(2): Geometrical objects

We have introduced oriented three-dimensional Euclidean space, denoted E_3, and understood as a three-layer structure: 3D affine space (a set of points on which translations can be performed), plus a dot product, plus an orientation. We now look at denizens of this universe. We'll pay attention to which of these structures they really depend on, and review their use as descriptors of physical entities, with emphasis on the notion of *force*, which itself ushers the electric field and the magnetic field, conceived as differential forms.

2.1 Vectors

Vectors we know well. They belong to the "vector space" structure (V_3) and represent translations: for any two points x and y, there is a unique vector v such that $y = x + v$. (It's indeed the "free" vector v, an element of V_3, not the bound vector $\{x, v\}$ we are speaking about: For it's the same v-translation that will send x', say, to $y' = x' + (y - x)$.) For vectors so to stand for translations is totally in line with their abstract definition. But precisely, what we want to discuss this time is the vector as representing something else than itself, the vector *as a proxy* for some entity of physical interest.

For instance, vectors are often used to represent *position*. Vector r (as it's often denoted) is then assimilated with the point $0 + r$, which involves some arbitrariness, since such a representation depends on a choice of origin. Granted, in many questions of physics, it does make sense to specialize a point to play origin: in atom dynamics, celestial mechanics, etc., there *is* such a privileged point. But otherwise, it's not such a good idea.

A more legitimate use is for *displacements from* a reference position, as one does in continuum mechanics (and now it's *bound* vectors we have in mind). Consider a moving mass of fluid, or a deformable body, represented by a set[1] B, elements of which are called material particles. The material particle sitting at point x at a reference instant ($t = 0$, usually) will be found at time t at a different point $x + \xi_t(x)$. The vector field $x \to \xi_t(x)$ is called *displacement* at time t, and its evolution in time describes what happens to the whole body, provided one knows[2] where each particle stands at $t = 0$. *Virtual* displacements, as one knows, may have to be taken in consideration, and are represented by (time-independent, of course) vector fields, the same way. Metric and orientation of space are irrelevant to such descriptions.

Vectors can also stand for *velocities*. Let's first consider a single particle, which passes at point x at time t. It will be at $x + \delta t\, v$, up to higher order terms, δt seconds later. The bound vector $\{x, v\}$ thus fully represents the particle's motion at time t. To describe its fate over some span of time $[t_1, t_2]$, one will resort to the notions of *trajectory* (a smooth map from the real interval $[t_1, t_2]$ into E_3) and of *field of tangent vectors*,

$$(1) \qquad v(t) = \lim_{\delta t \to 0} [x(t + \delta t) - x(t)]/\delta t,$$

one at each point $x(t)$. (Pause a moment to check that such a limit is well defined, without need for a metric on V_3.)[3] Now if instead of a particle we have the above extended body, a vector *field* will be able to describe its instant motion, while a time-dependent vector field will account for its evolution.

Although metric and orientation of space are there again irrelevant, this time one may object, "But is there not some metric element here, as betrayed by your reference to the *second*, the unit of time? No metric on E_3, all right; but you need a metric (and an orientation, to

[1] B is not a naked set: in order to account for the notion of "material continuity", it must be endowed with a topology (and a bit more: the right structure is that of smooth manifold, actually).

[2] The mathematical device by which such information can be encoded is, of course, a *map*, u_0 say, from B to E_3: the material particle $b \in B$ sits at point $x = u_0(b)$. Such a map is called a *placement*. Note how the initial placement u_0 and the *dis*-placement ξ_t combine to give the placement at time t.

[3] The trajectory is more than its supporting curve: it's this curve plus a specific way to run along it (just as a graded ruler is more than a plain ruler). Notice how the supporting curve is oriented (it's *inner* orientation) by the law of motion.

boot) on the time-axis." Right on! But please make allowance for the necessity to proceed step by step in this deconstruction process we have initiated. We focus on the structure of *space*, for the moment, and other fundamental categories such as time, energy, electric and magnetic charge, etc., will have to wait in line. So we take seconds, joules, coulombs, and webers[4] for granted. (It's meters and inches that are under attack!) Yet, let's acknowledge that when it comes to velocities, vector fields cannot do the job *alone*, some extra structure (here, a chronometry) is needed.

Vectors are also frequently cast in the role of *rotations*. We shall dwell for a while on this example, which will lead to the construction of a new geometric object.

2.2 "Axial vectors"

You know the trick: When a solid has one of its points anchored at a fixed position a, its velocity field v is given by

$$(2) \qquad v(x) = s \times (x - a),$$

where the *spin vector*[5] s may depend on time. The instantaneous axis of rotation is then the line supporting s, and the norm $|s|$ is a measure of the rate of spin. As we know, defining the cross product requires a metric and an orientation of space. One cannot object about metric, since the very notion of rotation depends on it. But orientation? Look at a spinning baseball; its velocity field (the v of (2)) is what it is, and exists independently of any orientation convention. Yet s, in (2), *does* depend on orientation, since $\times$ itself does! Change the orientation and you need to change s into $-s$ in (2) in order to obtain the same velocity field.

