(2): Network constitutive laws

Where we stand

Our objective, in this series, is to solve the Maxwell equations in a closed cavity, in presence of a given current density, starting from no field at time zero. What we aim at is a *numerical scheme*, an algorithm, so that the numbers issued by the number-cruncher as the computation proceeds can be converted into usable information about the evolution of the field. This task is what one calls, in common parlance, the "discretization" of the equations.

In this respect, what we achieved so far is a "spatial" discretization of the Faraday and Ampère relations,

$$\partial_t b + \mathrm{d}e = 0, \qquad -\partial_t \tilde{d} + \mathrm{d}\tilde{h} = \tilde{j},$$

in the form of a system of differential equations (eqs. (9) and (11) in the previous issue):

$$(1) \qquad \partial_t \mathbf{b} + \mathbf{R}\mathbf{e} = 0, \quad (2) \quad -\partial_t \tilde{\mathbf{d}} + \mathbf{R}^t \tilde{\mathbf{h}} = \tilde{}.$$

There, **e** and **b**, $\tilde{\mathbf{d}}$ and $\tilde{\mathbf{h}}$, are "DoF-arrays": arrays of real numbers, the so-called "degrees of freedom", associated each with a specific geometrical element (edge or facet) of two interlocked "cellular pavings". Figure 1 reminds this association: assuming, here, tetrahedra as primal cells, and a barycentric construction of the dual paving, we assign the degrees of freedom $\mathbf{e}_e$ and $\mathbf{b}_f$ to the edge labelled e and to the facet labelled f, while $\tilde{\mathbf{h}}_f$ and $\tilde{\mathbf{d}}_e$ are affixed to the "dual edge" $\tilde{f}$ and to the "dual facet" $\tilde{e}$. Single letters with bold face, like for instance **b**, denote arrays such as $\{\mathbf{b}_f : f \in \mathcal{F}\}$, indexed over sets of so-called "p-cells", like here the set $\mathcal{F}$ of primal facets ($p = 2$). Same interpretation for $\tilde{} = \{\tilde{}_{\tilde{e}} : e \in \mathcal{E}\}$, indexed over edges, except that $\tilde{}$ is not a DoF-array but an array of *data*, the intensities across the dual facets. These are easily computed from the current density, which we considered as a given function of time, in the model problem we have in view.

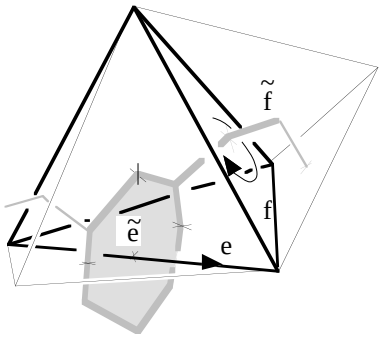

Figure 1. A few typical cells, in the case of what we called last time the "star construction" of a dual mesh, based on a simplicial primal mesh. Each primal edge or facet has its own inner orientation, which induces an outer orientation of its dual associate: for instance, the forward direction along edge e is taken as "crossing direction" of the dual facet $\tilde{e}$, etc.

These DoFs are what will eventually be known about the fields, once the computation is over: $\mathbf{e}_e$ will be the electromotive force[1] along edge e, $\tilde{\mathbf{d}}_e$ will be the flux of displacement current across the dual facet $\tilde{e}$, etc.

As we saw last time, the "network equations" (1) and (2) are not enough to determine **e**, **b**, $\tilde{\mathbf{h}}$, and $\tilde{\mathbf{d}}$. They need to be complemented by "network constitutive laws", that would relate **e** to $\tilde{\mathbf{d}}$ and **b** to $\tilde{\mathbf{h}}$. Finding such laws, i.e., discretizing the constitutive laws

$$b = \mu \tilde{\star} \tilde{h}, \qquad \tilde{d} = \epsilon \tilde{\star} e,$$

is the order of the day.

2.1 A generalized Yee scheme

Let's write these desired discrete relations as follows:

$$(3) \qquad \mathbf{b} = \tilde{\star}_\mu \tilde{\mathbf{h}}, \qquad (4) \qquad \tilde{\mathbf{d}} = \tilde{\star}_\epsilon \mathbf{e}.$$

[1] or rather, the best estimate of this voltage that we can achieve by using these meshes. Of course, some error occurs, and we shall not avoid the issue. We pointed out last time that everything one wants to know about the field can be obtained from the degrees of freedom, and that pointwise values of the field have secondary interest. Yet, one may have need for them, so we shall not dodge this issue either.

Here, $\tilde{\star}_\mu$ and $\tilde{\star}_\epsilon$ denote square matrices, of respective dimensions F and E (the numbers of active primal facets and edges[2]), which stand as finite-dimensional approximations of the above $\mu\tilde{\star}$ and $\epsilon\tilde{\star}$. Building them is our objective, but we shall carry on for a while *as if* we knew them. This will point to a number of desirable properties of the discrete Hodges, and thus help in their construction.

