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Differential Geometry

for the student of numerical methods in Electromagnetism

August 1991

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Introduction

0.1 Prerequisite and notations

Beyond calculus, and a previous encounter with Maxwell equations, some familiarity is assumed with what could be called "the functional point of view" in mathematics: that things like functions and vector fields can be considered as points in some abstract space. The reader should therefore know about distances, norms, linear operators, integrable functions, Hilbert space, etc. Some notions about vector and affine spaces, elementary but undertaught, are recalled below.

Notations are classical, except perhaps for a few idiosyncrasies:

First, *all* functions are a priori partial: if f goes from X to Y, it is defined on a part of X, denoted dom(f), its *domain*[1], which in general is not all of X. The set of possible values f(x), called here *codomain* of f (instead of the more familiar *range*), is denoted cod(f). The set of all partial functions from X into Y is denoted $X \rightarrow Y$, and one will write

$$f \in X \rightarrow Y$$

to assert that f is such a function (one will say that f is "of type $X \rightarrow Y$"). "Injective" will refer to a function f such that a point of cod(f) is the image of a single point of dom(f). Then its reciprocal $f^{-1} \in Y \rightarrow X$ is defined, $\text{dom}(f^{-1}) = \text{cod}(f)$, and $\text{cod}(f^{-1}) = \text{dom}(f)$. "Mapping" and "function" will be synonymous. If two functions have the same expression by formulas but different domains, they are deemed distinct.

Next, a construct like

(1) $$x \rightarrow E(x),$$

where E is a Y-valued expression depending on x, denotes a function f of type $X \rightarrow Y$. Since (1) and f then denote the same object of $X \rightarrow Y$, one will feel authorized to write

[1] *Italics* are used either for emphasis, or to warn that a definition of the italicized word is implied or suggested by the context. The distinction between both uses should be easy in all cases.

$$f = x \to E(x)$$

as a definition of f. This is non-ambiguous if $\mathrm{dom}(f) = X$. (Otherwise, one may write, according to the same principle,

$$f = x \in A \to E(x),$$

where A is a subset of X, which is then $\mathrm{dom}(f)$. But this is heavy notation, and we shall try to avoid it.) As an example of the use of this formalism, take the following example: The potential of a charge distribution q can be considered as a function of position x,

$$\varphi = x \to (4\pi)^{-1} \int q(y)\, |x - y|^{-1}\, dy,$$

but as well (and the point of view is then quite different) as an operator which associates φ with q. If G is this operator, one may define it by writing

$$G = q \to (x \to (4\pi)^{-1} \int q(y)\, |x - y|^{-1}\, dy).$$

Provided some precautions are taken, like being generous with parentheses when there is risk of ambiguity, this notation is very helpful.

Last, one will use E_3, or simply E, to denote the Euclidean three-dimensional affine space, and the dot " $\cdot$ " for the scalar product of two vectors of the associated vector space (more on these concepts in Section 0.2). A field of normals is always denoted n. Differentiation is always denoted with ∂, never with a prime. All vector spaces will be real, i.e., with $\mathbb{R}$ as underlying field. One uses $L^2(D)$ for the Hilbert space of square integrable real functions on a domain D of space, $(f, g) = \int_D f\, g = \int_D f(x)\, g(x)\, dx$ for the scalar product, and the norm in this space is $|f| = (f, f)^{1/2} = [\int_D |f(x)|^2\, dx]^{1/2}$.

One has tried to adopt a *geometrical* style, that would avoid confusion between abstract objects and their various concrete representations, and a few words of warning about this may perhaps be helpful. If v is a vector, belonging to a vector space V_n of dimension n, the list $\{v^1, \ldots, v^n\}$ of its components in a given basis, denoted $\underline{v}$, is not the same object as v: $\underline{v}$ is also a vector, but one which belongs to $\mathbb{R}^n$ (the Cartesian product of $\mathbb{R}$ by itself, n times), and though it represents v, it should not be confused with it. Indeed, if the basis is changed, v will be represented by a *different* element of $\mathbb{R}^n$. In the same spirit, one distinguishes between vectors, elements of a vector space V_n, and *covectors*, elements of its dual $V_n{}^*$. A covector is thus a *function* on V_n, linear, and real-valued.

One well knows that a linear function $c \in V_n \rightarrow \mathbb{R}$ can be represented, *after having selected a basis* in V_n, by a vector of $\mathbb{R}^n$, since

(2) $$c(u) = \Sigma_{i=1,\ldots,n} c_i v^i ,$$

thus $\underline{c} = \{c_1, \ldots, c_n\}$ does represent c. But one should not confuse c with $\underline{c}$, or with the vector of V_n represented by $\underline{c}$. In other words, if the choice of a basis allows one, thanks to (2), to establish a bijection between V_n and V_n*, it does not warrant their identification: V_n and V_n* are isomorphic, as vector spaces, but nothing beyond that, and the isomorphism depends on the basis. It is not, as one says, "canonical", i.e., determined by the sole vector space structure of V_n.

The distinction between vectors and covectors is rarely stressed, even less often illustrated by graphical means. Burke [26] has promoted a very natural and not widely enough known convention to this effect (Fig. 1), which seems to come from Schouten [88]. He represents covectors by two parallel straightlines (two parallel planes, in dimension 3), one of them through the origin, the other one a bit farther away, capped with an arrowhead. These two lines (or planes) are meant to represent two level lines (or surfaces) of the function c: the one through the origin is the locus of the v's such that c(v) = 0, the other one of the v's such that c(v) = 1. The closer these two level sets, the *larger* the covector (beware!). For vectors, Burke uses arrows, as we all do.

This convention has many good points for it. First, the action of covector c on vector v can be read off the picture (Fig. 2): it's a ratio of two lengths measured along the same line, a well defined number (independent of the direction of this line), which makes sense without any reference to notions like distance, or angle, which have no meaning in V_n. Next, it provides a very natural graphical rendering of the notion of "tangent" covector to a surface, which is ubiquitous in physics, where displacements are generally vectors, and forces, covectors. The electric field, for instance, is rightfully represented by a covector at each point of space, since it makes itself being felt by the force it exerts on charged particles. This covector is tangent to conductive surfaces (Fig. 3): this property characterizes such surfaces. Remark the invariance of Figs. 2 and 3 with respect to affine transformations: whatever the position of your eye, you see covectors as tangent to the surface (whereas the right angle between a surface and its normal does not project as a right angle in general).

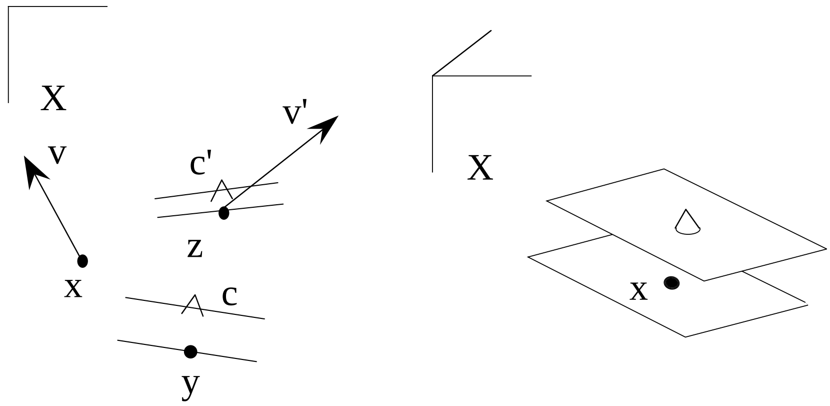


Figure 1. Vectors and covectors according to Burke [26]. On the right, a covector in spatial dimension 3.

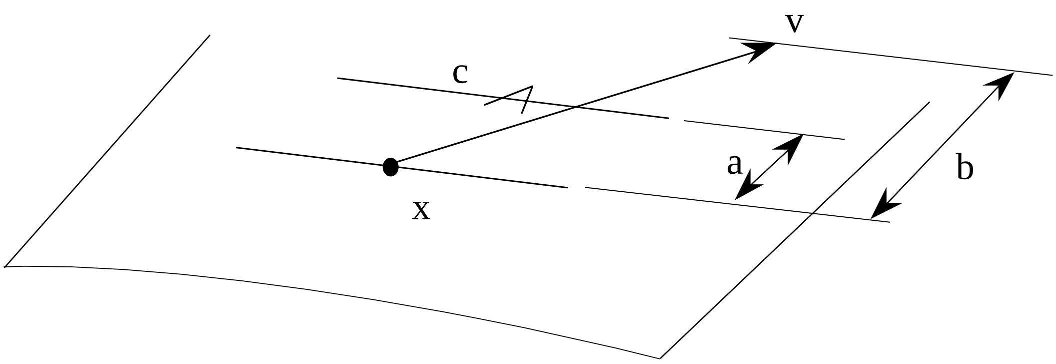


Figure 2. The effect of covector c on vector v, that is c(v), is the ratio b/a. (This ratio is an affine entity, which does not need a metric in order to be defined.) Cf. [26, 27].

When a scalar product is defined on V_n, one may pair vectors and covectors in a more canonical way. One will denote the scalar product in V_n with a dot. Let thus

$$\cdot \in V_n \times V_n \rightarrow \mathbb{R}$$

be such a scalar product, i.e., a bilinear, symmetric function, such that $v \cdot v > 0 \Leftrightarrow v \neq 0$. If c is a covector, there exists a unique vector u_c of V_n such that

$$c(v) = u_c \cdot v.$$

Thus one may define an electric field "vector", a force "vector", etc. But this is taking advantage of an additional structure on V_n (the one conferred on it by the operation $\cdot$) which may not exist, or be quite fortuitous when it does exist. For instance, the position of a pendulum can be specified by two angles θ_1 and θ_2 (Fig. 4). Let $\delta\theta = \{\delta\theta_1, \delta\theta_2\}$ be a displacement of its bob and $\delta\theta'$ another displacement. Which physical meaning can be attributed to the scalar product $\delta\theta_1\, \delta\theta'_1 + \delta\theta_2\, \delta\theta'_2$? None whatsoever. On the contrary, the expression $c_1\, \delta\theta_1 + c_2\, \delta\theta_2$ can be interpreted as the effect of a covector c on the displacement $\delta\theta$ (**Exercise 1**: what is the physical meaning of the c_is? and of the full expression?). With the latter scalar product, one may always associate a vector with c. But what sense would it make to identify something which physically is a torque with something which looks rather like an angle, or the variation of an angle?

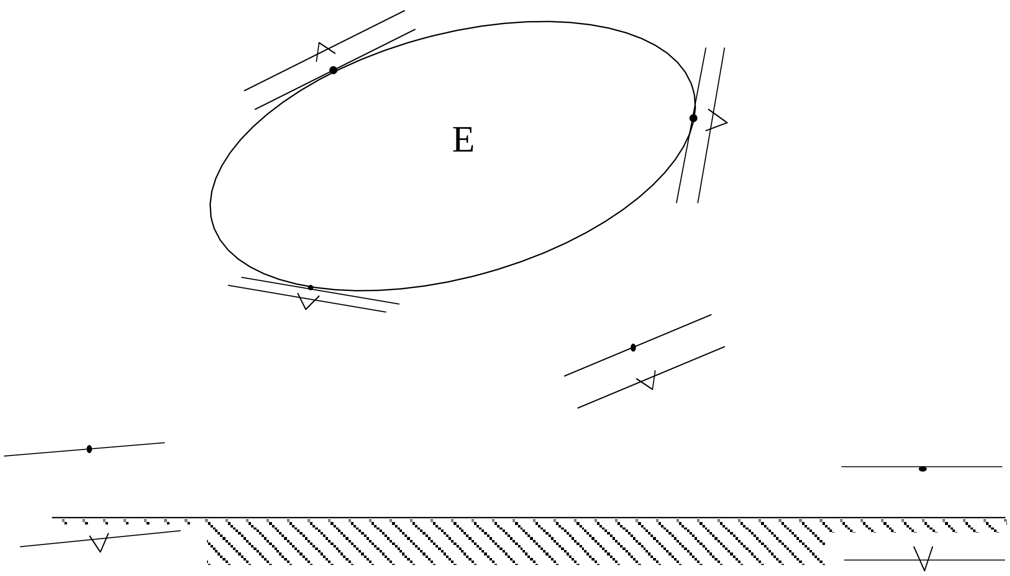


Figure 3. The "electric field" covector at a few points of the space lying between an electrode E at potential 1 and the ground.

So we shall not confuse V_n with its dual. However, there are cases in which a distinguished scalar product exists on V_n. One then calls *Euclidean space of dimension* n the pair $\{V_n, \cdot\}$, i.e., V_n endowed with the structure which stems from this scalar product (including the notions of distance, angle, area, volume, etc.), and one reserves the notation E_n for it. Ordinary space is E_3, as we said earlier.

We shall not confuse *vector* and *affine* space either. An affine space (whose elements are then called "points") is a vector space "deprived from its origin", so that one cannot add two points, or multiply a point by a scalar. But one can still consider the midpoint of the segment linking two points, and more generally the barycenter w.r.t. to real weighting coefficients, and take the ratio b/a of Fig. 2.

The difference of two points is a vector (belonging to a vector space which is said to be *associated* with the affine space), and this is what gives meaning to

(3) $$x = \Sigma_{i = 0, \ldots, n} \lambda^i x_i$$

where the x_is are points, and $\Sigma_i \lambda^i = 1$. For (3) reads as

$$\Sigma_{i = 0, \ldots, n} \lambda^i (x_i - x) = 0,$$

meaning that x is the barycenter of the x_is with weights λ^i. The set of points of the form (3) is what is called an *affine subspace*. If it happens to be the whole space, and if all points x_i are necessary for this to be true, n is the *dimension* of the space. In that case, the λ^is of (3), considered as functions of x, are the *barycentric coordinates* of x in the *basis* of the x_is. A function (of the variable x) which is linear with respect to the barycentric coordinates of x, in some basis, is said to be *affine*. (This property then holds in any basis.)

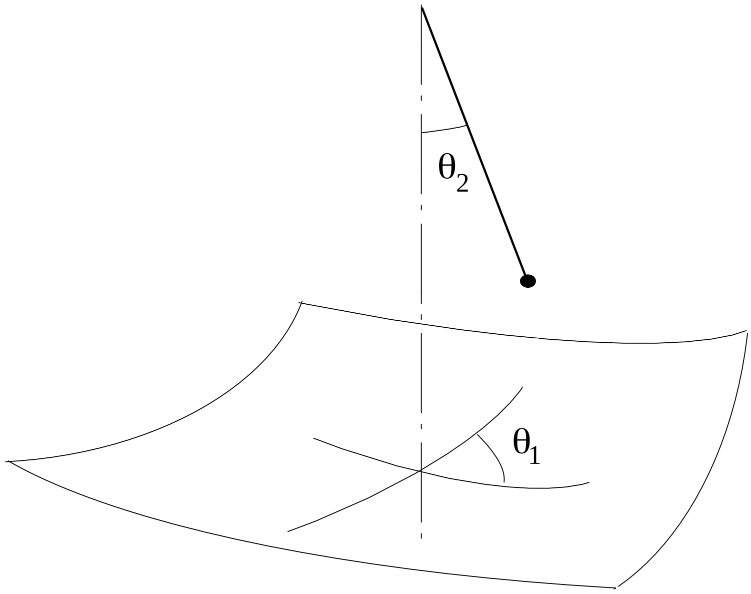


Figure 4. Configuration parameters for a pendulum.

