(4): "Maxwell's house"

Last time, we were able to express Faraday's law and the Lorentz force in terms of new mathematical abstractions, namely the differential forms e and b, defined on affine 3D space without any metric or orientation. We now extend this approach to the full system of Maxwell equations. The metric of space will not be irrelevant. But we shall see exactly where and why it is needed, thus completing our layer-by-layer analysis of geometrical structures underlying Maxwell's equations. A synoptic presentation of the Maxwell equations ("Maxwell's house") can then be proposed.

4.1 The way ahead

In the previous article, we stopped at the point where Faraday's law could be expressed by $\partial_t b + \mathrm{d}e = 0$, instead of the familiar $\partial_t \mathbf{B} + \mathrm{rot}\, \mathbf{E} = 0$. In the newly arrived-at viewpoint, b and e are seen as more legitimate representatives of the (physical) electromagnetic field than their "vector proxies" **B** and **E**, because the latter need, in order to be defined, a metric and an orientation of ambient space: Both structures we recognized as irrelevant when discussing Faraday's law, whose physical content owes nothing to them. Lorentz force, also, proved representable by an affine object, the 2-covector $e - i_V b$. The correspondences $e = {}^1\mathbf{E}$, $b = {}^2\mathbf{B}$, and $i_V b = -{}^1(V \times \mathbf{B})$, where the superscripts 1 and 2 implicitly refer to a specific metric and orientation, made possible the passage from one formalism to the other.

We also acknowledged the existence of an intermediate formalism, in which ambient space need not be oriented, but still must be equipped with a metric. It accounts for Faraday's law in the rather awkward form $\partial_t \tilde{\mathbf{B}} + \tilde{\mathrm{rot}}\, \mathbf{E} = 0$, where $\tilde{\mathbf{B}}$ is a *twisted*, or *axial vector*, also acting as proxy for b, and $\tilde{\mathrm{rot}}$ is an orientation-independent variant of the curl operator, which maps ordinary (or *polar*) vectors to twisted ones.

Now it seems natural to go forward and to subject all other elements of the Maxwell system of equations to a similar transcription process. We'll find two main obstacles along the road: (1) It's not the same kind of integrals that appear in the integral forms of Faraday's and Ampère's law, (2) Metric reenters the stage when it comes to expressing constitutive laws such as $\mathbf{B} = \mu \mathbf{H}$, or $\mathbf{J} = \sigma \mathbf{E}$, and so on. Both obstacles will be overcome by the promotion of appropriate new entities ("twisted" forms, and the Hodge operator), and when we are through, a novel geometrical framework, called in jest "Maxwell's house", will be seen standing erect. We'll verify that this framework is "home", indeed, to all geometrical objects which contribute to the description of the electromagnetic field and to its dynamics.

4.2 Currents, twisted forms

Let's begin with the equation $\partial_t q + \mathrm{div}\, \mathbf{J} = \mathbf{0}$, which expresses the conservation of electric charge.

One is tempted to say, "Well, vector **J** looks very much like the proxy for a differential form j, namely $j = {}^2\mathbf{J}$; therefore, by mere imitation of what was done for Faraday's law, we guess that $\partial_t q + \mathrm{d}j = 0$ is the sought-for affine expression for charge conservation, provided we see q as a 3-form." But then what about the *integral* form of this law on some 3D domain D, that would be $\partial_t \int_D q + \int_{\partial D} j = 0$, thanks to the Gauss theorem? The integral $\int_{\partial D} j$ cannot immediately be recognized as one of the two quantities that would make physical sense, namely, the flux of charge *exiting from* D and the flux *entering* D, because integrals of this kind, as we saw last time, are only defined over *inner*-oriented surfaces, whereas such qualifiers as "exiting from" or "entering" imply a crossing direction through the boundary.

More generally, given a surface S (not necessarily closed), what we wish to capture is the notion of *intensity*, which relates to the question "how much electric charge crosses S, *in some definite direction*, per unit of time?" Intensity, therefore, refers to what we called last time an outer-oriented surface, that is, a surface endowed with a crossing direction. Note that ∂D, above, can be given one: either from inside to outside, or the other way round, as we wish. It's a matter of convention on which we shall return.

So if we want an affine object that would properly represent intensities, it has to be something whose integral makes sense over outer-oriented surfaces. A two-form, which as we saw last time is intended to be integrated on inner-oriented surfaces, cannot do the job, unless—and this points to the solution—one has a way to infer an inner orientation of S from the given crossing direction. If ambient space is oriented, the crossing direction does imply an inner orientation of S, as we saw repeatedly (Fig. 1). But of course "exiting intensity" or (in the case of Fig. 1) "intensity from region $-$ to region +", as physical concepts, have objective meaning, which cannot depend on a mere convention about orientation. Therefore, the geometrical object entrusted to code the information about intensities across all possible outer-oriented surfaces must bring with it a tool to convert outer orientation into inner orientation.