The first, fairly obvious, requirement is that $\tilde{\star}_\mu$ and $\tilde{\star}_\epsilon$ should be *regular* matrices. If so, eqs. (1)–(4) can be rewritten as

$$\partial_t \mathbf{b} + \mathbf{R}\tilde{\star}_\epsilon^{-1}\tilde{\mathbf{d}} = 0, \qquad -\partial_t \tilde{\mathbf{d}} + \mathbf{R}^t \tilde{\star}_\mu^{-1}\mathbf{b} = \tilde{j},$$

a system of ODE's in terms of $\mathbf{b}$ and $\tilde{\mathbf{d}}$. This, plus the initial conditions $\mathbf{b}(0) = 0$ and $\tilde{\mathbf{d}}(0) = 0$ at time $t = 0$, constitutes a discretization "in space" of the original equations, to which we have thus substituted a finite-dimensional *dynamical system*. Obvious equivalent forms of this system will come to mind, since there are four ways to eliminate one variable out of two in each group, **e**–$\tilde{\mathbf{d}}$ and **b**–$\tilde{\mathbf{h}}$. For instance, the following:

$$\partial_t \mathbf{b} + \mathbf{R}\mathbf{e} = 0, \qquad -\partial_t \tilde{\star}_\epsilon \mathbf{e} + \mathbf{R}^t \tilde{\star}_\mu^{-1}\mathbf{b} = \tilde{j},$$

which we shall adopt for definiteness.

The next step is a discretization "in time". Introducing a time-step $\delta t > 0$, let's approximate the time-dependent DoF array $\mathbf{b}$ by linear interpolation between successive values at times $t_k = k\delta t$, which we denote by $\mathbf{b}^k$, with $k = 0, 1, \ldots$. This way, the quotient $(\mathbf{b}^{k+1} - \mathbf{b}^k)/\delta t$ constitutes a natural approximation of the time-derivative $\partial_t \mathbf{b}$ at time $t_{k+1/2} = (k+1/2)\delta t$, which instructs us to approximate $\mathbf{e}$ by linear interpolation between values at such "half-integer times". These values we denote by $\mathbf{e}^{k+1/2}$. The obvious thing to do, now, is to let the $\mathbf{b}^k$s and $\mathbf{e}^{k+1/2}$s satisfy the following system of equalities,

$$\mathbf{b}^0 = 0, \quad \mathbf{e}^{-1/2} = 0,$$

$$(5) \quad \frac{\mathbf{e}^{k+1/2} - \mathbf{e}^{k-1/2}}{\delta t} = \tilde{\star}_\epsilon^{-1}(\mathbf{R}^t \tilde{\star}_\mu^{-1}\mathbf{b}^k - \tilde{j}^k),$$

$$(6) \quad \frac{\mathbf{b}^{k+1} - \mathbf{b}^k}{\delta t} + \mathbf{R}\mathbf{e}^{k+1/2} = 0,$$

for successive values $k = 0, 1, \ldots$, of the "discrete time" parameter. Clearly, this "leapfrog scheme" is an *algorithm* to solve eqs. (1)(4): it produces the $\mathbf{b}^k$s and $\mathbf{e}^{k+1/2}$s, step by step, when fed with the succession of known values of $\tilde{j}^k$. These, of course, are obtained by taking the flux of the known current density $\tilde{j}(t)$ at time t_k across dual facets.

Note that we start with $k = 0$, so the first required value of $\tilde{j}$ is $\tilde{j}^0$, which as a rule will be zero, but may not be: the algorithm can cope with sudden jumps of the current density (unphysical as these may be). More interestingly, let's remark that if $\tilde{j}(t)$ was a succession of steps, with $\tilde{j}(t)$ equal to some steady current density between times $(k-1/2)\delta t$ and $(k+1/2)\delta t$, the solution of (1)–(4) would be *exactly* what one obtains by linear interpolation in time between the successive $\mathbf{b}^k$s and $\mathbf{e}^{k+1/2}$s, as output by the numerical scheme, namely,

$$\mathbf{b}_{\delta t}(t) = [(t_k - t)\mathbf{b}^{k-1} + (t - t_{k-1})\mathbf{b}^k]/\delta t$$

for $t \in [t_{k-1}, t_k]$. This, though not making a proof, strongly suggests that the time-varying DoF arrays $\mathbf{b}_{\delta t}$ and $\mathbf{e}_{\delta t}$ thus built do converge towards the solution[3] $\{t \to \mathbf{b}(t), t \to \mathbf{e}(t)\}$ of (1)–(4) when δt tends to zero.

This property is a well-known feature (see., e.g., [MS]) of the classical Yee scheme [Ye], also known as the "finite difference in time-domain (FDTD) method", to which (5)(6) reduces when both pavings are made of brick-shaped cells, with facets parallel to the coordinate planes of an orthogonal Cartesian frame. Figure 2 is a reminder of the way vector components were assigned to grid-points in Yee's approach, 30 years ago, in the simplified 2D framework one had to assume in this age of limited computer resources.[4] (We consider

[2] This difference in dimensions explains why we have $\tilde{\star}_\mu$ and $\tilde{\star}_\epsilon$ instead of $\mu\tilde{\star}$ and $\epsilon\tilde{\star}$. There is no *single* discrete Hodge operator $\tilde{\star}$, but one discrete Hodge for each constitutive law. In fact, we should also replace $\mu\tilde{\star}$ and $\epsilon\tilde{\star}$ by $\tilde{\star}_\mu$ and $\tilde{\star}_\epsilon$ in the case of *anisotropic* materials, as discussed earlier (*JSAEM, 6*, 4, 1998, p. 325).

[3] I seize this pretext to recall that "$t \to f(t)$" means "the function that maps t to the value $f(t)$". (Cf. (*JSAEM, 6*, 2, 1998, p. 119.) This notation often helps to lift some ambiguities, as in the present case, where **b**, **e**, and their approximations $\mathbf{b}_{\delta t}$ and $\mathbf{e}_{\delta t}$ are not conceived as DoF-arrays, but as functions of time, whose instant values are DoF-arrays.

