# (3): Integration, Stokes, Faraday's law

Where do we stand? We have identified *three* systems of mathematical entities, closely related but distinct, by which the physical electromagnetic field can be represented: (1) The pair of vector fields **E** and **B**, (2) The pair of differential forms $e$ and $b$, (3) The pair consisting of **E** plus the *axial* vector field $\tilde{\mathbf{B}}$—the definition of which will briefly be recalled. Now, we show how Faraday's law is expressed in these systems.

### 3.1 Three systems of representation

In the previous chapters, we have found a minimalist description of the EM field. With the barest equipment in background (affine 3D space), it was possible to display geometric objects which tell about forces felt by electric charges: a field $x \to e_x$ of covectors, denoted $e$, which gives the Coulomb force, and a field $x \to b_x$ of 2-covectors, denoted $b$, which gives the part of the Lorentz force proportional to the particle's velocity. (We'll say the "dynamic part" of the Lorentz force, for shortness, the "static part" being of course Coulomb force.)

Since the EM field was defined as "what pushes loose electric charges", the concept of electric charge being taken as a given, the above two geometric objects (known as "differential forms", the 1-form $e$ and the 2-form $b$, both time-dependent) contain all the relevant information about the field. Neither the orientation of space nor its metric played any role in it. Of course, one may have to introduce them later, in order to deal with other aspects of physics. But up to now, and to say it briefly, we have been able to describe the electromagnetic field via *affine* objects.[1]

As suggested last time, it's like having at each point of space a machine with two slots and two dials. Insert a vector $v$ in the first slot, and the first dial displays the (virtual) work yielded by the field in the virtual displacement $v$ of a particle of charge unity standing at this point, hence the "static" force that a passing charged particle will feel. This is the electric side of the machine, so to speak. It has a magnetic side, too: To exploit it, insert the actual speed $V$ of such a particle in the second slot, and read off the corresponding virtual work (still with respect to $v$), on the second dial. If we agree to denote by $i_u b$ a covector[2] of the form $v \to b(u, v)$, the "dynamic" force is therefore the covector $-i_V b$, and the Lorentz force is the covector $e - i_V b$.

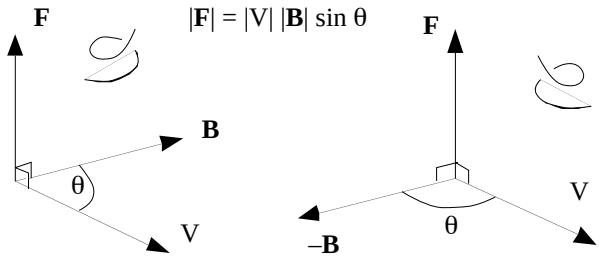


**Figure 1.** Construction of the vector proxy for the "dynamic" force, here denoted **F**, from the velocity vector $V$ and the magnetic-induction vector proxy **B**. Metric information (lengths of $V$ and **B**, and the angle they form) is enough to know the length of **F** and its supporting line, but an orientation is needed to know which way **F** points. If one decides to change the orientation of space (right side of the figure), one must represent the same magnetic induction by $-\mathbf{B}$ instead of **B**, since **F** doesn't change. So, for a given metric, the same 2-form $b$ has *two* vector proxies, **B** and $-\mathbf{B}$, one for each orientation of ambient space.

The vector formalism replaces these pretty complex machines by simple pairs of vectors: **E** and **B**, one pair at each point. The relation with $e$ and $b$ is given by $e = {}^1\mathbf{E}$ and $b = {}^2\mathbf{B}$, in last installment's notation.[3] Now the above virtual work is $(\mathbf{E} + V \times \mathbf{B}) \cdot v$, which

[1] One can do with even *less* than an affine structure: Since $p$-covectors make sense at each point of a differentiable manifold (they act on tangent vectors), differential forms make sense too. But the generality thus gained would not compensate for the increased conceptual difficulty. Differential manifolds are apt to model our intuitive notion of "three-dimensional continuum" that ambient space seems to have, at least at our scale. Some physicists currently speculate about the possibility of not even assuming such continuity [Bw].

[2] This is called the *inner product* or *contraction* of the 2-covector $b$ by the vector $u$. We'll use it later.