So here again, the spin vector cannot do its job alone. It needs extra structure in background. But whereas one part of this structure, the metric, is clearly relevant (as was the chronometry in the velocity example), another part, orientation, seems artificially introduced here, since it cancels out in (2). To say the same thing in a different way, what truly represents the rotatory motion is *not* the spin vector s alone, but a composite object, the *pair* $\{s, Or\}$ where Or is the chosen orientation for ambient space. Moreover, since the opposite pair $\{-s, -Or\}$ represents the same instantaneous velocity field, we have two *equivalent* descriptors for the same instant motion (and only two: no other pair will do). Formally, we may denote this as

$$(3) \qquad \{s, Or\} \equiv \{-s, -Or\},$$

an equivalence between the two pairs.

In mathematics, when objects are equivalent in some respect, we often bundle them together, putting like with like, and start considering each of the "equivalence classes" thus obtained as a new object in its own right. This is how, to recall only one well-known example, rational numbers are defined as equivalence classes of pairs of nonzero integers: We consider two ordered pairs of signed integers $\{m, n\}$ and $\{m', n'\}$ as equivalent if $mn' = m'n$, and we dump all equivalent pairs in the same class q. (Then we justify the abuse of notation $q = m/n = m'/n'$, and happily go ahead.)

In the spirit of such tradition, what follows, which Voigt[6] first did around 1910, appears as a rather natural move. Let's consider an equivalence class for relation (3) as a geometrical object in its own right, that we shall call (very provisionally) a *rotator*, and denote by $\tilde{s}$. So, formally,

$$(4) \qquad \tilde{s} = \{\{s, Or\}, \{-s, -Or\}\}.$$

Thus defined as an equivalence class, the rotator $\tilde{s}$ is *not* the same kind of object as s or $-s$. But it can be *represented* by one of these, in a very definite way: If space is oriented once and for all, this establishes a one-to-one association between vectors and rotators by which, as one sees, s stands for $\tilde{s}$ if the chosen orientation is Or.[7] But let's insist again on the fact that $\tilde{s}$

[4] Grams, no. Can you see why?

[5] It's a vector at point a, thus a *bound* vector. So the "axial" objects we are about to define will also be point-bound. But I stop insisting on this distinction from now on. Whether bound or free objects are meant should be obvious from the context each time.

[6] Cf. [Po], which points to [Vg]. Post credits Voigt for the introduction of the term "tensor", too.

[7] The representative would of course be $-s$ if the chosen orientation was $-Or$. (The notation $\tilde{s}$ is not unimpeachable: it betrays a bias in favor of Or as the symbol for the standard orientation.)

exists in its own right, just as s, whether space be oriented or not.[8]

Of course "rotator" is not a good name, since such geometrical objects can serve for other things than velocity fields in a rotatory motion. (To quote only one, of which the reader will have been aware already, and to which we shall of course return, the magnetic induction field **B** can be represented by a field of such "rotators", one at each point.) But the name chosen by Voigt for objects like $\tilde{s}$, *axial vectors*, appears much worse, in retrospect.

For again (remember, we had the same trouble with "bound vector", last time), "axial vector" must be understood as a single, unbreakable label for this new kind of geometrical object. "Axial" is *not*, definitely not, *an adjective* that would point to some quality possessed by either $\tilde{s}$ or s, some "axiality". Such a thing cannot exist, anyway. It can't be an attribute of "the vector" $\tilde{s}$, if only because $\tilde{s}$ is not a vector. Neither can it be an attribute of s, which just *plays the role* of $\tilde{s}$. So there is here a quite unfortunate choice of terms,[9] aggravated by the habit to call "polar vectors", for contrast, the "ordinary" vectors.

Remark. Alternative denominations exist: e.g., *twisted vector,* which has the same drawbacks (but also some advantages, to be discussed in due time), or *pseudo vector,* which is a bit better (since a pseudo X is not supposed to be an X). But be careful: authors may use such names for again slightly different objects. $\Diamond$

Let us check that this geometrical object is indeed able, by itself and without the pre-existence of an orientation, to represent rotatory motion. Given a metric, consider an axial vector $\tilde{s}$ at point a. Let's select one of the two elements of the class, say $\{s, Or\}$. Metric and orientation Or define a cross product $\times$. Now, the instantaneous velocity field is $x \rightarrow s \times (x - a)$. Had we chosen the other element of the class (4), two signs would change (one in front of s, one in the definition of the cross product), and we would get the same field.

Figure 1 shows how to design a convenient icon for axial vectors. (Figure 2 gives the 2D version.) The idea is simply to replace symbols by icons in (4), and to do some stylizing (since the two arrows do double duty, keep only one, etc.). Hence the icon for the axial vector $\tilde{s}(x)$ at point x: it consists of a plain segment (not a vector), with x at midpoint, and of an outer orientation of its supporting line. This is satisfactorily suggestive of the notion of "turning around" the rotation axis, at a speed proportional to the length of the segment—the very notion we wanted to capture in the first place.

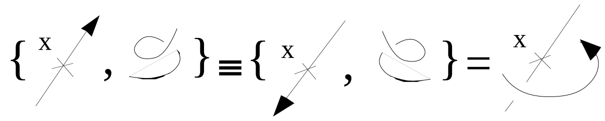

Figure 1. Merging the icons for vector and orientation to produce an icon for the axial vector.