No particular notation has been reserved for affine spaces: the context (the fact that we have been speaking of points or of vectors) should be enough to tell whether we mean the affine or the vector space. Actually, when working in V_n or E_n, both structures are often needed simultaneously, for physics needs not only vectors and covectors, but "bound vectors", which are pairs consisting of a point x and a vector v (one will then say, with some abuse, that v is a "vector at x").

0.2 Why study differential geometry ?

While the intrusion of differential geometry in eddy-currents theory is a recent phenomenon [7, 33, 50, 56], Electromagnetism in the large has for long made a substantial use of its concepts, especially differential forms (cf. e.g., [27], [89]). This modern point of view was anticipated by Maxwell himself [70], and by Kelvin [95]. Moreover, physics as a whole, nowadays, undergoes geometrization, and in some areas, like mechanics of the continua, the use of differential geometry is much more intensive than what we shall try to foster here (cf., e.g., [1], [68], [39]).

But eddy-currents theory, and their computation, are parts of engineering science, and the latter seems to be less concerned, up to now, by this geometrizing trend. One easily understands why. Engineering sciences are less interested in understanding phenomena than in predicting them, in precise quantitative terms. Hence they require computation, which implies a representation of abstract geometric objects with the help of numbers. Measuring a magnetic field about a point x, for instance, will yield three numbers, corresponding to the intensity of the field (or rather, of the magnetic induction) along three directions. These three numbers being all one needs to know about this induction (and its effects), the temptation is strong to identify them with the induction at x (call it $b(x)$). We may well argue that they are only a particular concrete representation of $b(x)$, that this object $b(x)$ is of a very different nature than a mere triple of numbers, that it is, as we shall see, a "2-covector". Such a discourse has no urgent appeal to an engineer, who has other and more pressing things to care about. Only the proof that this new viewpoint brings *computational* advantages can divert the attention of engineers and convince them to take the time to study it.

There is a historical precedent: vector calculus. Strange as it may appear today, it is only during the Fifties that notions like "vector space", "linear transformation", etc., have become commonplace in engineering science. (In France, they did not enter the curricula of so-called "preparatory classes", where candidates to engineering schools are trained, before about 1960.) When this happened, it was clearly due to the realization of the power of the matrix formalism as a computing tool (enhanced, as it then was, by electronic computers), not to some late recognition of the conceptual simplification brought into science by the notions of vectors and of linearity in general, which was obvious since the end of 19th century.

Time does marvels. One is so fond of vectors today that students protest when you strip them of these so pretty arrows, straight or curved as the case may be, and that some Journals set them in a special face (and insist on your compliance to such conventions).

One could jeer for pages on such inertia phenomena, observe that physicists have not yet fully adopted Schwartz's distributions, a tool custom-made to fit their needs, or quote from Kron, persecuted all his life by international institutions of electric science and their mandarins, who maliciously tried to force him to substitute "matrix" for "tensor" everywhere in his papers [60]. That nowadays, among eddy-current specialists, one still prefers to see h and b, for instance, as vector fields and not as differential forms of degree 1 and 2 respectively, should be blamed on this kind of inertia. But this is besides the point. If we are right in thinking that novel mathematical objects enter the toolkit of engineers only when they lend themselves to *computation,* we must consider whether things are ripe, from this point of view, as far as differential forms are concerned.

The answer is not obvious. On the one hand, yes, there exists a calculus based on differential forms. The classical formulas—Green, Ostrogradskii, etc.—, or vector analysis identities like $\text{rot rot} = \text{grad div} - \Delta$, all have much simpler expressions in terms of differential forms. From this point of view, we do have there a workable computing tool, even better than vector analysis. The understandable objection that "computers can perhaps understand real numbers, but not differential forms" does not hold water: one may *code* numerical methods based on differential geometric concepts, thanks to elementary objects (in both senses: mathematical objects and program objects) called Whitney forms [104], which are to differential forms what shape-functions are to functions in finite element theory [6].

But on the other hand, no, differential forms cannot exclude vector fields from current usage. Consider, for instance, the two Green's formulas[1]

(4) $$\int_D \operatorname{div} b\ \varphi + \int_D b \cdot \operatorname{grad} \varphi = \int_{\partial D} n \cdot b\ \varphi,$$

(5) $$\int_D \operatorname{rot} h \cdot a - \int_D h \cdot \operatorname{rot} a = \int_{\partial D} n \times h \cdot a.$$

They are special cases of a single formula, which applies in dimension n for all integers p from 1 to $n - 1$. Here $n = 3$, so there are only two possible values of p, hence the two above formulas. But for this reason, there is also a symmetry, a duality between the cases $p = 1$ and $p = 2$, which are particular to dimension 3, and which play an essential rôle. Quite often, to be "forced" by conventional vector notation to write twice the "same" formula will be illuminating, by emphasizing this duality. One may find there a good reason to stick with the "old" notations rot, div, etc.

[1] I find convenient to call them that, but it's an abuse (cf. the index of [50]). But a mild one, since there are already so many Green's formulas around . . .

Actually, as we shall see in Chap. 5, all differential forms in dimension 3 are representable either by functions or by vector fields. (In dimension 4, things already go differently: the electromagnetic field tensor F_{ij} is a form of degree 2, and it has no representation as a vector field in general.) Thus, everything that can be done with forms can be done with vector fields as well, and often more simply[1]. The advantage of differential forms, in this context, is that they help understand what one is doing: They explain some formal analogies (like between (4) and (5) above) which otherwise would look fortuitous, they suggest interesting symmetries.

So: the conceptual interest of differential forms is certain, but their benefits to computation are not so obvious if one does not go beyond dimension 3.

A reasonable stand at the present time could therefore be: talk vectors, work with functions and vector fields, but while being fully aware of their geometric nature as differential forms, and being able to make it explicit when needed, especially when such a move helps understand symmetries and analogies. The present lecture notes should be enough from this point of view, even if they fall short from what should be requested of a development which would frankly rely on differential geometry[2]. (There is no shortage, anyway, of texts of such a nature [32, 33, 50, 73, 89, 97, 103, etc.].)

The first three Chapters proceed along the same path as most treatises (cf., e.g., [62]): notion of manifold, construction of manifolds, tangent vectors, tangent space and its dual, differential forms, orientation and integration. (More space than usual, however, is devoted to orientation-related notions: "twisted" differential forms, etc.) All this can be done without introducing more structure than that of differentiable manifold. The structures added in Chap. 4: "standard density", "metric", then allow us to make the connection with standard objects of vector analysis. One thus arrives, in Chap. 5, to three-dimensional Euclidean space, where all the familiar notions: operators grad, rot, div, Green formulas, etc., are waiting to be revisited.

We do however acknowledge the right to be reluctant to follow this classical itinerary, which is unavoidably wearing, even if most mathematical technicalities are left aside, as we tried to do. It's a fact, a bit paradoxical but inherent in the nature of mathematical apprenticeship, that the richer the structures, the easier they are to

[1] But not always: in some field-computation problems in spatially periodic structures, one meets exotic three-dimensional manifolds, non orientable, on which the "translation" in terms of vector fields may become exacting.

[2] The most salient omission is the "Lie derivative", indispensable to the student of electromagnetic forces.

understand and to handle: E_3 is subjectively "simpler" than the underlying three-dimensional manifold, and closer to our intuition. For the reader who would therefore prefer to directly embark on Chap. 5, one has tried to make it logically independent. In this chapter, one does not forgo the project to emphasize distinctions which are blurred by elementary geometry: vectors vs. covectors, etc., quite the contrary. But thanks to the presence of the strong structures of E_3, the essential definitions of the first four chapters can be recast in much simpler form.

A possible reading strategy may thus consist in beginning with Chap. 5. One will find there frequent cross-referencing to Chaps. 1 to 4, which one will probably want to follow up, in order to absorb this material on a piecemeal basis. The reader doing so is however advised to neglect, at first reading, all mentions of "twisted forms" and of orientation-related problems.

Chapter 1

Manifolds

A manifold is a set equipped with some structure which makes it look like $\mathbb{R}^n$ in the vicinity of any of its points: a closed surface, for instance, looks like $\mathbb{R}^2$ locally, the set of all possible rotations of a solid with respect to one of its points locally looks like $\mathbb{R}^3$, etc.

The concept of manifold is intended to model the somewhat fuzzy idea of "multi-dimensional continuum", as encountered in physics. The Earth surface, for instance (from the point of view of geodesy) is a bidimensional continuum: two coordinates are needed to specify a location. The variety of colors that a normal human eye can perceive is, it seems, a three-dimensional continuum. The set of all possible configurations of a car, from the point of view of the driver trying to enter a tight parking slot, is a continuum in four dimensions: two for the position of the centre of the car, one for its orientation, one for the angle of the front wheels. Etc. The mathematical concept of manifold is designed to serve in the modelling of situations where such continua play a rôle.

1.1 Definitions

For this, it must reflect the intuitive image we have of such continua: besides the possibility of specifying points by giving their coordinates, which will be taken into account by the notion of chart, the concept of manifold should incorporate some flavor of homogeneity and regularity. For instance, the sets pictured in Fig. 5 are not manifolds, for lack of homogeneity. The surface of a cube, for lack of regularity at the corners, is a "topological" manifold, not a "differentiable" one. The definition will either discard them, or make the distinction precise.

1.1.1 Differentiable manifolds

Definition 1 (Fig. 6): *A* manifold *of dimension* n *is the assembly of the following elements*: 1°- *a set* X, 2°- *a family of functions* ψ_α, *of type* $X \to \mathbb{R}^n$, *the so-called* charts *(their collection,* $\{\psi_\alpha : \alpha \in \mathcal{A}\}$, *being called the* atlas), *with the following properties*:

(a) $\mathrm{cod}(\psi_\alpha)$ *is a connected open set of* $\mathbb{R}^n$ *(non empty),*

(b) ψ_α *is injective,*

(c) X *is the set-union of the* $\mathrm{dom}(\psi_\alpha)$,

(d) *The* ψ_α*s are "compatible" (as will be explained).*

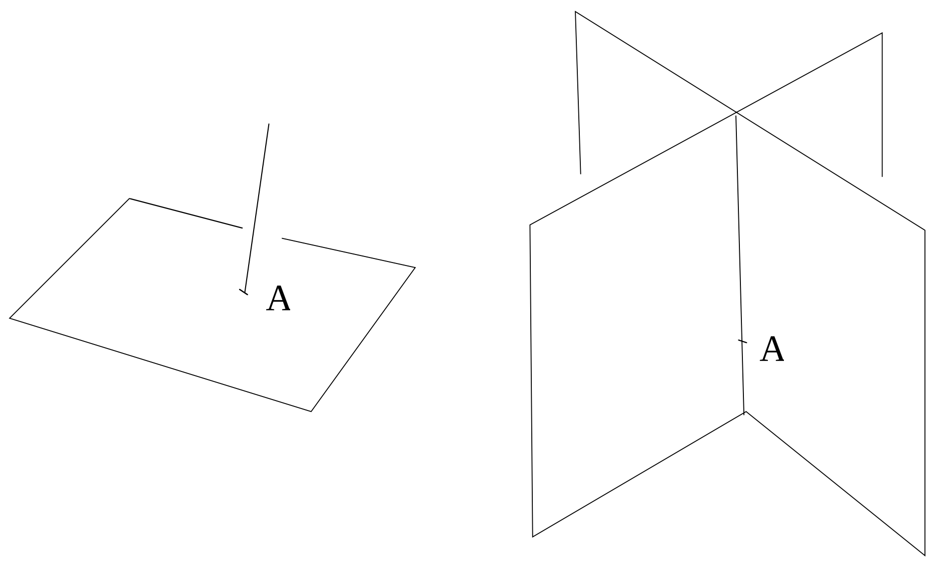


Figure 5. The union of two planes, or of a half-line and a plane, is not a manifold: no neighborhood of A looks like a chunk of $\mathbb{R}^n$, whatever n. ("Neighborhood" should here be understood in the sense of the natural topology of these sets, the one induced by $\mathbb{R}^3$.)

If X is not empty, there is at least one chart, according to point (c). A single chart may sometimes be enough: if for instance X is a vector space, and if $\psi(x) = \{x^1, \ldots, x^n\}$, where the x^i are the components of the vector x in some frame of basis vectors, then ψ turns X into a manifold of dimension n (the one dubbed V_n in the Introduction). Similarly, if X is an affine space, n barycentric coordinates (out of n + 1) constitute a chart. The charts are what physicists call "reference frame", or "system of coordinates".

When one looks at two different charts in a real-life atlas, for instance those of Europe and of former USSR, one can tell them as "compatible": the Russia of both charts is the same territory, only with different scales, shapes and orientations. Condition (d) is crafted in order to grasp this notion of compatibility: Let us set, for two charts α and β, with overlapping domains,

$$\psi_{\alpha\beta} = \psi_\alpha|_{\mathrm{dom}(\psi_\beta)},\ \psi_{\beta\alpha} = \psi_\beta|_{\mathrm{dom}(\psi_\alpha)},$$

i.e., for each of these charts, its restriction to the domain of the other one, and

(6) $$\gamma_{\alpha\beta} = \psi_{\beta\alpha} \circ \psi_{\alpha\beta}^{-1}.$$

Then the *transition function* $\gamma_{\alpha\beta}$ is of type $\mathbb{R}^n \to \mathbb{R}^n$ (its domain is a part of cod(ψ_α), cf. Fig. 7) and its continuity, its differentiability, etc., make sense (whereas such notions are meaningless as regards the charts themselves). According to the geographical analogy, $\gamma_{\alpha\beta}$ should at least be *continuous*. Hence the following complement:

Definition 1 (continued): *The* ψ_α *are* C^k*-compatible, meaning that, for some* k ≥ 0, *the* $\gamma_{\alpha\beta}$ *of* (6) *are all of class* C^k *(i.e., with open domain and* k *times continuously differentiable).*

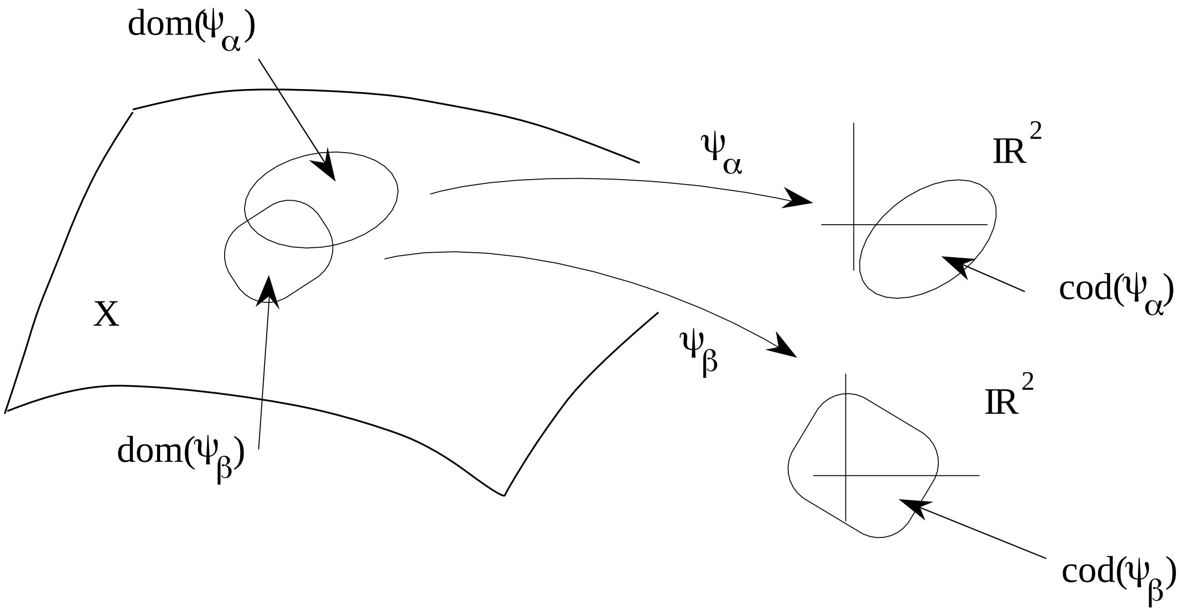


Figure 6. Concept of manifold.