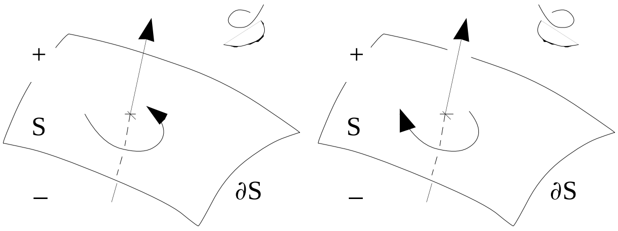

Figure 1. When a crossing direction is assigned to a surface (here from region "$-$" below to region "+" above), no inner orientation results, unless an ambient orientation of space is given. Then an inner orientation of S is induced. The figure displays the two possibilities in this respect.

And objects thus equipped we know about: *twisted* two-forms are it. A twisted 2-form $\tilde{j}$ is an equivalence class of pairs $\{$*ordinary* 2-*form*, *orientation*$\}$: We take all pairs $\{j, Or\}$, where j is a 2-form and Or one of the two orientations possible in ambient space, decree that pair $\{j, Or\}$ is equivalent to pair $\{-j, -Or\}$, and to none other, and denote by $\tilde{j}$ the equivalence class $\{\{j, Or\}, \{-j, -Or\}\}$ thus obtained. Of course, we may envision $\tilde{j}$ also as a field of twisted 2-covectors: the value of $\tilde{j}$ at point x is the twisted 2-covector $\tilde{j}(x)$, which is itself a class $\{\{j(x), Or\}, \{-j(x), -Or\}\}$ of two equivalent pairs of type $\{$*ordinary* 2-*covector*, *orientation*$\}$.

Hence new geometrical entities which are duly integrable over outer-oriented surfaces: indeed, to compute $\int_S \tilde{j}$, pick one of the pairs which compose $\tilde{j}$, the pair $\{j, Or\}$ say, use Or to derive an inner orientation of S from its given outer one, as Fig. 1 explains, then integrate the ordinary form j over S, thus inner oriented. Hence a real value. Taking the other pair $\{-j, -Or\}$ would give the same value, because of two changes of sign which cancel out, so the number $\int_S \tilde{j}$ is well defined by this recipe.

This is quite satisfying, for it corrects an imbalance which could be felt since we first characterized differential forms as "objects that can be integrated over inner-oriented manifolds". This left open the question "but then what about *outer*-oriented manifolds?", which we can answer now: It's twisted p-forms,

$$\tilde{\omega} = \{\{\omega, Or\}, \{-\omega, -Or\}\},$$

that can be integrated over outer-oriented p-manifolds. The integral of $\tilde{\omega}$ over the outer-oriented M is defined as $\int_M \omega$, with on M the inner orientation induced by its outer orientation in conjunction with Or.

In particular, when M is a domain of E_3, its outer orientation is a sign, + or $-$, and the inner orientation induced by Or is $+Or$ or $-Or$, depending on this sign. Hence $\int_M \tilde{\omega} = \pm \int_M \omega$, the outer orientation providing the sign.

Remark. An abstract manifold, though not necessarily orientable, can always be assigned an outer orientation (with respect to itself as its own "ambient space", so to speak), since this amounts to selecting a sign, + say. So, twisted forms directly defined over such a manifold can be integrated without ado. The weight of a Möbius band, for instance, is the integral of an appropriate 2-form, representing the paper's density. Another name for twisted forms of maximal degree is, appropriately, "densities". $\Diamond$

One may feel that the symmetry between ordinary forms and twisted ones is not as neat as the symmetry between inner- and outer-oriented manifolds, because of this cumbersome definition via classes.[1] But scales are balanced

[1] Earlier, we compared a p-covector with a machine with p numbered slots, in which one inserts vectors, v_1 in slot 1, v_2 in slot 2, etc., to get a real number in

again when one realizes that ordinary differential forms can be defined as equivalence classes, too: the form ω can be identified with the class $\{\{\tilde{\omega}, Or\}, \{-\tilde{\omega}, -Or\}\}$. One should therefore consider ordinary forms and twisted forms as objects of similar complexity.

Or should I say, as objects of similar *simplicity?* For we see now the possibility of defining differential forms, straight or twisted, in the most simple fashion, without having to introduce covectors first, as we did. Why not say that a (standard, twisted) p-form is just a *linear map* from (inner-oriented, outer-oriented) p-manifolds to real numbers? Only tradition, which nowadays reserves the label "differential form" to smooth fields of covectors, prevents us to do that.

Let me stress how easily this notion accounts for the notion of (physical) current density. For if we want a mathematical object that would describe electrical current, what do we need, *a minima*? Just to be able to tell which quantity of electric charge flows, per unit of time, across any given surface. This requires a machine of type *OUTER_ORIENTED_SURFACE* $\rightarrow$ *REAL*, a mapping that should obviously be linear. A twisted 2-form, in the sense of the last paragraph, fits this description. Nothing else, metric, orientation, or whatever, is needed, and the mathematical representation of the physical entity "current" could hardly be made simpler.

