(4): From degrees of freedom to fields

Where we stand

"Building a finite-dimensional 'Maxwell's house'", so was the subtitle of this series. The goal is now in sight, as we have in hand a systematic discretizing procedure for all equations derived from Maxwell's. Solving the algebraic system that results, we can find time-dependent arrays of degrees of freedom (DoF arrays), which give an approximate picture of the fields by telling us about fluxes through faces, about emf's along edges, etc. What we don't know yet is how to "interpolate" from DoF's, i.e., how to reconstruct fields from computed DoF arrays.

4.1 Why interpolants?

One may wonder, *must* this be done, really? After all, as we argued earlier, all one can hope to measure about the electromagnetic field is, precisely, fluxes and circulations, so what's wrong with a method that would provide only those? First answer: We may want them at a much smaller scale than what the computational mesh provides, hence the necessity of interpolation. Post-processing, in the same spirit, may require the determination of the fields inside mesh cells. But these reasons are not the most compelling ones. As we began to see last time, field reconstruction is needed, basically, to *assess the validity* of the numerical method.

To stress this point anew, let's review the discretization process. Having built a mesh, with its incidence matrices **G**, **R**, and **D**, we assign unknowns to primal or dual cells, according to their geometrical nature (Fig. 1), and replace fields, in the equations, by arrays **e**, **b**, $\tilde{\mathbf{d}}$, $\tilde{\mathbf{h}}$, of such DoF's. We replace the operators rot and div by **R** and **D** (or by $\mathbf{R}^t$ and $\mathbf{G}^t$, according to what Fig. 1 suggests), the time-derivatives by finite differences (at integer or half-integer time-steps), and the material coefficients ϵ and μ by our so-called "discrete Hodge operators", the square matrices denoted $\tilde{\star}_\epsilon$ and $\tilde{\star}_\mu$ in past issues. Hence algebraic equations, that a computer can solve for us.

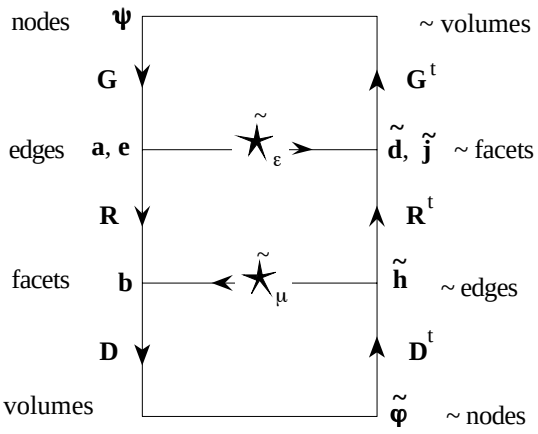

Figure 1. A discretizing machine for Maxwell's equations, showing where degrees of freedom sit ("∼" is short for "dual") and which matrices link the various DoF arrays. (Time discretization, not shown, is done via the replacements $\partial_t \mathbf{b} \simeq (\mathbf{b}^{k+1} - \mathbf{b}^k)/\delta t$ and $\partial_t \tilde{\mathbf{d}} \simeq (\tilde{\mathbf{d}}^{k+1/2} - \tilde{\mathbf{d}}^{k-1/2}/\delta t$.) Compare with Fig. 6, p. 326, *JSAEM, 6,* 4 (1998).

We realized that building these matrices $\tilde{\star}_\epsilon$ and $\tilde{\star}_\mu$ is the central problem in this approach: Assessing their quality mandates a convergence proof. We argued that it was enough to have one in the static case, and we identified two steps in this proof: checking *consistency*, i.e.,

(1) $\qquad \tilde{\star}_\mu r_{\mathtt{M}} - r_{\mathtt{M}} \tilde{\star}_\mu \to 0$ when $\mathtt{M} \to 0$

(and a similar thing about $\tilde{\star}_\epsilon$),[1] and showing *stability*, a property of the approximation

[1] See *JSAEM, 7,* 4 (1999), for the notation. Let's recall that $r_{\mathtt{M}}$ denotes the operator which maps a field to a DoF array of the appropriate kind. For instance, in the case of $\tilde{h}$, it maps to the array $r_{\mathtt{M}}\,\tilde{h}$ of mmf's $\tilde{\mathbf{h}}_f = \int_{\tilde{f}} \tilde{h}$ along dual edges, indexed over the set $\mathcal{F}$ of primal faces. As $b = \tilde{\star}_\mu \tilde{h}$, the array $r_{\mathtt{M}}\,\tilde{\star}_\mu \tilde{h}$ contains the facet-fluxes of b. This is not quite the same as $\tilde{\star}_\mu r_{\mathtt{M}}\,\tilde{h}$, hence $\tilde{\star}_\mu r_{\mathtt{M}} - r_{\mathtt{M}}\,\tilde{\star}_\mu \neq 0$. But we made sure, in defining $\tilde{\star}_\mu$,

scheme which we wrote, in the case of **b**, as

$$\alpha\,|p_{\mathtt{M}}\,\mathbf{b}|_\nu \le |\mathbf{b}|_\nu . \tag{2}$$

On the right, there,[2] the "discrete energy" of the DoF array **b** (up to squaring and halving). On the left, the magnetic energy of the approximation $p_{\mathtt{M}}\,\mathbf{b}$ to b that one can produce, given **b**, thanks to the interpolation operator $p_{\mathtt{M}}$. We had, last time, such an operator ready-made, thanks to the fact that **Db** = 0.[3] But that would not work for volumes with more than four facets. Besides, to go beyond magnetostatics, we need (2) for other fields than b, for which such a lucky accident as this relation **Db** = 0 will not occur.