If k = 0, we are dealing with a *topological* manifold, and with a *differentiable* manifold if k ≥ 1. The required differentiability may depend on the situation. We shall agree once and for all that all our manifolds are *smooth*, i.e., of class C^k for all k, or as one says, C^∞. (*Smooth* also refers to the transition functions themselves. Note that one might be interested in other properties of these functions: their linearity, their analyticity, etc., hence as many specialized notions of manifolds.)

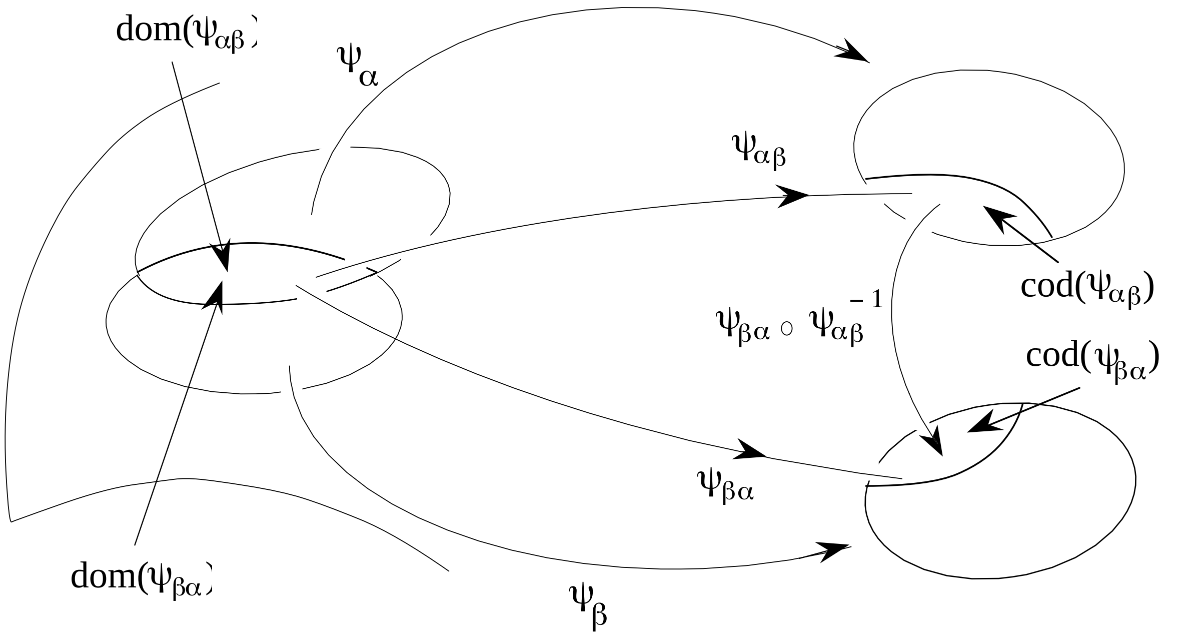


Figure 7. Compatibility of two charts.

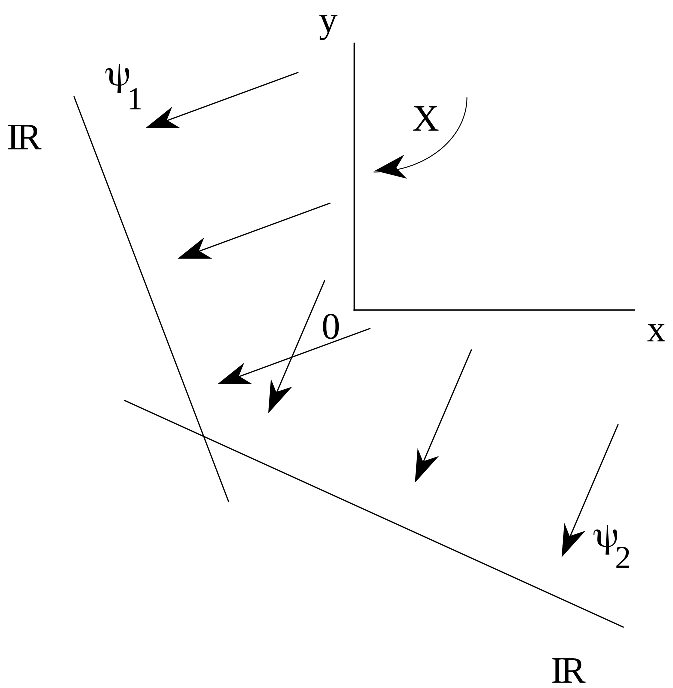


Figure 8. By orthogonal projection of X (the set-union of two half-axes) onto non parallel lines of the plane, one gets charts of X (each with domain X), which are not C^1-compatible.

Exercise 2: Consider the manifold made of the subset $\{\{x, y\} : (x = 0$ and $y \geq 0)$ or $(x \geq 0$ and $y = 0)\}$ of the plane $\mathbb{R}^2$, with the two charts suggested by Fig. 8. Show that it is of class C^0, but no more. Discuss the above reference to the surface of a cube.

Thus X, which, stripped of its charts, would be an amorphous set, inherits from them a very rich structure. For instance, X has a topology, the one for which open sets are the preimages of the open sets of $\mathbb{R}^n$ under the ψ_α (they do satisfy the axioms for open sets, thanks to the compatibility condition). So one is entitled to speak of a continuous function from one manifold into another one, of a homeomorphism, etc. But there is more: If X and Y are two manifolds of class C^1, of respective dimensions m and n, one may speak of the differentiability of a function f of type $X \to Y$: one refers for this to the maps of type $\mathbb{R}^m \to \mathbb{R}^n$ obtained by composition of f with appropriate charts. (One will call *regular* a function which can be differentiated indefinitely[1].) Two manifolds are *diffeomorphic* if they admit of a one-to-one mapping, differentiable in both directions. Then, their dimensions are the same. (Later, we shall see what the derivative of a function of type $X \to Y$ is.)

This structure, however, does not allow one to talk about a "distance" on X. If one has a need for this, one must endow X with additional structure, as we shall do later. It is not sufficient either to decide whether X has the *Hausdorff separation property*, i.e., whether non-intersecting neighborhoods can always be found around two distinct points. This is an independent hypothesis, which is generally understood: all our manifolds will be Hausdorff, unless this is explicitly denied. Similarly, X has no reason to be *separable* (i.e., to possess an enumerable set of open sets from which all open sets can be obtained by union operations), but all our manifolds will be supposed to have this property.

Exercise 3. Under which conditions is a manifold *connected?*

To which extent does this structure on X, as provided by charts, depend on these charts? The geographical analogy, again, suggests the answer. Consider two atlases of Britain, $\mathcal{A}_1$ and $\mathcal{A}_2$: one can tell they chart the same territory from the fact that any chart from the first one is compatible with any chart from the second one (when their domains do overlap), which allows one to merge the two atlases as a single one. One will therefore say that two manifolds $\{X, \mathcal{A}_1\}$ and $\{X, \mathcal{A}_2\}$, on the same set X, are *equivalent* if all charts of $\mathcal{A}_1$ are compatible with all those of $\mathcal{A}_2$. This suggests that the mathematical objects we really want, i.e., those which can serve as models for the intuitive idea about continua we started from, are in fact not the manifolds in the somewhat narrow sense of Def. 1, but their equivalence classes with respect to the relation just defined. Each of these classes contains a distinguished representative, the atlas of which is the collection of *all* mutually compatible charts. This atlas is said to be *complete* (or *maximal*). So when we

[1] or at least, as many times as required by the situation.

shall mention a manifold, we shall be referring to the structure conferred on set X by the complete atlas (even if as few as one or two charts may be enough to describe it, and to perform computations when necessary).

1.1.2 Manifolds with boundary

Now, an objection. With the previous definition, all points of a manifold are similar, to the extent that their neighborhoods all look, in the precise sense we just elaborated, like a part of $\mathbb{R}^n$. This is not always satisfactory. For instance, if a car is blocked against the kerb, or if its steering wheel is locked, the car is clearly "at the boundary" of its configuration set: the neighborhood of such a configuration does not look like an open set of $\mathbb{R}^4$. Many multidimensional continua do have, in this way, a boundary. The corresponding mathematical notion is that of *manifold with boundary*, which is obtained by allowing X to look like a closed half-space of $\mathbb{R}^n$, instead of $\mathbb{R}^n$, in the neighborhood of some points. Fig. 9 should be enough to convey the idea (and one may refer to [46] or [84], for instance, for precise definitions). A manifold in the former narrow sense (i.e., one without boundary) then becomes a special case of manifold with boundary. (In the sequel we shall omit the words "with boundary", unless this is required for the sake of clarity.)

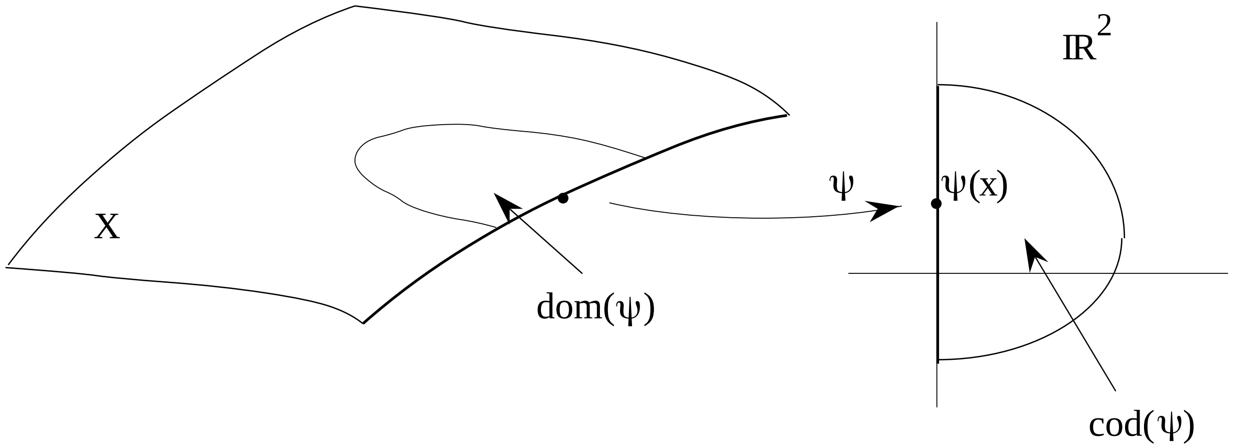


Figure 9. By convention, heavy lines correspond to the boundary, and thin lines are not part of X.

Let us concede that even this broadened definition is not completely satisfactory, for it does not discriminate between various kinds of boundary points: edges, corners, etc. There does not seem to have been much interest in Mathematics in the task of working out the concepts necessary to deal with such fine distinctions, but it could be done if really needed (cf., e.g., the concepts of "pseudo-manifold" and "pseudo-boundary" in [90], vol. 2, pp. 148 and 158), and we shall rest on this.

Exercise 4: A color is often specified by giving three intensities of primary colors. In another system, one makes use of three variables, called luminance, hue and saturation. Show that these systems can be understood as two charts on the same "manifold of colors". Describe it. Allow for the possibility of continuously going from red to purple by *two* essentially different routes. (For an account of the "theories of color", cf. [40], which refers to the classics: Aristotle, Newton, Gœthe, Grassmann, Maxwell . . . Cf. [47] for a precise description of one of these "theories", i.e., from the present point of view, one of the charts which have been proposed for the colors manifold.)

Exercise 5: Normal vectors (of all lengths) to a surface form a manifold (not to be confused with the set-union of lines normal to the surface!). Provide an atlas for it.

Exercise 6: In a given plane, the set of all equilateral triangles of unit side-length has a manifold structure. Describe it (dimension? charts? other, diffeomorphic manifold(s)?).

Exercise 7: Give the set of all triangles inscribed in the unit circle, non degenerated, and *isosceles*, a manifold structure.

Which manifolds can one come across with in numerical electrotechnics? First of all, regions on which one may have to compute fields: parts of E, open or closed, or (in the case of, e.g., the computation of eddy-currents on thin conductive sheets) surfaces embedded in E, with or without a boundary. But that is not the end of it. When one wants to compute a spatially periodical field, as e.g., in an alternator, the computational domain can be reduced to some fraction of space, that may be called the "symmetry cell". But the underlying manifold is not this part of space, it is what is obtained by suitable identification of opposite sides of the symmetry cell (more about this later, Section 1.4.2). Finally, other kinds of continua than "spatial" ones (as were all the previous ones) may claim consideration. For instance, when one measures a magnetic field in some spatial region, one is really roaming inside a manifold of dimension six (three for the position x, three for b(x), so that each measurement result is described by six parameters).

We shall therefore examine how such non-elementary manifolds can be constructed from simpler manifolds. There are basically two ideas: *1°*- gluing, *2°*- forming products, which find their synthesis in the notion of "fibered manifold", or "bundle".

1.2 Construction of manifolds: gluing

To illustrate the notion of gluing, let us start from a manifold (with boundary) like the unit square C^2 of Fig. 10. Let us introduce the relation

$$A \sim A' \ \underline{\text{if}}\ (y(A') = 1,\ y(A) = 0\ \underline{\text{and}}\ x(A) = x(A')).$$

By assuming that $A \sim A$ and that $A' \sim A$ if $A \sim A'$, one obtains an equivalence relation over C^2. Let C be the set of equivalence classes, or *quotient* of C^2 with respect to this relation. One will easily see how to provide C with charts in order to turn it into a manifold (with boundary) of dimension 2 (**Exercise 8:** describe such a chart in the neighborhood of the point $A \equiv A'$ of C). Clearly, there is a surjection $f \in C^2 \rightarrow C$, with $f(A) = f(A')$, which respects the manifold structure except at points like A or A'.