But let's end this digression. Next issue is the d of twisted forms and Stokes' theorem.

return. A twisted p-covector, in this spirit, is a *pair* of similar machines, set to yield opposite numbers when fed the same way, and which one is actually used would depend on ambient orientation. Thus described, straight and twisted covectors look like very dissimilar objects, indeed. But there is another metaphor for a p-covector: a machine with (1) a bin in which the vectors are thrown, unsorted, (2) a lever with which one selects an inner orientation of the p-dimensional subspace spanned by these vectors. Since inserting vectors in a definite order does point to an inner orientation (the one for which they form a positive frame), the two machines are easily seen to be equivalent. Now, the twisted p-covector machine has (1) a bin, of the same design, (2) a lever to select an *outer* orientation of the spanned subspace. The symmetry is thus restored. To give a concrete example, in dimension 3, a twisted 2-covector machine will accept two vectors (whose order doesn't count), plus a crossing direction of the parallelogram they form, and yield a number (think of an intensity, for instance). We'll return to this with Fig. 5, below.

4.3 The d, Stokes, and charge

Recall that an inner-oriented manifold M induces an orientation of its boundary ∂M (cf. Fig. 2). For shortness, we'll say that orientations of M and ∂M "match" when the orientation of ∂M is the one induced by M. Then, given a p-form ω, where p is the dimension of ∂M, one has $\int_M \mathrm{d}\omega = \int_{\partial M} \omega$. The d was defined, back in §3.4, in order to have this result, which holds whatever the orientation of M, provided the boundary's one does match.

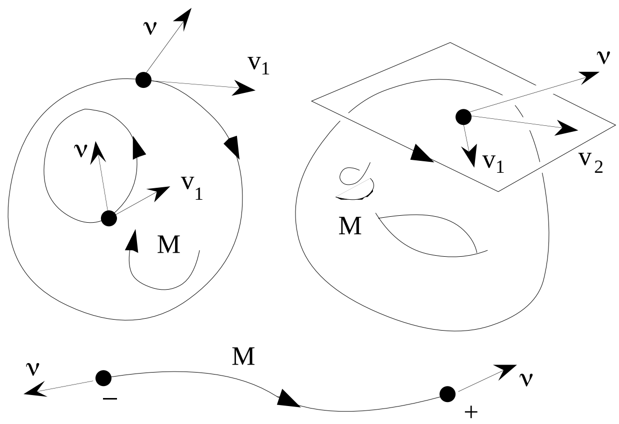

Figure 2. How the orientation of M induces one on ∂M. (M's dimension $p + 1$ is 2 on the left, 3 on the right, 1 at the bottom.) To know whether a frame of p vectors tangent at a point of ∂M is direct or skew, take a vector ν which points outwards with respect to M, list behind it the p vectors, and check whether the $(p + 1)$-frame $\{\nu, v_1, \ldots, v_p\}$ thus obtained is direct or skew with respect to the orientation of M. Note that ∂M may well be disconnected (left). Also notice the special icons (+ or −) for inner orientation of points.

Remark. As the caption of Fig. 2 explains, the matching rule relies on having chosen "inside to outside" as crossing direction through ∂M. This is the usual convention. Should one want to reverse it, there would be a change of sign in the Stokes theorem, then $\int_M \mathrm{d}\omega + \int_{\partial M} \omega = 0$. ♢

By definition, $\mathrm{d}\tilde{\omega}$ is the equivalence class $\{\{\mathrm{d}\omega, Or\}, \{-\mathrm{d}\omega, -Or\}\}$, which means, informally, "given a twisted form $\tilde{\omega}$, select a representative, whence a standard form to which one applies the d, and an orientation which one puts back, thus obtaining a representative of $\mathrm{d}\tilde{\omega}$".

This is engineered in such a way that Stokes' theorem be valid for twisted forms, too, when the outer orientations of M and ∂M match. We don't need a new convention to define this new kind of matching: Just select an orientation for ambient space, then outer orientations of M and ∂M induce inner orientations, which do match or don't. If they do, it's evidence that the given outer orientations did. Figure 3 explains this.

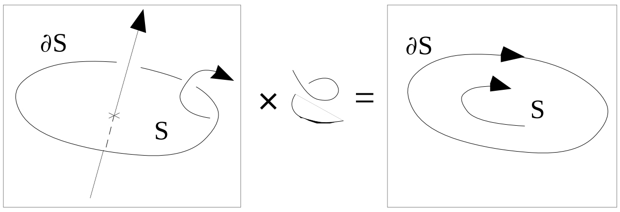

Figure 3. Matched outer orientations for a surface S and its boundary ∂S. To check that they do match, choose an ambient orientation, and derive from it inner orientations, which match, as one can see (cf. Fig. 2). (The symbol × may be understood as "compose the orientations on the left with" the icon that follows, which figures the orientation of ambient space.)

