, one finds

$$\text{grad}\, \varphi = x \to \frac{1}{4\pi}\int_S q(y)\, \frac{y-x}{|x-y|^3}\, dy,$$

and this time the convergence of the integral when $x \in S$ is by no means certain. On the other hand, the real-valued integral

(34) $$(Hq)(x) = \frac{1}{4\pi}\int_S q(y)\, n(x) \cdot \frac{y-x}{|x-y|^3}\, dy$$

does converge when $x \in S$: For if x is a point of regularity of S, R a positive real value, and $|q|_R$ an upper bound for $q(y)$ on the set $S_R(x)$, the contribution of $S_R(x)$ to the integral is bounded by



$$\frac{|q|_R}{4\pi} \int_{S_R(x)} n(x) \cdot \frac{y-x}{|x-y|^3}\, dy, \tag{35}$$

a quantity which tends to 0 when R tends to 0 (take polar coordinates originating at x, remark by looking at the inset that $|x - y| \sim r$ and $n(x) \cdot (y - x) \sim r^2$). Hence another integral operator H, of the same type as K.

The form of Hq could suggest that it is the restriction to S of the function $n \cdot \text{grad}\, \varphi$ (compact notation for $\xi \to n(\xi) \cdot \text{grad}\, \varphi(\xi)$). But this is not so, for $n \cdot \text{grad}\, \varphi$, contrary to φ, is *not* continuous across S, but has a jump, equal to q, as we presently see. (Recall the definition of the jump as

$$[n \cdot \text{grad}\, \varphi]_S = n_+ \cdot (\text{grad}\, \varphi)_+ + n_- \cdot (\text{grad}\, \varphi)_-,$$

with the notation of Fig. 7.4. This can be denoted by $[\partial\varphi/\partial n]_S$, or better, $[\partial_n \varphi]_S$.)

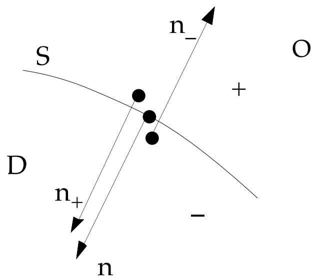


FIGURE 7.4. Notations for Proposition 7.5.

Proposition 7.5. *Let* φ *be the function defined in* (33). *One has*

$$[n \cdot \text{grad}\, \varphi]_S \equiv n_+ \cdot (\text{grad}\, \varphi)_+ + n_- \cdot (\text{grad}\, \varphi)_- = q,$$

$$n_+ \cdot (\text{grad}\, \varphi)_+ - n_- \cdot (\text{grad}\, \varphi)_- = Hq.$$

Proof. All these are well-defined functions, for φ is C^∞ outside S. Let d and R be fixed. Let us sit at point $\xi = x + \alpha\, d\, n(x)$, where α is meant to eventually converge to 0, and let $\beta = |\alpha|^{1/2}$. The contribution of the set $S - S_{\beta R}(x)$ to the integral

$$n(\xi) \cdot (\text{grad } \varphi)(\xi) = (4\pi)^{-1} \int_S dy\, q(y)\ |x - y|^{-3}\, n(\xi) \cdot (y - \xi)$$

has a well-defined limit (namely, $(Hq)(x)$) when ξ tends to x. So let us examine, according to a standard technique in singular integral computations, the contribution of $S_{\beta R}(x)$, whose limit will depend on the sign of α. Up to terms in $o(\alpha)$, this is

$$(4\pi)^{-1}\, q(x)\, n(x) \cdot \int_{S_{\beta R}} dy\, q(y)\ |\xi - y|^{-3}\, (y - \xi) \approx$$
$$(4\pi)^{-1}\, q(x)\, \alpha\, d \int_0^{\beta R} 2\pi\, r\, dr\, (r^2 + \alpha^2 d^2)^{-3/2}\, .$$