The upper and lower edges of C^2, can be glued in another way, the relation then being

$$A \sim A' \ \underline{\text{if}}\ (y(A') = 1\ \underline{\text{and}}\ y(A) = 0\ \underline{\text{and}}\ x(A) = 1 - x(A')).$$

The manifold thus obtained is of course something else entirely. (It's the Möbius strip, denoted MS.)

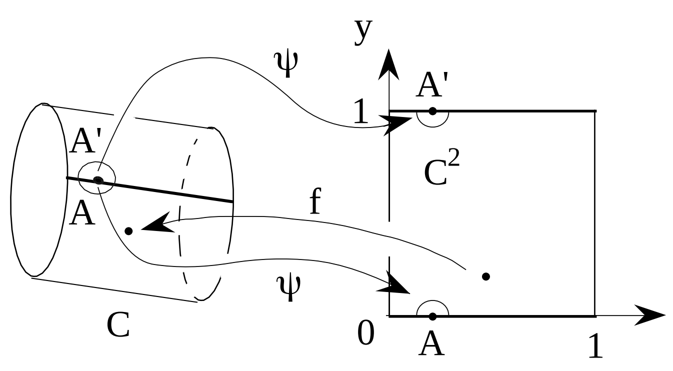


Figure 10. Manifold C obtained by identification of the upper side and lower side of a square.

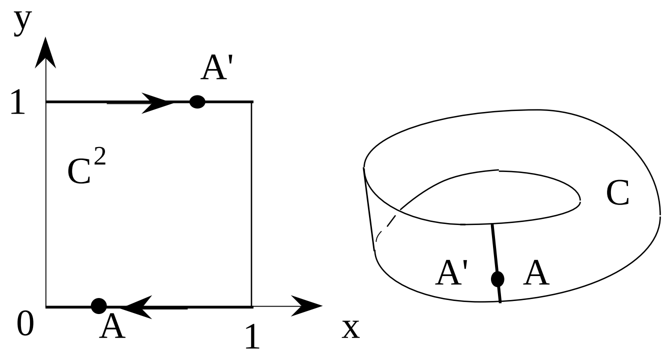


Figure 11. Möbius strip obtained by identifying upper side and lower side after reversal of one of these.

A warning, at this stage: Fig. 10 (or Fig. 11) shows *more* than the manifold C (or MS), it shows this manifold as embedded in space E, thanks to a perspective view. Let it be well understood that this is a tribute, not mandatory on principle grounds, to the taste of many of us for the visualization, plane or spatial, of mathematical objects. A manifold should actually be conceived in abstracto, for itself, not as a part of some "ambient space". For instance, let us think of the manifold each point of which is a line of E_3 passing through the origin. This is a two-dimensional manifold which can be conceived without any reference to any of its possible representations as a surface immersed in E_3. Its name is *projective plane*. Similarly, Klein' bottle of Fig. 12 is a quite simple manifold of dimension 2. What makes the quaint charm of such geometric objects is not their intrinsic structure as manifolds but the complexity of their representations in E_3: by playing with scissors and a Möbius strip, one does not actually study the manifold MS but rather its various possible immersions into E_3. (Cf. [8, 25, 41, 54, 81], among others, for games of this kind, sometimes actually quite serious [91].) In fact, according to a general result due to Whitney, a separable manifold of dimension n can always be embedded into $\mathbb{R}^{2n+1}$ (the words "embed", "immerse", etc., have a precise meaning, that will be disclosed later: cf. Def. 9, p. 63). But there is no particular physical interpretation to this encompassing manifold. For instance, the configuration space of a double pendulum oscillating in a given vertical plane is the surface of a torus, but the three-dimensional space in which one can visualize this torus is conceptually irrelevant: it has no particular physical meaning.

Exercise 9: What is the configuration space of the pendulum of Fig. 4 (p. 6)?

Exercise 10: What is the configuration manifold of a car with locked front-wheels *1*°- on dry ground? *2*°- on ice?

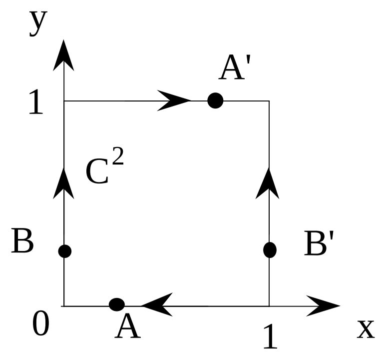


Figure 12. Klein bottle. The equivalence relation A ~ A' is defined by ((y(A) = (0 or 1) and x(A) + x(A') = 1) or (x(A) = (0 or 1) and y(A) = y(A'))).

Exercise 11: Weld two by two the edges of a square in order to get a torus.

Exercise 12: Describe the projective plane with three charts. Can *two* charts be enough?

Exercise 13: Make a projective plane out of a disk, by gluing the boundary onto itself. Try to draw the result as immersed in E_3.

Exercise 14: Weld two by two the edges of a square in order to get a projective plane.

Exercise 15: Show how to glue two Möbius strips into a Klein bottle.

Our gluing moves, so far, yielded manifolds, but this is not always so. For instance, let us start from $\mathbb{R}$ and let us identify $x = 1$ with $y = -1$ by putting them into the same equivalence class (each of the remaining point being a class by itself). One gets a topological space this way, but not a manifold, for the neighborhoods of the welding point cannot be assimilated to neighborhoods in $\mathbb{R}$ (Fig. 13). Same thing about the equivalence relation $x \sim y \Leftrightarrow (|x| \geq 1$ and $y = -x)$. There is no simple general criterion, aside from the definition itself, saying whether the result will be a manifold or not: one has to check that the charts around points to be identified do match properly. (See [46] for a few tempering examples.)

So far we have made our gluing job by identifying two parts of the same manifold. One may as well work with two manifolds X and Y by identifying a part A of X and a part B of Y, provided there exists an injective mapping $f \in X \to Y$, with $\text{dom}(f) = A$ and $\text{cod}(f) = B$. One first takes the manifold $X \cup Y$ (over the set $X \cup Y$, with the union of the two atlases for its atlas), then the relation $y = f(x)$ between pairs of points of $X \cup Y$, and one goes on as above.

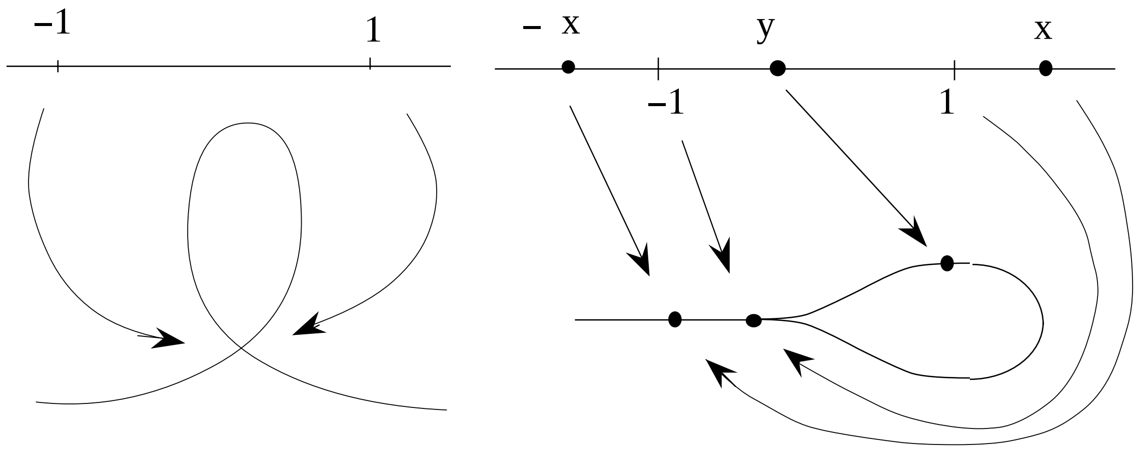


Figure 13. Ways of gluing which do not yield manifolds

One may for instance glue two half-planes (Fig. 14) and get a plane. Fig. 14 represents an injection of it into space, which is not an embedding (not an immersion either). Such situations are not to be excluded in physical applications: suppose one has to compute direct currents (not induced currents) on a conductor made out of two conductive sheets welded together, as in Fig. 14. The geometric singularity at the junction of the two sheets, being physically irrelevant, should not be a concern in the mathematical modelling process. It all goes as if one had to work on the manifold of dimension 2 obtained by (mathematical) gluing, without any regard to which way it is injected into E.

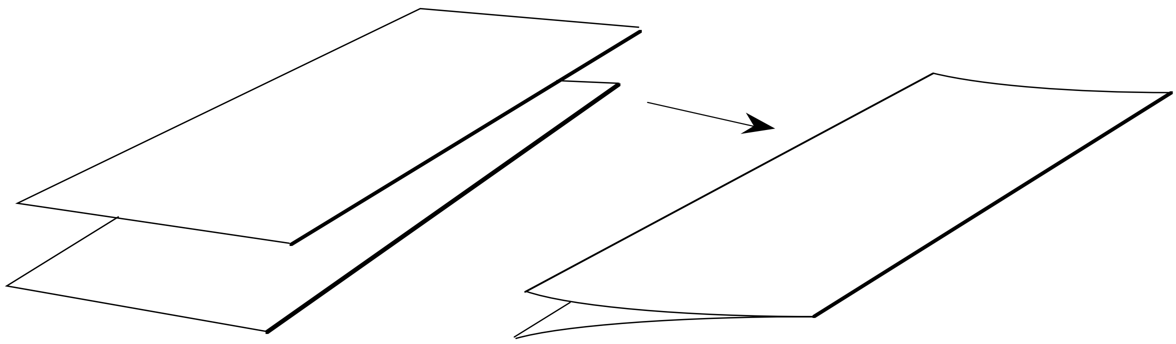


Figure 14.

In the same spirit, Fig. 15 represents a "wild" injection into E of a manifold with boundary of dimension 2 obtained quite regularly by gluing. One has taken a rectangle (manifold with boundary), removed two disks (hence, again, a manifold with boundary), then glued the edges of the holes together in the way indicated by the figure (instead of the other possible one). One will check (by describing a chart around a point of the edge of the hole) that this procedure does yield a manifold. The latter is not orientable (a concept on which we shall have more to say), just like a Möbius strip, but it's not MS. (It's a Klein bottle minus a disk.) There, again, one may very well have to compute currents on a contraption like that of Fig. 15, for instance in order to determine its ohmic resistance.

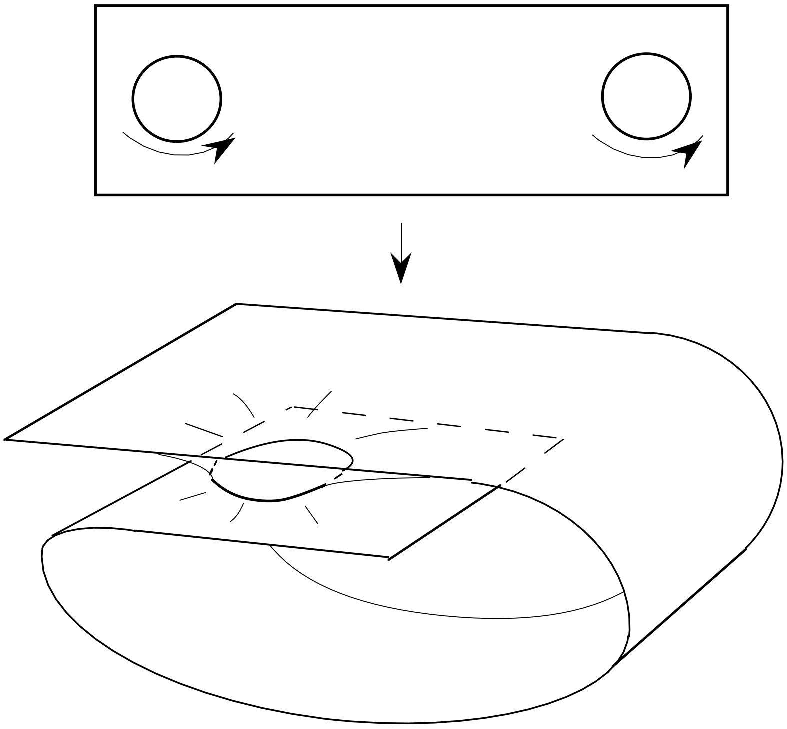


Figure 15.

As a slightly more complex variation, Fig. 16 shows a manifold of dimension 2 obtained by gluing a cylinder (the manifold with boundary of Fig. 10) to the edges of two holes left by the removal of two disks from a rectangle. One may easily imagine an eddy-currents problem on such a surface, or on an even more complex one. In all these cases, the domain of computation is therefore a manifold of dimension 2 with boundary, not necessarily orientable.

Such manifold constructions are not made in the mind only. When one designs a workpiece with the help of a CAD system, one is actually charting some manifold. There are three differences, however. First (the less consequential one) the charts "go the other way", in general: from a part of $\mathbb{R}^3$ to the manifold to be constructed. Next, objects constructed this way are not always manifolds, for some of the "monsters" previously barred by the definition (cf. Fig. 5) might be relevant, and should be describable by the system. The right mathematical concept does exist: that of "cell complex" (cf. [44], p. 134, or [53], Chap. 7, or [3], or [84]), but such complexes are more general than manifolds. Last, these software systems are designed to describe manifolds embedded in three-space only, and they don't take the concept of atlas into account: each part of the workpiece is described by a single chart. One may view these characteristics as weaknesses of such systems, and think that some differential geometry could help in improving them.

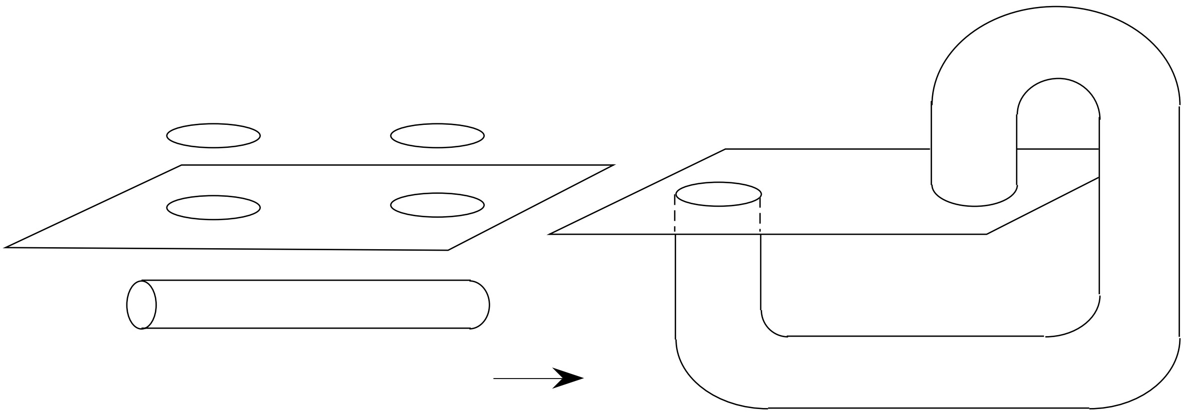


Figure 16. Non orientable surface obtained by surgical procedures: dissecting, rearranging, gluing back...