Remark. Notice how an axial vector confers an *outer* orientation on its supporting line, whereas a polar vector gives it *inner* orientation. This suggests an alternative way to define these objects in affine space. A (bound) polar vector is made of (1) a point, (2) a line through this point, (3) a real number (the length of the vector), (4) an *inner* orientation of the line, that is, a sense or "pointing direction". An axial vector is made of the same items, except for (4), which is an *outer* orientation of the line, that is, a sense or "turning direction". $\Diamond$

Note that no ambient orientation is suggested by the icon. *None should be,* because the axial vector does *not* depend on orientation. (This was the whole point in defining it the way we did.) Like the bound vector, it's an "affine object", meaning that, of the three layers of structure that make E_3, only the affine structure is necessary to its existence. Yet axial vectors differ a lot from vectors, and I can't resist quoting Burke's elegant argument (Fig. 2) to show to which extent they do so.

[8] Maybe an analogy can help, for what it's worth. Think of E_3 (the world ...) as a stage. Orientation, metric, etc., are elements of the set; $\tilde{s}$ is a character in the play; s and $-s$ are actors, who may alternate in the part, depending on which set (Or or $-Or$) is installed.

[9] Though one tries hard to avoid such things in mathematics, they do happen. A "*signed* measure" is not a *measure*, "*free* Abelian groups" are not Abelian groups that would happen to be "free", and so on. So much for logic!

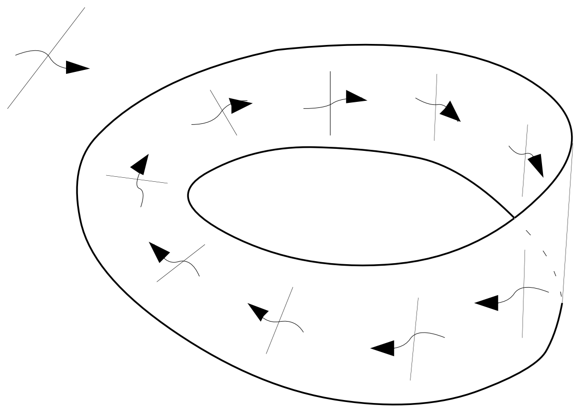

Figure 2. 2D icon for axial vectors, and how it's used in Burke's graphic illustration [Bu]: "To appreciate that a twisted vector is an independent notion, consider the problem of finding a continuous nonzero vector field on the Möbius strip which is everywhere transverse to the edge. No such vector field exists, but a twisted vector field with these properties does."

Axial vectors can be subjected to stretching, turning, mirroring, etc., i.e., to all geometric transforms, just like vectors.[10] Since a skew transform M (such as a mirror reflection) changes the orientation class of frames, it sends the representative $\{s, Or\}$ of $\tilde{s}$ to $\{Ms, -Or\}$, not to $\{Ms, Or\}$. Hence the different behavior of axial and polar vectors under mirror reflection (Fig. 3). This is well seen by using icons, for one need only apply the transformation to all graphical elements that compose the icon[11] in order to visualize the transform of an object.

Exercise. Study Fig. 4 and comment on its meaning.

There is nothing simple in what precedes, so one may wonder whether representing spin by a vector was such a good idea in the first place. Thinking about dimension n instead of 3 may help, there. In all dimensions n, rotations are represented, via an orthonormal basis, by $n \times n$ orthogonal matrices. Instantaneous velocity fields in such rotations (or if one prefers, "infinitesimal rotations") are then represented by $n \times n$ skew-symmetric matrices, which depend on $n(n-1)/2$ parameters, and this happens to equal n when $n = 3$. So it's only when $n = 3$ that vectors can stand for infinitesimal rotations,[12] thanks to the cross-product trick[13] of (2). It's a spurious association. No wonder we had so much trouble!

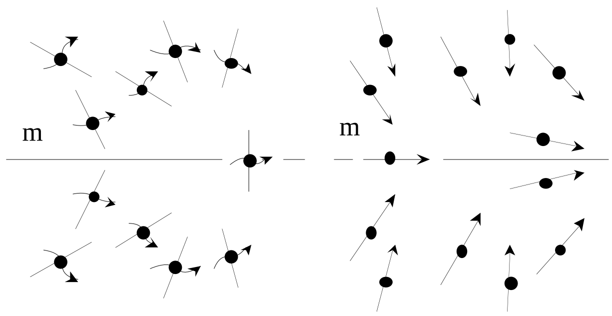

Figure 3. Two fields (suggested by a few scattered icons), each invariant by mirror reflection, but one made of axial vectors, the other one of polar vectors.

We gained something of value, however, by this brush with axial vectors: awareness that vectors and vector fields, these workhorses of calculus, are not always the best tool for the job at hand. What we want to challenge next is their use to represent *forces*.

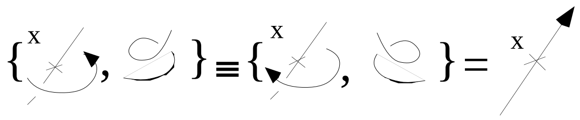

Figure 4. How polar vectors, too, could be defined in terms of axial ones.

2.3 Covectors

Is force a vector? Or, to be precise about the meaning of "is" (we shall feel free to abuse the language from now on, so let it be the last time I fuss like that), "is this physical manifestation we call force properly described by this (well understood) mathematical abstraction, the vector in Euclidean space?"