Studying this integral is a classical exercise:[17] Its limit is $\pm\, q(x)/2$ according to whether α tends to 0 from above or from below. The limit of $n(\xi)$, in the same circumstances, is n_+ or n_-. Thus $n_\pm \cdot (\text{grad } \varphi)_\pm = q/2 \pm Hq$. Hence, by addition and subtraction, the announced equalities. ◊

Here follows a first implication of Prop. 7.5. Let q' be another charge density, and φ' its potential. According to the divergence integration by parts formula, one has

$$(36) \qquad \int_{E_3} \text{grad } \varphi \cdot \text{grad } \varphi' = \int_D \text{grad } \varphi \cdot \text{grad } \varphi' + \int_O \text{grad } \varphi \cdot \text{grad } \varphi'$$
$$= \int_S \varphi\, [\partial_n \varphi] = \int_S \varphi\, q' = \int_S Kq\, q',$$

and in particular, $\int_{E_3} |\text{grad } \varphi|^2 = \int_S Kq\, q$. The operator K is thus self-adjoint and (strictly) positive definite on its domain, which we restricted up to now to regular functions. Note the formal similarity with P, exterior or interior: K is a bilateral Dirichlet-to-Neumann operator, so to speak.

This suggests the following extension of the definition of K, which is a variant of the “extension by continuity” of A.4.1. Suppose $q \in H^{-1/2}(S)$ given. The problem *find* $\varphi \in \Phi$ *such that*

$$\int_{E_3} \text{grad } \varphi \cdot \text{grad } \varphi' = \int_S q\, \varphi' \quad \forall\, \varphi' \in \Phi$$

is well posed, since $\varphi' \in H^{1/2}(S)$, with continuity of the trace mapping, and the map $q \to \varphi_S$ is therefore continuous from $H^{-1/2}(S)$ into $H^{1/2}(S)$. As it constitutes an extension of K, it is only natural to also denote it by K. (The K thus extended is an isometry between $H^{-1/2}(S)$ and $H^{1/2}(S)$, again for a different norm than in the previous two cases.) Then, after (36),

$$\int_{E_3} \text{grad } \varphi \cdot \text{grad } \varphi' = \langle q, Kq' \rangle,$$

[17]This is the same computation one does in electrostatics when studying the field due to a uniform plane layer of electric charge.

hence $\int_{E_3} \text{grad } \varphi \cdot \text{grad } \varphi' = \int_S q\ Kq'$ when $q \in L^2(S)$, so that K, now considered as an operator from $L^2(S)$ into itself, is self-adjoint and positive definite.

A second implication is the following formula, which explicitly gives the normal derivative of φ in terms of the charge q:

$$\partial_n\varphi(x) \equiv (n \cdot \text{grad } \varphi)(x) = \frac{1}{2}\, q(x) + \frac{1}{4\pi}\int_S q(y)\, n(x) \cdot \frac{y - x}{|x - y|^3}\, dy,$$

that is,

(37) $\partial_n\varphi = (1/2 + H)\, q.$

Since the mapping $q \to \partial_n\varphi$ is linear continuous from $H^{-1/2}(S)$ into itself, and since $Hq = \partial_n\varphi - q/2$ when q is regular, after (37), we may extend the operator H to $H^{-1/2}(S)$ in the present case, too.

We may now, at last, give an explicit form to the operator P. (Let's revert to our usual convention that φ_S denotes the trace of φ.) Since $\partial_n\varphi = P\varphi_S$, by definition, and $\varphi_S = Kq$, one has $PK = 1/2 + H$, after (37), hence the result we were after:

(38) $P = (1/2 + H)\, K^{-1}.$

Yet we are not through, far from it, for (38) must be discretized, and this cannot be done simply by replacing the operators H and K by their matrix equivalents **H** and **K**, whatever they are (we'll soon give them): The matrix $(1/2 + \mathbf{H})\, \mathbf{K}^{-1}$ thus obtained would not be symmetrical, contrary to our wishes.