So, to sum up: For each kind of DoF array, we need an operator, generically denoted by $p_{\mathtt{M}}$, which maps it to a differential form (DF) of the appropriate kind: $p_{\mathtt{M}}\,\mathbf{e}$, starting from an edge-based DoF array **e**, should be a 1-form; $p_{\mathtt{M}}\,\mathbf{b}$, obtained from a facet-based **b**, should be a 2-form, and so forth. We want the emf of $p_{\mathtt{M}}\,\mathbf{e}$ along edge e, that is to say $(r_{\mathtt{M}}\,p_{\mathtt{M}}\,\mathbf{e})_e$, to equal $\mathbf{e}_e$, for each edge, therefore $r_{\mathtt{M}}\,p_{\mathtt{M}} = 1$ should hold. On the other hand, if the array, **u** let's say, comes from a DF u by $\mathbf{u} = r_{\mathtt{M}}\,u$ (i.e., by taking fluxes or circulations of u, as the case may be), we don't expect $p_{\mathtt{M}}\,\mathbf{u}$ to equal u, but the difference should be small in the proper energy norm, and the finer the mesh, the smaller it should become. So $p_{\mathtt{M}}$ should behave as follows in relation with $r_{\mathtt{M}}$:

$$r_{\mathtt{M}}\,p_{\mathtt{M}} = 1, \quad p_{\mathtt{M}}\,r_{\mathtt{M}} \to 1 \quad \text{when } {\mathtt{M}} \to 0. \tag{3}$$

As for the stability property (2), we saw last time that it would automatically be satisfied in the case of a *uniform* family of meshes, defined as one in which only a finite number of cell-shapes can exist, for both the primal and the dual[4] mesh. Our objective, therefore, is to satisfy (3) when building $p_{\mathtt{M}}$ for all possible kinds of DoF arrays.

4.2 Interpolating from nodal values

We know a solution to this problem in the case of (primal-) *node-based* DoF arrays, at least for a simplicial primal mesh.[5] Such arrays correspond to straight DF's of degree 0, i.e., to functions, and interpolating a function from its nodal values is what *finite elements* are about, so we are on familiar ground there. To capitalize on this knowledge, we shall need to look at this kind of interpolation from an unusual angle, however.

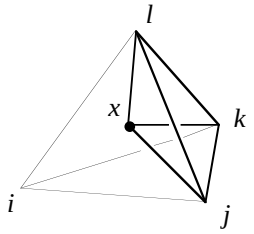

Figure 2. The weight $\lambda^i(x)$ is the relative volume of the tetrahedron $\{x, j, k, l\}$ (cf. (5) below).

Let's first consider a single tetrahedron in 3D space (Fig. 2), with nodes labelled i, j, k, l, located at points x_i, x_j, x_k, x_l (not all of them in the same plane). As one knows, there are four functions, $\lambda^i, \lambda^j, \lambda^k, \lambda^l$, such that

$$\sum_{n\in\{i,j,k,l\}} \lambda^n(x)\,(x - x_n) = 0 \tag{4}$$

for all points x. Known as the *barycentric coordinates* of x, they satisfy the equality $\sum_n \lambda^n = 1$, and have the familiar geometric

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that they *were* the same when b is uniform, a condition to which one tends, locally, when the mesh is refined; hence the ease with which we proved (1) for the specific discrete Hodge operator of last issue, the one with entries μ area(f)/length($\tilde{f}$) on the diagonal. ($\tilde{\star}_\nu$ is the inverse of $\tilde{\star}_\mu$.)

[2] Again, let's recall that $|\mathbf{b}|_\nu$ (with boldface vertical bars), called the "ν-norm" of **b**, is the square root of the energy-related quantity $\langle \mathbf{b}, \tilde{\star}_\nu \mathbf{b}\rangle$, where $\langle \mathbf{b}, \tilde{\mathbf{h}}\rangle$ stands for the sum $\sum_{f\in\mathcal{F}} \mathbf{b}_f \tilde{\mathbf{h}}_f$. The similar quantity $|b|_\nu$ was defined as the square root of $\int_D \mu^{-1}|\mathbf{B}|^2$, where **B** is the vector proxy of b.

[3] We simply took the proxy **B** uniform inside each tetrahedron, by adjusting three of its facet-fluxes to the given $\mathbf{b}_f$s. The fourth DoF is automatically right, because of the linear constraint **Db** = 0 (which also entails div**B** = 0).

[4] This implies that the centers of primal cells, as used in the orthogonal construction, occupy the same relative position within all cells of similar shape. This property *was* used last time (*JSAEM, 7,* 4 (1999), p. 407) to show that the α of (2) is > 0, but I should have put more emphasis on this important point.

