(5): The "Galerkin hodge"

Where we stand

With the "Whitney map" of last issue, we have a way to pass from degrees of freedom (DoF) to fields, in the case of a simplicial mesh. In particular, we may construct a vector field

$$\mathbf{E} = \sum_{e \in \mathcal{E}} \mathbf{e}_e \mathbf{W}^e \qquad (1)$$

from a DoF array $\mathbf{e}$, edge-based, thanks to the "edge element"

$$\mathbf{W}^e = w^m \nabla w^n - w^n \nabla w^m. \qquad (2)$$

In this formula, m and n are the endpoints of edge e (cf. Fig. 1), and w^n is the "hat function" of standard finite element theory (equal to 1 at node n, 0 at other nodes, and linearly interpolating in between). $\mathbf{W}^e$ is a rightful finite element for the electric field, because it has tangential continuity across element interfaces, thus automatically conferring to $\mathbf{E}$ this essential physical property. Its circulation along edge e' is 1 if $e' = e$, else 0, which ensures, thanks to (1), that each DoF $\mathbf{e}_e$ is indeed the circulation of $\mathbf{E}$ along edge e.

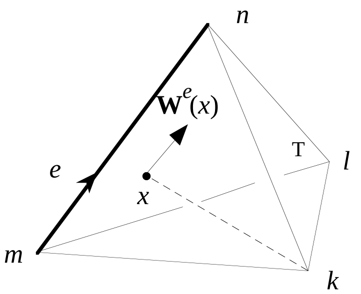

Figure 1. The edge element. One has $\mathbf{W}^e(x) = kl \times kx/(6\,\mathrm{vol}(\mathrm{T}))$, which makes it easy to visualize the field. Recall however that in spite of this apparently metric-dependent expression, the edge element is an affine object: $\mathbf{W}^e$ is just the vector proxy of the Whitney form $w^e = w^m \mathrm{d}w^n - w^n \mathrm{d}w^m$.

All this designates $\mathbf{W}^e$ as a suitable finite element for vector-valued entities associated with lines, such as the fields $\mathbf{E}$ and $\mathbf{H}$. So why not use it as such for problems involving this kind of fields, by following the Galerkin variational approach? That was indeed the viewpoint 20 years ago, when the edge element made possible the solution in dimension 3 of eddy current problems, with $\mathbf{H}$ as unknown field. But nowadays we tend to see things in a different light: The Galerkin method using edge elements can be interpreted as a way—one among several possible ways—to build a discrete Hodge operator, which is what will occupy us in this installment of the series.

5.1 Model problem

The model problem this time, for a change, will be *electrostatics*, in the same cavity as usual (Fig. 2). Given a time-independent charge density q, find $\mathbf{D}$ and $\mathbf{E}$ such that

$$\begin{aligned} &\operatorname{div}\mathbf{D} = q,\ \mathbf{D} = \epsilon\,\mathbf{E},\ \operatorname{rot}\mathbf{E} = 0, \\ &\nu \cdot \mathbf{D} = 0 \text{ on } S^h,\ \nu \times \mathbf{E} = 0 \text{ on } S^e. \end{aligned} \qquad (3')$$

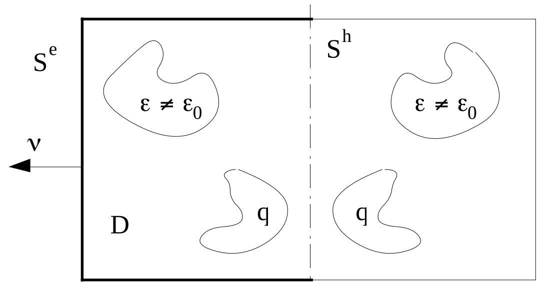

Figure 2. Position of the model problem: Same metallic cavity as before (*JSAEM*, *7*, 1999, p. 151), but a steady cloud of electric charge where we formerly had an antenna. Symmetry of the cavity, and of the charge distribution, allow us to compute in domain D (the left half), with the boundary condition $\nu \cdot \mathbf{D} = 0$ on the symmetry plane S^h, where ν denotes the outward-directed unit normal vector.

In differential geometric form, this is

$$\begin{aligned} &\mathrm{d}\tilde{d} = \tilde{q},\ \tilde{d} = \tilde{\star}_\epsilon e,\ \mathrm{d}e = 0, \\ &t\tilde{d} = 0 \text{ on } S^h,\ te = 0 \text{ on } S^e, \end{aligned} \qquad (3)$$

where $\tilde{q}$ stands for the twisted 3-form the scalar proxy of which is q.