[10] If an object $\tilde{o}$ is defined as a class of equivalent tuples $\{o_1, o_2, \ldots\}$, and if some transform T can act on all components o_i (which in general are objects of different types), then $T\tilde{o}$ is defined as the class of $\{To_1, To_2, \ldots\}$, provided the latter compound be equivalent to $\{To'_1, To'_2, \ldots\}$ when $\{o_1, o_2, \ldots\}$ and $\{o'_1, o'_2, \ldots\}$ are equivalent. Only transforms for which this condition holds are legitimate.

[11] That's the rationale for good icon design [Al]: if $I(o)$ is the icon of object o, and T a geometric transform, the icon $I(T(o))$ of the transformed object $T(o)$ should be $T(I(o))$.

[12] The "right" geometrical objects for infinitesimal rotations in all dimensions are bivectors, which we shall soon encounter. Cf. [He], where Hestenes has recast a large part of classical physics in the language of multivectors. A good summary of his views can be found in [Hs]. It's fascinating and recommended reading, even if he overstates his case at times.

[13] A binary operation with the properties of the cross product can exist only in 3 and 7 dimensions [Ec]. Case $n = 3$ we know about. I don't know what the implications of the case $n = 7$ are.

The answer seems obvious: force has a magnitude, right? A direction, right? Hence, it's a vector, what else?

Well, it's no so obvious that force has direction. If you kick a golf ball, yes, it goes along with your shove. (Doesn't it?) But this is a simple, point-like object. What of a rugby ball? Should we take the direction in which the body moves as the direction of the thrust we exert? Playing a few minutes with a gyroscope is enough to cast this in doubt. It's easy to see that, in all cases where we can assign a definite direction to a force, this is the direction taken by material *objects* to which the force is applied, and since this direction is determined in part by the shape and structure of such objects, it cannot be *attributed to* the force. Force has no *intrinsic* pointing direction.

So let's take the question from another angle. Consider a physical force field, such as the gravitational field, or the electric field. (Assume, for simplicity, a static field, so time is no concern.) We want to describe this empirical reality by some mathematical abstraction. How do we know about the force field? By the deformations it causes on material structures, by the *displacements* it imparts on loose objects placed in the field. And since we can represent displacements by vectors, there lies our handle, in which one will recognize the time-honored *principle of virtual work*: Imagine (to treat the case of the electric field), an electric charge χ coulombs strong, placed at point x, where χ is a scalar factor meant to tend to zero[14], and let's displace it to $x + \lambda v$, where v is a vector (the "virtual displacement") and λ another "vanishing" scalar factor. The work involved in this displacement, or virtual work, is a smooth function of $\lambda\chi$ and v, which one can Taylor-expand in $\lambda\chi$. The leading term of this expansion is of the form $\lambda\chi f_x(v)$, where $v \to f_x(v)$ is some *linear* map (this is the key point). Now *if we know the map* $v \to f_x(v)$, *we know the force* at point x. This is as complete a characterization of force as one may desire.

In this description, the force at point x "is", therefore, a linear map of type[15] *VECTOR* $\to$*REAL*, that is to say—by the very definition of dual space—an element of the *dual* of the vector space V_3. This dual, V_3^*, is a three-dimensional vector space too, so its elements would deserve the generic name of vectors. But they are not of the same type as the vectors of V_3, so we call them *covectors*, instead. Force, as we see, is a covector.

The force *field*, now, is a field of covectors, one at each point. Calling that a *covector field* would make perfect sense. But it happens that fields of covectors have another name: one calls them *differential forms* (DF) for reasons that will little by little become apparent. Anyhow, such a covector field, or DF, appears as the right geometric object by which to represent a force field.

As a corollary, this will give us the right mathematical representation of the electric field. Humankind, by a protracted process of experimentation and theorization, recognizes the existence of a particular substance called "electric charge" and of a physical manifestation, called the "electromagnetic (EM) field", which affects the space around us in ways which are revealed by, precisely, the behavior of electrically charged objects. More specifically, a moving particle of (vanishingly small) charge q appears subject to a force, the *Lorentz force*, which (1) is proportional to q, (2) depends in part on the velocity of the particle. We thus distinguish two parts in this force, the static one (which a nonmoving particle feels) and the dynamic one (due to motion), and hence, we also distinguish two aspects, two facets, of the EM field: the electric one and the magnetic one. The electric field is this part of the EM phenomenon that is revealed by forces on nonmoving charged particles.

So, assuming by convention a unit charge, the electric field is akin to a force field. *The electric field is a covector field.* It "is", more precisely, the mathematical object we may denote as follows

$$e = x \to (v \to e_x(v)), \tag{5}$$

a compact and spiked (but convenient, as you will see) expression, that should be parsed as follows: e (the differential form that represents the electric field) is the field $x \to e_x$, where e_x is a covector at x, which itself is the linear map $v \to e_x(v)$, where the real number $e_x(v)$

[14] since actually putting a *finite* charge there would alter the field.

[15] When a function f sends all or part of a set X to another set Y, we say that "the type of f is $X \to Y$".

is interpreted as a virtual work. More precisely, the work yielded by the field when a charge χ at x is pushed to $x + \lambda v$ would be $\lambda \chi e_x(v)$, up to terms of higher order in $\lambda \chi$.