[5] The case of hexahedral 3-cells, also well understood, won't be discussed here for the sake of brevity.

interpretation that Fig. 2 recalls:

$$\lambda^i(x) = \frac{\mathrm{vol}(\{x,j,k,l\})}{\mathrm{vol}(\{i,j,k,l\})}. \tag{5}$$

Clearly, at least one $\lambda^n(x)$ is negative for x outside the tetrahedron.

Remark. There is a unique set of "barycentric weights" $\lambda^n(x)$ for which (4) holds, so (4) does define the λ^ns, which are therefore *affine* objects. Formula (5), which needs the extraneous notion of volume to make sense, is not their definition but a consequence. There is no measure of volumes in an affine space, yet *ratios* of volumes are a meaningful affine concept. (Check that the ratio (5) is invariant with respect to a change of metric, i.e., the same for two different dot products.) ♢

Remark. Actually, volume in an affine space can be defined without a dot product in background. It suffices to introduce a reference tetrahedron (a reference d-simplex, in dimension d), which by convention has volume 1. Then, any tetrahedron T can be mapped onto the reference one by some affine transformation, the determinant of which is, by definition, vol(T). (It's a signed number; take its absolute value to get a positive volume; chop bodies of more general shape into tetrahedra to extend the notion of volume to them.) The mapping T $\to$ vol(T) is called "a volume". Possible volumes, in this sense, differ by a multiplicative constant. Note that giving a volume says nothing about areas or lengths: In this respect, the structure "affine space + volume" lies between "(naked) affine space" and "Euclidean space" (affine space + dot product). ♢

Since $\sum_n \lambda^n(x) = 1$ for all x, another way to write (4) can be, with a very mild notational abuse,

$$x = \sum_{n\in\{i,j,k,l\}} \lambda^n(x)\, x_n, \tag{4'}$$

which represents x as the barycenter of the vertices x_n, with weights $\lambda^n(x)$.

Next, if instead of a single tetrahedron we have a tetrahedral paving of some domain D (this will be our primal mesh, M), then a barycentric function λ^n_{T} can be defined for each node n and each tetrahedron T. Set[6]

$$w^n(x) = \max\{\lambda^n_{\mathrm{T}}(x) : \mathrm{T} \in \mathcal{T}\}. \tag{6}$$

This is a piecewise affine function (i.e., affine over each individual T), continuous in x, non-negative all over D (contrary to the λ^n_{T}s), and positive inside the set-union of tetrahedra that have n as one of their nodes. One will have recognized the standard[7] "hat function" of finite element theory.

On the model of (4′), we may write

$$x = \sum_{n\in\mathcal{N}} w^n(x)\, x_n, \tag{7}$$

again a fairly legitimate abuse of notation: (7) makes x appear as a weighted sum of *all* nodes of the mesh, but with all weights null, except for the nodes of the simplex that contains[8] x. This is a key observation, which we shall soon exploit.

Hat functions do serve as interpolants, as one knows: Starting from a node-based DoF array $\{\psi_n : n \in \mathcal{N}\}$, one may set

$$(p_{\mathtt{M}}\,\psi)(x) = \sum_{n\in\mathcal{N}} w^n(x)\boldsymbol{\psi}_n \tag{8}$$

to obtain a piecewise affine function whose value at x_n be $\boldsymbol{\psi}_n$. Hence a $p_{\mathtt{M}}$ for which $r_{\mathtt{M}}\, p_{\mathtt{M}} = 1$ holds. On the other hand, if we start from some continuous function ψ, and set $\boldsymbol{\psi} = r_{\mathtt{M}}\,\psi$, that is, $\boldsymbol{\psi}_n = \psi(x_n)$, we get an approximation $p_{\mathtt{M}}\, r_{\mathtt{M}}\,\psi$ to ψ, with the property that $\int_D |p_{\mathtt{M}}\, r_{\mathtt{M}}\,\psi - \psi|^2 \to 0$ when $\mathtt{M} \to 0$. (No need to prove this well-known fact here. See [2] for a proof in line with the present approach.)

[6] Recall that we denote by $\mathcal{N}, \mathcal{E}, \mathcal{F}, \mathcal{T}$ the sets of nodes, edges, etc., of the primal mesh—*all* of them: the restriction to "active" cells, introduced in previous columns, is irrelevant this time.

[7] Except for the unusual symbol w, that one may understand as a mnemonic for "weight". Later, we'll find another interpretation.

[8] It may be a p-simplex with $p < 3$. Remember that our primal cells are not supposed to contain their boundaries, unless $p = 0$. So if we say that "x belongs to edge $e = \{m,n\}$", for instance, we understand that $w_m(x) > 0$, $w^n(x) > 0$, and $w_i(x) = 0$ for all other nodes.

Well, what have we done there, that could show the way to generalization? Compare (8) to (7). The value we attribute to $p_{\mathtt{M}}\,\psi$ at point x is the weighted sum of the nodal values ψ_n, *the weights being the same as those by which x is expressed as a sum of nodes*. That's our clue: if we were somehow able to express a line as a weighted sum of mesh-edges, a surface as a weighted sum of mesh-facets, etc., we could generate a 1-form from an edge-based DoF array, a 2-form from a facet-based array, and so on.