We learned how to produce a discretization, at least in the case when an orthogonal dual mesh can be built. This gave us a diagonal matrix $\tilde{\star}_\epsilon$, whose entries, indexed over the set $\mathcal{E}$ of "active" edges (those not in S^e), were $\tilde{\star}_\epsilon^{ee} = \epsilon\, \text{area}(\tilde{e})/\text{length}(e)$, where $\tilde{e}$ is the 2-cell dual to e. Hence the numerical scheme:

$$(4) \qquad -\mathbf{G}^t\tilde{\mathbf{d}} = \tilde{\mathbf{q}},\ \tilde{\mathbf{d}} = \tilde{\star}_\epsilon \mathbf{e},\ \mathbf{Re} = 0,$$

with $\mathbf{e} = \{\mathbf{e}_e : e \in \mathcal{E}\}$ and $\tilde{\mathbf{d}} = \{\tilde{\mathbf{d}}_e : e \in \mathcal{E}\}$, the electric fluxes across the $\tilde{e}$s. The data $\tilde{\mathbf{q}} = \{\tilde{\mathbf{q}}_n : n \in \mathcal{N}\}$ is obtained by integration of q over each dual 3-cell $\tilde{n}$. All we need to do, to get (4), is to replace each element of ($3'$) by its discrete counterpart: **E** by **e**, rot by **R**, div by $-\mathbf{G}^t$, etc., and ϵ by the discrete Hodge operator $\tilde{\star}_\epsilon$. Of course, one must have done the practical work leading to $\tilde{\star}_\epsilon$.

The Ritz–Galerkin approach proceeds differently, and one certainly does not recognize it in (4) at first glance. Yet, in depth, there are strong homologies. To argue this point, I shall first sketch a possible presentation of the Galerkin method, using standard finite elements, to a classroom of advanced students in computational electromagnetism. Then a crucial modification (cf. Prop. 1 below) will be introduced, leading back to (4).

5.2 The Galerkin method, with node-based scalar finite elements

The method is based on the so-called *weak formulation* of the equation div$\mathbf{D} = q$ and (both things in one stroke) the boundary condition $\nu \cdot \mathbf{D} = 0$ on S^h:

$$-\int_D \mathbf{D} \cdot \nabla\psi' = \int_D q\,\psi' \text{ for all } \psi' \text{ in } \Psi,$$

where Ψ is a space of *test functions*, characterized, in addition to a few technical requirements,[1] by the condition $\psi' = 0$ on S^e. Using the fact that the equations about **E**, i.e., rot$\mathbf{E} = 0$ and $\nu \times \mathbf{E} = 0$ on S^e, are equivalent to "$\mathbf{E} = -\nabla\psi$ for some ψ in Ψ", one is led to *find ψ in Ψ such that*

$$(5) \qquad \int_D \epsilon\,\nabla\psi \cdot \nabla\psi' = \int_D q\,\psi'\ \forall\,\psi' \in \Psi.$$

This comprehensive weak formulation of the original problem is equivalent, as one easily shows, to minimizing[2] over Ψ the energy-related quantity

$$W(\psi') = \int_D \epsilon\,|\nabla\psi'|^2 - 2\int_D q\,\psi'.$$

At this stage, one points out that a function such as

$$(6) \qquad \psi = \sum_{n \in \mathcal{N}} \boldsymbol{\psi}_n w^n$$

does belong to Ψ, since $\psi = 0$ at nodes of S^e, which have been excluded from the set $\mathcal{N}$. Such functions span a subspace of Ψ of finite dimension (equal to the number N of active nodes), which we denote by $\Psi_{\mathcal{M}}$, since it depends on the mesh. When "$\mathcal{M} \to 0$" (in the precise sense introduced last time, including uniformity of the family of meshes), $\Psi_{\mathcal{M}}$ "tends to fill-out" Ψ, which motivates the replacement of (5) by the approximation *find $\psi_{\mathcal{M}}$ in $\Psi_{\mathcal{M}}$ such that*

$$(7) \qquad \int_D \epsilon\,\nabla\psi_{\mathcal{M}} \cdot \nabla\psi' = \int_D q\,\psi'\ \forall\,\psi' \in \Psi_{\mathcal{M}}.$$

This—one then proceeds to show—is actually a linear system with respect to the N components of the DoF array $\boldsymbol{\psi} = \{\boldsymbol{\psi}_n : n \in \mathcal{N}\}$. Indeed, introduce the matrix elements

$$(8) \qquad \mathbf{A}^{nm} = \int_D \epsilon\,\nabla w^n \cdot \nabla w^m$$

and the N-vector **b**, say (nothing to do with the induction field), whose components are $\mathbf{b}^m =$

[1] They must belong to the Sobolev space $L^2_{\text{grad}}(D)$ of square-summable functions whose grad, too, is in $L^2(D)$. Then, $\nabla\psi$ belongs to the space $L^2_{\text{rot}}(D)$ that we briefly encountered in the first column (*JSAEM, 7*, 1999, p. 152). The necessary recourse, if one wants to be thorough, to such difficult notions of functional analysis has acted as a serious deterrent against the popularization of the method.