[4] A fully 3D modern avatar is the MAFIA code [W&], which can handle millions of DoFs on staggered cellular grids. (The "Finite Integration Technique" of [W&] is what we described last time when deriving eqs. (1) and (2).)

here the case of a horizontal electric field **E** = $\{\mathrm{E}^x, \mathrm{E}^y\}$ and of a vertical magnetic induction, whose single scalar component is denoted B.)

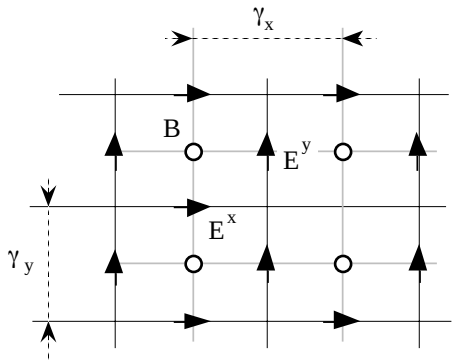

Figure 2. Assignment of field components to grid-points, in a 2D "transverse electric" Yee scheme. (We assume a uniform primal grid. Then all 2-cells have the same size, $\gamma_x \times \gamma_y$.)

It takes little imagination to see the relation between these components and our edge- and facet-based degrees of freedom. (It may be slightly easier in three dimensions, cf. Fig. 3, which corresponds to Fig. 1 in [Ye].) Yee's staggered grids are an instance of what we called last time the "orthogonal construction", with the additional feature that edges pierce their associated facets in their exact middles, and are divided by them into equal parts. All primal edges are oriented along the coordinate axes. All facets are oriented according to the usual conventions, so that for instance, facet f of Fig. 3, which lies in a horizontal plane, has the standard counterclockwise orientation.

There is an inessential difference, however: The degree of freedom $\mathbf{b}_f$ of facet f (cf. the caption of Fig. 3) is *not* the component B^z at f's center, i.e., at point $\{i+1/2, j+1/2, k+1\}$, but corresponds to the flux $\gamma_x\gamma_y\mathrm{B}^z$, and $\mathbf{e}_e$ is $\gamma_z\mathrm{E}^z$. Thus taking the vector-proxy components as DoFs makes the analogue of our network equations a bit cumbersome, because of the appearance of the edge-lengths γ_i in formulas, and of the heavy labelling (e.g., our $\mathbf{b}_f$ for facet $\{i + 1/2, j + 1/2, k + 1\}$ is $\gamma_x\gamma_y\mathrm{B}^z_{i+1/2,j+1/2,k+1}$). In compensation, network constitutive laws are most natural in Yee's scheme: Just set

$$\mathrm{B}^z_{i+1/2,j+1/2,k+1} = \mu\,\mathrm{H}^z_{i+1/2,j+1/2,k+1}, \tag{7}$$

and so forth.

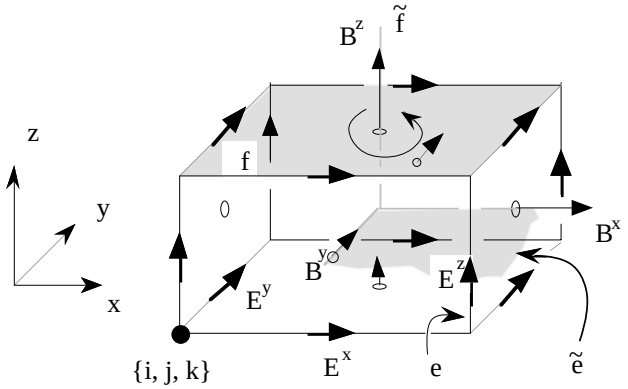

Figure 3. Connection with the present approach, in 3D. One face f and its dual $\tilde{f}$ are highlighted, as well as one edge e and a part of its dual facet $\tilde{e}$. It's convenient, with such grids, to label primal nodes with integer triples $\{i, j, k\}$, hence a natural labelling for all cells: For instance, the f of this figure is facet $\{i + 1/2, j + 1/2, k + 1\}$ (the label that one would stick to its center), edge e is $\{i + 1, j, k + 1/2\}$, etc.

Since $\gamma_x\gamma_y$ is the area of f and γ_z the length of its dual edge $\tilde{f}$, we may rewrite (7) in our notation as follows:

$$\mathbf{b}_f = \mu\,\frac{\mathrm{area}(f)}{\mathrm{length}(\tilde{f})}\,\tilde{\mathbf{h}}_f. \tag{8}$$

So, we have with (5)(6) a generalization of Yee's scheme, which only differs from it in details,[5] mostly notational, as far as the network equations are concerned. But—and here lies, from the point of view adopted here, Yee's achievement—both instances of *the discrete Hodge operator take on an ideally simple form*, in FDTD, thanks to the adoption of a system of staggered uniform grids. They are expressed by *diagonal* matrices $\tilde{\star}_\mu$ and $\tilde{\star}_\epsilon$, whose entries are given by simple formulas[6] such as (8).

Looking back at (5), we may appreciate how essential to the efficiency of the scheme this property of diagonality can be. Because of well-known concerns about stability on which we soon return, δt has to be small. Having to solve one or two linear systems at each time

[5] Among these: In [Ye], **E** was evaluated at integer times, and **B** at half-integer times.

[6] If μ is non-uniform but smooth, just replace it in (8) by $\mu_{i+1/2,j+1/2,k+1}$, with obvious notation. We shall return on the case when point $\{i + 1/2, j + 1/2, k+1\}$ happens to be at a material interface. Note, on the other hand, that anisotropic tensors μ and ϵ *can* be accommodated if their principal directions go along coordinate axes.

step, as (5) may seem to require, would therefore make the computation too slow. Thanks to diagonality of the Hodges, Yee's scheme is *explicit*, not requiring any linear system solution. Diagonality of $\tilde{\star}_\mu$ and $\tilde{\star}_\epsilon$ is thus, we see, highly desirable.