4.3 Chains

To see how, suppose for a moment that we have (to be specific) a facet-based DoF array **b**, from which we want to build a (straight) 2-form $p_{\mathtt{M}}\,\mathbf{b}$. Straight 2-forms map inner-oriented surfaces to real numbers. So we will know $p_{\mathtt{M}}\,\mathbf{b}$ if we know its integrals $\int_S p_{\mathtt{M}}\,\mathbf{b}$ for all inner-oriented surfaces S. These we do know when S is one of the primal facets (remember they have inner orientation), since then, $\int_f p_{\mathtt{M}}\,\mathbf{b} = \mathbf{b}_f$. (This is what $r_{\mathtt{M}}\,p_{\mathtt{M}} = 1$ means in that case.) Now, suppose we have a sensible way to represent S as a sum of facets:

$$S \simeq \sum_{f\in\mathcal{F}} w^f(S)\,f, \tag{9}$$

with appropriate weights $w^f(S)$ (compare with (7)). Then, owing to the additivity of the integral,

$$\int_S p_{\mathtt{M}}\,\mathbf{b} = \sum_{f\in\mathcal{F}} w^f(S) \int_f p_{\mathtt{M}}\,\mathbf{b} \equiv \sum_{f\in\mathcal{F}} w^f(S)\,\mathbf{b}_f,$$

so it's just natural to set

$$\int_S p_{\mathtt{M}}\,\mathbf{b} = \sum_{f\in\mathcal{F}} w^f(S)\,\mathbf{b}_f. \tag{10}$$

Thus our problem is solved if we can make some sense out of formula (9).

Not an easy task, it seems, for the term on the right in (9) has no obvious meaning, on the face of it (what would be the result of "multiplying a facet by a number", and then, what would it mean to "add" such objects?) This is why we can't use an equal sign, hence the "$\simeq$", meaning "similar to". Yet we can give status to this expression $\sum_{f\in\mathcal{F}} w^f(S)\,f$, by considering it as just another way to denote the array of real values $\{w^f(S) : f \in \mathcal{F}\}$. Such an array (quite alike a DoF array, but conceptually different) is called a (simplicial) 2-*chain*. Needless to say, there are p-chains for all dimensions p of the underlying simplices.[9]

If $\mathbf{c} = \{c^f : f \in \mathcal{F}\}$ is such a 2-chain, writing it as a formal sum, $\mathbf{c} = \sum_{f\in\mathcal{F}} \mathbf{c}^f\,f$, is a convenient device, which makes chain addition, for instance, defined as $\mathbf{c}_1 + \mathbf{c}_2 = \sum_{f\in\mathcal{F}}(\mathbf{c}_1^f + \mathbf{c}_2^f)\,f$, look natural: just follow the rules of algebra. Further evidence of its usefulness will come.

So that's what we have at the right-hand side of (9): a 2-chain. Could S itself considered as a 2-chain in some way?

This is not so unlikely. A single facet f can be viewed as the chain with all weights 0, except $\mathbf{c}^f$, equal to 1. Therefore, an inner-oriented surface which is made of an assembly of facets can be viewed as a chain, too, that we shall call the *associate* chain: its coefficients are ± 1 for each facet of the assembly, with sign + or $-$ depending on whether orientations match or not, and 0 for all other facets (Fig. 3). For brevity, we shall call "M -surfaces" those composed of such assemblies. (There is a similar notion of "M -line", and we'll say "M -manifold" to cover all cases, p = 0 to 3.) If S in (9) is an M -surface, the pseudo equality begins to make sense: on the right, we have the 2-chain associated with S, with weights ± 1 or 0.

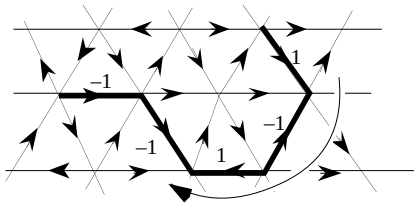

Figure 3. An M -line, oriented, embedded in a 2D-mesh, and the coefficients of its 1-chain associate. (Arrows indicate the orientation of each edge.) Imagine that for p = 2 in dimension 3.

Yet the two concepts don't coincide. Not all surfaces are M -surfaces, and coefficients of an arbitrary chain can take other values than 0 and

[9] Chains make sense for all kinds of cell-pavings, not only the simplicial ones to which we restrict here.

± 1. So it's better to use distinct notation for the $\mathtt{M}$ -surface S and for its associate: We shall denote the latter by $p^t_{\mathtt{M}}\, S$. (Why this symbol will soon be clear.) Now we may rewrite (9) as $p^t_{\mathtt{M}}\, S = \sum_f w^f(S)\, f$, with a legitimate equality, and the task at hand becomes better defined: To any inner-oriented surface S, made of mesh-facets or not, associate a 2-chain $p^t_{\mathtt{M}}\, S$. Then use the coefficients of this chain as weights in (10).