[2] This constitutes a *variational principle*, as so often met in physics. If there is such a variational characterization of the solution, one can derive a weak formulation from it. But many important problems don't correspond to the minimization of anything, and still can be cast in an appropriate weak form that makes them eligible to application of the Galerkin method. This is why the variational aspects of the method, although historically decisive, are downplayed in the presentation suggested here.

$\int_D q\, w^m$. Then, using (6) and replacing ψ' by the test function w^m (eligible, when $m \in \mathcal{N}$, since it then belongs to $\Psi_{\mathcal{M}}$), one derives from (7) the N algebraic equations

$$(9) \qquad \sum_{n \in \mathcal{N}} \mathbf{A}^{mn} \psi_n = \mathbf{b}^m \text{ for all } m \text{ in } \mathcal{N},$$

or $\mathbf{A}\psi = \mathbf{b}$, in compact form. Matrix $\mathbf{A}$, obviously symmetric and positive definite, is called the "stiffness matrix" of the problem, due to a mechanical analogy of little concern for us.

Assuming one has already covered a matrix algebra curriculum, including algorithms to solve $\mathbf{A}\psi = \mathbf{b}$, the last practical topic to treat is the effective computation of the stiffness matrix. This process, called "assembly" of $\mathbf{A}$, consists in first evaluating the "element matrices" $\mathbf{A}_{\mathrm{T}}$ defined, for each tetrahedron T, by $\mathbf{A}_{\mathrm{T}}^{nm} = \int_{\mathrm{T}} \epsilon \nabla w^n \cdot \nabla w^m$, then to form $\mathbf{A} = \sum_{\mathrm{T}} \mathbf{A}_{\mathrm{T}}$ by looping over T. Each $\mathbf{A}_{\mathrm{T}}$ is easily computed. There are *few* nonzero $\mathbf{A}_{\mathrm{T}}^{nm}$s, for one thing, and if ϵ is taken uniform inside T, then, after (8),

$$(10) \qquad \mathbf{A}_{\mathrm{T}}^{nm} = -\epsilon \cot(\theta_{kl}^{\mathrm{T}}) \text{ length}(kl)/6$$

(Fig. 3), as some elementary geometry shows.

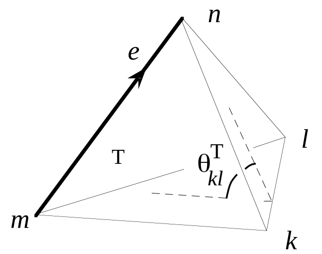

Figure 3. Notation for formula (10): θ_{kl}^{T} is the dihedral angle in front of edge $e = \{m, n\}$.

So the assembly problem is solved by formulas (8) and (10). These, however, hide some structural features of $\mathbf{A}$ which, though one can ignore them in a first course, are of some interest.

5.3 A reinterpretation

Let's introduce, for two edges e and e' of the mesh, the number

$$(11) \qquad \mathbf{M}^{ee'} = \int_D \epsilon\, \mathbf{W}^e \cdot \mathbf{W}^{e'},$$

and call "mass matrix" (again, due to an analogy) the square $(E \times E)$-matrix $\mathbf{M}_1(\epsilon)$, abbreviated as $\mathbf{M}$, as a rule, that these numbers form.[3] Now, $\mathbf{G}$ being the nodes-to-edges incidence matrix as usual,

Proposition 1. $\mathbf{A} = \mathbf{G}^t \mathbf{M} \mathbf{G}$.

Proof. This is a consequence of a major structural property of the Whitney complex, which appeared last time in the general form $\mathrm{d} p_{\mathcal{M}} = p_{\mathcal{M}} \mathbf{d}$ (*JSAEM, 8,* 2000, p. 109, formula (22)). Its avatar of interest here is $\nabla w^n = \sum_{e \in \mathcal{E}} \mathbf{G}_{en} \mathbf{W}^e$ (the gradient of a hat function is a linear combination of edge elements, those of the edges that abut on the node, with weights ± 1 according to orientation). Bringing that into (8), we do get $\mathbf{A}^{nm} = \sum_{e,e'} \mathbf{G}_{en} \mathbf{M}^{ee'} \mathbf{G}_{e'm}$, thanks to (11). $\diamondsuit$