The point of view of "inverted charts" (let's say, more elegantly, of "parametric representations") can be systematized. Let U and V be two open sets of $\mathbb{R}^n$, and $f \in U \rightarrow V$ a bijection (between dom(f) $\subset$ U and cod(f) $\subset$ V), differentiable in both directions. By gluing according to f, one gets a manifold. This can be generalised to a family of open sets U_i and to "gluing functions" $f_{ij} \in U_i \rightarrow V_j$, which must be compatible, in a sense which can easily be made precise. This does correspond well to the idea of a continuum which locally looks like $\mathbb{R}^n$, since it was built by patching up pieces of $\mathbb{R}^n$. One could give of manifolds a definition different from Def. 1 (although equivalent) by working from this point of view, which can be qualified as *constructive*, or *synthetic:* one builds a manifold by patching pieces together, whereas the point of view of Def. 1 was rather *analytic:* given a manifold, one scans it piece by piece, with the help of charts.

Exercise 16: Fig. 17 describes two manifolds of dimension 1 obtained by gluing two copies of the segment]0, 2[. One of them is not Hausdorff. Why? (See [9] for other examples of "unreasonable" manifolds.)

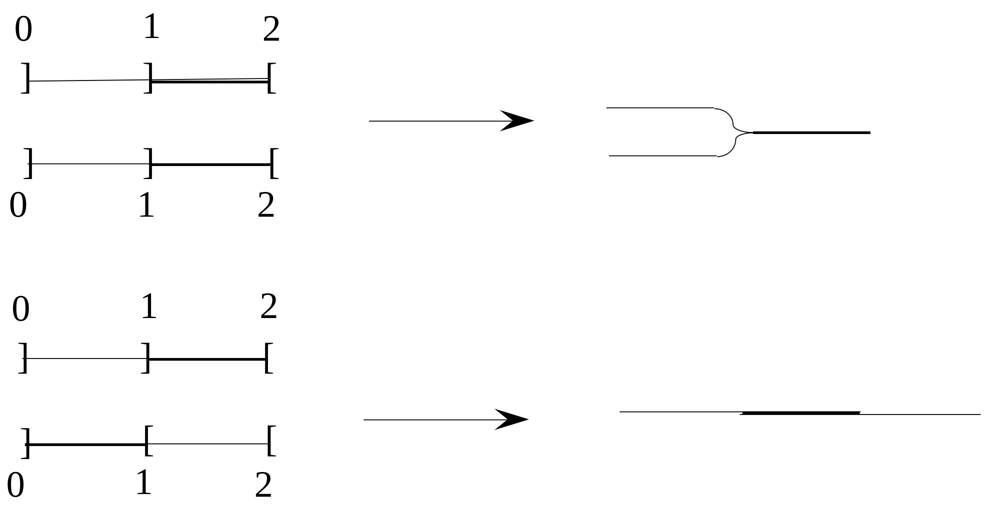


Figure 17. Gluing according to the bijections $x \in]1,2[\to x$ above and $x \in]1,2[\to x - 1$ below

Exercise 17: Show that by gluing two copies of $\mathbb{R}^n$ according to a bijection f between two open sets U and V (thus dom(f) = U and cod(f) = V), one gets a Hausdorff manifold if and only if neither f nor f^{-1} admits of a continuous continuation to a larger open set.

1.3 Construction of manifolds: bundles

The second way to make manifolds consists in taking products. For instance: on a surface B, the "base", one may consider tangent vectors. The continuum formed by all these vectors (each considered as attached to some point of the surface) can be assimilated, locally at least, to the product of B by the vector space V_2. One says this is a *fibered* manifold or *bundle,* of *base* B, of *fibre* V_2. The set of all pairs $\{x, v\}$, where v is a tangent vector at point x, forms the *fibre above* x.

The reader who wishes to arrive quickly to the notions of tangent vector and of differential form can safely jump to Chap. 2 right now.

1.3.1 Bundles

There is no problem in defining the Cartesian product of two manifolds $\{X, \mathcal{A}\}$ and $\{Y, \mathcal{B}\}$: it's $X \times Y$, with the following collection of charts as atlas:

$$\{x, y\} \in X \times Y \to \{\varphi_\alpha(x), \psi_\beta(y)\} \in \mathbb{R}^n \times \mathbb{R}^m.$$

Indeed, as one may check, all these charts are compatible two by two if the φ_αs were, as well as the ψ_βs. But the notion of bundle does not reduce to the notion of product. For instance, the ring of Fig. 10 (p. 18) is a product: that of S_1 (the unit circle) by the segment [0, 1]. But the Möbius strip of Fig. 11 is not one (otherwise, it would have the same global topological structure as the ring, which clearly is not the case). But locally, both the ring and MS do look like the product of $\mathbb{R}$ by [0, 1]. So what makes the difference? We shall try and understand this point in this Section on bundles.

Other example: the manifold of all vectors tangent to a sphere embedded in E_3 (Fig. 18). Again, it looks locally like a product, but is not one. Its points are pairs consisting of a point of the sphere and a vector, based at this point, lying in the tangent plane. Its dimension is 4. If one just looks at tangent vectors whose tails are in a small chunk U of the sphere, this piece of manifold is clearly identifiable with the product $U \times \mathbb{R}^2$. The whole manifold (which we shall meet again under the name of TS_2) looks locally like a Cartesian product. But if it was one, one might assign to each point of the sphere a tangent vector, continuously depending on this point, and nowhere vanishing (cf. Exer. 18). But this is a notorious impossibility (it's the problem of "combing the hedgehog"), ruled out by a celebrated theorem of Brouwer ([5], p. 110, [38], p. 131).

Exercise 18: Make the above argument precise by showing that the mapping $x \to \{x, f\}$ of S_2 into $S_2 \times \mathbb{R}^2$, where $f \neq 0$ is a fixed vector of $\mathbb{R}^2$, is continuous.

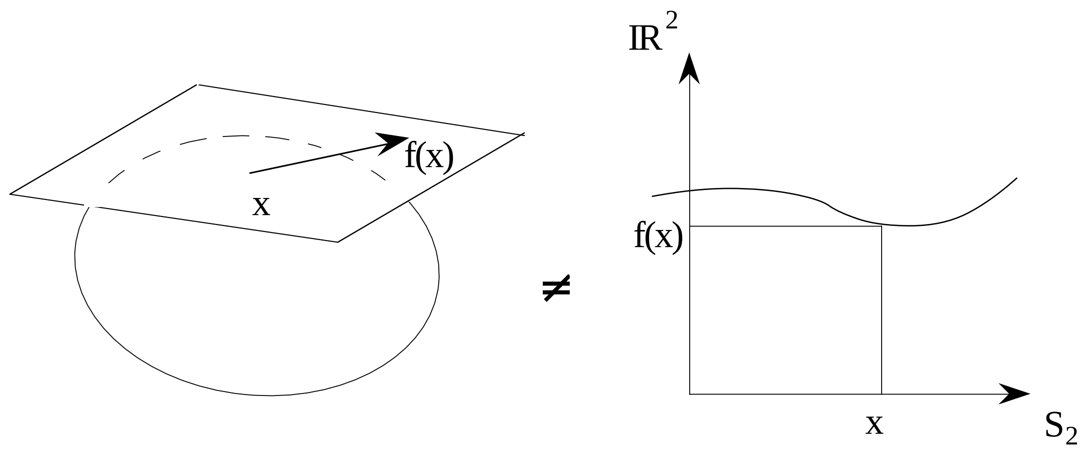


Figure 18. TS_2 is not the Cartesian product of S_2 and $\mathbb{R}^2$

This example may help grasp the deep difference between a continuum like TS_2 and (for instance) the one obtained by assigning to each point on the Earth the local values of pressure and temperature. One must not confuse a vector field on a two-dimensional surface with a pair of scalar fields: they are objects of different types, they are, more specifically, two "sections" of two different "bundles" on the same "base" S_2. It is now time to define these concepts.

Just as when defining manifolds, one may here adopt the analytic or the synthetic point of view. We shall begin with the latter, which is more intuitive.

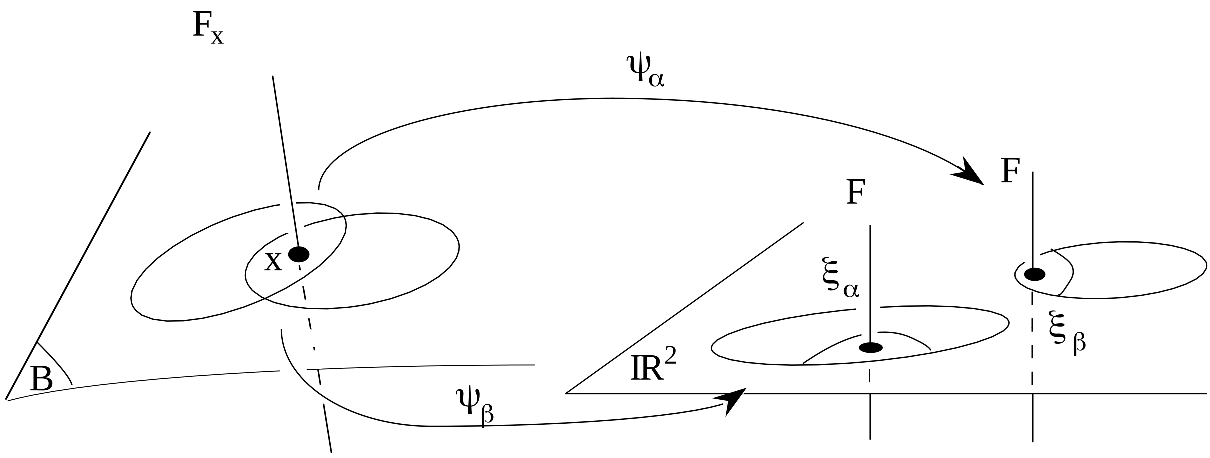


Figure 19. The fibre F_x "above" x is obtained by identifying two copies of F, one above $\xi_\alpha = \psi_\alpha(x)$, one above ξ_β. But this identification is not necessarily the identity mapping.

Let thus F be a manifold, the *fibre*, and B another one, the *base*. For simplicity, we assume F is described by a single chart. For each chart of B, say $\psi_\alpha \in B \to \mathbb{R}^n$, let us build the product manifold $\mathrm{cod}(\psi_\alpha) \times F$ and let's try to patch these products into a whole. So, consider two charts ψ_α and ψ_β with overlapping domains (Fig. 19). The first idea which comes to mind is to glue $\mathrm{cod}(\psi_\alpha) \times F$ and $\mathrm{cod}(\psi_\beta) \times F$ by identifying the pairs $\{\xi_\alpha, f\}$ and $\{\xi_\beta, f\}$ of $\mathbb{R}^n \times F$ if and only if

(7) $$\psi_\alpha^{-1}(\xi_\alpha) = \psi_\beta^{-1}(\xi_\beta).$$

But what one obtains this way is the product $B \times F$, since a class of equivalent pairs $\{\xi, f\}$ in the sense of (7) is characterised by a point of B (the preimage $x = \psi_\alpha^{-1}(\xi_\alpha) = \psi_\beta^{-1}(\xi_\beta)$) and a point f of F. So this assembly rule (which consists in identifying the fibre above $\psi_\alpha(x)$ with the one above $\psi_\beta(x)$) is too restrictive.

What more flexible rule could one adopt? The example of MS will give a clue in this respect (Fig. 20). Let $F = [-1, 1]$ be the fibre, S_1 the base, conceived as the unit circle in the plane, a point of S_1 being specified by its polar angle. Two charts are enough to cover it, with domains (and codomains as well, cf. Fig. 20)

$$\mathrm{dom}(\psi_\alpha) =]\,-\varepsilon, \pi + \varepsilon\,[, \quad \mathrm{dom}(\psi_\beta) =]\,\pi - \varepsilon, 2\pi + \varepsilon\,[$$

with $0 < \varepsilon < \pi$. Then $dom(\psi_\alpha) \cap dom(\psi_\beta)$ consists of two open segments U and V. One gets a Möbius strip by gluing fibre to fibre "without flipping" above U but "with flipping" above V. The equivalence is thus (7) above U, but above V the identification is made according to the non-trivial rule:

$$\{\xi_\alpha, f\} \sim \{\xi_\beta, f_\beta\} \Leftrightarrow (\psi_\alpha^{-1}(\xi_\alpha) = \psi_\beta^{-1}(\xi_\beta) \ \underline{\text{and}} \ f_\alpha = -f_\beta).$$

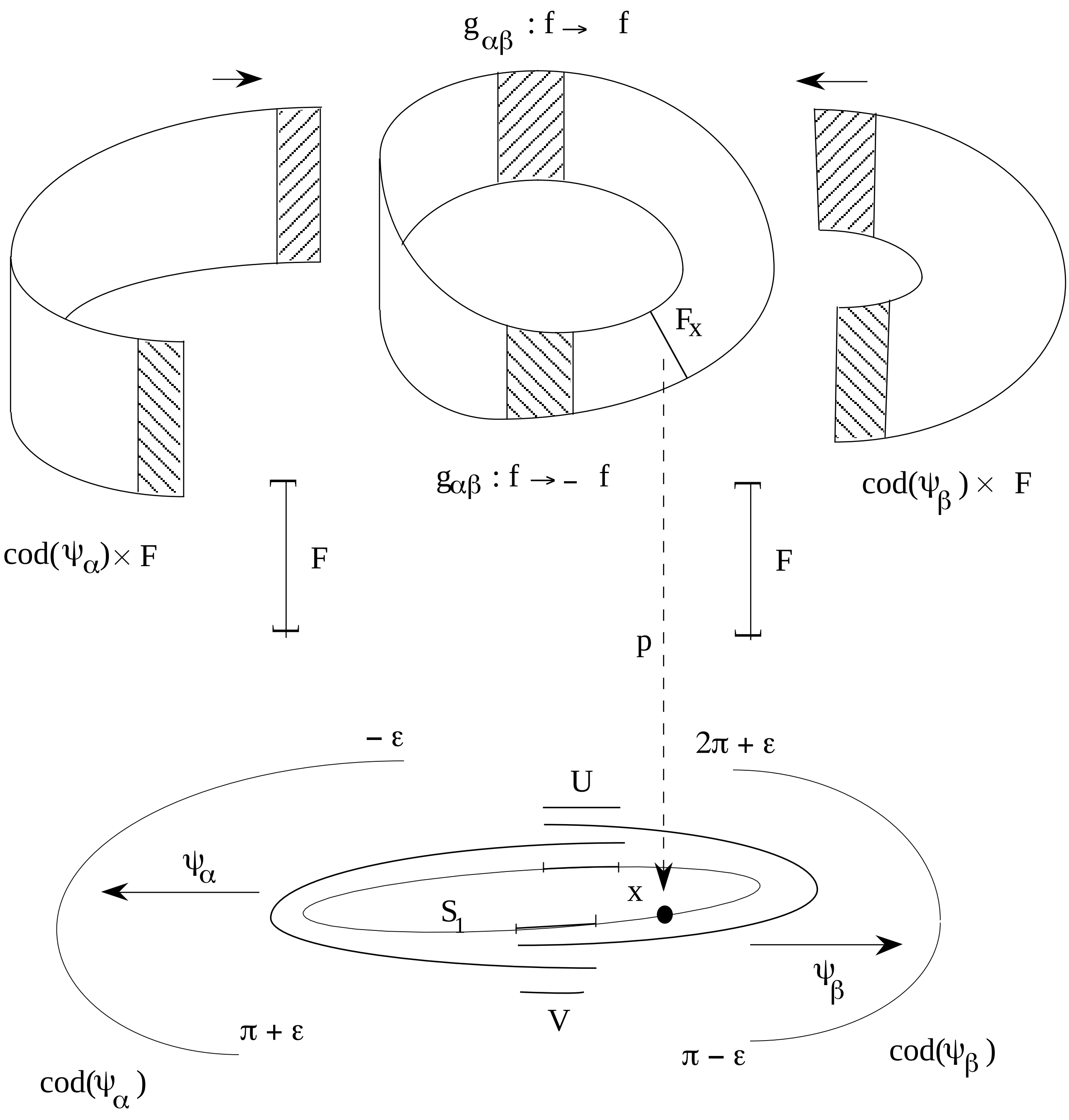


Figure 20. Patching two rectangles into a Möbius strip. (Remark the notational abuse which consists in giving identical names to $dom(\psi_i)$, which is a part of S_1, and $cod(\psi_i)$, a part of $\mathbb{R}$.)