Remark. It will be convenient to write the value $e_x(v)$ as $\langle e_x, v \rangle$, with the covector on the left and the vector on the right (that's Dirac's "bra–ket" notation), and to informally refer to this number as "the effect of e_x on v", since it's the value of a map, e_x, that acts on vectors. Note that we can say "the effect of v on e_x" as well, since a vector can be seen as a linear map over covectors, by reflexivity of the duality relationship (the dual of V_n^* is V_n). $\diamondsuit$

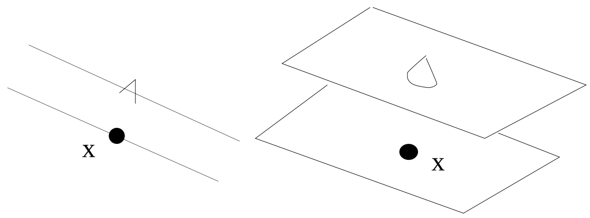

Figure 5. Icons for a covector at x, in 2D and 3D. (Their origin can be traced back to [VW].) The length of the segment or the area of the plane patches are not meaningful.

We said that force should not be construed as pointing towards some direction, the way a vector does. This doesn't imply that force has no directionality at all, for spatial directions are not all alike with respect to a force covector. The map $v \to e_x(v)$ has a *kernel*, made of vectors at x such that $e_x(v) = 0$. They define, in 3D, a plane containing x, along which virtual work is zero, to first order. (So this plane is tangent to the equipotential surface of the force field through x, if there are[16] such surfaces.) Under the virtual work interpretation, the force at x should, actually, be visualized as a pair of parallel planes (Fig. 5). One is the previous null plane, passing through x. And—just as the tip of the velocity vector of a particle was the point reached after one second of movement, assuming no change in velocity—the other plane is the one reached by releasing one joule of virtual work, again assuming uniformity of the force field.

[16] Which is not necessarily the case. One may imagine force fields such that the virtual work involved in pushing a particle form x to y depends on the trajectory followed, not only on the end-points. It means that the above "null planes" can't be quilted together to envelop surfaces. A central result of differential geometry, the *Frobenius theorem* (see, e.g., [Sc], p. 82) tells when they can.

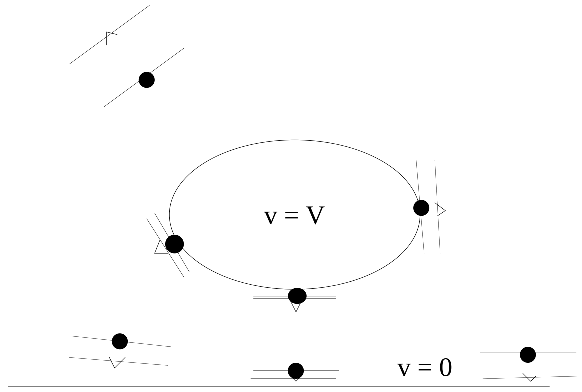

Figure 6. The electric field, between an electrode at potential V, and the ground.

Remember this hall for electrostatic experiments in the first installment ([B1], p. 19)? See on Fig. 6 how such icons nicely visualize the electric field around a charged isolated conductor, and near the ground. This pictorial representation of the force field does not depend on a metric. In particular, the notion of orthogonality of field *lines* (which have no status so far) with respect to the conducting surfaces is irrelevant. Note how Fig. 6 can be looked at from any angle, and retain its meaning. (Same remark about Fig. 5 of last issue [B1].)

It should not be felt as counterintuitive that the larger the covector (which measures the intensity of the electric field), the thinner its icon, i.e., the closer the two planes which compose it. It's because higher intensity means closer equipotentials. Figure 7 should make that clear. (Since a covector ω is a linear map from vectors to reals, it makes sense to ask the question, "given a covector ω and a vector v, what is the value of $\langle \omega, v \rangle$?", which Fig. 7 answers.)

We are now prepared to introduce the notion of gradient of a function. If a smooth function f maps A_n (the affine space) to reals, we may expand it in the vicinity of a point x, thus obtaining $f(y) = f(x) +$ <a linear part in $y - x$> + <higher order terms>. The "linear part" here, considered as a function of $y - x$, is a covector, which we may denote as $\mathrm{d}f(x)$, and call the *differential* of f *at* point x. Hence a covector field $\mathrm{d}f$ (the map $x \to \mathrm{d}f(x)$), i.e. a differential form, which it would be natural to call the differential of f. It's more common to call it its *gradient*, however. No surprise here: we expected the force field to be the gradient of the potential function, when there is one. But a great risk of confusion, because "gradient" may mean something else when a metric structure is present. Let's address this delicate issue.

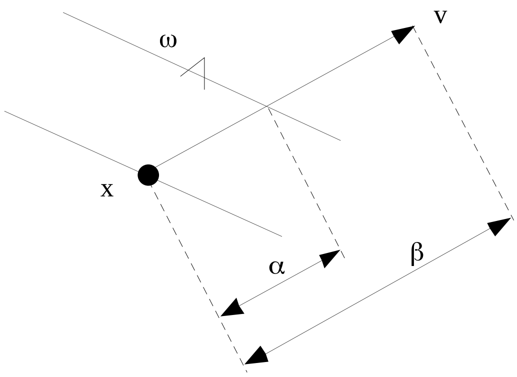

Figure 7. Since $\langle\omega, v\rangle = 1$ when the tip of vector v lies on the "arrowed front" of the covector ω, the value of $\langle\omega, v\rangle$ for the vector v of this figure is the ratio β/α, by linearity. (This ratio is an *affine* notion, since only one direction is concerned. No metric involved.) Large ratios, i.e., big covectors, thus correspond to closely spaced planes in the covector's icon.