Let's not misunderstand this result: The point is *not* to compute $\mathbf{A}$ by first computing $\mathbf{M}$, then using Prop. 1. Standard assembly, using (10), is cheaper. Here is the point: If we set $\mathbf{e} = -\mathbf{G}\psi$ and $\tilde{\mathbf{d}} = \mathbf{M}\mathbf{e}$, the equation $\mathbf{G}^t \mathbf{M} \mathbf{G} \psi = \mathbf{b}$ is equivalent to the system

$$(12) \qquad -\mathbf{G}^t \tilde{\mathbf{d}} = \mathbf{b}, \ \tilde{\mathbf{d}} = \mathbf{M}\mathbf{e}, \ \mathbf{R}\mathbf{e} = 0,$$

which looks very much like (4), and suggests to interpret $\mathbf{M}$ as a discrete Hodge operator similar to $\tilde{\star}_\epsilon$. Indeed, $\mathbf{M}$ has the right size (the number of active edges), and $\mathbf{e}$ is the array of edge-circulations of $\mathbf{E}$, as we remarked in the Introduction. But it's not so obvious that component $\tilde{\mathbf{d}}_e$ of $\tilde{\mathbf{d}}$ corresponds to the flux of $\mathbf{D}$ through some dual cell associated with e. As we shall see, this is indeed so, and $\tilde{\mathbf{d}}_e = \int_{\tilde{e}} \mathbf{D}$, where $\tilde{e}$ is the "barycentric dual" of edge e. There is also a sense in which the $\mathbf{b}$ of (12) deserves to be denoted by $\tilde{\mathbf{q}}$.

This will take some preparation (next Sections, 5.4 and 5.5). Meanwhile, let's remark that the mixed systems one can derive from the symmetric formulation (12), that is,

$$(13) \qquad \begin{pmatrix} -\mathbf{M} & \mathbf{R}^t \\ \mathbf{R} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \tilde{\mathbf{h}} \end{pmatrix} = \begin{pmatrix} -\tilde{\mathbf{d}}^q \\ 0 \end{pmatrix}$$

(where $\tilde{\mathbf{d}}^q$ is a DoF array such that $-\mathbf{G}^t \tilde{\mathbf{d}} = \mathbf{b}$, and $\tilde{\mathbf{h}}$ a kind of electric vector potential), and

$$(14) \qquad \begin{pmatrix} \mathbf{M}^{-1} & \mathbf{G} \\ \mathbf{G}^t & 0 \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{d}} \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ -\mathbf{b} \end{pmatrix},$$

[3] The subscript refers to the simplices' dimension, 1 for edges. Similarly, there is a mass matrix $\mathbf{M}_2(\mu^{-1})$, with entries $\int_D \mu^{-1}\, \mathbf{W}^f \cdot \mathbf{W}^{f'}$.

are less similar than their analogues in magnetostatics were (see (8) and (9) in *JSAEM, 7*, 1999, p. 403), because $\mathbf{M}^{-1}$ is here a full matrix. For the same reason, the facet-based "electric vector potential" formulation, $\mathbf{R}\mathbf{M}^{-1}\mathbf{R}^t\tilde{\mathbf{h}} = -\mathbf{R}\mathbf{M}^{-1}\tilde{\mathbf{d}}^q$ (compare with eq. (6), same source) has little appeal here.

5.4 A remarkable formula

We now introduce a new notion: "dyadic products", or simply "dyads". Remember the metaphor of a covector ω as a machine with one slot, in which one slips a vector v, to get in return a real number, denoted $\langle \omega, v\rangle$? We generalized that to p-covectors (machines with p slots, each of them meant to receive one vector). Now, our purpose is to define other similar machines, whose slots can accept geometrical objects of various types.

The simplest case is that of a machine with two slots, one on the left that can receive a covector, one on the right that can receive a vector, and a central dial that displays a real number, with the habitual linearity properties with respect to both arguments. Denoting by M the machine, we shall write the dial's reading in the form $\langle \omega, M, v\rangle$.

What can be the inner structure of such a machine? Note that if ω stays permanently in the left-hand slot, the machine behaves like a covector "when operated, single-handedly, from the right", and vice-versa. Hence the idea to build a machine of this kind by using as inner components a fixed vector and a fixed covector, w and η let's say, and to make the dial indicate $\langle \omega, w\rangle\langle \eta, v\rangle$ when ω and v are slipped into the slots, since this rule has the required linearity properties. Let's denote the machine thus obtained by the symbol $w\rangle\langle\eta$, and call that the *dyadic product* of w and η. So if we substitute $w\rangle\langle\eta$ for M in the above expression $\langle \omega, M, v\rangle$, we get this:

$$(15) \qquad \langle \omega, w\rangle\langle \eta, v\rangle = \langle \omega, w\rangle\langle \eta, v\rangle \ \forall\, \omega, v,$$

which in spite of looking like a notational joke is a bona-fide *definition* of $w\rangle\langle\eta$! It would not be difficult to show that the structure of the general machine—the right name for which, by the way, is *tensor*—is $M = \sum_i w_i\rangle\langle\eta_i$, but we'll dispense with the proof. (**Hint:** Take a basis, represent M as a matrix, and think of its decomposition as a sum of matrices of rank 1.) We shall not insist either on the generalization to dyadic products of p-vectors by q-covectors.