Yet, we can be content with less, for the main point is to keep the number of arithmetic operations in steps (5) and (6) as low as possible. In (6), we have to multiply the E-dimensional vector $\mathbf{e}^{k+1/2}$ by a sparse matrix $\mathbf{R}$, anyway. A similar multiplication, by $\mathbf{R}^t$, intervenes in (5). So if both matrices $\tilde{\star}_\mu^{-1}$ and $\tilde{\star}_\epsilon^{-1}$ can be made sparse to a comparable degree, it will be an acceptable state of affairs. This should be achievable, since the operators these matrices are meant to approach, the inverses of $\mu\tilde{\star}$ and $\epsilon\tilde{\star}$, have a *local* character.

2.2 Wedge product, energy

So, a regular matrix which purports to approximate a Hodge operator should, if not diagonal, at least be sparse, or have a sparse inverse. A connection with the notion of *energy* of the field will suggest that it should be *symmetric*, too.

This is the place to go deeper into the notion of wedge product of forms, that was only treated incidentally up to now. Let ω and η be a p-covector and a q-covector. We shall define a new covector, of degree $p+q$, denoted $\omega \wedge \eta$, and call it their *wedge product*, or external product. The notion of wedge product for p- and q-forms, i.e., for fields of covectors, will then follow naturally.

Recall that a p-covector ω was conceived as a machine with p slots, in which p vectors $\{v_1, \ldots, v_p\}$ can be inserted, in a definite order. The machine then outputs a real number, which linearly depends on all factors (ω is a "multilinear" map), and changes sign if two of them are interchanged ("alternating" map). Now, have two machines of this kind, with p and q slots respectively, and a sequence of $p+q$ vectors, $\{v_1, \ldots, v_p, v_{p+1}, \ldots, v_{p+q}\}$, to process. As these arrive, we may assign them to one or the other machine, so that p of them go to the first machine and the remaining q to the other one. Hence two numbers, of which we take the product. This satisfies the multilinearity condition. But since there is no criterion to allot vectors to one machine or the other, we must consider *all* ways to do that, and add the results, with appropriate sign changes in order to satisfy the alternation condition. For instance, if $p = q = 1$, this suggests the following, which we already know,

$$(9) \quad (\omega \wedge \eta)(v_1, v_2) = \omega(v_1)\eta(v_2) - \omega(v_2)\eta(v_1),$$

as definition of $\wedge$. If $p = 1$ and $q = 2$, the same idea leads to

$$(10) \quad (\omega \wedge \eta)(v_1, v_2, v_3) = \omega(v_1)\eta(v_2, v_3) + \omega(v_2)\eta(v_3, v_1) + \omega(v_3)\eta(v_1, v_2).$$

To comfortably generalize that, let's denote by σ a map from the set of integers $[1, \ldots, p]$ into the larger set $[1, \ldots, p+q]$, such that $\sigma(i) < \sigma(i+1)$ for all $i < p$. The set of positions not occupied in $[1, \ldots, p+q]$ determines in an obvious way (Fig. 4) a *complementary* map ς with similar properties. Now, we define

$$(11) \quad (\omega \wedge \eta)(v_1, \ldots, v_{p+q}) = \sum_\sigma \mathrm{sgn}(\sigma)\, \omega(v_{\sigma(1)}, \ldots, v_{\sigma(p)})\, \eta(v_{\varsigma(1)}, \ldots, v_{\varsigma(q)}),$$

where $\mathrm{sgn}(\sigma) = \pm 1$, as explained on Fig. 4, the sum being taken with respect to all possible σs. (Note there are as many such σs as ways to choose p integers out of $p+q$.)

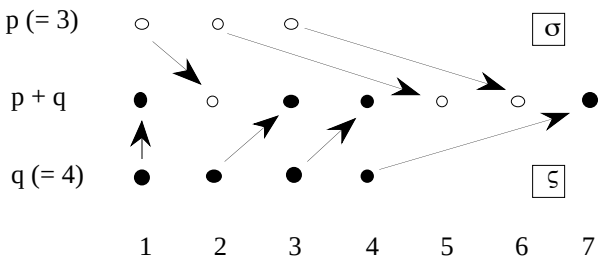

Figure 4. An increasing injection σ from $[1, \ldots, p]$ into $[1, \ldots, p+q]$ and its complement ς. We call *signature* of σ (or of ς), denoted sgn, the parity of the number of swaps between black and white spots that will bring all the white ones to the left (or all the black ones to the right). Here, this takes 7 swaps, so $\mathrm{sgn}(\sigma) = \mathrm{sgn}(\varsigma) = -1$ in this example.

We won't really use this formula, given here for the sake of completeness. It's enough to know that covectors can thus be wedge-multiplied two by two,[7] and hence, differential forms, which are fields of covectors, have wedge-products, too. One or the factors, or

[7] The operation is associative. As for commutativity, one has $\eta \wedge \omega = (-1)^{pq}\, \omega \wedge \eta$, as detailed examination of (11) will show.

both, can be a *twisted* covector (*JSAEM, 6*, 2, 1998, p. 123). Rules to this effect are obvious, and tedious.[8]

As an immediate application of this remark, it makes sense to wedge-multiply the electric field e, a 1-form, by the current-density $\tilde{j}$, a twisted 2-form, thus obtaining a 3-form $e \wedge \tilde{j}$ which, being a *twisted* form, can be integrated over 3D space, irrespective of ambient orientation. As our previous discussion of the Lorentz force should make clear, the number thus obtained is, up to sign, the rate of work involved in moving electric charges in the antenna the prescribed way, which entails working against the electric field created by these very charges. With correct sign, $\int e \wedge \tilde{j}$ is the *power yielded by* the electromagnetic field to the rest of the world.