2.4 Vectors as proxies for covectors

Suppose that a dot product has been defined. Then, each linear map $v \to \langle\omega, v\rangle$, that is to say, each covector ω, has an associated *Riesz vector*, defined as the unique vector w such that $w \cdot v = \langle\omega, v\rangle$ for all v. Conversely, of course, each vector w generates a linear map, that is a covector. (Hence an isomorphism, non-canonical, between V_n and V_n^*.) A vector field, therefore, generates a differential form. To save on notation, I will denote the DF thus associated with a vector field w as ${}^1w$, where the 1 refers to something called the "degree" of the DF (later to be defined, but of no importance right now). Same notation, of course, for the covector ${}^1w$ (or ${}^1w(x)$ if we need to refer to its location), generated by the vector w at point x.

So if a force field is described by a DF ω, it can as well be described by the vector field w such that ${}^1w = \omega$. By the very definition of the iconic planes in Fig. 5, w is orthogonal to them, and hence the field lines of w are orthogonal to the equipotentials of the field. This restores the sense of pointing directionality of force that we shunned a moment ago, and also gives status to the notion of field lines. These lines support the test-particle trajectories, so there is no doubt that metric is physically relevant here. But what it tells about is the structure of *space*, as revealed by the dynamics of simple particles,[17] not the structure of the over-imposed electric field.

The best way to make that obvious is to imagine two metrics that would only differ by the chosen unit of length, say $\cdot_i$ for inches and $\cdot_m$ for meters. The *same* electric field e is then represented by two *different* vector fields $\mathbf{E}_i$ and $\mathbf{E}_m$, linked[18] by the equality

$$(6) \qquad \mathbf{E}_i(x) \cdot_i v = \mathbf{E}_m(x) \cdot_m v$$

for all test vectors v. This is reflected in the choice of units: volts per inch for $\mathbf{E}_i$, volts per meter for $\mathbf{E}_m$, whereas the DF itself is, so to speak, in "volts per vector". One should imagine, at each point of space, a machine where one can insert a vector, to then see a dial give the number of joules (recall that joule = volt × coulomb) available by letting a unit charge (one coulomb) drift from x to $x + v$ (all that, of course, to first order and virtually).

The proxy fields $\mathbf{E}_i$ and $\mathbf{E}_m$, therefore, cannot do their job alone. Both require a specific metric, irrelevant to the virtual work available at each point, which reflects the real nature of the electric field: a differential form.

Now, let's go back to the notion of gradient. A pressure gradient, for instance, is routinely given in millibars per kilometer, in technical meteo reports, showing that what people have in mind there is the vector field representative, not the differential form. We must commit ourselves to some nonvarying use, so we shall denote the differential form by df and reserve the notation $\mathrm{grad}\, f$ to the proxy vector field with respect to a background metric which, hopefully, will be fixed once and for all. This way, therefore, d$f = {}^1(\mathrm{grad}\, f)$.

Remark. If one applies the DF df to a specific vector field v, one gets a scalar field

$$(7) \quad x \to \langle \mathrm{d}f(x), v(x)\rangle \equiv (\mathrm{grad}\, f)(x) \cdot v(x),$$

which is the one often denoted by $\partial f/\partial v$, or better $\partial_v f$, that is, the *derivative of f along*

[17] Note that a particle with complex inner structure, like a spinning top charged off-center, would behave differently.

[18] The differential form e is the same in both cases, so $e = {}^{1_i}\,\mathbf{E}_i = {}^{1_m}\,\mathbf{E}_m$, as we would be forced to write if we insisted on having the metric explicitly appear in the notation. Just keep in mind that this left upperscript "1" implicitly points to the metric.

v.[19] (The notion of *normal derivative,* $\partial_n f$, is a case in point.) An *affine* notion, as one sees, despite the appearance of a dot product on the right of (7). By a reversal of viewpoint, a vector field can thus be seen as an *operator,* acting on functions, which has all the formal properties of a *derivation* (linearity, $\partial_v(f+g) = \partial_v f + \partial_v g$, and Leibniz rule, $\partial_v(fg) = f\partial_v g + g\partial_v f$). *Vector fields are derivations.* We won't make much of this important observation for the time being, except justify the use of an otherwise bizarre notation, ∂_i, for the fields of basis vectors in what follows. ◇

2.5 Components

We avoided using frames and components up to now, and indeed, one can do much mileage without them. They can't be ignored, however, if only to make contact with other work using them.

So let's assume a smoothly varying field of frames over A_n : at each point, n independent bound vectors $\{\partial_1(x), \partial_2(x), \ldots, \partial_n(x)\}$, forming a frame at x, that we may denote $\partial(x)$, and also forming n vector fields ∂_1, ∂_2, ..., ∂_n. The field $x \to \partial(x)$ is called a *reference frame.*[20] Any vector field v can then be written

$$(8) \qquad v = \sum_{i=1,\ldots,n} v^i \partial_i,$$

with of course $v(x) = \sum_i v^i(x)\partial_i(x)$ at each point. The v^is are the *components* of v in this reference frame. We define an associated *dual* frame, or *coframe,* by introducing the covectors $d^i(x)$ such that

$$(9) \qquad \langle d^i(x), \partial_j(x)\rangle = 0 \text{ if } i \neq j, 1 \text{ if } i = j.$$

A covector field ω can then be written $\omega(x) = \sum_i \omega_i(x) d^i(x)$ at each point, that is[21]

$$(9) \qquad \omega = \sum_{i=1,\ldots,n} \omega_i d^i.$$

As a result, the scalar field $\langle \omega, v\rangle$ is equal to

$$(10) \qquad x \to \sum_{i=1,\ldots,n} \omega_i(x) v^i(x),$$

which is of course generally abbreviated as $\sum_i \omega_i v^i$ or even $\omega_i v^i$ (the Einstein convention). Be careful, *this is not a scalar product,* but a so-called "duality product".

