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# Foreword

This book is the second volume in the Academic Press Electromagnetism Series, written by Professor Alain Bossavit, one of the most active researchers in the area of electromagnetic field calculations. Professor Bossavit is well known and highly regarded in the electromagnetic community for his seminal contributions to the field of computational electromagnetics. In particular, he has pioneered and strongly advocated the use of edge elements in field calculations. These elements, which are now widely accepted by engineers, have become indispensable tools in numerical analysis of electromagnetic fields. His work on the use of symmetry in numerical calculations, computational implementation of complementarity, and evaluation of electromagnetic forces have also been extremely important for the development of the field.

This book reflects the unique expertise and extensive experience of the author. It is written with a strong emphasis on comprehensive and critical analysis of the foundations of numerical techniques used in field calculations. As a result, the book provides many valuable insights into the nature of these techniques. It contains information hardly available in other sources and no doubt will enrich the reader with new ideas and a better conceptual understanding of computational electromagnetics. The material presented in the book can be expected to contribute to the development of new and more sophisticated software for electromagnetic field analysis.

The book is distinctly unique in its original style of exposition, its emphasis, and its conceptual depth. For this reason, it will be a valuable reference for both experts and beginners in the field. Researchers as well as practitioners will find this book challenging, stimulating, and rewarding.

Isaak Mayergoyz, *Series Editor*



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