7.4.5 Discretization

In this subsection, we simply write φ for φ_S, and Φ for the space $H^{1/2}(S)$. Let Q denote the space $H^{-1/2}(S)$ where q lives. After (38), written in weak form, operator P is such that

(39) $\langle P\varphi, \varphi'\rangle = \langle q, \varphi'\rangle/2 + \langle Hq, \varphi'\rangle \quad \forall\, \varphi' \in \Phi$

for each couple $\{\varphi, q\}$ linked by the relation

(40) $\langle \varphi, q'\rangle = \langle Kq, q'\rangle \quad \forall\, q' \in Q.$

A mesh *m* of D, and thus of S, being defined, let us denote by Φ_m and Q_m mesh-dependent approximation spaces for Φ and Q, to be constructed. We have a natural choice for Φ_m already: the trace on S of $W^0_m(D)$.

The choice of Q_m, for which we have no obvious rationale yet, is deferred for a while.

On the sight of (39) and (40), a discretization principle suggests itself: We look for P_m, an operator of type $W^0_m(S) \to W^0_m(S)$, that should be symmetrical like P and such that

(41) $\quad <P_m\varphi, \varphi'> = <q, \varphi'>/2 + <Hq, \varphi'> \quad \forall\, \varphi' \in \Phi_m$

for all couples $\{\varphi, q\} \in \Phi_m \times Q_m$ linked by the relation

(42) $\quad <\varphi, q'> = <Kq, q'> \quad \forall\, q' \in Q_m$.

The representations $\varphi = \sum_{n \in \mathcal{N}(S)} \boldsymbol{\varphi}_n w_n$ and $q = \sum_{i \in \mathcal{J}} \mathbf{q}_i \zeta_i$ (where the set $\mathcal{J}$ and the basis functions ζ_i have not yet been described), define isomorphisms between the spaces Φ_m and Q_m and the corresponding spaces $\boldsymbol{\Phi}$ and $\mathbf{Q}$ of vectors of DoFs. Let us denote by $\mathbf{B}, \mathbf{H}, \mathbf{K}$ the matrices defined as follows, which correspond to the various brackets in (41) and (42):

(43) $\quad \mathbf{B}_{n\,i} = \int_S dx\, w_n(x)\, \zeta_i(x),$

(44) $\quad \mathbf{K}_{i\,j} = (4\pi)^{-1} \iint_S dx\, dy\, (|y-x|)^{-1} \zeta_i(x)\, \zeta_j(y),$

(45) $\quad \mathbf{H}_{n\,i} = (4\pi)^{-1} \iint_S dx\, dy\, (|y-x|)^{-3}\, n(x) \cdot (y-x)\, \zeta_i(y)\, w_n(x).$

According to the foregoing discretization principle, we look for the symmetric matrix $\mathbf{P}$ (of order $N(S)$, the number of nodes of the mesh on S), such that

(46) $\quad (\mathbf{P}\,\boldsymbol{\varphi}, \boldsymbol{\varphi}') = (\mathbf{B}\,\mathbf{q}, \boldsymbol{\varphi}')/2 + (\mathbf{H}\,\mathbf{q}, \boldsymbol{\varphi}') \quad \forall\, \boldsymbol{\varphi}' \in \boldsymbol{\Phi},$

for all couples $\{\boldsymbol{\varphi}, \mathbf{q}\} \in \boldsymbol{\Phi} \times \mathbf{Q}$ linked by

(47) $\quad (\mathbf{B}^t\,\boldsymbol{\varphi}, \mathbf{q}') = (\mathbf{K}\,\mathbf{q}, \mathbf{q}') \qquad \forall\, \mathbf{q}' \in \mathbf{Q}$

(the bold parentheses denote scalar products in finite dimension, as in 4.1.1). Since (47) amounts to $\mathbf{q} = \mathbf{K}^{-1}\mathbf{B}^t\boldsymbol{\varphi}$, we have

(48) $\quad \mathbf{P} = \mathrm{sym}((\mathbf{B}/2 + \mathbf{H})\,\mathbf{K}^{-1}\,\mathbf{B}^t)$

after (46), with t for "transpose" and sym for "symmetric part".