Now suppose there is a metric, defined by the dot products $g_{ij} = \partial_i \cdot \partial_j$ (there, position dependent). Then, the Riesz vector of ω is w such that $\langle \omega, v\rangle = \sum_{i,j} g_{ij} w^j v^i = \sum_i \omega_i v^i$, hence $\omega_i = \sum_j g_{ij} w^j$. The components ω_i of ω are then called the *covariant components* of ... w! These components do not depend on w and the basis only (as do its *contravariant* components, i.e., the w^is in the expansion $w = \sum_i w^i \partial_i$), but also on the metric, to which the w^is owe nothing, so it's a misleading symmetry that is suggested by this unfortunate terminology.

Where does it come from, by the way? If the basis is changed to $\bar\partial_i$, with the new basis vectors given as $\bar\partial_i = \sum_j A_i^j \partial_j$ in terms of the old ones, the new components $\bar v^i$ must satisfy $\sum_i \bar v^i \bar\partial_i = \sum_i v^i \partial_i$, hence $v^i = \sum_j A_j^i \bar v^j$: they transform "the other way" with respect to the basis vectors. A similar calculation, based on (9)(10), shows that "covariant components" of w transform just as basis vectors do: $\bar\omega_i = \sum_j A_i^j \omega_j$.

This is why vectors and covectors are sometimes called "contravariant vectors" and "covariant vectors". One may find this debatable. (After all, defining a change of frames by the way the ∂_is change is an arbitrary choice. One might as well give the new basis covectors in terms of the old ones. Then, it's the ω_is that would be "contra"!) But much worse, covectors (and all other sorts of tensors) are sometimes *defined* by their behavior under frame changes, and that is really old-fashioned, a remembrance of the time when vectors were not conceived as

[19] So one has $\langle \mathrm{d}f, v\rangle = \partial_v f$, exhibiting a duality that would better be seen by writing $\langle \mathrm{d}f, v\rangle = \langle f, \partial v\rangle$. But no notational system can satisfy all needs with equal success.

[20] Be well aware—this is a nasty little trap, that very few authors warn about, [Sc] being one of the exceptions—that whether a *coordinate system,* that is, a set of n functions $\xi^i : A_n \to \boldsymbol{R}$ that map points to n-tuples of coordinates, induces basis vectors (∂_i is just the one corresponding to the i-th partial derivative), *the converse is not true,* even if one restricts attention to the neighborhood of a point. This is why we talk of *components,* not of *coordinates.*

[21] The symbol d^i thus stands with advantage for the "dx^i" of the physics literature, a badly thought-of notation in many respects.

autonomous objects, but as frame-related sets of numbers.[22]

Remark. Components of $\mathrm{d}f$ are the partial derivatives $\partial_i f$. So $\partial_v f = \sum_i \partial_i f v^i$. Components $(\mathrm{grad}\, f)^i$ of $\mathrm{grad}\, f$, such that $\mathrm{grad}\, f = \sum_i (\mathrm{grad}\, f)^i \partial_i$, verify $\partial_i f = \sum_j g_{ij} (\mathrm{grad}\, f)^j$.$\Diamond$

2.6 Gyroscopic forces, bi-covectors

Now, let's deal with the "magnetic part" of the Lorentz force, the one due to motion. It has the experimentally demonstrable property of being *gyroscopic*, that is, to depend on the *actual* velocity vector V in such a way that the *virtual* work for a v parallel to V is null. (Electrons in a steady magnetic field twirl around field lines in complex motion, but neither lose nor acquire energy.[23])

A *gyroscopic force field*, therefore, can be characterized as a *covector-valued map*, which at each point sends the actual velocity vector V of the particle passing there to a covector $\omega(V)$, with the essential property that

$$(11) \qquad \langle \omega(V), V \rangle = 0 \text{ for all } V.$$

By linearity,

$$0 = \langle \omega(v+V), v+V \rangle = \langle \omega(v), V \rangle + \langle \omega(V), v \rangle$$

for any pair of vectors $\{v, V\}$. This suggests to define a new entity, denoted b, acting on such pairs to yield a real number, which we shall denote $\langle b; v, V \rangle$, and define as

$$\langle b; v, V \rangle = \langle \omega(V), v \rangle$$

where $\omega(V)$ is the force covector. So we have here (at each point) a mapping of type *VECTOR* $\times$ *VECTOR* $\to$ *REAL*, which is linear with respect to both arguments, and *alternating*, meaning that $\langle b; v, V \rangle = -\langle b; V, v \rangle$ whatever v and V. Such an object is called a *bicovector*, or 2-covector. (They form a vector space, which has a dual, elements of which are *bivectors*.) A field of 2-covectors is a *differential form of degree* 2, or 2-form for short, in reference to the number of vectors acted upon. So *the magnetic part of the EM field is a 2-form*, called "magnetic induction". It's like having a distribution of machines with two slots (to be filled in this precise order), one for the virtual displacement, one for the actual velocity, the dial then giving the virtual work. Figure 8 displays a suitable icon [Bu].