Rather than offer a proof of this, which would lead us astray, let's cash in on our knowledge that the density of such power is the dot product $\mathbf{E} \cdot \mathbf{J}$. So it's a matter of proving that, if $\mathbf{E}$ and $\mathbf{J}$ are the vector proxies of a covector e and a 2-covector j, one has $e \wedge j = \mathbf{E} \cdot \mathbf{J}\, vol$, where vol is the 3D volume form, $vol(v_1, v_2, v_3) = v_1 \cdot (v_2 \times v_3)$.

Indeed, let's apply (10) to e and j, knowing that $e(v) = \mathbf{E} \cdot v$ and $j(v, w) = \mathbf{J} \cdot v \wedge w = vol(\mathbf{J}, v, w)$, by the very definition of vector proxies. What comes out is $(e \wedge j)(v_1, v_2, v_3) = \mathbf{E} \cdot v_1 vol(\mathbf{J}, v_2, v_3) + \ldots$, with circular permutation, an expression in which one will recognize $\mathbf{E} \cdot \mathbf{J}$ times $vol(v_1, v_2, v_3)$. In compact form, using the notation introduced in *JSAEM, 6*, 3 (1998), p. 233,

$$^1\mathbf{E} \wedge {}^2\mathbf{J} = {}^3(\mathbf{E} \cdot \mathbf{J}).$$

The case $p = q = 1$ is even easier: Applying (9) to $e = {}^1\mathbf{E}$ and $h = {}^1\mathbf{H}$, we get $(e \wedge h)(v, w) = (\mathbf{E} \cdot v)(\mathbf{H} \cdot w) - (\mathbf{E} \cdot w)(\mathbf{H} \cdot v)$, equal to $(\mathbf{E} \times \mathbf{H}) \cdot (v \times w)$, by a well-known formula. So

$$^1\mathbf{E} \wedge {}^1\mathbf{H} = {}^2(\mathbf{E} \times \mathbf{H}).$$

Hail to thee, Poynting vector!! Indeed, taking account of the orientation issues neglected in what precedes, what plays the role of "Poynting field" in the geometric approach is the twisted 2-form $e \wedge \tilde{h}$, whose integral over a closed surface (outer-oriented from inside to outside) is the exiting power.

Now let us consider a product such as $\mathbf{B} \cdot \mathbf{H}$, which is, up to a factor 2, the density of magnetic energy. We see that ${}^3(\mathbf{B} \cdot \mathbf{H}) = b \wedge \tilde{h}$, and since $b = \mu\, \tilde{\star} \tilde{h}$, this density is the twisted 3-form $\mu/2\, \tilde{\star} \tilde{h} \wedge \tilde{h}$. One may verify, as an easy exercise,[9] the Poynting theorem:

$$(12) \quad d_t W_\Omega(e, \tilde{h}) + \int_{\partial\Omega} e \wedge \tilde{h} = -\int_\Omega \tilde{j} \wedge e,$$

where $W_\Omega(e, \tilde{h}) = \int_\Omega (\mu/2\, \tilde{\star} \tilde{h} \wedge \tilde{h} + \epsilon/2\, \tilde{\star} e \wedge e)$ is the part of the field energy ascribed to region Ω. This expresses energy conservation.

The equality $b \wedge \tilde{h} = {}^3(\mathbf{B} \cdot \mathbf{H})$ has something else to tell us. Replace $\tilde{h}$ by a different field $\tilde{h}'$, and set $b = \mu\, \tilde{\star} \tilde{h}$. Then $\mu\, \tilde{\star} \tilde{h} \wedge \tilde{h}' = {}^3(\mu\, \mathbf{H} \cdot \mathbf{H}')$, a symmetrical expression. Hence something we didn't pay attention to so far: the *symmetry* of the Hodge operator acting on forms, as expressed by $\int \mu\, \tilde{\star} \tilde{h} \wedge \tilde{h}' = \int \mu\, \tilde{\star} \tilde{h}' \wedge \tilde{h}$ for all $\tilde{h}, \tilde{h}'$. Moreover,[10] $\int \mu \tilde{\star} \tilde{h} \wedge \tilde{h} = \int \mu \mathbf{H} \cdot \mathbf{H} > 0$ for $\tilde{h}$ not identically 0, so $\tilde{\star}$ is positive definite. The matrices $\tilde{\star}_\mu$ and $\tilde{\star}_\epsilon$ by which we purport to approximate $\mu\, \tilde{\star}$ and $\epsilon\, \tilde{\star}$ should, consistently, be symmetric and strictly positive definite, too.

2.3 "Discrete" energy, and stability

To discuss the implications of this remark, let's introduce a notation for such sums as $\sum_{f \in \mathcal{F}} \mathbf{b}_f \tilde{\mathbf{h}}_f$, which one might construe as a dot product between two vectors, $\mathbf{b}$ and $\tilde{\mathbf{h}}$, of common dimension F. Rather than using a dot, however, we shall denote this[11] by

$$\langle \mathbf{b}, \tilde{\mathbf{h}} \rangle = \sum_{f \in \mathcal{F}} \mathbf{b}_f \tilde{\mathbf{h}}_f.$$

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[8] For instance, if the pair $\{\omega, Or\}$ represents the twisted covector $\tilde{\omega}$, its wedge product by the straight η is the twisted covector represented by the pair $\{\omega \wedge \eta, Or\}$. The product with $\tilde{\eta}$ = (the class of) $\{\eta, Or'\}$ is (the class of) $\{\omega \wedge \eta, Or\, Or'\}$. Observe that the product of two twisted covectors is straight.