Outer-orienting a surface, therefore, induces a way to circle around its rim (not the same as turning around the surface along the rim!). Outer-orienting a 3D domain induces a crossing direction for (each of) its bounding surface(s). This outer orientation, being the inner orientation of a zero-dimensional vector space, is just a sign, plus or minus. If the sign is plus, the induced crossing direction is the conventional one (inside to outside). On the other hand, if we outer-orient the boundary from outside to inside, the outer orientation of D that matches this is the one with the minus sign. Please think all this over, because it's essential to the discussion we'll have in a moment about whether electric charge is, as physicists have it, "a pseudo-scalar or a true scalar" [Br]. Another useful exercise is to think about both kinds of orientation of the endpoints of a line (Fig. 4).

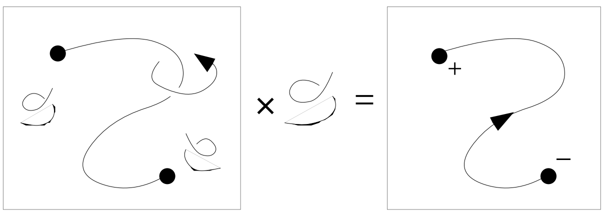

Figure 4. Matched outer orientations for a line and its endpoints (cf. Fig. 2). Outer orientation of a point is inner orientation of its 3D neighborhood, hence the icons attached to the endpoints.

Now, we may return to our equation and take a step forward. Since current density must be represented by the twisted form $\tilde{j}$, and knowing what we know about the Stokes theorem for twisted forms, charge conservation must be expressed by

(1) $$\partial_t \tilde{q} + \mathrm{d}\tilde{j} = 0,$$

where the charge density $\tilde{q}$ must be a *twisted* 3-form, for consistency. Is it all right? Yes, if we define the *charge inside* D as $\int_D \tilde{q}$, with + for outer orientation of D. Then the matching outer orientation of ∂D is from inside to outside, so $\int_{\partial D} \tilde{j}$ is the outgoing current, and the expected conservation law, $\partial_t \int_D \tilde{q} + \int_{\partial D} \tilde{j}$, does result from (1) by Stokes.

What we need to fathom, now, is the relation between this $\tilde{q}$ and the ordinary charge density q, understood as a function. Recall that, if one assumes a metric and an orientation (call it Or), a function q generates a 3-form ${}^3q$, but in an orientation-dependent way, so there are, for a given metric structure, *two* opposite straight 3-forms coming from q, one for Or, one for $-Or$. In contrast, there is only one twisted 3-form associated with q, which is $\tilde{q} = \{\{{}^3q, Or\}, \{-{}^3q, -Or\}\}$, and the other way around. So, given a metric, there is a well defined function q associated with $\tilde{q}$, the twisted 3-form that stands for charge density. This function, which does not depend on the orientation of space, and is therefore a "true scalar", is what we understand by electric charge density usually. Its Lebesgue integral over a domain D, which is the same as $\int_D {}^3q$ when D's inner orientation is Or, coincides with the integral of $\tilde{q}$ over D when D's outer orientation is +.

So the "scalar proxy" of a twisted 3-form is a function. An ordinary 3-form can also be represented by a function, but the sign of the latter depends on orientation. We know how to give status to such a "pseudo scalar", as physicists say in their confusing lingo: Define a *twisted function* $\tilde{f}$ as the class $\{\{f, Or\}, \{-f, -Or\}\}$ of pairs of type $\{$*ordinary function, orientation*$\}$. In metricized, but non-oriented 3-space, the scalar proxy of a straight 3-form is a twisted function.

Remark. Electric charge is div **D**. By analogy, magnetic charge, if such a thing existed,[2] would be div **B** (and we would have to add some "magnetic current", $-$**K** say, on the right-hand

[2] which seems unlikely, as early observations of such charges were disconfirmed later [GT].

side of Faraday's law to account for nonzero values of div **B**). Since **B** stands for a 2-form b, magnetic charge is the straight 3-form db (let's call this m), and its scalar proxy would be a twisted (or pseudo, or axial, ...) function, so magnetic charge is not akin to electric charge in this respect. Its conservation would be expressed by

$$\partial_t m + \mathrm{d}k = 0, \tag{2}$$

of which db = 0 is a consequence when $k = 0$ and $m = 0$ at $t = 0$. There is a theory which sees the origin of electric charge in small-scale topological twists of space [So], based on an argument which applies to twisted forms but not to straight ones. Maybe, as it has been speculated, this is why magnetic monopoles don't show up, in spite of "grand unification" theories that seem to require them [GT]. ◊

4.4 The Maxwell equations

Next issue is Ampère's theorem. To simplify the discussion a little, let's first ignore displacement currents, and address the equation rot **H** = **J**, which is closer to the practice of low-frequency electrical engineering anyway.