[3] For the reader's convenience, let's recall this notation: given a vector $u$, ${}^1u$ denotes the covector $v \to u \cdot v$ and ${}^2u$ the 2-covector $\{v, w\} \to u \cdot (v \times w)$. The metric is involved in both mappings. Moreover, since the cross product $\times$ is orientation-dependent, the correspondence

is another way of saying that the Lorentz force is the covector $^1$(**E** + $V \times$ **B**), the same covector as $e - i_V b$. Owing to long familiarity with the expression **E** + $V \times$ **B**, one may consider the {**E**–**B**}-machines as simpler than the $\{e$–$b\}$-ones, but this is an illusion. The {**E**–**B**}-machines cannot work without a whole infrastructure in background: the dot product and the orientation, both necessary to give sense to the cross product operation (see Fig. 1). The $\{e$–$b\}$-machines, more basic, can work in autonomy. It's in that sense that our description deserved to be called "minimal".

From this point of view, the "{**E**–$\tilde{\mathbf{B}}$}-machines" stand in between, for this mode of representation necessitates only a metric in background, no orientation (cf. Fig. 2). At each point, there is a regular (or "polar") vector **E** and a "twisted" (or "axial") vector $\tilde{\mathbf{B}}$. The force vector **F** is of course the same, but is obtained by a slightly different rule, a kind of "orientation-free" variant of the cross product, which Fig. 2 displays.

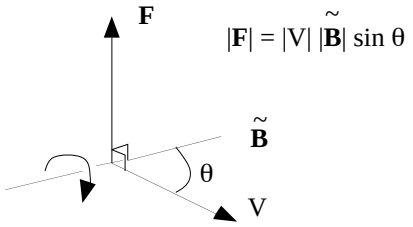


**Figure 2.** Construction of the dynamic-force vector **F** from the velocity vector $V$ and the magnetic-induction *twisted*-vector proxy $\tilde{\mathbf{B}}$. This time, orientation of ambient space is irrelevant: the outer orientation of the support of $\tilde{\mathbf{B}}$, which $\tilde{\mathbf{B}}$, as a twisted vector, brings with it, suffices to show which way **F** must point: so that turning it 90° around $\tilde{\mathbf{B}}$ will bring it in the same plane as $V$ and $\tilde{\mathbf{B}}$.

Is the simplification thus brought in by the use of an axial vector worth the trouble? The issue will be more clearcut after we have compared the expressions of Faraday's law within each of the three systems. For this, we need to know how to integrate differential forms first.

**Exercise.** Both figures 1 and 2 can be read as geometric constructions of the icon of the Lorentz-force vector proxy, assuming a metric in background. Similarly, it should be possible to build the Lorentz-force *co*-vector's icon from those of $V$ and of the 2-covector $b$ (see last issue for iconic conventions). Can you propose such a construction? (You may find it a rather involved and cumbersome recipe. Decide by yourself, however, to which extent this judgment is biased by long-time familiarity with vectors' manipulation.)

between $u$ and $^2u$ also depends on orientation. (Exercise: Check that $i_v{}^2u = -^1(v \times u)$.)

### 3.2 Integration of forms

Integrating differential forms is a simple matter: they are, as we presently see, geometrical objects which are *meant* to be integrated. More precisely, $p$-forms have canonically defined integrals over $p$-dimensional *inner* oriented manifolds. Cases $p$ = 1 and 2, that is, lines and surfaces, are the ones of interest for us.

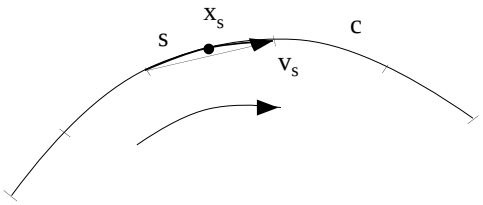


**Figure 3.** Subdividing an oriented curve $c$ in order to set up a Riemann sum such as (1); shown, a generic "curved segment" $s$, and the associated point and vector.

Suppose we are given the electric field as a smooth 1-form $e$ and are being asked to find the corresponding electromotive force along an oriented smooth curve $c$ (Fig. 3). Let's chop the curve into a finite family $\mathcal{S}$ of adjacent curve-segments and pick a point $x_s$ in each of them. To each such short curve $s$, we may associate a vector $v_s$ which joins its endpoints; we make this vector point in the direction indicated by the curve's orientation. Let the covector $e(x_s)$ at point $x_s$ act on $v_s$, yielding the real number $\langle e(x_s), v_s \rangle$. Summing up these numbers, we obtain the following "Riemann sum"

$$(1) \qquad I(c; \mathcal{S}, e) = \sum_{s \in \mathcal{S}} \langle e(x_s), v_s \rangle .$$

Now, by definition, the integral of $e$ over $c$ is the limit

$$\int_c e = \lim_{\mathcal{S} \to \infty} I(c; \mathcal{S}, e),$$

where "$\mathcal{S} \to \infty$" means that the number of curved segments tends to infinity as the subdivision is repeatedly refined.