[9] Start from (1) and (2), wedge-multiply by $\tilde{h}$ and $-e$, add, and integrate over Ω, using Stokes and the formula $\mathrm{d}(e \wedge \tilde{h}) = \mathrm{d}e \wedge \tilde{h} - e \wedge \mathrm{d}\tilde{h}$.

[10] The integral concerns the whole region of space under consideration, i.e., the whole domain D in our model problem. Note (as a follow-up to a previous remark, *JSAEM, 7*, 2, 1999, p. 154) that $\int \mu\, \tilde{\star} \tilde{h} \wedge \tilde{h}'$ can now be understood as a *scalar product* on the functional space of (twisted) 1-forms, which can thereby be turned into a Hilbert space.

[11] Because this bra–ket notation traditionally connotes duality products between objects of *different* types, as was the case for instance in $\langle \omega, v \rangle$, which we used

Similarly, $\langle \tilde{\wr}, \mathbf{e} \rangle$ is defined as $\sum_{e \in \mathcal{E}} \tilde{\wr}_e \mathbf{e}_e$. One checks that

$$\langle \mathbf{Re}, \tilde{\mathbf{h}} \rangle = \sum_{e,f} \mathbf{R}_{ef} \mathbf{e}_e \tilde{\mathbf{h}}_f = \langle \mathbf{e}, \mathbf{R}^t \tilde{\mathbf{h}} \rangle,$$

a key formula in what follows.

Next, let's do what was suggested in Note 9, but applied to the dynamical system (1)–(4) instead of to the original equations: Take the $\langle\ ,\ \rangle$-product of (1) with $\tilde{\mathbf{h}}$, of (2) with $-\mathbf{e}$, add, apply the previous formula, replace $\mathbf{b}$ by $\tilde{\star}_\mu \tilde{\mathbf{h}}$ and $\tilde{\mathbf{d}}$ by $\tilde{\star}_\epsilon \mathbf{e}$. What results,

$$d_t(\,{}^1\!/_2 \langle \tilde{\star}_\epsilon \mathbf{e}, \mathbf{e} \rangle + {}^1\!/_2 \langle \tilde{\star}_\mu \tilde{\mathbf{h}}, \tilde{\mathbf{h}} \rangle) = -\langle \tilde{\wr}, \mathbf{e} \rangle$$

looks so much like (12)[12] that we can't avoid calling ${}^1\!/_2 \langle \tilde{\star}_\epsilon \mathbf{e}, \mathbf{e} \rangle$ and ${}^1\!/_2 \langle \tilde{\star}_\mu \tilde{\mathbf{h}}, \tilde{\mathbf{h}} \rangle$ the *electric* and *magnetic discrete energy*, respectively, and $\langle \tilde{\wr}, \mathbf{e} \rangle$ the *discrete power* leaving the system. But of course (don't forget we have *no* explicit definition of $\tilde{\star}_\mu$ and $\tilde{\star}_\epsilon$ yet!), this has no justification till we establish some link between discrete and continuous energy or power (for instance by proving that discrete power and energy converge, in an appropriate sense, towards their continuous counterparts).

Now what about the *time-discretized* dynamical system (5)(6)? Would it conserve discrete energy too? No such luck. Start from (5)(6), where we suppose all $\tilde{\star}$s = 0, to simplify a little. (In compensation, suppose a nonzero initial situation $\{\mathbf{b}^0, \mathbf{e}^{-1/2}\}$.) Use $\tilde{\mathbf{h}}$ for $\tilde{\star}_\mu^{-1} \mathbf{b}$. Right-multiply (5) by $\mathbf{e}^{k+1/2}$, (6) by $\tilde{\mathbf{h}}^k$, subtract to check that

$$\langle \tilde{\star}_\epsilon (\mathbf{e}^{k+1/2} - \mathbf{e}^{k-1/2}, \mathbf{e}^{k+1/2} \rangle + \langle \tilde{\star}_\mu (\tilde{\mathbf{h}}^{k+1} - \tilde{\mathbf{h}}^k, \tilde{\mathbf{h}}^k \rangle$$

vanishes, repeat this with k changed to $k-1$ in (6), add the results, to finally obtain

$$\langle \tilde{\star}_\epsilon \mathbf{e}^{k+1/2}, \mathbf{e}^{k+1/2} \rangle + \langle \tilde{\star}_\mu \tilde{\mathbf{h}}^{k+1}, \tilde{\mathbf{h}}^k \rangle =$$

$$\langle \tilde{\star}_\epsilon \mathbf{e}^{k-1/2}, \mathbf{e}^{k-1/2} \rangle + \langle \tilde{\star}_\mu \tilde{\mathbf{h}}^k, \tilde{\mathbf{h}}^{k-1} \rangle.$$

So it's not discrete energy which is conserved. Only a quantity which looks like it (imagine δt tending to 0) happens not to depend on k.