No choice here: This translates as $\mathrm{d}\tilde{h} = \tilde{\jmath}$, in which $\tilde{h}$ must be a twisted form, like $\tilde{\jmath}$, and of degree 1. A magnetomotive force, therefore, is the result of integrating on an outer-oriented line (let's be careful not to say "along" the line, which was all right for electromotive forces, but connotes inner orientation). Figure 3, left, well illustrates how the crossing direction for currents and the "way of turning around" the line must match for the integral version of the theorem, $\int_S \tilde{\jmath} = \int_{\partial S} \tilde{h}$, to hold.

Finally, let's introduce displacement current. Being alike total current, it must be represented by a twisted 2-form, denoted $\tilde{d}$. The complete version of Ampère's theorem, in local differential form, is then $-\partial_t \tilde{d} + \mathrm{d}\tilde{h} = \tilde{\jmath}$. Applying d to both sides, and integrating in time, we get $\mathrm{d}\tilde{d} = \tilde{q}$, the transcription of div **D** = q in the new language, and the expected expression for the electric charge in terms of the electric induction.

So now we know them all! All the fields appearing in Maxwell equations, as formulated in oriented Euclidean space, have been replaced by differential forms in an orderly way: **E** and **B** by straight forms e and b of degrees 1 and 2, **H** and **D** by twisted forms $\tilde{h}$ and $\tilde{d}$ of degrees 1 and 2, **J** and q by twisted forms $\tilde{\jmath}$ and $\tilde{q}$ of degrees 2 and 3. Magnetic current and charge would be, if they existed, a 2-form $-k$ and a 3-form m, both straight. It's just as easy to guess about the potentials one may be led to use: Vector potential **A**, similar to **E**, becomes the 1-form a, electric potential is the 0-form ψ, and all this fits well with the representation $e = -\partial_t a - \mathrm{d}\psi$. Magnetic potential is a twisted 0-form $\tilde{\varphi}$, such that $\tilde{h} = \mathrm{d}\tilde{\varphi}$ holds (locally, at least ...) in current-free regions. Outside of such regions, one can introduce an electric vector potential—the one denoted by **T** in the context of the so-called "**T**– method" [Ca], here the twisted 1-form $\tilde{\tau}$, such that $\tilde{h} = \tilde{\tau} + \mathrm{d}\tilde{\varphi}$.

Of all these objects, only the twisted 0-form $\tilde{\varphi}$ is new to us. It is, as usual, an equivalence class $\{\{\varphi, Or\}, \{-\varphi, -Or\}\}$, where φ is a straight 0-form. As 0-forms are in direct correspondence with functions (no metric needed), we see that the magnetic potential is an instance of "pseudo scalar", one of these functions that, mysteriously, "change sign with orientation". We are now in a position to take a global view of this kind of phenomena. For this, let's round up all the geometrical objects that contribute to the field's description, in each of the three systems of representation, and display the relevant equations again.

In naked affine space, we have, in order of increasing degree, the straight forms ψ, e and a, b and k, and magnetic charge m, on one side, and the twisted forms $\tilde{\varphi}$, $\tilde{h}$ and $\tilde{\tau}$, $\tilde{d}$ and $\tilde{\jmath}$, and electric charge $\tilde{q}$, on the other side. The basic equations are

$$-\partial_t \tilde{d} + \mathrm{d}\tilde{h} = \tilde{\jmath}, \tag{3}$$

$$\partial_t b + \mathrm{d}e = -k, \tag{4}$$

where k, always null for all we know, is just a false window for symmetry. Conservation equations (1) and (2) derive from that, if one sets $\tilde{q} = \mathrm{d}\tilde{d}$ and $m = \mathrm{d}b$. Lorentz force on a unit electric charge with velocity V is the covector $e - i_V b$. (The reader is invited to work out a formula for the force that a hypothetical magnetic monopole would feel.)

When space is equipped with a metric and an orientation, we have vector and scalar proxies for all these entities, in terms of which these equations rewrite as

$$(3') \qquad -\partial_t \mathbf{D} + \operatorname{rot} \mathbf{H} = \mathbf{J},$$

$$(4') \qquad \partial_t \mathbf{B} + \operatorname{rot} \mathbf{E} = -\mathbf{K},$$

and conservation equations as

$$(1') \qquad \partial_t q + \operatorname{div} \mathbf{J} = 0,$$

$$(2') \qquad \partial_t m + \operatorname{div} \mathbf{K} = 0.$$

The Lorentz force has $\mathbf{E} + V \times \mathbf{B}$ for vector proxy. If one chooses to work with the opposite orientation, one will describe the same physics by changing the signs of **B**, **H** (and hence, of **T** and φ, if they are used), **K**, and m, because rot and $\times$, both sensitive to orientation, "change sign" in the process, too.