The two copies of the fibre above a point are indeed identified via a bijection from F onto itself, but this bijection is not necessarily the identity.

This should be enough to motivate the following construction rule:

Definition 2: *Given,*

1°- A *manifold* B*, the* base*, of dimension* n*, with an atlas* $\{\psi_\alpha : \alpha \in \mathcal{A}\}$,

2°- *A manifold* F*, the* fibre*,*

3°- *A family* G *of diffeomorphisms of* F*,*

4°- *For each pair* $\{\alpha, \beta\}$*, a* transition function $g_{\alpha\beta}$*, of type* $B \to G$*, of domain* $\mathrm{dom}(\psi_\alpha) \cap \mathrm{dom}(\psi_\beta)$*,*

the bundle made out of these elements is the manifold V *obtained by identifying the pairs* $\{\xi, f\} \in \mathbb{R}^n \times F$ *according to the following rule*: $\{\xi_\alpha, f_\alpha\}$ *and* $\{\xi_\beta, f_\beta\}$ *are equivalent if, on the one hand,*

(8) $$\psi_\alpha^{-1}(\xi_\alpha) = \psi_\beta^{-1}(\xi_\beta),$$

i.e., if ξ_α *and* ξ_β *are the images of the same* $x \in B$*, and if, on the other hand,*

(9) $$f_\alpha = g_{\alpha\beta}(x)\, f_\beta.$$

Condition (8) tells how to glue the $\mathrm{cod}(\psi_\alpha)$ together in order to get B, and (9) tells how to assemble the fibres. The definition is still incomplete, because a transition function cannot be just any function. First, from (9), $g_{\alpha\beta} \circ g_{\beta\alpha}$ is the identity. By the same argument, if x belongs to the domains of three distinct charts, one has $g_{\alpha\beta} \circ g_{\beta\gamma} = g_{\alpha\gamma}$. So the values of the $g_{\alpha\beta}(x)$ form, taken all together, a group: therefore, one will require that G be a group of diffeomorphisms of the fibre. Moreover, for v a point of V, i.e., an equivalence class of $\{\xi_\alpha, f_\alpha\}$, the functions $v \to \{\xi_\alpha, f_\alpha\}$ are charts about v, which must be compatible. By writing down explicitly the correspondence $\{\xi_\alpha, f_\alpha\} \to \{\xi_\beta, f_\beta\}$, we see that the function

$$\{\xi, f\} \to \{\psi_\beta \circ \psi_\alpha^{-1}(\xi), g_{\beta\alpha}(\psi_\alpha(\xi))\, f\}$$

must be differentiable. So the $g_{\beta\alpha}$ themselves must have this property, and for this to make sense, G has to wear a structure of differentiable manifold. Groups which are also manifolds (and in such a way that group operations be differentiable) are called *Lie groups*. So, finally,

Definition 2 (continued): G *is a Lie group, called* structural group *of* V, *and the transition functions are differentiable.*

Thus we have finally formalized the notion of smooth patching of the fibres that we wanted.

"The" group G is not, in fact, uniquely determined by the structure of the fibre. One has some leeway in choosing it, and one generally takes it as small as possible, so it is often finite[1]. (In the case of MS, it contains two elements: the identity and the "flip" $f \to -f$.) In that case, the end of Def. 2 is redundant, the G-valued transition functions being piecewise constant.

If it happens, when one builds a bundle, that the fibre has more structure than a plain manifold (like, for instance, a linear space), one naturally tries to preserve this structure, in such a way that the fibre F_x above x inherits from it. So transition functions must themselves respect the structure of F, and one has to choose the group G accordingly. For instance, if F is a vector space of dimension n, G will be the group GL_n of isomorphisms of F, i.e., the linear invertible mappings of F onto itself. (Thus, in the case of TS_2, the structural group is GL_2.) What one gets this way is called a *vector bundle*. Most of those we shall encounter are of this kind.

According to (8), to each $v \in V$ (an equivalence class of $\{\xi_\alpha, f_\alpha\}$) corresponds a point x in the base, the one such that $\psi_\alpha(x) = \xi_\alpha$ for all charts about x. This point is the *projection* of v, denoted $x = p(v)$ (cf. Fig. 20). The preimage

$$F_x = p^{-1}(x),$$

called *fibre above* x, inherits any structure belonging to the fibre F: if F is a linear space, F_x is one, etc. Mappings which transform fibres into fibres while respecting whatever structure they have are called *bundle maps*. So if $u \in V \to V'$ is such a mapping, it sends a fibre F_x above x onto a fibre $F_{x'}$ above x' (and the restriction of u to F_x respects the structure of F: it is linear when F is a linear space, etc.). Moreover, there exists $g \in B \to B'$ such that the diagram

[1] A finite set, once equipped with the discrete topology, bears a manifold structure, thus finite groups are Lie groups (as also are groups like $\mathbb{Z}$, $\mathbb{Z}^n$, etc.).

$$\begin{array}{ccc} V & \xrightarrow{\;u\;} & V' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{\;g\;} & B' \end{array}$$

where p and p' are the projections, commute. The bundle maps play with respect to the structure of bundle the same rôle as held by continuous—or linear, or differentiable, etc.—functions with respect to the structures of topological space—or linear space, or manifold, etc.

1.3.2 Sections

Now, a very important notion:

Definition 3: *One calls* section *of a bundle* V *of base* B *any function* $s \in B \rightarrow V$ *such that*

(10) $\quad p(s(x)) = x \quad \forall\, x \in \mathrm{dom}(s).$

Thus a section assigns to each point x within its domain in the base a point of the fibre F_x above x (Fig. 21). (Note that a bundle map transforms sections into sections.)

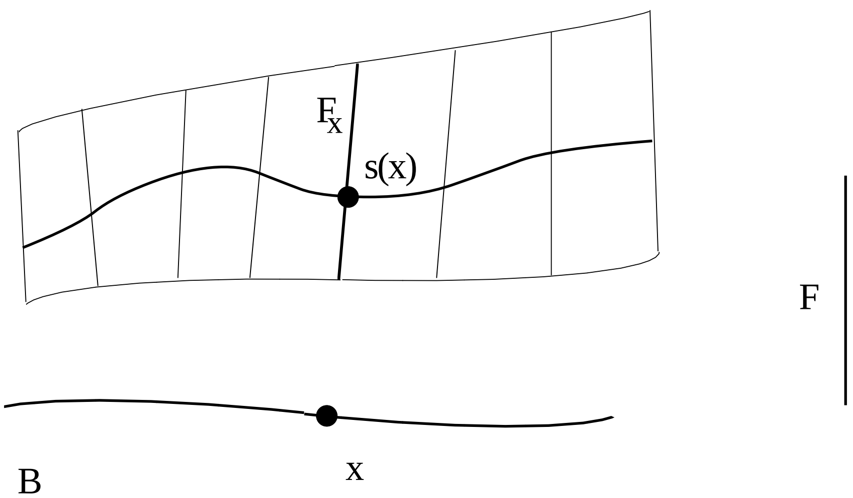


Figure 21. Notion of section.

One is strongly tempted to say "s is (thus) an F-valued function over B". But this is the wrong idea. Section s is not an object of type $B \rightarrow F$, but an object of type $B \rightarrow V$ which satisfies condition (10). The distinction is clear-cut in the case of TS_2: an example of $\mathbb{R}^2$-valued function over S_2 is the mapping {position} → {temperature, pressure}, whereas a section of TS_2 is a field of tangent vectors, and we have already noticed the difference. We shall now analyze it in full generality.

By the very definition of V, we have charts about s(x), i.e., mappings such as $s(x) \to \{\xi_\alpha, f_\alpha\}$. Consider such a chart of V, which assigns to s(x) the pair $\{\xi_\alpha, f_\alpha\}$, and let $f_\alpha = \varphi_\alpha(s(x))$. Locally, thanks to this chart, we can study the continuity, the differentiability, etc., of f_α as a function of x. By compatibility of charts, these are intrinsic properties of a section. So we should like to call f_α the "fibre component" of s(x), just as x is its "base projection". But can we? To say, for instance, that this fibre component f_α is "constant" in the neighborhood of x is saying something which is valid in this particular chart, but not in another one, and thus cannot be attributed to the section. To speak of a "constant" section is thus meaningless. More generally, there is no way in which the "fibre components" of s(x) and s(y) can be compared when $x \neq y$, their possible equality being a chart-dependent phenomenon, devoid of any intrinsic meaning.

A bundle, thus, is only fibered "vertically" (Fig. 22). The notion of "horizontal strata", or of "sections parallel to the base", does not exist. When there is a need for it, one must endow the bundle with an additional element of structure (called a "connection" [12, 55]).

Sections of bundles are the right objects by which to model physical fields. When for instance one is studying conduction on a metallic surface, or elastic deformation of the same, there is no intrinsic way—and no need—to compare the current density vectors, or the stress tensors, at two points remote from each other: such a comparison would not make physical sense. Other example, the field of displacements of an elastic structure. In all these instances, as one knows, it pays to make use of local frames, i.e., not in any fixed relationship one with respect to the other when the shape of the body under study is changing: such a practice is tantamount to considering said fields as sections of some bundle, the base of which is some reference configuration of the body, and the fibres, vector spaces of various dimensions. No comparison of "values" of the field at remote points is called for, and there is no need, when modelling the situation, to choose a richer mathematical structure than necessary. To the contrary, excess structure can be a nuisance (as some cumbersome treatments of elastic shells theory testify).

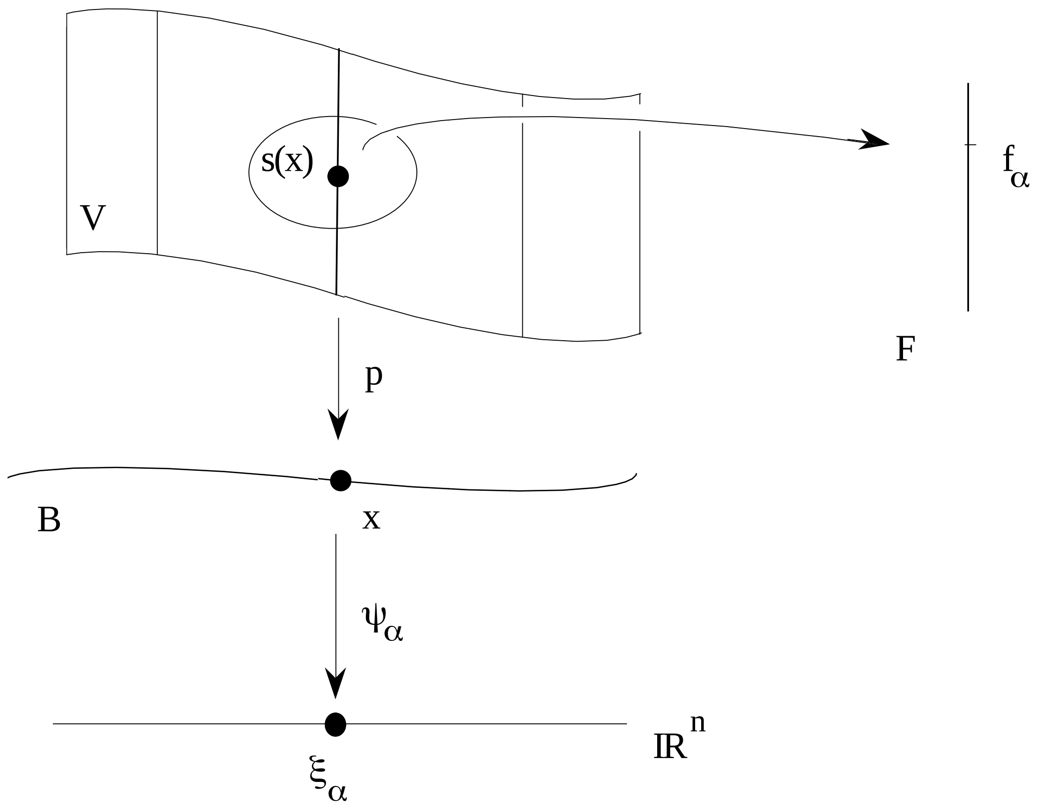


Figure 22. The "component in the fibre" f_α, contrary to the projection x, is chart-dependent, and has no intrinsic meaning.

1.4 Coverings

Midway between the two manifold construction methods we have examined (gluing and fibre assembly), there is an intermediate case: when the fibre is a set of isolated points. The bundle and its base, then, are manifolds of equal dimensions.

Everything we have said is valid in this case, since a set of isolated points has a manifold structure (charts are functions $\psi \in F \to \mathbb{R}^0$, where $\mathbb{R}^0$ is by convention reduced to a single point (point 0), and each point f of F contributes one chart ψ_f, for which $\text{dom}(\psi_f) = \{f\}$). But owing to the fact that the structural group is a permutation group, more precisely, a subgroup of the group of permutations acting on F, there are special properties.

1.4.1 The notion of covering

Let's begin with two examples (Fig. 23). The base is the circle S_1, the fibre is a set of two points. Fig. 23 shows the two possible bundles. As one may notice, the preimage of a small enough neighborhood of x consists in two non-intersecting neighborhoods, and the restriction of p to each of them is a local diffeomorphism. This is, by definition, the characteristic property of *coverings*. They are also requested to be *connected*, which eliminates the case of Fig. 23, left. Another example (Fig. 24): the base is S_1, the fibre $\mathbb{Z}$ (the set of signed integers), and the group $\mathbb{Z}$ as well, acting on itself via the operations $g_n = m \rightarrow n + m$. The bundle, as one sees, is nothing else than the real line. When a covering is, as in the present case, *simply* connected, one calls it "the" *universal* covering of the base [67]. This terminology is supported by a theorem asserting existence and uniqueness, up to diffeomorphism, of this universal covering [67].