Before facing this task, a last concept relative to chains: boundaries. Taking the boundary of a chain is an algebraic operation, defined in order to correspond, as closely as possible, to taking the boundary of a manifold. The boundary of a p-chain $\mathbf{c}$ is a $(p-1)$-chain $\boldsymbol{\partial}\mathbf{c}$, with the following properties: First, linearity, $\boldsymbol{\partial}(\mathbf{c}_1 + \mathbf{c}_2) = \boldsymbol{\partial}\mathbf{c}_1 + \boldsymbol{\partial}\mathbf{c}_2$. Second, if M is an $\mathtt{M}$ -manifold with boundary ∂M, the boundary of the chain associated with M is the chain associated with the boundary of M ... Oh well, rather use symbols:

$$p^t_{\mathtt{M}}\,(\partial M) = \boldsymbol{\partial}(p^t_{\mathtt{M}}\, M), \tag{11}$$

that is, $p^t_{\mathtt{M}}\,\partial = \boldsymbol{\partial} p^t_{\mathtt{M}}$, another instance of $conjugacy$. Thanks to (11), $\boldsymbol{\partial}$ is known for all chains, by linearity, if it is known for those associated with simplices.

But this information is precisely what incidence matrices convey: For instance, $\boldsymbol{\partial} p^t_{\mathtt{M}}\,(f) = \sum_e \mathbf{R}_{fe}\, e$, where $\mathbf{R}$ is the edge-to-facets incidence matrix. From this, the boundary of a 2-chain $\mathbf{c}$ is seen to be $\mathbf{R}^t\mathbf{c}$. Matrix representations of $\boldsymbol{\partial}_p$, therefore, are $\mathbf{G}^t$, $\mathbf{R}^t$, $\mathbf{D}^t$, for p = 1, 2, 3. Remember that, as regards DoF arrays, we had a generic operator $\mathbf{d}_p$, realized as $\mathbf{G}$, $\mathbf{R}$, $\mathbf{D}$, for p = 0, 1, 2. ($JSAEM$, 7, p. 156.) So $\boldsymbol{\partial}_{p+1} = \mathbf{d}^t_p$. More and more, chains appear as $dual$ objects with respect to what we have called up to now DoF arrays.[10]

4.4 Interpolating from edge values

We know how to associate a 0-chain with a point: that's what formula (7) does, although we should rewrite it, in full rigor, $p^t_{\mathtt{M}}\, x = \sum_{n\in\mathcal{N}} w^n(x)\, x_n$ if the right-hand side is seen as a 0-chain, not as a barycenter. We won't bother with that, however, and drop the $p^t_{\mathtt{M}}$ in what follows.

Next step is p = 1, curves. How on earth can a curve c be expressed as a 1-chain? Easy: Chop it into small parts, replace them by straight segments, assign chains to these, sum all these chains, and go to the limit. So it boils down to being able to associate a 1-chain to any oriented segment, going from point x to point y (we shall write it xy). We may as well (Fig. 4) suppose that x and y belong to the same tetrahedron, $\{i, j, k, l\}$ say. (Otherwise, break xy into subsegments.)

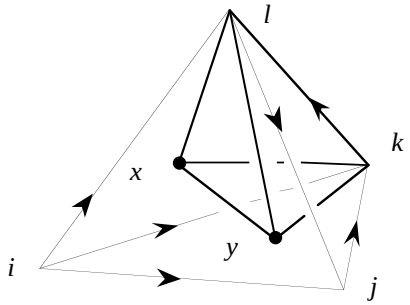

Figure 4. The weight of xy relative to $x_i x_j$ (see (13) below).

We now indulge for a while in a heuristic derivation (for which sloppy notation can be tolerated). Since $x = \sum_n \lambda^n(x)\, x_n$ and $y = \sum_n \lambda^n(y)\, x_n$ (where n spans $\{i, j, k, l\}$), one has, by deliberately confusing oriented segments such as xy with vectors such as $y - x$,

$$y - x = y - \sum_n \lambda^n(x)\, x_n = \sum_n \lambda^n(x)\,(y - x_n)$$

(since $\sum_n \lambda^n(x) = 1$), hence

$$\begin{aligned} xy &= \sum_n \lambda^n(x)\,(x_n - \sum_m \lambda^m(y)\, x_m) \\ &= \sum_n \lambda^n(x) \sum_m \lambda^m(y)\, x_m x_n, \end{aligned} \tag{12}$$

and that's a step forward: we have the segment xy as a weighted sum of segments which coincide with edges, but either with one or the other orientation. So each of the relevant edges (those marked by orientation arrows on Fig. 4) appears twice, with opposite signs, in the above

[10] Indeed, the received name for DoF arrays is "cochains". The duality pairing between a 2-chain $\mathbf{c}$ and a 2-cochain $\mathbf{b}$ yields the number $\sum_{f\in\mathcal{F}} \mathbf{b}_f \mathbf{c}^f$. Eq. (10) says this will coincide with $\int_S p_{\mathtt{M}}\, \mathbf{b}$ when $\mathbf{c} = p^t_{\mathtt{M}}\, S$. Here we see the rationale for the symbol $p^t_{\mathtt{M}}$: in ad hoc notation, $\langle p_{\mathtt{M}}\, \mathbf{b}, S\rangle = \langle \mathbf{b}, p^t_{\mathtt{M}}\, S\rangle$ —a transposition.

sum. Grouping these pairs of terms, we find

$$xy = (\lambda^i(x)\lambda^j(y) - \lambda^j(x)\lambda^i(y))\, x_i x_j + \ldots, \tag{13}$$

where the dots stand for five similar expressions for the other edges—hence our weights, and the 1-chain which represents xy.