If we keep the metric and shun the orientation, the equations become

$$(3'') \qquad -\partial_t \mathbf{D} + \operatorname{rot} \tilde{\mathbf{H}} = \mathbf{J},$$

$$(4'') \qquad \partial_t \tilde{\mathbf{B}} + \widetilde{\operatorname{rot}}\, \mathbf{E} = -\tilde{\mathbf{K}},$$

$$(1'') \qquad \partial_t q + \operatorname{div} \mathbf{J} = 0,$$

$$(2'') \qquad \partial_t \tilde{m} + \operatorname{div} \tilde{\mathbf{K}} = 0,$$

where **E**, **D**, **J** are vector fields, $\tilde{\mathbf{H}}$, $\tilde{\mathbf{B}}$, $\tilde{\mathbf{K}}$, twisted vector fields. Electric charge density q is here a function, magnetic charge density $\tilde{m}$ a twisted function (alike magnetic potential $\tilde{\varphi}$). The "twisted curl" $\widetilde{\operatorname{rot}}$ sends vectors to twisted vectors, while rot (which is thus subtly different from the rot of (3′)) does the converse. The Lorentz force is the "polar" vector $\mathbf{E} + V \tilde{\times} \tilde{\mathbf{B}}$, where $\tilde{\times}$ denotes this orientation-independent cross product that was described in Fig. 2 of last installment.

As one sees, there is a clear correspondence between the two latter systems (equations numbered as (n') and (n'')): things denoted with a tilde in the orientation-free framework, that is, twisted (or axial, etc.) vectors and scalars, correspond to those fields in the standard framework (oriented Euclidean space) whose sign must be changed when one decides to change orientation.

Unfortunately, rules about the correspondence between equations in (n') and equations in (n), those of the metric-free system, are not so clearcut. In particular, and because of the vagaries of the curl operator, it is not true that proxies for twisted forms are, systematically, twisted vectors or scalars. The proxies of ψ and $\tilde{q}$ are scalar fields, while those of m and $\tilde{\varphi}$ are twisted scalar fields. It doesn't mean that there is no simple rule (there is one, as will be obvious on Fig. 6), but the tildes are no reliable mnemonics in this respect. I readily concede that calling "twisted" all geometrical objects that carry orientation with them may not have been such a good idea in the first place. Perhaps one should speak of twisted and straight *forms*, and of axial and polar *fields*? No accepted terminology has yet emerged, although one would think that the available vocabulary (odd vs even, axial vs polar, twisted vs plain or straight, pseudo versus true, and so forth) is rich enough for the needs of a rational taxonomy of our zoo of geometrical objects. Meanwhile, be wary of authors of papers or books who, misled by their own terminology, may misclassify some electromagnetic entities [BH]—and exert such caution against the present writer, too.

4.5 The Hodge operator

Let's now tackle constitutive laws. In the non-oriented Euclidean framework, we have

$$(5'') \qquad \tilde{\mathbf{B}} = \mu \tilde{\mathbf{H}}, \qquad \mathbf{D} = \epsilon \mathbf{E}.$$

Ohm's law would be $\mathbf{J} = \sigma \mathbf{E}$. These are relations between objects of the same type—which is at it should be, since μ, ϵ, and σ are scalar entities.

Or are they? Let's not go too far and characterize such coefficients as scalar *invariants*, which they are not: they have dimension, they change value when the metric is changed, and besides, there is such a thing as anisotropy. So maybe the real nature of these parameters is still hidden by the formalism.

So let's see what their status can be in the minimal framework of differential forms—and it looks like a mess: We certainly *can't* write $b = \mu \tilde{h}$ and $\tilde{d} = \epsilon e$, or $\tilde{\jmath} = \sigma e$, because that would be trying to establish a proportionality relationship *between objects of different types*—different on two counts: forms of unequal degrees, which differ in orientation status (straight and twisted). So μ, ϵ and σ cannot be scalar multipliers, even if endowed with physical dimension. They have to be *operators*, linking objects of different types. It happens that classical differential geometry has such an operator in store, that will prove perfectly apt to the task.

This so-called *Hodge operator*, or *star* operator in some countries, is a linear machine which maps p-covectors of one kind (twisted or straight) to $(n-p)$-covectors of the other kind, where n is the dimension of the underlying vector space. In affine space, a smooth field of similar machines, one at each point, will therefore map p-forms of one kind to $(n-p)$-forms of the other kind, which seems to be exactly what we need.

So let V_n be a real vector space of dimension n, endowed with a dot product "$\cdot$" and an orientation Or. Let a p-covector ω be given. We denote by $\star\omega$ the $(n-p)$-covector such that, if the family of vectors $\{v_1, v_2, \ldots, v_n\}$ makes a *direct orthonormal frame*, then