This leaves the selection of "basis charge distributions" ζ_i to be performed. An obvious criterion for such a choice is the eventual simplicity of the computation of double integrals in (43–45), and from this point of view, taking ζ_i constant on each triangle is natural: thus $\mathcal{J}$ will be the set of surface triangles, and one will define ζ_i for $i \in \mathcal{J}$ as the characteristic function of triangle i (equal to one over it and to 0 elsewhere), divided by the area. This was the solution retained for the Trifou eddy-current code (cf. p. 225), and although not totally satisfactory (cf. Exer. 7.3), it does make the computation of double integrals simple.

Simple does not mean trivial however, and care is required for terms of **K**, which are of the form

$$K_{TT'} = \frac{1}{4\pi} \int_T dy \int_{T'} dx\ |x - y|^{-1},$$

where T and T' are two non-intersecting triangles in generic position in 3-space. The internal integral is computed analytically, and the outer one is approximated by a quadrature formula, whose sophistication must increase when triangles T and T' are close to each other. Any programmer with experience on integral or semi-integral methods of some kind has had, at least once in her life, to implement this computation, and knows it's a tough task. Unfortunately, the details of such implementations are rarely published (more out of modesty than a desire to protect shop secrets). Some indications can be gleaned from [AR, Cl, R&].

Remark 7.3. As anticipated earlier, the "naive" discretization of (38), yielding $\mathbf{P} = (1/2 + \mathbf{H})\mathbf{K}^{-1}$, would be inconsistent (the dimension of **K** is not what is expected for **P**, that is, N(S)). But the more sophisticated expression $(\mathbf{B}/2 + \mathbf{H})\ \mathbf{K}^{-1}\ \mathbf{B}^t$ would not do, either, since this matrix is not symmetric, and the symmetrization in (48), to which we were led in a natural way, is mandatory. ◊

Exercise 7.3. Show, by a counter-example, that matrix **P** may happen to be singular with the above choice for the ζ_i.

7.5 BACK TO MAGNETOSTATICS

We may now finalize the description of the "finite elements and integral method in association" method of 7.3.3, in the case when the unknown is the scalar potential. Let $\Phi(D)$ be the space of restrictions to D of the scalar potentials in space Φ. We denote again by φ_S the trace of φ on S. Thanks to the operator P, the Euler equation (15) is equivalent to the following problem: *find* $\varphi \in \Phi(D)$ *such that*

$$(49) \qquad \int_D \mu\,(h^j + \text{grad}\,\varphi) \cdot \text{grad}\,\varphi' + \mu_0 \int_S (n \cdot h^j + P\varphi_S)\,\varphi'_S = 0 \quad \forall\, \varphi' \in \Phi(D).$$

Let m be a mesh of D. Then $\Phi_m(D) = \{\varphi : \varphi = \sum_{n \in \mathcal{N}} \boldsymbol{\varphi}_n w_n\}$ is the natural approximation space for $\Phi(D)$. Hence the following approximation of (49), *find* $\varphi \in \Phi_m(D)$ *such that, for all* $\varphi' \in \Phi_m(D)$,

(50) $\int_D \mu\,(h^j + \operatorname{grad}\varphi)\cdot\operatorname{grad}\varphi' + \mu_0\int_S (n\cdot h^j + P\varphi_S)\,\varphi'_S = 0.$

Let $\boldsymbol{\varphi}$ be the vector of degrees of freedom (one for each node, including this time those contained in S), and $\boldsymbol{\Phi} = \mathbb{R}^{\mathcal{N}}$ (there are N nodes). We denote

$$\boldsymbol{\eta}^j_n = \int_D \mu\, h^j\cdot\operatorname{grad} w_n + \mu_0\int_S n\cdot h^j\, w_n,$$