What of vector proxies? There's an obvious alternating bilinear map that one can associate with a vector u: it's

$$(12) \qquad \{v, V\} \to u \cdot (v \times V),$$

but this time *both* metric and orientation are necessary to build this associate bicovector, that I shall denote ${}^2u$. (Thus, by definition, $\langle {}^2u; v, V \rangle = u \cdot (v \times V)$, or $(V \times u) \cdot v$.) In the stage analogy, therefore, we can have a vector field playing the part of the magnetic induction field, but only if metric *and orientation* have previously been set, say g and Or. Then there is a vector field **B** which describes the magnetic part of the EM field, in the precise sense that the Lorentz force *vector* (mind that!) on a unit charge of velocity V is $V \times \mathbf{B}$. Of course, if one substitutes $-Or$ to Or, it's $-\mathbf{B}$ which acts.[24]

One can do without metric and orientation, thanks to the concept of 2-form: **B** is just a proxy for the 2-form $b = {}^2\mathbf{B}$, which is a purely affine object. One can keep the metric and do without the orientation by introducing an axial vector $\tilde{\mathbf{B}}$, the one represented by the pair $\{\mathbf{B}, Or\}$. Hence the oft-repeated assertion that "magnetic field is an axial vector". But is there any wisdom in thus disposing of the orientation while keeping the metric? Especially when it calls for as difficult a concept as "axial vector"?

[22] It goes this way: Build composite objects $\omega = \{\partial_1, \ldots, \partial_n;\ \omega_1, \ldots, \omega_n\}$, made of n vectors and n real numbers. Say ω and $\bar{\omega}$ are equivalent if there is a regular matrix A such that $\bar{\partial}_i = \sum_j A_i^j \partial_j$ and $\bar{\omega}_i = \sum_j A_i^j \omega_j$, and call the equivalence class a "covariant" vector. (Note that, in order to define vectors in the first place by such a method, you would have to deal with classes of $n(n+1)$-tuples of numbers. No wonder "old tensor" calculus is so dreaded.) With this approach, an axial vector is a (class of) similar sets of numbers, but with the last n components a^i transforming as $a^i = \mathrm{sign}(\det(A)) \sum_j A_j^i \bar{a}^j$.

[23] The energy build-up in synchrotrons is due to an *electric* field, which changes direction at each half-turn.

[24] And if one substitutes inches for meters. . . But no need to repeat this.

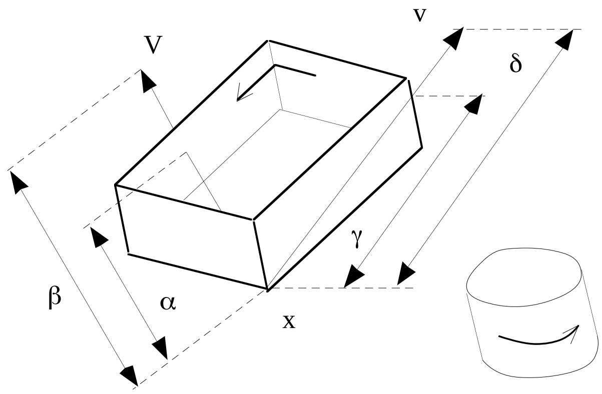

Figure 8. A 2-covector at x needs two pairs of parallel planes (with a way to tell which is first, here the caret in front of x). So it has a definite associated spatial direction (its kernel, to which its vector proxy will be parallel). The length of the icon along this direction is not meaningful. Note the suggestion of a flux tube. (Strong fields correspond to narrow tubes.) The effect on the pair of vectors is the ratio $\beta\delta/\alpha\gamma$ (again an affine notion), but you need to shear the cross-section first, to let one plane absorb one of the vectors. Rules of this kind make such icons too cumbersome to be useful as such. On the right, a simplified one.

2.7 Twisted covectors

The same way we defined axial vectors, we may introduce new affine objects,

$$\tilde{\omega} = \{\{\omega, Or\}, \{-\omega, -Or\}, \qquad (13)$$

where ω is a covector. But "axial covectors" would be a poor name for them, wrongly suggesting the existence of some axis which is nowhere in sight on the icon (Fig. 9). We shall call them *twisted* covectors, and unify the convention by thus referring to all objects which "carry orientation in their bag", including axial vectors, from now on *twisted* vectors. Of course vector proxies for twisted covectors are twisted vectors. Having no immediate use for such objects, however, we shall not deal with them right now, but return to the analysis of Maxwell's equations, beginning with *Faraday's law*, in search for motivation.

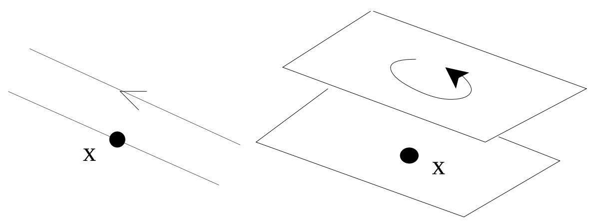

Figure 9. Icons for a *twisted* covector [Bu], in 2 and 3 dimensions.

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