After (5), it's a good bet that

$$\lambda^i(x)\lambda^j(y) - \lambda^j(x)\lambda^i(y) = \frac{\text{vol}(\{x, y, k, l\})}{\text{vol}(\{i, j, k, l\})},$$

as suggested by Fig. 4. To check this, it will be convenient to place the origin at x_k, which allows us to use the symbol x for the vector $x - x_k$, and thus to write $x = \lambda^i(x)\, x_i + \lambda^j(x)\, x_i + \lambda^l(x)\, x_l$. Let's also introduce a metric, via a dot product "$\cdot$", and an orientation, which makes a cross product "$\times$" available. The volume of $\{x, y, k, l\}$ is then $1/6(x \times y) \cdot x_l$, i.e. (terms in $\lambda^l x_l$, which would contribute nothing, are left out), one sixth of

$$((\lambda^i(x)\, x_i + \lambda^j(x)\, x_j) \times (\lambda^i(y)\, x_i + \lambda^j(y)\, x_j)) \cdot x_l$$

$$\equiv (\lambda^i(x)\lambda^j(y) - \lambda^j(x)\lambda^i(y))(x_i \times x_j) \cdot x_l,$$

which equals $6\,\text{vol}(\{i, j, k, l\})$ when $x = x_i$ and $y = x_j$, hence the above expression of the weight as a ratio of volumes. Of course, this result does not depend on the metric used.

Yet, while we have a metric, let's use it to express (13) a little differently. Since the gradients $\nabla\lambda^n$ are uniform fields, one has $\lambda^n(y) - \lambda^n(x) = \nabla\lambda^n \cdot xy$. Therefore,

$$\begin{aligned} &\lambda^i(x)\lambda^j(y) - \lambda^j(x)\lambda^i(y) = \\ &\lambda^i(x)(\lambda^j(y) - \lambda^j(x)) - \lambda^j(x)(\lambda^i(y) - \lambda^i(x)) \\ &= (\lambda^i(x)\nabla\lambda^j - \lambda^j(x)\nabla\lambda^i) \cdot xy, \end{aligned}$$

also equal to $(\lambda^i(y)\nabla\lambda^j - \lambda^j(y)\nabla\lambda^i) \cdot xy$, by symmetry between x and y, and to

$$(\lambda^i(z)\nabla\lambda^j - \lambda^j(z)\nabla\lambda^i) \cdot xy$$

for all z on the segment xy, thanks to the affine character of the λ^ns. Averaging in z, one sees that $\lambda^i(x)\lambda^j(y) - \lambda^j(x)\lambda^i(y)$ equals the circulation along xy of the vector field $\lambda^i\nabla\lambda^j - \lambda^j\nabla\lambda^i$, in which one recognizes the familiar form of the "edge element".

Remark. Another useful form of it (easy to check: just dot-product by xy) is

$$(\lambda^i\nabla\lambda^j - \lambda^j\nabla\lambda^i)(x) = \frac{xx_k \times xx_l}{(x_i x_k \times x_i x_l) \cdot x_i x_j}.$$

(The denominator is 6 vol($\{i, j, k, l\}$), a constant, but be careful with its *sign*, orientation-dependent. In programming, the formula as given is safer.) Note that $xx_k \times xx_l = (x_k - x) \times (x_l - x) = (x_l - x_k) \times (x - x_k)$, hence the Nedelec $A + B \times x$ representation [4] of a vector field generated by edge elements,[11] the one used in early implementations [3]. ♢

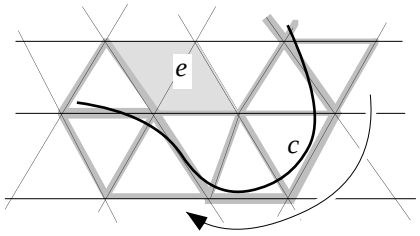

Figure 5. Edges with nonzero weight (highlighted) in the chain associated with c. The weight is null, for instance, for edge e, since the support of its Whitney form (shaded) is not traversed by c.

Back to where we started, a curve c inside D, its weight, with respect to an edge $e = \{m, n\}$ which goes from node m to node n, is the limit of the Riemann-like sum of contributions of small segments analogous to xy in which c is partitioned. So it's the circulation $w^e(c) = \int_c \tau \cdot \mathbf{W}^e$ along c (oriented by the choice of unit tangent vectors τ) of the vector field

$$\mathbf{W}^e = w^m\nabla w^n - w^n\nabla w^m. \tag{14}$$

Only edges of the tetrahedra traversed by c contribute, actually (Fig. 5), and $w^e(c)$ only depends on the part of c lying in the support of $\mathbf{W}^e$, i.e., in the cluster of tetrahedra that have edge e in common.

[11] Here, A and B denote 3D vectors, one pair for each tetrahedron, hence 6 parameters to match the six edge circulations. That was neat, at the time. But this expression looked so much like the field of velocities in the rotation of a solid (cf. a previous column, *JSAEM, 6*, 2 (1998), p. 115) that years were lost on this false track before the *affine* nature of the edge element (cf. (15) below) was recognized. Metric notions, such as "rigid-motion fields" or "orthogonal subspaces", though often met in discussions of edge elements, are irrelevant.