We shall not attempt to make this "$\mathcal{S} \to \infty$" notion more precise. This would raise a series of technical points (for instance, make sure that no finite part of the curve escapes subdivision),

and thus hide the main ideas. Moreover, there is no big difference, in how one would tackle the passage to the limit, with respect to the Riemann integration theory for functions, to which the reader has probably been exposed once (and once is enough ...).

What does make an important difference, on the other hand, is the absence here of any reference to the *lengths* of the vectors $v_s$. Such lengths are not defined anyway, since we did not assume a metric on the ambient space. Had we done that, there would be a notion of length of lines, of area of surfaces, of volume of 3D regions: In one word, a *measure*[4] (in the sense of classical measure theory [Ha]), induced by the metric, on manifolds of any dimension. Here, there is no need for such a measure: the differential form $e$ carries with itself, so to speak, all it needs to be integrated over an oriented line.

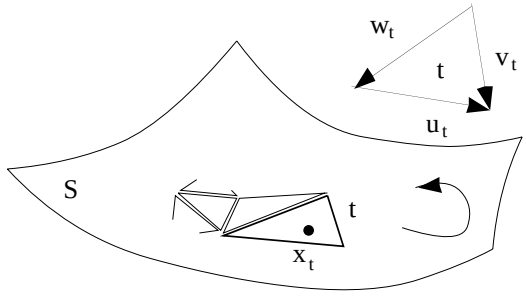


**Figure 4.** Subdividing an inner oriented surface $S$ into triangular patches. A generic triangle $t$, and its assigned point $x_t$ in $S$. Top right: three vectors associated, in an arbitrary way, with $t$'s boundary.

*Oriented* is the keyword, as we shall see more clearly in dimension 2. Now, it's a 2-form $b$ that we want to integrate over a surface $S$ (Fig. 4), and the idea is to approximate $S$ by a finite family $\mathcal{T}$ of tiny adjacent triangles, each with an associated surface point $x_t$ and boundary edge vectors $u_t, v_t, w_t$. These have arbitrary orientation, but the inner orientation of $S$ intervenes in the way we now build each term of the Riemann sum: the rule is to select any two vectors out of three, to *order them* so that they make a *positive frame* (this point is essential), and to let the covector $b(x_t)$ act on them. For instance, in the case of Fig. 4, a number which complies with this prescription is $\langle b(x_t); v_t, u_t\rangle/2$ (the 2 is a dimensional factor), and one easily sees that the same number would pop up if we had chosen vector-pairs $\{w_t, u_t\}$ or $\{w_t, v_t\}$, thanks to the properties of $b$ as a bilinear alternating mapping. This number is thus a characteristic of the oriented triangle $t$, so we shall abbreviate it as $\langle b(x_t); t\rangle$. The rest is straightforward: form the Riemann sum

$$I(S; \mathcal{T}, b) = \sum_{t\in\mathcal{T}} \langle b(x_t), t\rangle,$$

and set $\int_S b = \lim_{\mathcal{T}\to\infty} I(S; \mathcal{T}, b)$.

Generalizing to a connected[5] $p$-manifold $M$ is easy: To get the integral $\int_M \omega$, divide $M$ into a family $\mathcal{S}$ of $p$-simplices, pick a point $x_s$ in each of them, build vectors along the edges of $s$, select any $p$ of them which form a *direct* frame, let $\omega(x_s)$ act on them, and divide by $p!$ to get $\langle \omega(x_s), s\rangle$. Then $\int_M \omega = \lim_{\mathcal{S}\to\infty} \sum_{s\in\mathcal{S}} \langle \omega(x_s), s\rangle$, as above.

The reader may want to work out the case $p = 3$, in 3D affine space. When $p = 0$, of course, the integral of a 0-form $f$ over a 0-dimensional connected manifold, that is to say, a single point[6] $x$, is $f(x)$.

**Remark 1.** One should not confuse the integral of a 0-form, just defined, with the integral of the function $f$, in the common acception of the word, over a line, a surface, etc. This doesn't make sense in the present context, where there is no underlying measure. What may make sense, on the other hand, is the integration of some differential form *generated by* the function. We return to this point in a moment. ♢

### 3.3 "Old" vs "new" integration

We need to review the relationships between this new concept of integration and the standard one when dealing with differential forms in oriented Euclidean 3D space. From last installment (cf. Note 3), we know four kinds of such objects: the 0-form and 3-form ${}^0f$ and ${}^3f$

[4] Called the *Lebesgue measure*.