Among machines like M, one is special, the *unity*, denoted by 1, and defined by $\langle \omega, 1, v\rangle = \langle \omega, v\rangle$ for all ω and v.

With this, we are ready for

Proposition 2. $\sum_{e\in\mathcal{E}}\ e\rangle\langle w^e(x) = 1,$

where e is not only a label for edge e, but stands for the vector along edge e, while w^e is the associated Whitney form.

No proof is needed, for Prop. 2 is actually what we obtained last time (*JSAEM, 8*, 2000, p. 107): The Whitney forms w^e were constructed in such a way that a vector v anchored at x (then denoted xy), be equal to the sum $\sum_e\langle w^e(x), v\rangle\, e$ over edges of the mesh. This equality between vectors is equivalent to

$$\langle \omega, \sum_e \langle w^e(x), v\rangle\ e\rangle = \langle \omega, v\ \rangle$$

for any covector ω, and hence, by linearity, to

$$\sum_e \langle \omega, e\rangle\langle w^e(x), v\rangle = \langle \omega, v\ \rangle\ \ \forall\, \omega, v$$

which after (15) is exactly what Prop. 2 says. One may argue that the very definition of Whitney 1-forms (edge elements) was engineered in order to obtain Prop. 2.

Since similar considerations dictated the construction of Whitney forms for simplices of all dimensions, edges here have no privilege. There is such a formula for simplices of all dimensions: For facets, one has $\sum_{f\in\mathcal{F}}\ f\rangle\langle w^f(x) = 1$, where f is interpreted as a 2-vector. For nodes, $\sum_{n\in\mathcal{N}}\ n\rangle\langle w^n(x) = 1$. This one, like all formulas of the family, says two things: that $x = \sum_n w^n(x)\, x_n$ (formula (7) of last paper), and that $\sum_n w^n(x) = 1$, whatever x, the "partition of unity" property of hat functions. That's what is so remarkable about the formula of Prop. 2, and similar ones: They express the fact that *Whitney forms make a partition of unity*, for all degrees.

Remark. Some time ago (*JSAEM, 6*, 1998, pp. 121–ff), we discussed basis vectors ∂_i and basis covectors d^i, and noticed that $\langle \omega, v\rangle = \sum_i \omega_i v^i$, in terms of the components. With the dyadic notation, we can rewrite this as $\sum_{i=1,\ldots,n} \partial_i\rangle\langle \mathrm{d}^i$ = 1, where n is the space dimension. A partition of unity, again, which makes Prop. 2 less surprising, and points at a deep analogy between "Cartesian frames" on the one hand, and the (local) "barycentric frames" provided by a simplicial mesh on the other hand. Pursuing this would lead us too far astray. $\Diamond$

Dyadic products are affine objects, but the idea of vector proxies also applies to them. For instance, if **H** is the proxy for η, the proxy for the dyad $w\rangle\langle\eta$ is the so-called "vector dyadic product" $w\otimes\mathbf{H}$, which we define by the formula

$$(15') \quad \Omega \cdot (w \otimes \mathbf{H}) \cdot v = (\Omega \cdot w)(\mathbf{H} \cdot v) \;\; \forall\, \Omega, v,$$

where both Ω and v are vectors, this time. The slot for Ω may be left empty, which results in $(w\otimes\mathbf{H})\cdot v = (\mathbf{H}\cdot v)\, w$ for all v. Alternatively, v may be left out. Note that if Ω is considered as the proxy for covector ω, $(15')$ is just an avatar of (15). Prop. 2 translates as $\sum_{e\in\mathcal{E}} e\otimes\mathbf{W}^e(x) = 1,$ which amounts to

$$(16) \qquad \sum_{e\in\mathcal{E}} (\mathbf{W}^e(x)\cdot v)\; e = v \;\; \forall\, v.$$

It will be a bit easier for us to use that, rather than the literal version of Prop. 2, for the computations that follow.

5.5 Matrix M and the dual mesh

So in this section, we shall use a definite metric (the one implicitly assumed here since the beginning, for which the edge element has the form (2)), and a definite orientation of ambient space (the "right-hand rule" one).