But we know better than resting with (14): the weight $w^e(c)$ must not depend on the metric. What we have in (14) is just a vector proxy for the real thing, the differential form

$$w^e = w^m \mathrm{d}w^n - w^n \mathrm{d}w^m, \tag{15}$$

whose integral $\int_c w^e$ is the weight $w^e(c)$. (Note that $\int_e w_e = 1$.) This is called a "Whitney form", and we have one for each edge. (In retrospect, w^n, a zero-form, was the Whitney form of node n, so now we may expect each simplex, whatever its dimension, to have a Whitney form of its own.)

Thus we have solved the problem of associating a chain with c: the associate is

$$p^t_{\mathtt{M}}\, c = \sum_e w^e(c)\; e \tag{16}$$

(compare with (7)). Correlatively, we have solved the problem of interpolating from edge values: what interpolates from the edge-DoF array **a** is the (straight) 1-form

$$p_{\mathtt{M}}\, \mathbf{a} = \sum_{e\in\mathcal{E}} \mathbf{a}_e w^e. \tag{17}$$

Note that $\int_{e'} w_e = 0$ for $e' \neq e$, because both w^n and w^m in (14) or (15) vanish on other edges than $e = \{m, n\}$. This shows (integrate both sides of (17) along e') that $r_{\mathtt{M}}\, p_{\mathtt{M}} = 1$ holds, again in this case. The convergence property, $p_{\mathtt{M}}\, r_{\mathtt{M}} \to 1$ when $\mathtt{M} \to 0$, also holds,[12] under the condition of mesh uniformity.

4.5 The complex of Whitney forms

How to associate a chain with a tiny triangle xyz, and hence, how to interpolate from facet values, is now an easy guess.

First, we notice a pattern in (4′) and (12), which suggests

$$xyz = \sum_{m,n,q} \lambda^m(x)\lambda^n(y)\lambda^q(z)\, x_m x_n x_q$$

as the next item in the sequence. Here, indices m, n, and q span the set $\{i, j, k, l\}$ independently, so it makes 64 terms, but many of them, like e.g., the term in $x_i x_i x_j$, don't correspond to facets, and thus "count for nothing". Others, such as $x_i x_j x_k$, $x_j x_k x_i$, $x_k x_j x_i$, etc., correspond to the same facet, but with both possible orientations. Grouping them, we find that the weight of xyz with respect to facet ijk is

$$\lambda^i(x)\lambda^j(y)\lambda^k(z) + \ldots = \frac{\mathrm{vol}(\{x, y, z, l\})}{\mathrm{vol}(\{i, j, k, l\})},$$

where the dots stand for 5 other terms, obtained by index permutation and (for the three odd permutations) sign-change. If this sounds like a determinant, no surprise: indeed, $\mathrm{vol}(\{x, y, z, l\})$ is one sixth of the determinant of vectors $x_l x$, $x_l y$, $x_l z$. The consistency, so far (Figs 2, 4, and 6), of these interpretations of weights as "relative volumes" shows that we have the general rule, as summarized by Fig. 6.

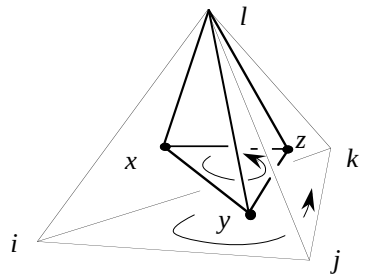

Figure 6. Representing a "small" p-simplex (xyz, here, $p = 2$) as a weighted sum of p-faces of the d-simplex $\{i, j, k, l, \ldots\}$ (spatial dimension $d = 3$ here). Note how the very ordering of points, x, y, z, inner-orients xyz. Its weight with respect to $\{i, j, k\}$ is the ratio of the volumes of $\{x, y, z, l, \ldots\}$ and of $\{i, j, k, l, \ldots\}$. (Recall that volumes can be negative, and that a *ratio* of volumes is a purely affine notion.)

Next, a manipulation similar[13] to what led us to the above form $\lambda^i \nabla\lambda^j - \lambda^j \nabla\lambda^i$ of the vector proxy leads this time to

$$\mathbf{W}^{ijk}(x) = 2\,(\lambda^i(x)\nabla\lambda^j \times \nabla\lambda^k + \ldots + \ldots)$$

[12] Refs [9] and [17] of last issue contain a general proof, valid for all dimensions p. Let's just sketch an elementary one for $p = 1$, using vector proxies. The task reduces to evaluate $\int_{\mathrm{T}} |\mathbf{A} - \mathbf{A}_{\mathtt{M}}|^2$, where $\mathbf{A}$ and $\mathbf{A}_{\mathtt{M}}$ are proxies for a and $p_{\mathtt{M}}\, r_{\mathtt{M}}\, a$, for each tetrahedron T. There, one has $\mathbf{A}_{\mathtt{M}}(x) = A_{\mathrm{T}} + B_{\mathrm{T}} \times (x - x_{\mathrm{T}})$, where x_{T} is some point inside T. The Taylor expansion of $\mathbf{A} - \mathbf{A}_{\mathtt{M}}$ about x_{T} contains terms of degree 2 and higher, which contribute to the integral a term in $\gamma_{\mathtt{M}}^4\,\mathrm{vol(T)}$ (where $\gamma_{\mathtt{M}}$ is the grain of the mesh), and the gradient of a quadratic function which one may assume *vanishing at nodes* (by adjusting A_{T}). Thanks to this, and to uniformity (necessary, as the counter-example in [1] shows), its contribution is in $\gamma_{\mathtt{M}}^2\,\mathrm{vol(T)}$, hence $\int_D |\mathbf{A} - \mathbf{A}_{\mathtt{M}}|^2 \leq C(\mathbf{A})\gamma_{\mathtt{M}}$.