[5] No essential restriction there: The integrals can be evaluated over each connected component, then summed up. Note however that orientations could be chosen independently for each connected component of $M$.

[6] Which we assume, by convention, to be positively oriented. A point *can* be oriented: this amounts to give it a sign, + 1 or −1. But having no need for it, we don't consider this possibility. If however $x$ was negatively oriented, the "integral" of ${}^0f$ would be $-f(x)$, by convention.

generated by a function $f$, and the 1-form and 2-form ${}^1u$ and ${}^2u$ generated by a vector field $u$.

We just saw what the integral of ${}^0f$ is. What comes next is the integral of ${}^1u$ on a smooth oriented curve $c$. Let's introduce a field $x \to \tau(x)$ of vectors tangent to $c$, all of length 1, all pointing in the direction of positive orientation. Terms of the Riemann sum (1) are of the form $\langle {}^1u(x_s), v_s\rangle = u(x_s)\cdot v_s \simeq u(x_s)\cdot\tau|v_s|$. Since the norm $|v_s|$ is also the Lebesgue measure of the segment $s$, this sum can be rewritten as $\sum_{s\in\mathcal{S}} \tau\cdot u(x_s)|v_s|$, which can be viewed as a Riemann sum for the classical integral of the *function* $x\to\tau\cdot u(x)$, on $c$, with respect to the metric-induced measure. Hence, at the limit,

$$\int_c {}^1u = \int_c \tau\cdot u, \tag{2}$$

where the second integral is a classical one (again with the Lebesgue measure understood), called the *circulation* of $u$ along the oriented curve c.

It's an easy guess that, similarly

$$\int_S {}^2u = \int_S n\cdot u, \tag{3}$$

the *flux* of $u$ through $S$. The field $x\to n(x)$ is made of vectors of length 1, orthogonal to $S$, and all oriented the same way. *Which* way? This is told by the orientation we had on $E_3$ in the first place: the crossing direction is the one which makes the intrinsic orientation of $S$ and the ambient orientation "match", in the sense of the Ampère observer's rule. One may feel surprised that $S$ needs an *inner* orientation at the left-hand side of (3), and (apparently) an *outer* orientation at the right-hand side. But this outer orientation is only meant to induce an inner one, in conjunction with the ambient orientation on $E_3$. And since the way $u$ generates ${}^2u$ also depends on ambient orientation, the two effects of the latter cancel out. *No crossing direction is really involved in* (3), which makes the "flux" terminology a bit misleading.

**Remark 2.** So we may anticipate that when we really need to deal with a specific crossing direction, as will be the case for intensities, the integral of a 2-form will not be the adequate concept any longer. ♢

**Remark 3.** Would the flux of an *axial* vector[7] make sense? It does, over an *inner* oriented surface, even if the ambient Euclidean space is not oriented itself. Select one of the two representatives of the twisted vector, $\{u, Or\}$ say, use $Or$ to determine which way $n$ should point, then $\int_S n\cdot u$ is the flux. Again, we see the advantage of using twisted vectors: no need to specify an ambient orientation that will, anyway, be irrelevant to the final result (the value of the integral). But this is a tiny advantage: we still need a metric from which to build a Lebesgue measure, even though this measure is irrelevant to the final result. ♢

To complete the series, the integral of ${}^3f$ over an oriented three-dimensional domain $D$ is $\pm\int_D f$, the integral in the standard sense of the function $f$, with the Lebesgue measure of volumes understood. The sign depends on whether the orientations of $D$ and $E_3$ match or not.

There are other possibilities to build differential forms from functions and fields, but they always reduce to the above four ones in some way. For instance, what is often denoted $\mathrm{d}x$ is the one-form $v\to v^x$, where $v^x$ is the $x$-component of vector $v$ in an $x$–$y$–$z$ system of local coordinates. The integral $\int_c \mathrm{d}x$ over a curve $c$ is simply the circulation along $c$ of the field of $x$-directed basis vectors.

Conversely, one could argue that any integral actually concerns some differential form: whatever one finds under a summation sign is, in some way, a form. For instance, the "double integral" $\iint_S f(x,y)\,\mathrm{d}x\,\mathrm{d}y$ of classical calculus is the integral of a 2-form, here denoted $f(x,y)\,\mathrm{d}x\,\mathrm{d}y$ (and sometimes, in a way which helps reinforce this interpretation, $f(x,y)\,\mathrm{d}x\wedge\mathrm{d}y$), built from $f$ and from the coordinate-related 1-forms $\mathrm{d}x$ and $\mathrm{d}y$. (By definition, $\langle \mathrm{d}x\wedge\mathrm{d}y\,;v,w\rangle = v^x w^y - v^y w^x$. This is called the "wedge product" of $\mathrm{d}x$ and $\mathrm{d}y$.)