The computation could have been done differently, keeping the same value of k in (6) and changing it in (5), this time showing the conservation of

$$\langle \tilde{\star}_\epsilon \mathbf{e}^{k+1/2}, \mathbf{e}^{k-1/2} \rangle + \langle \tilde{\star}_\mu \tilde{\mathbf{h}}^k, \tilde{\mathbf{h}}^k \rangle$$

(which actually can be proved equal to the previous quantity). Setting $\mathbf{e}^k = (\mathbf{e}^{k+1} + \mathbf{e}^k)/2$ and $\tilde{\mathbf{h}}^{k+1/2} = (\tilde{\mathbf{h}}^k + \tilde{\mathbf{h}}^{k+1})/2$, we see that what is conserved is the more symmetrical-looking

$$\langle \tilde{\star}_\epsilon \mathbf{e}^k, \mathbf{e}^{k+1/2} \rangle + \langle \tilde{\star}_\mu \tilde{\mathbf{h}}^k, \tilde{\mathbf{h}}^{k+1/2} \rangle,$$

but ... so what? If $k + 1/2$, there, could be replaced by k, we would conclude that the algorithm is *stable*,[13] *under the condition of positive definiteness* of the matrices $\tilde{\star}_\epsilon$ and $\tilde{\star}_\mu$. But this little difference voids the argument of any value.

So let's try harder. Take (6), with $\mathbf{b}$ written as $\tilde{\star}_\mu \tilde{\mathbf{h}}$, substitute $k-1$ for k, subtract the two equalities thus obtained, and use (5) (again, no $\tilde{\star}$ at the righ-hand side) to eliminate $\mathbf{e}$, hence

$$\tilde{\star}_\mu (\tilde{\mathbf{h}}^{k+1} - 2\tilde{\mathbf{h}}^k + \tilde{\mathbf{h}}^{k-1}) + \delta t^2\, \mathbf{R} \tilde{\star}_\epsilon^{-1} \mathbf{R}^t \tilde{\mathbf{h}}^k = 0.$$

The question is: can $\tilde{\mathbf{h}}^k$, defined by this recurrence, blow up for some initial conditions?

The *modal analysis* technique to answer it is well known. Let's use the generalized eigenmodes $\{\lambda_j, \tilde{\mathbf{w}}_j\}$, solutions of

$$(13) \qquad \mathbf{R} \tilde{\star}_\epsilon^{-1} \mathbf{R}^t\, \tilde{\mathbf{w}} = \lambda^2 \tilde{\star}_\mu\, \tilde{\mathbf{w}},$$

with $\lambda \geq 0$ real, as a basis for the E-dimensional space to which $\tilde{\mathbf{h}}$ belongs. *If* both $\tilde{\star}_\epsilon$ and $\tilde{\star}_\mu$ are symmetric and strictly positive definite, as we assume, there is such a basis (including all $\tilde{\mathbf{w}}_j$s for which $\lambda_j = 0$, which span the nullspace, i.e., the kernel of $\mathbf{R}^t$), made of "$\langle , \rangle_\mu$-orthogonal" vectors, in the sense that $\langle \tilde{\star}_\mu \tilde{\mathbf{w}}_i, \tilde{\mathbf{w}}_j \rangle = 0$ for $i \neq j$. One can then write $\tilde{\mathbf{h}}^k = \sum_j h_j^k \tilde{\mathbf{w}}_j$, bring that into the recurrence

earlier, whereas the two arguments in a dot product are of the *same* type. Here, indeed, $\mathbf{b}$ and $\tilde{\mathbf{h}}$ don't belong to the same type, due to their respective associations with inner- and outer-oriented cells, and we don't consider an expression such as $\langle \mathbf{b}, \mathbf{b} \rangle$ as legitimate. Dimensional analysis makes this plain: While $\langle \tilde{\wr}, \mathbf{e} \rangle$ is a sum of factors expressed in ampères × volts, and hence a power, $\langle \mathbf{b}, \mathbf{b} \rangle$ would be in "squared webers", a preposterous unit.

[12] There is no analogue of a surface term because, when defining the incidence matrices, we deleted from the boundary S of the domain all cells which, a priori, bear a null DoF, owing to boundary conditions. One might extend the theory in order to have such a "discrete Poynting flux".

[13] To check stability for linear systems of difference equations, one looks whether an initial state gets amplified in the absence of right-hand side, so it was all right to dismiss $\tilde{\wr}$.

relation, and obtain that $h_j^{k+1} = r_j h_j^k$, where r_j is a root of the characteristic equation

$$r^2 - (2 - \lambda_j^2 \delta t^2) r + 1 = 0. \qquad (14)$$

So it's all right (no blow up) if both solutions, whose product is 1, lie on the unit circle, which happens (cf. Fig. 5) when

$$\lambda_j \delta t < 2 \text{ for all } j.$$

This is the condition for stability of the generalized Yee scheme (5)(6).

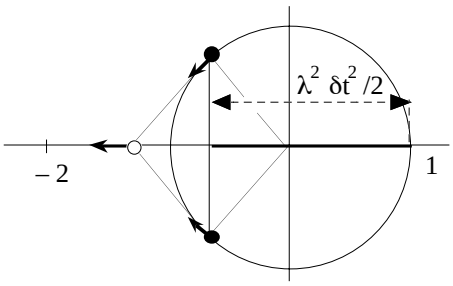

Figure 5. Why the $\lambda\delta t < 2$ condition. (The white spot lies at the sum of roots, i.e., $2 - \lambda^2 \delta t^2$. When it passes left to -2, as δt increases, the roots become real, one of them exiting the unit circle.)

In the case of the original Yee scheme, eigenvalues could explicitly be found, hence the well-know relation [Ye] between the maximum possible value of δt and the lengths of the cell sides. For general grids, we have no explicit formulas, but the thumbrule is the same: δt should be small enough for a signal travelling at the speed of light (in the medium under study) not to cross more than one cell during this lapse of time.

A contrario, having one of the matrices $\tilde{\star}_\mu$ or $\tilde{\star}_\epsilon$ *not* positive definite would destroy stability. Suppose this happens to $\tilde{\star}_\epsilon$ alone. Then, (13) will have imaginary solutions λ_j, for which $2 - \lambda^2 \delta t^2 > 2$ whatever δt, hence instability.