The dual mesh in consideration will be the one obtained by the barycentric construction (Fig. 4). In line with our convention to use e for the vector along edge e, we'll make $\tilde{e}$ serve not only as a label for the dual edge, but for its vectorial area, that is, the vector orthogonal to $\tilde{e}$, pointing in the same general direction as e, of length equal to the area of $\tilde{e}$. Generalizing that, we shall feel free to consider $\{k, l, m\}$, for instance, not only as a label for the facet spanned by these nodes, but as its vectorial area. (The vector points in the direction specified by the ordering of the nodes, in conjunction with Ampère's right-hand rule.)

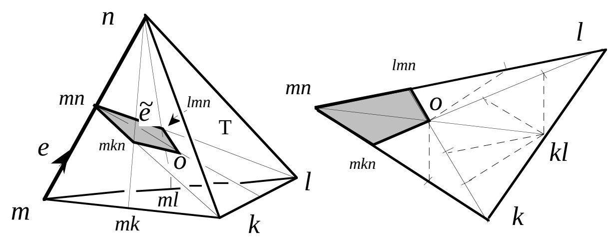

Figure 4. Left: The facet $\tilde{e}$ dual to edge e, in the barycentric construction. Labels such as mk, mkn, etc., point to centers of primal edges and facets. The tetrahedron's center is o. Right: The area of $\tilde{e}$ is one sixth that of $\{mn, k, l\}$. (All small triangles shown have the same area.)

Now, let us set $v = \epsilon\mathbf{W}^e$ in (16). Replacing the summation index e by e' to avoid confusion, we get

$$\sum_{e'\in\mathcal{E}} (\epsilon\mathbf{W}^e(x)\cdot\mathbf{W}^{e'}(x))\; e' = \epsilon\mathbf{W}^e(x),$$

which can be integrated over D, yielding, if one takes (11) into account,

$$\sum_{e'\in\mathcal{E}} \mathbf{M}^{ee'}\; e' = \int_D \epsilon\mathbf{W}^e.$$

Be careful here that we integrate a *vector*-valued function (the vector field $\epsilon\mathbf{W}^e$), so the result is a vector.

For the next step, let's suppose ϵ is the same all over D, and thus can be factored out, a simplifying hypothesis that we shall reconsider later. First,

Lemma 1. *The vector-valued integral of* $\mathbf{W}^e$ *over tetrahedron* T *is equal to the vectorial area of the part of* $\tilde{e}$ *contained in* T.

Proof. Let h^n be the length of the altitude falling from node n onto the opposite facet $\{k, l, m\}$. Since ∇w^n is a constant vector over T, of magnitude $1/h^n$, we have $\int_{\mathrm{T}} \nabla w^n$ = $\{k, l, m\}/3$. The average of w^m being 1/4, we get $\int_{\mathrm{T}} \mathbf{W}^e$ = $(\{k, l, m\} + \{k, l, n\})/12$. But this is $\{mn, k, l\}/6$, which, as Fig. 4 shows, is equal to the part of $\tilde{e}$ local to T. $\Diamond$

The result, obviously, does not depend on the particular arrangement of nodes on the figure: What counts is the fact that e and $\tilde{e}$ point in the same direction. Therefore, by adding contributions from all tetrahedra around edge e, we get this, where $\tilde{e}$ is now the whole dual facet:

Proposition 3. $\int_D \mathbf{W}^e = \tilde{e}$.

So when ϵ is constant, we arrive at the following relation:

$$(17) \qquad \sum_{e' \in \mathcal{E}} \mathbf{M}^{ee'}\, e' = \epsilon\, \tilde{e}.$$

Back to the general case, now. If ϵ is uniform in each individual tetrahedron, which we generally can assume, then $\int_{\mathrm{T}} \epsilon \mathbf{W}^e = \epsilon_{\mathrm{T}}\, \tilde{e}_{\mathrm{T}}$, where $\tilde{e}_{\mathrm{T}}$ is the part of $\tilde{e}$ local to T. By adding contributions of all tetrahedra, we still find (17), provided $\epsilon\, \tilde{e}$ is understood as the sum $\sum_{\mathrm{T}} \epsilon_{\mathrm{T}}\, \tilde{e}_{\mathrm{T}}$, which we may call the "ϵ-related vectorial area" of the dual facet $\tilde{e}$.

Remark. This is not ad-hoc fancy. As repeatedly mentioned here, the coefficient ϵ of $(3')$ (which could as well be a tensor ϵ_{ij}) is just a proxy for the Hodge operator $\tilde{\star}_\epsilon$. The constitutive law $\tilde{d} = \tilde{\star}_\epsilon e$, as expressed in terms of differential forms, has an intrinsic character which its equivalent in terms of the proxies, $\mathbf{D}^i = \sum_j \epsilon_{ij} \mathbf{E}^j$, lacks. Change the metric, $\tilde{\star}_\epsilon$ stays the same, but ϵ_{ij} has to change. Could therefore the metric be selected so as to make ϵ_{ij} as simple as possible, that is, unity?