### 3.4 The Stokes theorem

Knowing how to integrate forms, one may wish to be able to differentiate them, too.

Start from a system $\{v_1, ..., v_{p+1}\}$ of $p$+1 vectors at a point $x$. They form a $(p+1)$-simplex $s$, say a triangle in the case $p=1$, to which the order in which the vectors have been presented confers an inner orientation (Fig. 5). The boundary $\partial s$ of this simplex has an outer orientation, as we remarked earlier: "from inside to outside" defines a crossing direction. Using $s$

[7] Recall that I take "axial" and "twisted" as synonyms, but beware: other authors may use different conventions.

itself as "ambient space" in which $\partial s$ is embedded, we derive from that a canonical inner orientation for all *faces* of $s$, i.e., the $p$-simplices that together constitute $\partial s$. So we may integrate the given $p$-form, $\omega$ say, over $\partial s$, hence a number $\int_{\partial s} \omega$.

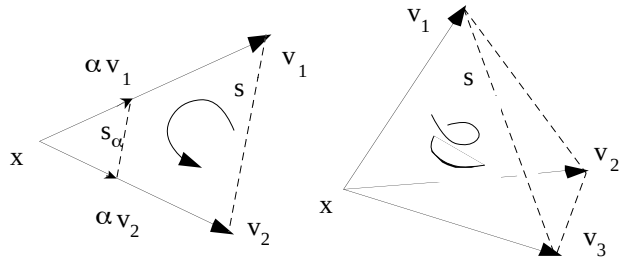


**Figure 5.** How $p + 1$ vectors, given in an ordered sequence, generate an inner oriented ($p$+1)-simplex. Left: construction of a homothetic triangle $s_\alpha$.

Now let's define a new covector at $x$, denoted $\mathrm{d}\omega$, by the following limit process:

$$\mathrm{d}\omega(v_1, \ldots, v_{p+1}) = \lim_{\alpha \to 0} \frac{(p+1)!}{\alpha^{p+1}} \int_{\partial s_\alpha} \omega,$$

where $s_\alpha$ is homothetic to $s$ as suggested by Fig. 5, and thus "shrinks to $x$" as $\alpha$ goes to 0. The new field $\mathrm{d}\omega$ of $(p+1)$-covectors thus obtained is called the *exterior differential* of the given $\omega$.

This way, we have the near-equality $\int_{\partial s} \omega \simeq \int_s \mathrm{d}\omega$ for a small enough simplex $s$. Summing up over all simplices of a subdivision $\mathcal{S}$ of some manifold $M$, we get

$$\int_M \mathrm{d}\omega \simeq \sum_{s \in \mathcal{S}} \int_s \mathrm{d}\omega \simeq \sum_{s \in \mathcal{S}} \int_{\partial s} \omega \simeq \int_{\partial M} \omega,$$

where the first two near-equalities stem from things we already know to be true: The first one comes from approximating $\int_M \mathrm{d}\omega$ by a Riemann sum, the second one from the very definition of d. If the third one can be justified, we shall have proven the *Stokes theorem*,

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega, \tag{4}$$

where $\omega$ is a $p$-form and $M$ an oriented $(p+1)$-manifold. All it takes is passing to the limit as $\mathcal{S} \to \infty$.

**Remark 4.** Equation (4) shows that d and $\partial$ are *adjoint* to each other, in some way. Note also this: since $\partial\partial M$ is empty, one has dd = 0 as a corollary. $\diamondsuit$

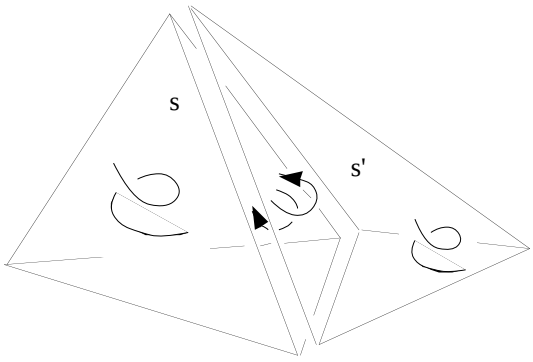


**Figure 6.** Because the orientation of $M$, whatever it is, induces compatible orientations on $s$ and $s'$, the two $p$-faces supported by the common triangle, which have opposite *outer* orientations, have opposite *inner* orientations, too.