[13] but a bit too lengthy to be included; the trick is to consider xyz as a vector (the vectorial area) and to express it as half the cross product $xy \times xz$.

(two other terms, by circular permutation on i, j, k), of which another form is

$$\frac{2\,x x_l}{(x_i x_k \times x_i x_l)\cdot x_i x_j} \equiv \frac{x x_l}{3\mathrm{vol}(\{i,j,k,l\})}.$$

As above, the weight of a surface S with respect to a facet $f = \{l, m, n\}$ is then the flux $w^f(S) = \int_S \nu \cdot \mathbf{W}^f$ of the vector field

$$\mathbf{W}^f = 2(w^l \nabla w^m \times \nabla w^n + \ldots + \ldots) \quad (18)$$

(again, two other terms, by circular permutation on l, m, n). Of course, $w^f(f') = 1$ if $f = f'$, 0 otherwise.

Remark. Be warned that the unit normal vector ν here is meant to $inner$-orient S, which it does in cooperation with the assumed orientation of 3D space. It does not correspond to an intrinsic, physically meaningful crossing direction. $\diamondsuit$

Finally, we infer the affine representation of the facet element from its vector proxy (18):

$$w^f = 2(w^l \mathrm{d}w^m \wedge \mathrm{d}w^n + \ldots + \ldots), \quad (19)$$

and from this, we infer the structure of Whitney forms in general: if $s = \{n_0, n_1, \ldots, n_p\}$ is a p-simplex (whose inner orientation is implied by the very order in which we list its nodes), its Whitney form [5] is

$$w^s = (-1)^i\, p! \sum_{i=0,\ldots,p} w^{n_i} \mathrm{d}w^{n_0} \wedge \ldots \langle i \rangle \ldots \wedge \mathrm{d}w^{n_p},$$

where the $\langle i \rangle$ means "omit the term $\mathrm{d}w^{n_i}$". But of course there is little use for such generality in the applications we have in view. Already when $p = 3$, in 3D, the "volume element" W^{T} for tetrahedron T is simply the function equal to 0 except on T, where it's equal to 1/vol(T). (This is the function proxy of a piecewise constant 3-form w^{T}, totally determined by the simple condition $\int_{\mathrm{T}'} w^{\mathrm{T}} = 1$ if $\mathrm{T} = \mathrm{T}'$, else 0.)

Anyway, we have enough to see the whole picture: To each primal p-simplex s (with p = 0 to d in d-dimensional affine space), there corresponds a Whitney form w^s of degree p, such that $\int_s w^s = 1$, and $\int_{s'} w^s = 0$ for other p-simplices of the mesh. The prolongator of a p-cochain $\mathbf{u}$ (the promised $Whitney\ map$) is

$$p_{\mathrm{M}}\,\mathbf{u} = \textstyle\sum_s \mathbf{u}_s w^s, \quad (20)$$

the p-chain associate of a p-manifold M is

$$p^t_{\mathrm{M}}\,M = \sum_s (\int_M w^s)\, s, \quad (21)$$

and (3) holds for a uniform family of meshes. Last, transposing (11) yields

$$\mathrm{d}p_{\mathrm{M}} = p_{\mathrm{M}}\,\mathbf{d}. \quad (22)$$

We shall adopt the notation $W^p(D)$ for the finite dimensional space generated by Whitney p-forms.

This quite satisfying uniformity of properties does not exhaust the subject. Remember that $\mathbf{d}_p \circ \mathbf{d}_{p-1} = 0$ and, in the case of a contractible[14] domain D, $\ker(\mathbf{d}_p) = \mathrm{cod}(\mathbf{d}_{p-1})$. Combining that with (22) we find, effortlessly, the following $structural\ property$ of the Whitney complex of forms: for p = 1 to d,

$$\mathrm{d}\,W^{p-1} \subset W^p, \quad \ker(\mathrm{d}\,; W^p) = \mathrm{d}\,W^{p-1}. \quad (23)$$

We'll see later how important this is (and hence, how important (11) is). But the next immediate task, now, is to use the above Whitney map for $Galerkin$-style discretization. We'll find, with not too much surprise, that this is just another way to derive discrete Hodge operators.

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[5] H. Whitney: **Geometric Integration Theory**, Princeton U.P. (Princeton), 1957.

[14] One in which any closed p-manifold can be shrunk to a point by continuous deformation. In dimension 3, this means a simply connected domain with connected boundary, or, more colloquially, "neither loops nor holes". (Holes, I understand, are aptly discussed by A. Kameari in *JSAEM*, *6*, 1 (1999), pp. 1-2.)