(6) $$\star\omega(v_{p+1}, \ldots, v_n) = \omega(v_1, \ldots, v_p).$$

This may look preposterous: does (6) really *define* a covector? Shouldn't we expect a formula that would give us the value $\star\omega(u_1, u_2, \ldots, u_{n-p})$ for *any* list of $n-p$ vectors? Such formulas can be given, but are not very instructive. Neither are they useful, for no actual computation is required. All we need is to make sure that $\star$ is well defined, and the above rule happens to be enough for that, thanks to the linearity of covectors with respect to their arguments and their alternation property (permute two factors, change the sign). Starting from a list of vectors $\{u_1, u_2, \ldots, u_{n-p}\}$, we may always apply the Gram–Schmidt orthogonalization method[3] to build a system of vectors all of length 1 and two-by-two orthogonal, and thus obtain a determinant, the value λ of which[4] we store. Call $\{v_{p+1}, v_{p+2}, \ldots, v_n\}$ the orthonormal system thus obtained. In the orthocomplement of the subspace it spans, pick p vectors $\{v_1, v_2, \ldots, v_p\}$ of length 1, orthogonal two-by-two, in such a way that the full list form a direct frame. Then $\star\omega(u_1, u_2, \ldots, u_{n-p}) = \lambda\omega(v_1, \ldots, v_p)$. The last objection, "but we could have selected a different system $\{v'_1, v'_2, \ldots, v'_p\}$," is countered by noting (same argument as in Note 5) that the determinant of $\{v'_1, v'_2, \ldots, v'_p\}$ with respect to the basis $\{v_1, v_2, \ldots, v_p\}$ would then be 1, by the rules. Note that $\star$ is a linear operator, in an obvious way.

Next, observe that not only metric played a role there, but orientation too, because of the stipulation that the v_i's should make a *direct* frame. Had we taken $-Or$ as orientation, the operator defined by (6) would have been the opposite. As we see, the star operator behaves very much like curl and the cross product, in this respect. Having already obtained, with (1″)–(5″), an expression of Maxwell's equations which is manifestly orientation-free, we can be sure that such an orientation-sensitive operator is not the right tool. But we also know how to fix it: define $\tilde{\star}$ by

(7) $$\tilde{\star}\omega = \{\{\star\omega, Or\}, \{-\star\omega, -Or\}\}.$$

This maps p-covectors to twisted $(n-p)$-covectors, indeed. Finally, if a twisted p-covector $\tilde{\omega}$ is given, we select a representative $\{\omega, Or\}$ of $\tilde{\omega}$, apply to ω the $\star$ *as defined thanks to* Or (this is of course the key point), and $\tilde{\star}\tilde{\omega}$ is what results, a straight $(n-p)$-covector. Note that $\tilde{\star}$ is its own inverse, up to sign: one has $\tilde{\star}\tilde{\star}\omega = \pm\omega$, the sign depending on p and n.

This operator is the device by which we shall link b and $\tilde{h}$, $\tilde{d}$ and e, etc., like this:

(5) $$b = \mu\,\tilde{\star}\,\tilde{h}, \qquad \tilde{d} = \epsilon\,\tilde{\star}\,e.$$

Let's show it by examining Ohm's law, which I claim is well expressed by $\tilde{\jmath} = \sigma\,\tilde{\star}\,e$.

To do this, select a point x, understood in what follows. Let e and $\tilde{\jmath}$ be the electric field and current density in the vicinity of x. Take three vectors $\{v_1, v_2, v_3\}$ at x, all of length one, mutually orthogonal, select as orientation the one which turns them into a direct frame (Fig. 5), and let j be the 2-covector at x that represents $\tilde{\jmath}$ for this orientation. Imagine a cube of metal of conductivity σ built on the three vectors. The intensity across the bottom of the cube is $I = j(v_2, v_3)$, by the very interpretation of the 2-covector j as current density. (Note that the chosen crossing direction does orient the cube's basis in such a way that $\{v_2, v_3\}$ is direct for the induced inner orientation.) The voltage drop from bottom to top is $V = e(v_1)$. And the resistance V/I of this unit cube is $1/\sigma$, so we have $j(v_2, v_3) = \sigma e(v_1)$. Compare this

[3] unless the u_i's are not independent—but then, the value of $\star\omega$ for such a list must be 0.

[4] Note that λ does not depend on which way the Gram–Schmidt procedure is performed.

with (6): it amounts to saying that $j = \sigma \star e$. Removing the orientation scaffolding, we get

$$\tilde{\jmath} = \sigma \tilde{\star} e, \tag{8}$$

as promised.

So we are through, at last: Equations (1)–(5), plus Ohm's law (8) if needed, give a full description of electromagnetism (for linear materials and non-moving bodies). Orientation of space has been discarded entirely. As for metric, it's only at the level of constitutive laws that it has been invoked.

Is metric *necessary* at this level, or could we perhaps strip the framework even further? Apart from merging space and time, which is feasible and would lead us to the relativistic formalism of box (c) of Fig. 1 in the first paper of the series, there is little room left for such further improvement. For if we try to discard metric, the constitutive laws are still there and must somehow be described. We might directly introduce a Hodge operator as an affine entity, that would linearly map p-covectors to $(n-p)$-covectors. But then it can be proved [B3] that, as soon as we have such an operator in the case $p = 1$, it is the Hodge associated with some metric: *Hodge implies metric*.