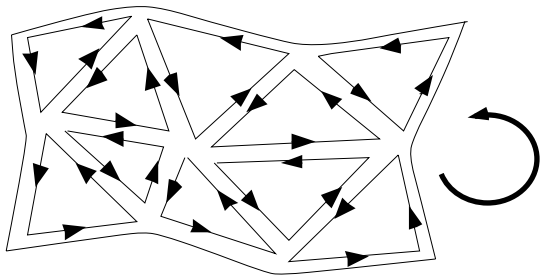

**Figure 7.** The same argument, in the case $p = 1$. Contributions of inner edges to the Riemann sum will cancel out.

The existence of an inner orientation of $M$, again, is the essential ingredient in this proof. First, because it gives sense to the first integral: if $M$ is oriented, the outer orientation of its boundary (from inside to outside) inner-orients $\partial M$, as we already argued. Second, because of the cancellation two by two of the integrals $\int_\sigma \omega$ at the right-hand side of the equality

$$\sum_{s \in \mathcal{S}} \int_{\partial s} \omega = \sum_{s \in \mathcal{S}} \sum_{\sigma \in \partial s} \int_\sigma \omega, \tag{5}$$

where $\sigma$ spans the set of faces of all simplices $s$ of the subdivision $\mathcal{S}$. A face of $s$ is an oriented $p$-simplex, whose orientation comes from that of $s$, the same way as above. But for two adjacent simplices of $\mathcal{S}$, say $s$ and $s'$ (Figures 6 and 7), it's two *opposite* orientations that are thus conferred on the two faces which coexist at the common boundary of $s$ and $s'$. So all terms $\int_\sigma \omega$ in (5) disappear, except those corresponding to faces $\sigma$ that belong to $\partial M$, and for these, the orientation is just right for

the corresponding terms to form a Riemann sum relative to $\int_{\partial M} \omega$.

Reviewing this proof, one will probably judge that the definition of the d operator was engineered *in order to* have Stokes' theorem hold. This is a quite defendable opinion. Indeed, the classical differential operators grad, rot, and div, are best defined in such a way, and we did that, earlier, bit by bit. So let's just review the whole thing systematically.

### 3.5 Grad, rot, and div, as avatars of d

Take $E_3$ as background, with metric and orientation Let be given a function $f$, and a curve $c$, represented by a parameterization[8] $t \to c(t)$, with $t \in [0,1]$. The gradient of $f$ is the vector field $\operatorname{grad} f$ such that $\int_c \tau \cdot \operatorname{grad} f = f(c(0)) - f(c(1))$. This latter expression happens to be $\int_{\partial c} {}^0f$, so using Stokes, i.e. $\int_c \mathrm{d}\, {}^0f = \int_{\partial c} {}^0f$, we have that

$$(6) \qquad {}^1(\operatorname{grad} f) = \mathrm{d}\, {}^0f \equiv \mathrm{d}f$$

(we may drop the zero superscript, for there is only one way to turn a function into a 0-form). *The vector-field* $\operatorname{grad} f$ *is a proxy for the* 1*-form* $\mathrm{d}f$, as we saw last time.

**Remark 5**. In particular, if $f$ is the function $\{x,y,z\} \to x$, in an $x$–$y$–$z$ Cartesian system, then $\mathrm{d}f$ is what we denoted $\mathrm{d}x$ a moment ago. Having a unique notation d for what looked to be different notions of differentiation seems more logical at present. So there is some method in our madness. $\diamondsuit$

Now, you guessed it:

$$(7) \quad {}^2(\operatorname{rot} u) = \mathrm{d}\, {}^1u, \qquad (8) \quad {}^3(\operatorname{div} u) = \mathrm{d}\, {}^2u.$$

To prove (7), just compare

$$\int_S {}^2(\operatorname{rot} u) = \int_S n \cdot \operatorname{rot} u = \int_{\partial S} \tau \cdot u$$

$$= \int_{\partial S} {}^1u = \int_S \mathrm{d}\, {}^1u,$$

where all equalities hold either by definition or owing to the Stokes theorem. The proof of (8) is left as an exercise.

[8] This is how it's done in practice. Note that $\tau$ is then the field $t \to (\partial_t c)(c(t))/|(\partial_t c)(c(t))|$, and that an inner orientation is implied: from $c(0)$ to $c(1)$, that is, for increasing $t$, by convention.

What's nice about it is that it works both ways: you can define the d of a 1-form ${}^1u$ by (6) if you have the curl already, say by its definition in coordinates, or define $\operatorname{rot} u$ once you know what d is.