$\boldsymbol{\eta}^j = \{\boldsymbol{\eta}^j_n : n\in\mathcal{N}\}$, and let $\mathbf{G}$, $\mathbf{R}$, $\mathbf{M}_1(\mu)$ be the same matrices as in Chapter 5. Still denoting by $\mathbf{P}$ the extension to $\boldsymbol{\Phi}$ (obtained by filling-in with zeroes) of the matrix $\mathbf{P}$ of (48), we finally get the following approximation for (50):

(51) $(\mathbf{G}^t\mathbf{M}_1(\mu)\mathbf{G} + \mathbf{P})\boldsymbol{\varphi} + \boldsymbol{\eta}^j = 0.$

Although the matrix $\mathbf{P}$ of (38) is full, the linear system (51) is reasonably sparse, because $\mathbf{P}$ only concerns the “S part” of vector $\boldsymbol{\varphi}$.

Remark 7.4. The linear system is indeed an *approximation* of (50), and not its interpretation in terms of degrees of freedom, for $(\mathbf{P}\boldsymbol{\varphi}, \boldsymbol{\varphi}')$ is just an approximation of $\int_S P\varphi_S\,\varphi'_S$ on the subspace $\Phi_m(D)$, not its restriction, as in the Galerkin method. (This is another example of “variational crime”.) ◊

Remark 7.5. There are other routes to the discretization of P. Still using magnetic charges (which is a classic approach, cf. [Tz]), one could place them differently, not on S but inside D [MW]. One might, for example [Ma], locate a point charge just beneath each node of S. (The link between $\mathbf{q}$ and $\boldsymbol{\varphi}$ would then be established by collocation, that is to say, by enforcing the equality between φ and the potential of q at nodes.[18]) Another approach [B2] stems from the remark that interior and exterior Dirichlet-to-Neumann maps (call them P_{int} and P_{ext}) add to something which is easily obtained in discrete form, because of the relation $(P_{int} + P_{ext})\varphi = q = K^{-1}\varphi$. Since, in the present context, we must mesh D anyway, a natural discretization $\mathbf{P}_{int}$ of P_{int} is available, thanks to the “static condensation” trick of Exer. 4.8: One minimizes the quantity $\int_D |\operatorname{grad}(\sum_{n\in\mathcal{N}(D)}\varphi_n w_n)|^2$ with respect to the *inner* node values $\boldsymbol{\varphi}_n$, hence a quadratic form with respect to the vector $\boldsymbol{\varphi}$ (of surface node DoFs), the matrix of which is $\mathbf{P}_{int}$. A reasoning similar to the one we did around (46–47) then suggests $\mathbf{B}\,\mathbf{K}^{-1}\,\mathbf{B}^t$ as the correct discretization of K^{-1}, hence finally $\mathbf{P} \equiv \mathbf{P}_{ext} = \mathbf{B}\,\mathbf{K}^{-1}\,\mathbf{B}^t - \mathbf{P}_{int}$ (which ensures the symmetry of $\mathbf{P}_{ext}$, but does not eliminate the difficulty evoked in Exer. 7.3). And (lest we

[18] See, e.g., [KP, ZK]. These authors’ method does provide a symmetric $\mathbf{P}$, but has other drawbacks. Cf. [B2] for a discussion of this point.

forget . . .) for some simple shapes of S (the sphere, for example), P is known in closed form, as the sum of a series. ◊

A similar theory can be developed "on the curl side" [B2, B3]: Start from problem (16), introduce the exterior energy $\frac{1}{2}\int_{E_3 - D} \mu^{-1} |\text{rot } a|^2$, then the operator $\mathbb{P} = a_S \to n \times \text{rot } a$, where a satisfies $\text{rot rot } a = 0$ outside D. Instead ot the auxiliary charge q, one has an auxiliary current density borne by S. See [RR] for an application of this technique.