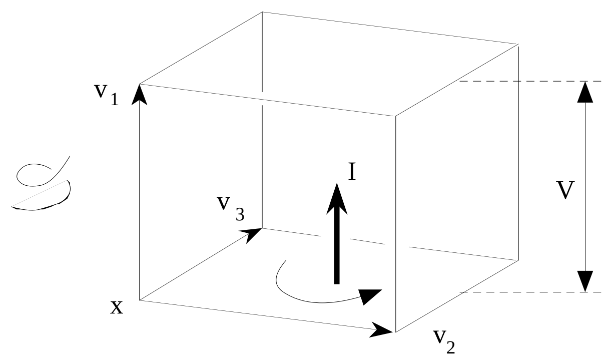

Figure 5. How $\tilde{\jmath} = \sigma \tilde{\star} e$ expresses Ohm's law (see text).

Still, since at least two such operators would be needed, one for μ, one for ϵ, the question would arise whether there is some *common* metric in which they would take the forms $\mu \tilde{\star}$ and $\epsilon \tilde{\star}$. When this is so, we say that the material is *isotropic*, though not necessarily homogeneous since μ and ϵ may depend on position. There is no space left here to address such issues, but what precedes hints at the potential usefulness of what we have been doing: Questions such as "what do we mean by isotropy, exactly?" and other similar ones related to *material symmetries*, as distinct from the symmetries *of the equations*, and to what remains of such symmetries when the material is strained, do benefit from this dissection of structures. It's also useful is the investigation of *forces*, as suggested in [B2].

4.6 A synoptic conclusion

Let's gather all our findings in graphic form (Fig. 6). Since all relevant objects are differential forms of degrees 0 to 3, straight or twisted, and since time derivatives and, occasionally (cf. the example of $\tilde{\tau}$ on Fig. 6), primitives in time may have to be considered, we can group them in four similar categories, shown as vertical pillars on Fig. 6. Each pillar symbolizes the structure made by spaces of forms of all degrees, linked together by the d operator. Straight forms are on the left and twisted forms on the right. Differentiation or integration with respect to time links each pair of pillars (the front pillar and the back pillar) forming the sides of the building. Horizontal beams symbolize constitutive laws.

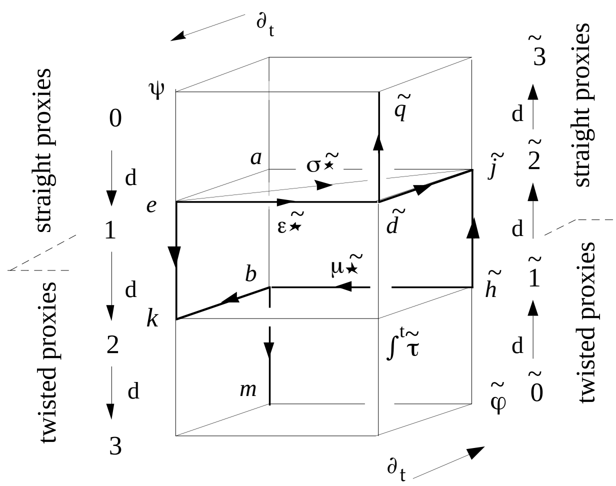

Figure 6. "Maxwell's house."

As one can see, each object has its own room in the building: b, a 2-form, at level 2 of the "straight" side, the 1-form a such that $b = \mathrm{d}a$ just above it, etc. Occasional asymmetries (e.g., the necessity to time-integrate $\tilde{\tau}$ before lodging it, the bizarre layout of Ohm's law ...) point to weaknesses which are less those of the diagram than those of the received nomenclature or (more ominously) to the incompatibility of Ohm's law with Einsteinian relativity. Most things mentioned up to now can be directly read off from the diagram, up to sporadic sign inversions. An equation such as $\partial_t b + \mathrm{d}e = -k$, for instance, is obtained by gathering at the location of k the contributions of all adjacent niches, including k's, in the direction of the arrows. Note how the rule about which scalar- or vector-proxies must be twisted or straight is now apparent.

But the most important thing is probably the neat separation, in the diagram, between "vertical" relations, of purely affine character, and "horizontal" ones, which depend on metric. If this was not drawing too much on the metaphor, one could say that a change of metric (due for instance to a change in the local values of μ, σ, etc., because of a temperature modification or whatever) would shake the building horizontally but leave the vertical panels unscathed.

This points to a methodology for *discretizing* the Maxwell equations: The orderly structure of Fig. 6 should be preserved, if at all possible, in numerical simulations. Hence the search for finite elements *which fit differential forms*, and thus would allow to build a similar "discrete" structure. This search is not over, in spite of the existence of differential-geometric objects (Whitney forms, see e.g., [B1]) which are remarkably efficient as finite elements for forms, because the simultaneous discretization of straight *and* twisted forms, on the same mesh, and the concomitant construction of discrete Hodge operators, is still an open field of inquiry.

So maybe we'll have more to say about such things in future columns.

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