Still, there are subtleties in (7) and (8), worth mentioning. First, the dependence on metric. Change the dot product $\cdot$ for a new one $\bullet$, the gradient changes: the vector fields $\operatorname{grad} f$ and $\mathbf{grad} f$ such that $\operatorname{grad} f \cdot v = \mathbf{grad} f \bullet v = \langle \mathrm{d}f, v \rangle$, for all $v$, are of course distinct. (The notational distinction, bold versus plain, is ad hoc, and won't be used again.) A similar thing happens with rot and **rot**, respectively associated with $\cdot$ and $\bullet$. In terms of the transition matrix $L$ such that $u \bullet v = Lu \cdot Lv$, one has $L^tL \operatorname{grad} = \mathbf{grad}$, obviously. If you solved the similar exercise regarding $\times$ (first paper of the series, §1.6), you won't find it too hard to work out the analogous relation between rot and **rot**, and to see why, as already mentioned, div = **div**.

The second remark is about the dependence on orientation. Observe that it intervenes twice in (8) (both superscripts 2 and 3 refer to orientation-dependent associations), once in (7), and not at all in (6). Therefore, rot is orientation-dependent, whereas neither grad nor div are. In more precise terms, if ${}^1u$ is the 1-form whose vector proxy is $u$, the vector proxy of its d, that is $\operatorname{rot} u$, depends on orientation.

Can that be cured? The illness comes from the fact that 2-forms, contrary to 1-forms, have two, opposite, vector proxies, one for each orientation. On the other hand, we know that a 2-form has a single *twisted* vector proxy. So let's represent the 2-form $\mathrm{d}^2u$ by a twisted vector field, denoted $\tilde{\operatorname{rot}} u$. This way, we have defined a differential operator, $\tilde{\operatorname{rot}}$, of a new type: it maps vector fields to *twisted* vector fields, instead of to plain vector fields. All this is so cumbersome that one may doubt the usefulness of such hair-splitting. So it's time to turn to a specific application: the various ways to mathematically describe (the ways to *encode*) Faraday's law.

### 3.6 Faraday's law

As established by Faraday's famous experiment, variations of the flux through a conducting loop create an electromotive force. A mathematical statement which is meant to express this law with maximum economy will there-

fore establish a link between the integral of $b$ (the induction 2-form) over a fixed surface $S$ and the integral of $e$ (the electric field 1-form) over its boundary $\partial S$. Here it is:

$$\partial_t \int_S b + \int_{\partial S} e = 0, \tag{9}$$

where $\partial_t$ denotes the time derivative. These numbers have dimension: webers for the first integral, volts for the second, which fits, since a volt is a weber per second.

**Remark 6.** Of course $S$ and $\partial S$ must be oriented (Fig. 8) for the integrals to make sense (and the orientation of $\partial S$ must be the induced one, as earlier—otherwise change + to $-$ in (9)). But which orientation of $S$ is selected is indifferent. (Physically, it's rather the orientation of $\partial S$ which is selected: it corresponds to the two ways a galvanometer can be inserted in the circuit here abstracted as $\partial S$. Hence the orientation of $S$.) $\diamondsuit$

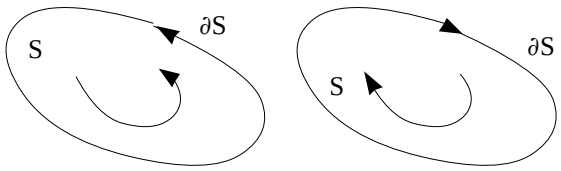


**Figure 8.** Both ways to orient $S$ and $\partial S$ in (9).

Putting metric and orientation in background (Fig. 9), and using vector proxies, we transform (9) into

$$\partial_t [\int_S n \cdot \mathbf{B}] + \int_{\partial S} \tau \cdot \mathbf{E} = 0, \tag{10}$$

the *integral* form of the law, saying that the rate of variation of the magnetic flux through a closed loop is compensated by the induced emf in this loop. By the Stokes theorem, (10) is equivalent to the *differential* version of the law:

$$\partial_t \mathbf{B} + \mathrm{rot}\mathbf{E} = 0, \tag{11}$$

and there we are in well charted territory.

But of course, one may as well apply Stokes to (9) directly, hence

$$\partial_t b + \mathrm{d}e = 0, \tag{12}$$

to be compared with (11). We have here a strictly *affine* expression of Faraday's law. Metric and orientation are not needed, and not invoked.