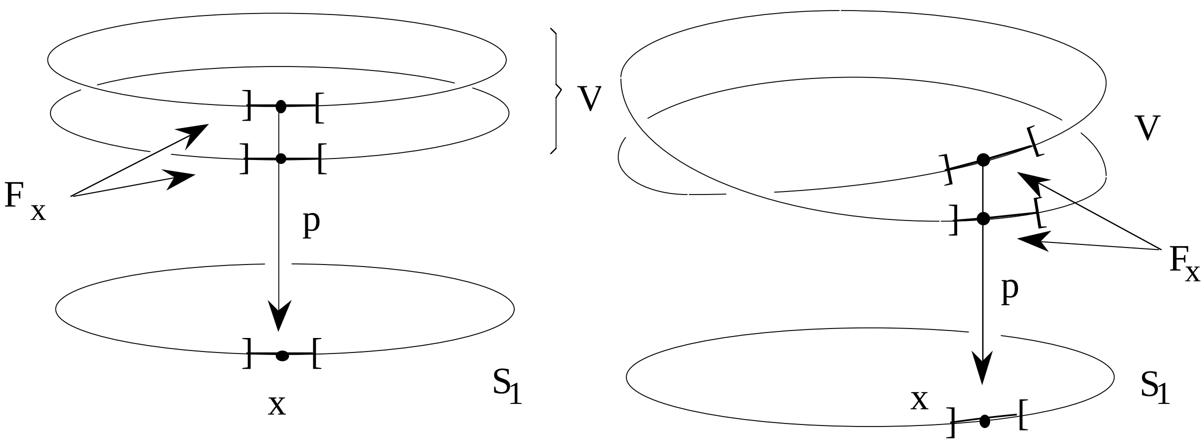
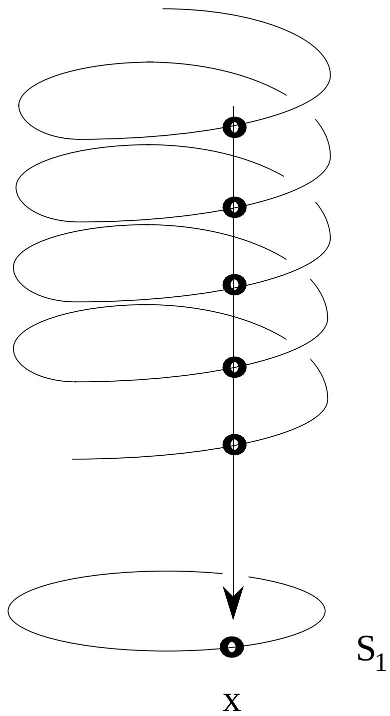


Figure 23. Two coverings of the circle. On the left, G reduces to the identity. On the right, G is the group of permutations of two objects.



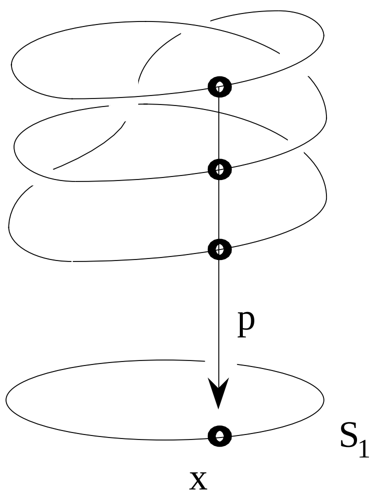


Figure 24. $\mathbb{R}$ as a covering of S^1.

Figure 25.

Exercise 19: With the two charts of Fig. 20 (p. 27) on S_1, describe in detail (i.e., by writing down the equivalence classes) the bundle of Fig. 23, right.

Exercise 20: Same problem, with as a fibre $F = \{0, 1, 2\}$, the structural group being $\mathbb{Z}_3$, i.e., the cyclic group with three elements. (Hint: Fig. 25.) This is a "three-sheet" covering.

Exercise 21 (Fig. 26): Cut out a paper ribbon of about 20 cm × 2 cm. Patch it into a Möbius strip. "Cover" it with a paper strip 45 cm long. Glue the ends of the latter together. Cut the MS and pull it off. What do you observe?

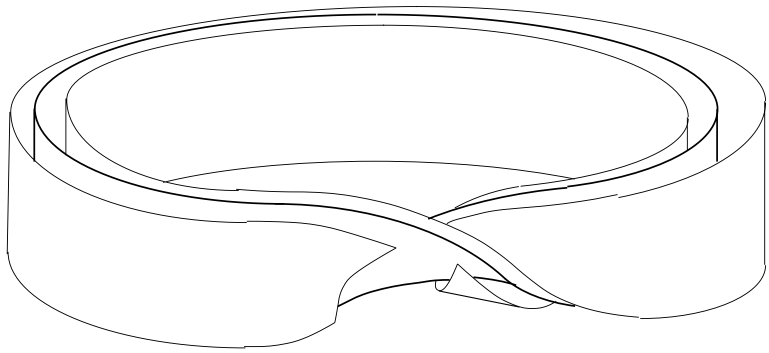


Figure 26. Two-sheet covering, orientable, of a Möbius strip.

1.4.2 Interest of the notion of covering

How relevant are such coverings to electrotechnics? How can they ever be? The fact is, they are, essentially in two ways: When discussing "multivalued potentials", and when symmetry is present.

Consider a ring in which flows a current of total intensity I (Fig. 27). Call B the open region around the ring, and h the magnetic field. Since $\text{rot}\, h = 0$ in B, there exists, locally, a potential φ such that $h = \text{grad}\, \varphi$. But since the circulation of h along a circuit like c (Fig. 27) is equal to I, φ is not globally defined over B. It's a mathematical freak, called a "multivalued function". At each point of B, there is not a single value, but an infinity of values of the potential, their differences two by two being multiples of I. Thanks to the concept of covering, this multivalued potential gains access to the status of a bona-fide function, living not on B, but on the universal covering of B (fibre $\mathbb{Z}$, group $\mathbb{Z}$).

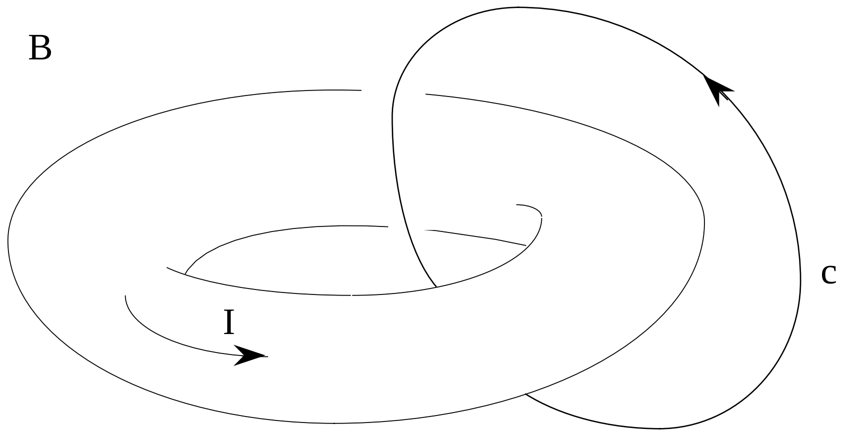


Figure 27. The magnetic potential outside the ring is a multivalued function.

In the same spirit, but with more complexity, the study of eddy-currents on conducting surfaces of convoluted shape, like e.g., tokamak shieldings [13, 74], or sheaths around alternator outputs [100], calls for multivalued functions whose natural home is a covering of the surface. Such questions are commonly treated by way of "cuts" of the surface, and by allowing the stream-function to be discontinuous across these cuts. But then the very determination of these cuts can be a non-trivial problem. Its solution requires a good understanding of certain notions of topology: first homology group, Betti numbers, which cannot be introduced here.

Remark 1. Making cuts is an old problem: in structural computations, it was identified, and the importance of the above notions acknowledged, decennials ago [45]. (Actually, Betti himself took interest in making cuts, cf. [10], as quoted in [82].) The problem was only recently solved in both a rigorous and *constructive* way. See on this the work by Kotiuga [57, 58], and the discussion hosted in 1990 by the IEE Journal [101, 18, 59]. ◊

Now about symmetries. Many structures, in electrotechnical applications, are *repetitive*, possibly at different levels: one may often generate a sizable part of the structure by suitable assembly of copies of a single element. If a field has to be computed in such a case, it is natural, at least at an early stage of the modelling, to pretend this repetitivity goes on indefinitely in all spatial directions. The problem then becomes one on an infinite domain with *periodicity* (with respect to space) of such physical properties as conductivity, permeability, etc. This spatial periodicity is also shared (in a sometimes not obvious way) by the field values. One may then [15, 16] limit the computation to a "symmetry cell" of the structure (Fig. 28).

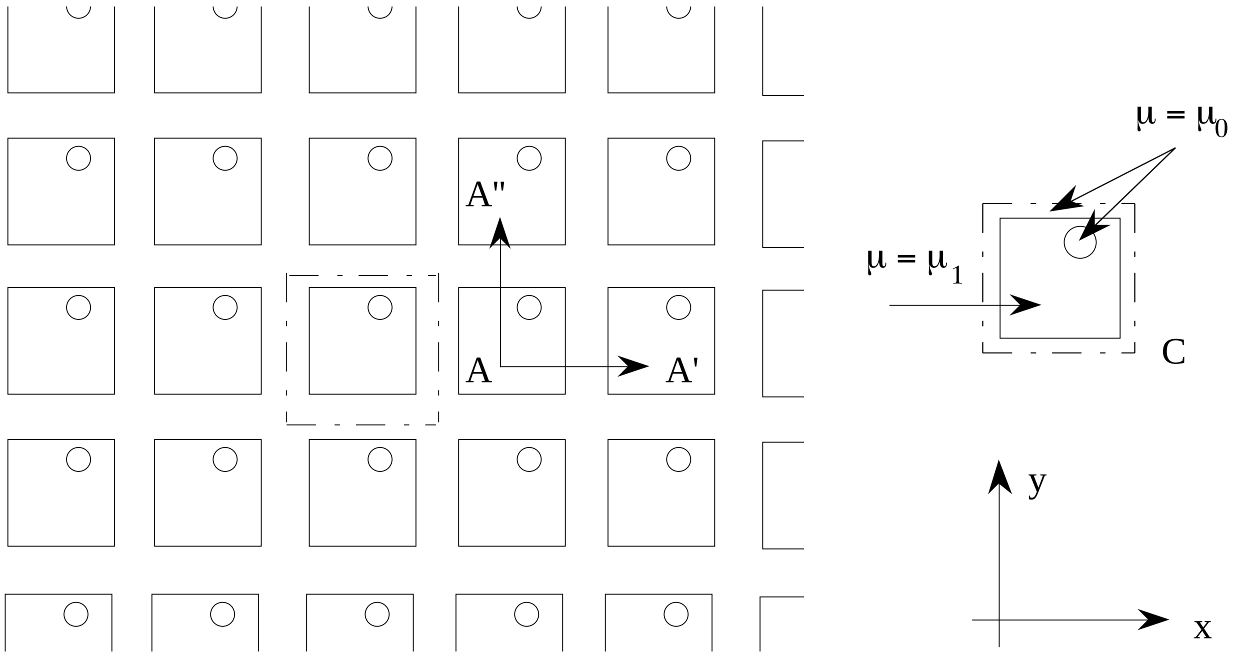


Figure 28. A repetitive structure and a periodicity cell for a bidimensional problem in magnetostatics.

To be specific, let us take the case of Fig. 28, where one wants to compute the perturbation to an initially uniform field due to a pattern of materials with two different permeabilities (i.e., μ periodical as a function of x and y). The field thus modified has the same periodicities[1] as the structure: h(A") = h(A') = h(A), thus one may compute its values on the symmetry cell C, with appropriate "periodicity" conditions (i.e., conditions imposed to the field components at homologous points on two opposite sides of the cell). But this amounts to solving the same equations on the manifold B obtained by gluing opposite sides of C, as was done in Exer. 11 (B is a torus).

This manifold can be obtained in another way. Let G be the group (with an infinity of elements) generated by the translations AA' and AA". One may identify the points of B with equivalence classes of points in the plane E_2, two points being considered as equivalent if one is sent to the other by one of the translations in the group. (One says that B is the *quotient* manifold of E_2 by the equivalence relation.) Clearly, now, the whole plane E_2 is a covering of B: one may thus conceive it as a bundle, with fibre G and group G. The fibre above x, which is the set of points

[1] Even when this is not so, spatial periodicity of the underlying medium can still be put to advantage, by a procedure which generalizes the Fourier decomposition method, provided the problem is a linear one. Cf. [16].

$$G_x = \{gx : g \in G\}$$

is called the *orbit* of x under the action of G.

Once the computation has been performed on the base, finding the field on the covering amounts to specifying a certain section of a bundle over B, with fibre $E_2 \times G$.

All of this works well provided all orbits are of the same kind, which is true when the group acts *freely*, i.e., no point is fixed by any group transformation other than the identity. But when there are reflection symmetries, this condition is not satisfied. For this reason, the notions of fibre, of coverings, etc., introduced so far, are not powerful enough to really account for what is done in the presence of symmetries. We already spotted, when discussing the notion of manifold with boundary, a few weaknesses of the run-of-the-mill mathematical apparatus, and this is another one. The "right" notions do exist (cf. [75], Chapter "Orbifolds"), but only small circles of specialists are familiar with them.

Exercise 22. The symmetry groups of "wall-paper patterns" like the one of Fig. 28 are all isomorphic to one of the groups of a list of seventeen, which can be found for instance in [14], or [66], [87], etc. Get the list of these 17 groups, then, for each of them, describe the analogue of B above. In which cases is it a differentiable manifold?

Chapter 2

Vector fields and differential forms

2.1 Vectors and covectors

The notion of tangent vector at x to a surface S seems familiar: one thinks of a "bound vector", at point x, whose supporting line is tangent to the surface at point x.

But this requires some ambient space, and if one is thinking for instance about the configuration manifold of a mechanical system, there is no natural ambient space to speak of, in general. So one should be able to define tangent vectors without any reference to such an ambient space. The mechanical notion of "velocity vector" will suggest how this can be done: we'll start from the idea of speed along a trajectory, in a manifold, a thing of obviously intrinsic character, and try to abstract out the right notion from there.

In a chart, the velocity vector is easily defined. In the case of the above-mentioned car, for instance, it has four components: two for the speed with respect to the ground, one for the speed of gyration around the vertical axis, one for that of the driving wheel. But there are other possible charts. In another one (with other coordinate axes on the ground, angles measured in degrees instead of radians, etc.), one would get a different set of four numbers. The velocity vector should be a chart-independent entity, only represented, in different charts, by such systems of four numbers. This entity, the "tangent vector", does exist, and we are about to define it.

A *trajectory*, in a manifold X, is a smooth function (cf. p. 13) of type $\mathbb{R} \rightarrow X$ whose domain is connected. This domain is therefore a segment of $\mathbb{R}$. A *scalar field* over X is a smooth function of type $X \rightarrow \mathbb{R}$. Its codomain is a segment of $\mathbb{R}$ (since X is connected). A trajectory g is *through* x ("at time 0" will always be understood) if $0 \in \text{dom}(g)$ and if $g(0) = x$. A scalar field f *vanishes at* x if $f(x) = 0$. Cf. Fig. 29.