The conclusion is neat: a good discrete Hodge operator is a symmetric, nearly diagonal, positive-definite one. It's time we show this can be achieved.

2.4 A diagonal, positive-definite Hodge

The idea (which has been independently developed by many people) is bold and simple: Use the orthogonal construction, and apply formula (8), the same as in FDTD. So diagonal entries of $\tilde{\star}_\mu$ are

$$\tilde{\star}_\mu^{ff} = \mu_f \frac{\text{area}(f)}{\text{length}(\tilde{f})}, \qquad (15)$$

where f is a primal facet, $\tilde{f}$ its dual edge (Fig. 6), and μ_f the value of μ at the meeting point, if well defined. (More below on this. For the moment, let us assume a uniform μ.) All other entries $\tilde{\star}_\mu^{ff'}$ are set to zero. By the virtues of the orthogonal construction, this is a diagonal positive definite matrix, the ideal situation. The construction of $\tilde{\star}_\epsilon$ is similar.

But why should they be good *as approximations* of $\mu\tilde{\star}$ and $\epsilon\tilde{\star}$? After all, one could imagine multiplying the above number $\tilde{\star}_\mu^{ff}$ by any arbitrary positive factor, and still satisfy the requirements. But let's consider a uniform field $\mathbf{H}$, and abuse the notation by also calling f the vectorial area of f, and $\tilde{f}$ the vector along $\tilde{f}$, which allows us to write

$$f = \frac{\text{area}(f)}{\text{length}(\tilde{f})} \tilde{f},$$

thanks to the orthogonality property. Then $\mathbf{b}_f = \mathbf{B} \cdot f$, and $\tilde{\mathbf{h}}_f = \mathbf{H} \cdot \tilde{f}$. Since $\mathbf{B} = \mu\mathbf{H}$, the ratio $\mathbf{b}_f / \tilde{\mathbf{h}}_f$ is $\tilde{\star}_\mu^{ff}$ *whatever* $\mathbf{H}$, so (15) is the right coefficient for uniform fields. This is the main point in favor of the orthogonal construction.

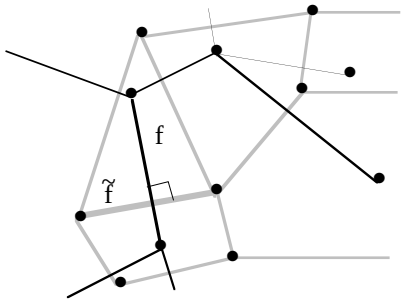

Figure 6. A piece of the pavings, in the case of the orthogonal construction (to be imagined in dimension 3). Essential features are that each dual edge $\tilde{f}$ is orthogonal to its associated primal facet f, does meet it, and that dual nodes are inside primal cells. Under these conditions, $\tilde{f}$ has positive length.

Since all smooth fields will appear uniform at the scale of a cell when the meshes are refined, one may imagine building on this an argument [To] that would lead to a comparison between discrete energy and energy. (Try it: the volume "controlled by" f and $\tilde{f}$ is $\text{area}(f) \times \text{length}(\tilde{f})/3$, the average projection of $\mathbf{H}$ onto a random unitary vector squares to $|\mathbf{H}|^2/3 \ldots$) But we are still far from that, which will require a serious convergence proof.

Finally, let's consider the case (Fig. 7) when two adjacent primal volumes T_1 and T_2, with

common facet f, have permeabilities μ_1 and μ_2, different. Call $\tilde{f}_1$ and $\tilde{f}_2$ the vectors along both parts of $\tilde{f}$. Then, instead of (15),

$$\tilde{\star}_\mu^{ff} = \frac{\mu_1\,\mu_2\,\text{area}(f)}{\mu_2\,\text{length}(\tilde{f}_1) + \mu_1\,\text{length}(\tilde{f}_2)}.$$

This is easily justified: let **u** and **v** be arbitrary vectors, normal and tangent to f respectively, and let $\mathbf{H}_1 = \mathbf{u} + \mathbf{v}$ in T_1. Transmission conditions across f determine a unique uniform field $\mathbf{B}_2 = \mu_1\mathbf{u} + \mu_2\mathbf{v}$ in T_2. Then $\mathbf{b}_f = \mu_1\, f \cdot \mathbf{u}$, and $\mu_2\tilde{\mathbf{h}}_f = \mu_2\,\tilde{f}_1 \cdot \mathbf{u} + \mu_1\,\tilde{f}_2 \cdot \mathbf{u}$. As f, $\tilde{f}_1$, and $\tilde{f}_2$ are collinear, **u** disappears from the quotient, as before.

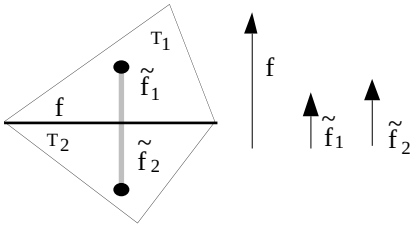

Figure 7. The case of a discontinuous permeability.

WHAT NEXT?

With this realization of the discrete Hodge operator, all elements of the theory have now a discrete counterpart, hence a "discrete Maxwell house". But is it safe to inhabit? This is the question of convergence. Before that, however, the main practical concern is, "can orthogonal meshes easily be produced?", and the answer, unfortunately, is "no, not always". So there is a need for alternatives. We'll see that the Galerkin method, which implies the use of finite elements as ways to reconstruct fields from DoF arrays, offers one. Finite elements will also be instrumental as regards convergence.

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