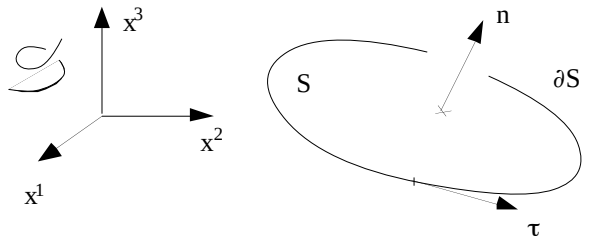


**Figure 9.** Notations used in formula (10). All vectors shown are of length 1.

By our earlier discussion of Lorentz force, we are prepared to see the third formalism, with no orientation but twisted vectors, stand in between. Indeed, in the "$\{\mathbf{E}$–$\tilde{\mathbf{B}}\}$-machinery", the differential expression of the Faraday law is

$$\partial_t \tilde{\mathbf{B}} + \tilde{\mathrm{rot}}\, \mathbf{E} = 0, \tag{13}$$

by using the above "twisted curl". Note the consistency: (13) is an equality between two objects of the same type—two twisted vector fields. There is of course an integral form (use Remark 3), which does not require, contrary to (10), an ambient orientation.

Is the simplification thus brought in by the use of an axial vector—the economy of the irrelevant ambient orientation—worth the trouble? Frankly, I don't think so. The metric, which is just as irrelevant, as (9) and (12) testify, is still present in (13). So why stop halfway? Either one decides to work in Euclidean space, with its metric and orientation, and the standard concept of vector field is enough, or one adopts the language of covectors and differential forms. We have not yet a compelling reason to take the latter stand (it's only when one deals with *deformable* conductors that one really appreciates the advantage of getting rid of the metric), but as for now, we find no outstanding advantage to the twisted vectors formalism.

Besides, these axial vectors pose notational problems when one wants to distinguish them from polar ones. This, which we managed to do here by using tildes, is often achieved by playing on the shape of the arrows above letters: thus $\breve{v}$ may denote an axial vector, whereas $\vec{v}$ is a polar one. Some electromagnetic treatises are graced at every page by such artwork. I am not very fond of the device: It's very difficult to use consistently, it generates cumbersome typesetting, and there are so many kinds of

geometrical objects to deal with (we have not exhausted the list!) that attempts to have a special graphism for each of them are hopeless. Better to explicitly declare the *type* of each entity when first introduced, and to use neutral typography.

We'll break here with our study of formalism, leaving the analogous discussion of Ampère's theorem for the next column. But before stopping, we need to discuss another issue: this restriction we made above by assuming a *fixed* surface $S$ in our study of the integral form (9) of the law is a matter of concern.

### 3.7 Faraday's law for moving circuits

Since (9) encodes, in mathematical terms, a physical *experiment*, and since (9) and (12) are mathematically equivalent, the local expression (12) subsumes Faraday's law: it's the one on which one may *always* rely.

In many circumstances, however, we need to know the circuital emf generated by flux changes which are due, in part or in totality, to the *movement* of the circuit, a case *not* covered by laws (9) or (10). Though this situation goes beyond the limits fixed to this series (a single reference frame, no motion), the issue cannot be ignored (see [Co, CL, RS] for relevant discussions).

So, if we want a flux–emf relation for moving circuits, we must *derive* it from (12), or from one of its equivalents (11) or (13). This amounts to evaluate $\partial_t \int_S b$ when $S$ moves, this motion being described by the velocity field $V$ at instant $t$. There are two parts in this rate of variation: one is $\int_S \partial_t b$, the other is the value that $\partial_t \int_S b$ would assume if $b$ was frozen at its instantaneous value $b(t)$, changes coming from the change of $S$ with time. Approximating the derivative by a finite difference, we are led to a comparison between $\int_{S(t+\delta t)} b$ and $\int_{S(t)} b$ for a fixed field $b$. This can be done by considering the volume enclosed by a surface composed of $S(t+\delta t)$, $S(t)$, and the extrusion of $\partial S$ in the direction of $V$, to which one applies Stokes' theorem, before letting $\delta t$ go to 0. The computation (which unrolls just as the well-known one in vector formalism, to be found e.g. in [Co]) results in

$$\partial_t \int_S b + \int_{\partial S} (e - i_V b) = 0,$$

where one recognizes the Lorentz force felt by charges sitting on $\partial S$.

Discussing this would lead us too far astray. But let's stress the conclusion that the simplest form of Faraday's law we have found during this investigation of formalisms, $\partial_t b + \mathrm{d}e = 0$, is also the one which may be trusted in all circumstances.

Next, we may expect a parallel treatment of the Ampère relation, $-\partial_t d + \mathrm{d}h = j$, but an important difference will emerge: it's *twisted* differential forms that appear in this formula. And a new geometric object, the "Hodge operator", will be introduced as the right tool to model constitutive laws.

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