EXERCISES

Exercises 7.1 to 7.3 are on pp. 205, 208, and 214, respectively.

Exercise 7.4. Show that (in spite of Exer. 7.3) the matrix of system (51) is regular.

Exercise 7.5. Given a smooth function q with bounded support, its *Newtonian potential* φ is

$$\varphi(x) = \frac{1}{4\pi}\int_{E_3} \frac{q(y)}{|x-y|}\, dy.$$

Show that $-\Delta\varphi = q$.

Exercise 7.6. Prove (22). Show that one can reduce the problem to the case where D is a half-space and S a plane, with φ compactly supported. Take Cartesian coordinates for which $D = \{x : x^1 \geq 0\}$, and use Fubini to show that the problem can be reduced to studying functions of one real variable t (here x^1) with values in a functional space X (here, $L^2(E_2)$). Work on functions $u \in C^1([0, 1]; X)$ and bound $\|u(0)\|^2$ by $\int_0^1 \|u(t)\|^2\, dt + \int_0^1 \|\partial_t u(t)\|^2\, dt$, using Cauchy–Schwarz.

HINTS

7.1. This is a poorly wrapped paradox, almost a mere play on words: Make sure you understand that "closure" means different things in the text of the exercise and in the sentence just before. (My apologies if you felt insulted.)

7.2. Take a sequence of points x_n in E_3 tending to infinity, and have φ supported by small neighborhoods of these points. It's easy to enforce $\int_{E_3} |\text{grad } \varphi|^2 < \infty$, although φ(x) does not tend to zero when $|x|$ tends to infinity, by construction.

7.5. Since differential operators commute with the convolution product, the problem reduces to showing that $-\Delta(x \to 1/|x|) = 4\pi\delta_0$, where δ_0 is the Dirac mass at the origin. Half of it was solved with Exer. 4.9. The delicate point is the computation of the divergence *in the sense of distributions* of the field $x \to -x/|x|^3$ (refer to A.1.9 for the arrowed notation).

SOLUTIONS

7.1. $C_0^\infty(D)$, which is dense in $L^2(D)$ with respect to the L^2-norm, is indeed dense also in the subspace $L^2_{grad}(D)$, *with respect to this same norm.* But with respect to the *stronger* norm put on $L^2_{grad}(D)$, which is the point, it's not, simply because there are *fewer* Cauchy sequences for this norm, so their limits form a smaller space, namely, $H^1_0(D)$. In short, the stronger the norm, the smaller the closure.

7.2. Let $\varphi_n = y \to n\,(1 - n^4\, |y - x_n|^+)$, where $+$ denotes the positive part of an expression. Then $\int |\text{grad}\ \varphi_n|^2$ is in n^{-2}, so $\varphi = \sum_n \varphi_n$ is in the Beppo Levi space, but $\varphi(x_n) = n$ for n large enough, and doesn't vanish at infinity.

7.3. See Ref. [CC].

7.5. The same computation as in Exer. 4.9 would give

$$(*) \qquad \text{div}(x \to x/|x|^3) = x \to 3/|x|^3 - 3\,x\cdot x/|x|^5 \equiv 0,$$

if it were not for the singularity at the origin. What was obtained there is only the "function part" of the *distribution* $\text{div}(x \to x/|x|^3)$, which is therefore concentrated at the origin. To find it, apply Ostrogradskii to a sphere of radius r centered at the origin, which gives the correct result, $\text{div}(x \to x/|x|^3) = -4\pi\,\delta_0$. Then, denoting by χ the kernel $x \to 1/(4\pi|x|)$, one has $\varphi = \chi * q$, and hence, $-\Delta\varphi = -\Delta(\chi * q) = (-\Delta\chi) * q = \delta_0 * q = q$.

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