The answer to that is *yes*. This is the "Hodge implies metric" result alluded to in *JSAEM, 6,* 1998, p. 325: Given a linear map $\tilde{\star}_\epsilon$ from 1-forms to twisted 2-forms, with adequate properties of symmetry and positivity, one may construct a metric the Hodge operator of which is precisely $\tilde{\star}_\epsilon$. Let's call that the "ϵ-related" metric. In this metric, vector proxies for $\tilde{d}$ and e are $\mathbf{D}_\epsilon$ and $\mathbf{E}_\epsilon$, different from $\mathbf{D}$ and $\mathbf{E}$, and they are linked by $\mathbf{D}_\epsilon = \mathbf{E}_\epsilon$. When one works out the exercise on "what is the ϵ-related vectorial area of $\tilde{e}$?", one does find the above expression $\sum_{\mathrm{T}} \epsilon_{\mathrm{T}}\, \tilde{e}_{\mathrm{T}}$, where ϵ_{T} may be understood as a tensor acting on the vector $\tilde{e}_{\mathrm{T}}$. $\diamondsuit$

Thus (17) has validity beyond the particular metric we use. Comforted by that, let's carry on with this particular metric to see the implications of this formula.

5.6 The Galerkin-induced discrete hodge

Suppose now a piecewise uniform vector field **E** (i.e., constant over each primal tetrahedron), not necessarily the physical solution, but a field which does have tangential continuity across facets. It's the proxy of some 1-form e. Let $\mathbf{D} = \epsilon\mathbf{E}$, also piecewise uniform by our assumptions about ϵ, and the proxy of $\tilde{d} = \tilde{\star}\, e$. Take the dot product of both sides with **E**. Then the array of circulations $\mathbf{e} = \{\mathbf{E} \cdot e \,:\, e \in \mathcal{E}\}$ is what we denoted $r_{\mathcal{M}} e$ in recent issues. The left-hand side thus yields the sum $\sum_{e'} \mathbf{M}^{ee'}\, \mathbf{e}_{e'}$, i.e., $(\mathbf{M} r_{\mathcal{M}} e)_e$. On the right-hand side, we get the flux of **D** through $\tilde{e}$, that is, the component at edge e of what we called $r_{\mathcal{M}} \tilde{\star}\, e$. We may therefore assert that $\tilde{\bigstar}_\epsilon r_{\mathcal{M}} e = r_{\mathcal{M}} \tilde{\star} e$ for any piecewise constant 1-form e when $\tilde{\bigstar}_\epsilon$ is taken equal to **M**.

This, we know, is the essential property for a would-be discrete hodge, which allows us to conclude, after a process of local Taylor expansion similar to what was done earlier,

$$(18) \qquad \tilde{\bigstar}_\epsilon r_{\mathcal{M}} - r_{\mathcal{M}} \tilde{\star}_\epsilon \to\ 0 \text{ when } \mathcal{M} \to\ 0,$$

the consistency property for $\tilde{\bigstar}_\epsilon \equiv \mathbf{M}(\epsilon)$.

So now, we are justified to consider the edge-element mass matrix $\mathbf{M}(\epsilon)$ of (11) as a realization of the discrete Hodge operator $\tilde{\bigstar}_\epsilon$. It links edge circulations with fluxes through dual facets, indeed, provided the latter are taken as the dual 2-cells in the barycentric construction (Fig. 4).

This "Galerkin hodge" gives a convergent scheme in statics, which is no news, but the interesting point is that it can be proven along the same lines as with the "orthogonal" hodge. Let's not go into this again. Note that stability is here a built-in property, because the (up to a factor 2) discrete energy $|\mathbf{e}|_\epsilon^2 = \langle \mathbf{M}\mathbf{e}, \mathbf{e} \rangle$ coincides here with the continuous energy $|p_{\mathcal{M}} \mathbf{e}|_\epsilon^2 = \int_D \epsilon |\mathbf{E}|^2$, by construction, hence $|p_{\mathcal{M}} \mathbf{e}|_\epsilon = |\mathbf{e}|_\epsilon$ (cf. (17) p. 406 in *JSAEM, 7,* 1999). As we invoked mesh uniformity earlier in order to assess stability, one may wonder whether uniformity is necessary in the Galerkin approach. It is: the convergence property $p_{\mathcal{M}} r_{\mathcal{M}} e \to e$ relies on it, and will be lost for pathological refinement methods [1].

For facets, the analogue of Prop. 3 is $\int_D \mathbf{W}^f = \tilde{f}$. This has an unexpected application to the interpetation of the standard "method of moments" [5] as a Galerkin method, for which I must refer to [3]. The analogue for nodes is worth looking at, too: it's $\int_D w^n = \mathrm{vol}(\tilde{n})$, with the consequence that if q is piecewise constant, then $\int_D q\, w^n$ is the total electric charge contained in the dual 3-cell $\tilde{n}$. JUst what we needed to identify the **b** of (12) and the $\tilde{\mathbf{q}}$ of (4): this is right when q is piecewise constant.

Remark. Be careful, it's *not* true that $\int_{\tilde{e}} \tilde{\star}_\epsilon w^{e'}$ = $\int_{\tilde{e}'} \tilde{\star}_\epsilon w^e$. Therefore, simply taking the flux of $\epsilon \mathbf{W}^e$ through the dual face $\tilde{e}'$ does not provide a discrete Hodge operator. ♢

The analogue of (17) for facets, derived from $\sum_{f\in\mathcal{F}} (\mathbf{W}^f(x)\cdot v)\ f = v\ \forall\, v$, is

$$\text{(19)} \qquad \sum_{f'\in\mathcal{F}} (\mathbf{M}_2(\mu^{-1}))^{ff'}\, f' = \mu^{-1}\, \tilde{f},$$

hence a Galerkin hodge that discretizes $\tilde{\star}_{\mu^{-1}}$.

5.7 Dynamics, again

Thus in possession of the two required hodges, we may use them to solve the antenna problem, with given current density. Substitute $\mathbf{M}_1(\epsilon)$ for $\tilde{\star}_\epsilon$ and $\mathbf{M}_2(\mu^{-1})$ for $\tilde{\star}_\mu^{-1}$ in eqs. (5) and (6), p. 295 of *JSAEM, 7,* 1999, hence the following scheme [6, 7]: Starting from $\mathbf{b}^0 = 0$ and $\mathbf{e}^{-1/2}$ = 0, find successive DoF arrays $\mathbf{b}^k$ and $\mathbf{e}^{k+1/2}$ such that, for $k = 0, 1, \ldots$, etc.,

$$\mathbf{M}_1(\epsilon)\frac{\mathbf{e}^{k+1/2} - \mathbf{e}^{k-1/2}}{\delta t} = \mathbf{R}^t\mathbf{M}_2(\mu^{-1})\mathbf{b}^k - \star^{k},$$

$$\text{(20)} \qquad \frac{\mathbf{b}^{k+1} - \mathbf{b}^k}{\delta t} + \mathbf{R}\mathbf{e}^{k+1/2} = 0.$$

This, which is the time-domain extension of an earlier proposal to use edge elements for the time-harmonic problem [2], has the advantage of avoiding the sometimes problematic construction of the orthogonal dual mesh. But the incurred penalty is heavy: Since $\mathbf{M}_1(\epsilon)$ is not diagonal, the scheme is not explicit. Fortunately [4],

Proposition 4. *The diagonal matrix* **H**, *indexed over edges, whose entries are, for each edge* $e = \{m, n\}$,

$$\mathbf{H}^{ee} = -(\mathbf{G}^t\mathbf{M}_1(\epsilon)\mathbf{G})_{mn} \equiv -\mathbf{A}_{mn}$$

(the **A** of Prop. 1), *verifies* $\mathbf{G}^t\mathbf{H}\mathbf{G}$ = $\mathbf{G}^t\mathbf{M}_1\mathbf{G}$.

Proof. $(\mathbf{G}^t\mathbf{H}\mathbf{G})_{mn}$ = $\sum_{e\in\mathcal{E}} \mathbf{G}_{em}\mathbf{H}^{ee}\mathbf{G}_{en}$. The only nonzero term in this sum obtains for edge e joining m to n, if there is such an edge, and then equals $-\mathbf{H}^{ee}$, since $\mathbf{G}_{em}\mathbf{G}_{en}$ = -1. On the other hand, $(\mathbf{G}^t\mathbf{M}_1\mathbf{G})_{mn}$ = 0 if m and n are not joined by an edge. (Note this is special to tetrahedral meshes. The proof would fail otherwise.) ♢

Thus **H** can replace **M** in statics, giving the same scheme. By a previously enunciated heuristic principle, we may substitute this "diagonally lumped" hodge **H** for $\mathbf{M}_1(\epsilon)$ in the dynamic scheme (20), and expect convergence. This works, as proven in [4].

Alas, there is a hitch: Nothing ensures that $\mathbf{H}_{ee} > 0$, which as we know is required for stability in (20). As formula (8) shows, this requires *acute* dihedral angles, a not so easily met condition. Nothing comes free!

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