A trajectory is thus, intuitively, a curve in X described according to some specific time-schedule, or as one may prefer, a graded and oriented curve. A scalar field f can be understood (cf. Fig. 29) as a partition of dom(f) into "level surfaces" $X_a = \{x \in X : f(x) = a\}$. ("Scalar", or "real" field, is of course meant here to contrast with "vector" field. We shall simply say "function" when no confusion is feared.)

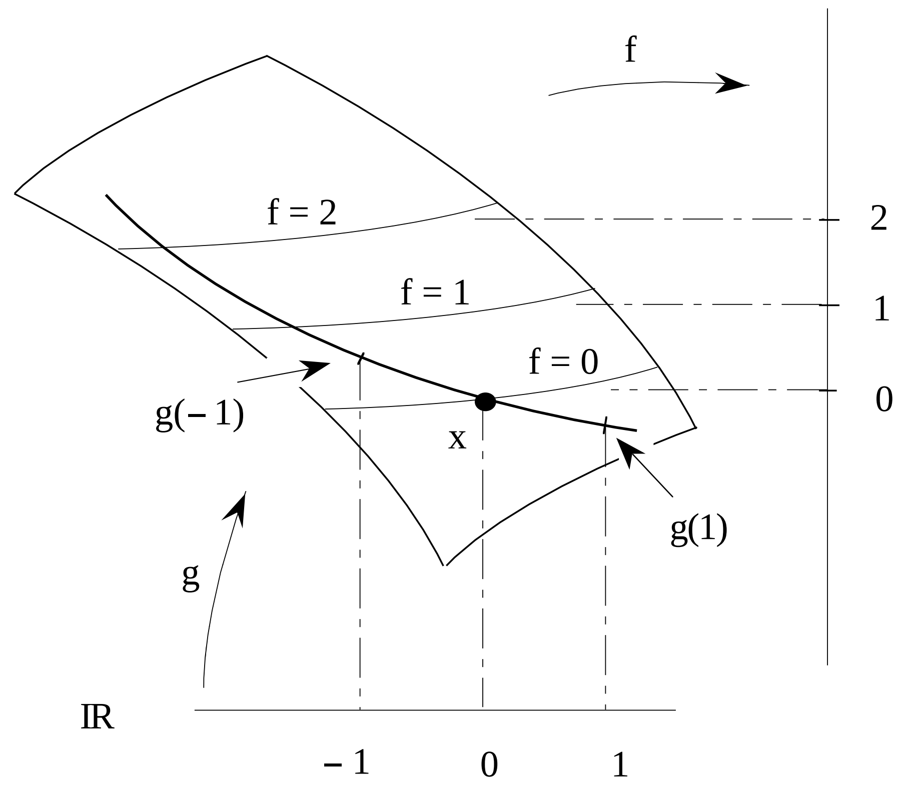


Figure 29. Trajectories and scalar fields.

Two trajectories g and g' through x are *tangent* ("at point x" being understood) if, for all charts ψ of a neighborhood of x,

(11) $$|\psi(g(t)) - \psi(g'(t))|/t = o(t)$$

(meaning: tends to 0 faster than t when $t \to 0$).

One will easily check that if (11) is valid for *any* chart about x, this is true in all C^1-compatible charts. (We are speaking here of *differentiable* manifolds, i.e., of class C^1 or better.) So this is an equivalence relation on trajectories.

Two functions f and f' vanishing at x will be said to be *tangent* (again, at x) if, for all $y \in \mathrm{dom}(\psi)$,

(12) $f(y) - f'(y) = o(|\psi(y) - \psi(x)|)$

(meaning: tends to 0 faster than the distance of x and y, as measured in the chart ψ, when $y \to x$). Here again, this property is chart-independent.

Following the lead of [27], we have emphasized the duality between the two notions (from which a duality between vectors and covectors will stem). Fig. 30 suggests what tangent trajectories or functions look like. One also says that they are in "contact of order one". (All this is part of a more general theory about the contact between mappings of type $X \to Y$, where X and Y are two manifolds.)

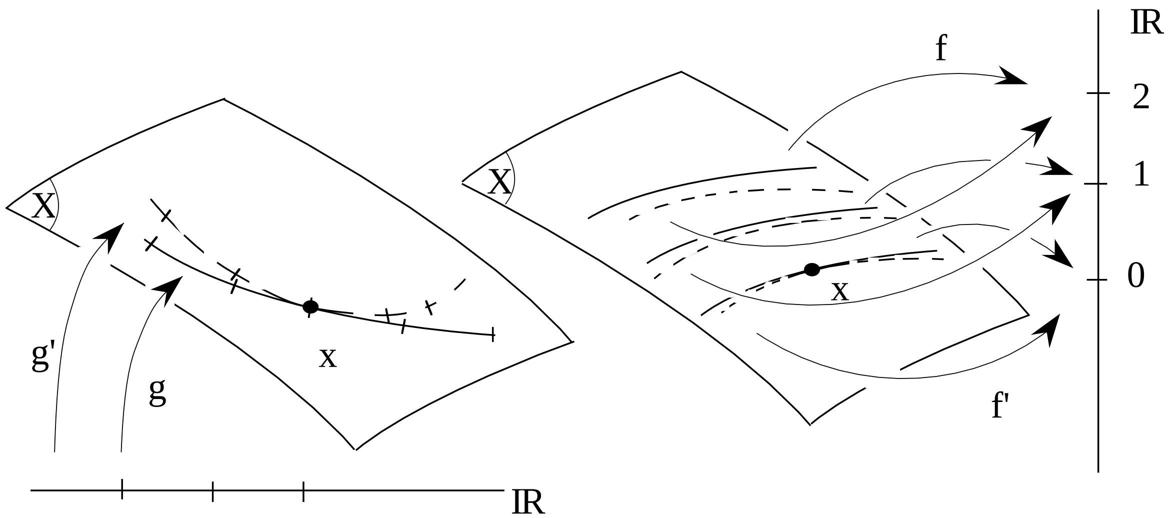


Figure 30. Tangent trajectories and tangent functions at point x.

Now,

Definition 4: *One calls* tangent vector *at* x *an equivalence class, in the sense of* (11)*, of smooth trajectories through* x.

Definition 5: *One calls* covector *at* x *an equivalence class, in the sense of* (12)*, of smooth functions vanishing at* x.

For reasons which should become clear below, I denote by g_* and f^* the equivalence classes of a trajectory g and of a function f at point x.

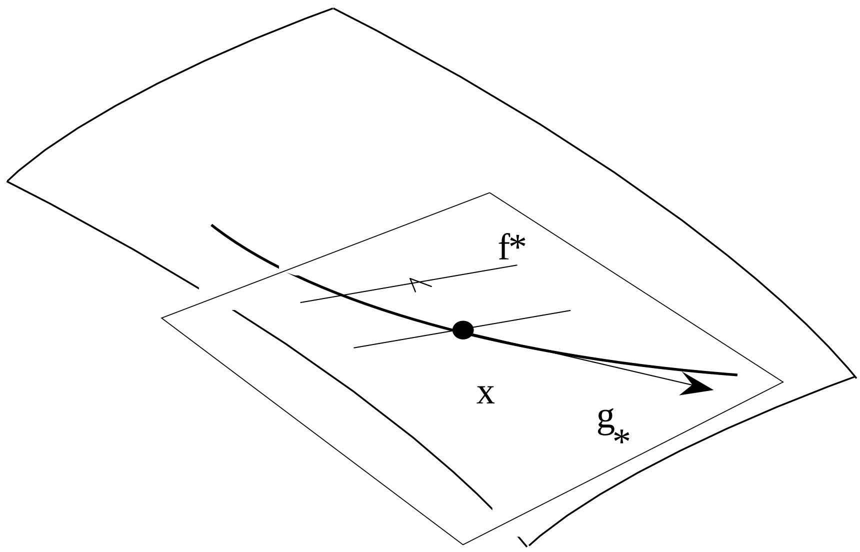


Figure 31. Vector and covector at point x. Vector and covector are supposed to lie in the tangent plane.

The intuitive meaning of Def. 4 is clear when X is V_n. For an equivalence class of tangent trajectories includes a particular, distinguished trajectory (or as one says, a "canonical representative"): the straight, uniform, trajectory (there is only one of this kind in the class). It can be characterized by a vector based at x, namely the velocity vector common to all trajectories of the class. Thus it is only natural to call the class itself a "vector" in the general case. (Why it should be qualified as "tangent" is clear if X is embedded in $\mathbb{R}^{n+1}$, cf. Fig. 31.)

Exercise 23: Show that when g and g' are equivalent in the sense of (11), either their images by all charts are tangent, or their class g_* is the one which contains the constant trajectory $t \to x$ (denoted $g_* = 0$).

As for Def. 5, the approach is the same: one gives to the whole class the name of the geometric object which best characterizes it in the case $X = V_n$, i.e., the covector at x associated with the one function of the class which is affine. The graphic representation of the covector introduced p. 4 consists, when n = 3, in drawing two parallel planes, tangent to the level surfaces of this function.

In the general case, there are neither "straight" nor "uniform" trajectories, but the same graphic symbolism can be used, hence Fig. 31.

2.2 Tangent and cotangent bundles, and duality

The existence of the "tangent plane" of Fig. 31 is not due to the fact that X is embedded in E_3 in this particular drawing. Such a plane has an independent reality. Indeed, as we shall now check, the set of tangent vectors at point x (denoted T_xX), and the set of covectors (denoted $T_x{}^*X$), both have a natural structure of vector space[1], of same dimension as X. Moreover, they are dual to each other in a way which also is natural, i.e., chart-independent.

The shortest path to this result takes the following detour: how do vectors and covectors transform when one maps a manifold to another one?

2.2.1 Tangent space

So let X and Y be two manifolds and $u \in X \to Y$ a smooth mapping. Let $y = u(x)$. If g is a trajectory through x, $u \circ g$ is a trajectory through y, which defines a tangent vector, for $u \circ g$ and $u \circ g'$ are equivalent (in the sense of (11)) if g and g' are, thanks to the differentiability of u. We shall denote this vector $u_* g_*$ (x understood). We just obtained a mapping $u_*(x)$ from T_xX to T_yY.

Similarly, if f is a function on Y (vanishing at y), $f \circ u$ is a function on X (vanishing at x), the corresponding covector can be denoted $u^* f^*$, and this defines a mapping $u^*(y) \in T_y{}^*Y \to T_x{}^*X$.

Note that, by construction, the following associativity rules hold when $u \in X \to Y$ and $v \in Y \to Z$:

$$(v \circ u)_* = v_* u_*, \qquad (v \circ u)^* = u^* v^*,$$

each expression being of course evaluated at x, $y = u(x)$ or $z = v(y)$, as the case may be[2]. (Mind the transposition of u and v in the second equality! The maps u^*, v^*, $(v \circ u)^*$, go *right to left*.)

Let us work out in detail (and once for all) what u_* and u^* are when X and Y are affine spaces: $X = V_m$, $Y = V_n$ and $u \in V_m \to V_n$. Let us arbitrarily select an origin in X, in order to match each point x of V_m (the affine space) with a vector x of V_m (the vector space). Same thing in Y.

[1] At least if $x \notin \partial X$. More on this point later (next Remark).

[2] One may dislike the notation, and prefer something like $(v \circ u) = v_* \circ u_*$, etc., but v_* and u_* are *linear* operators, and tradition wants their composition product written by simple juxtaposition, as for matrices.

Let e_J, $J = 1, \ldots, m$, be the basis vectors of V_m and e_i, $i = 1, \ldots, n$, those of V_n. (This convention, indices in small capitals on one side and in small case on the other, tends to become standard. Cf. for instance [68].) It is only natural to represent the *point* u(x), image of the *point* x, by listing the components u^i of *vector* u(x) as functions of the components of *vector* x. Thus,

$$x = \Sigma_{J = 1, \ldots, m}\ x^J e_J, \quad u(x) = \Sigma_{i = 1, \ldots, n}\, y^i e_i,$$

with $y^i = u^i(x^1, \ldots, x^m)$, where u^i is a function of type $\mathbb{R}^m \to \mathbb{R}$. Let us consider the trajectory $t \to x + t\,e_J$. Its velocity vector (at x) is equal to e_J, so one may name e_J also the vector of T_xX that is represented by this trajectory. Its image by u is a trajectory through u(x), namely $t \to u(x + t\,e_J)$, whose velocity vector at $t = 0$ is by definition $u_*(x)\, e_J$. One sees, by differentiating $t \to u(x + t\,e_J)$, that

$$(13) \qquad u_*(x)\, e_J = \Sigma_{i = 1, \ldots, n}\, \partial u^i/\partial x^J(x)\, e_i,$$

so $u_*(x)$ is in that case a rectangular matrix, indeed a familiar one: the *Jacobian* of the u^is.

In the same vein, let ε^J be the basis covectors of V_m: those are the linear functions $x \to x^J$ (so that $\varepsilon^I(e_J) = \delta^I_J$, i.e. 1 if $I = J$ and 0 otherwise). Let ε^i be those of V_n. A covector at $y = u(x)$ is (the class of) the affine function $y' \to \varepsilon^i(y' - u(x))$, and it is again natural to name this ε^i. By way of definition, $u^*(y)\, \varepsilon^i$ is the (class of the) function $x' \to u^i(x') - u^i(x)$, therefore (take the linear part of this difference with respect to $x' - x$):

$$u^*(y)\, \varepsilon^i = \Sigma_{J = 1,\ldots, m} \partial u^i/\partial x^J\, \varepsilon^J.$$

We just realized that if $X = V_m$ and $Y = V_n$, the mapping u_* (at point x) is the matrix of the $\partial u^i/\partial x^J$ and u^* (at point u(x)) is the transpose of this matrix. The whole point of the present development (which will not end before p. 49) is to show that in the general case also, u_* and u^* are two mutually transposed linear operators. For this, we have first to see that T_xX and T_yY are vector spaces.

In X of dimension n, let g_* be a vector at x and ψ a chart about x. Then $\psi_* g_*$ is a vector at $\psi(x)$. The mapping ψ_* is injective, for if two trajectories in V_n, say $\psi \circ g$ and $\psi \circ g'$, are tangent, g and g' are (it's what "tangent" means). It is thus legitimate to carry onto T_xX the vector space structure of V_n, which is done by defining the sum of two tangent vectors g_* and h_* as the tangent vector $\psi_*^{-1}(\psi_* g_* + \psi_* h_*)$. This move not only turns T_xX

into a vector space, but also gives a basis, dependent on ψ. The basis vectors are the classes of the trajectories $g^i = \psi^{-1} \circ \gamma^i$, with

$$\gamma^i = t \to \psi(x) + t\, e_i,$$

e_i being the i^{th} basis vector in V_n (Fig. 32). By associating the n components of $\psi(y)$ with the n components of $\psi_*(x)\, g_*(y)$ (a vector at $\psi(y)$), one gets a chart for the manifold of pairs $\{y, g_*(y)\}$, i.e., the manifold of all tangent vectors. The latter is thus of dimension $2n$. We shall denote it by TX.

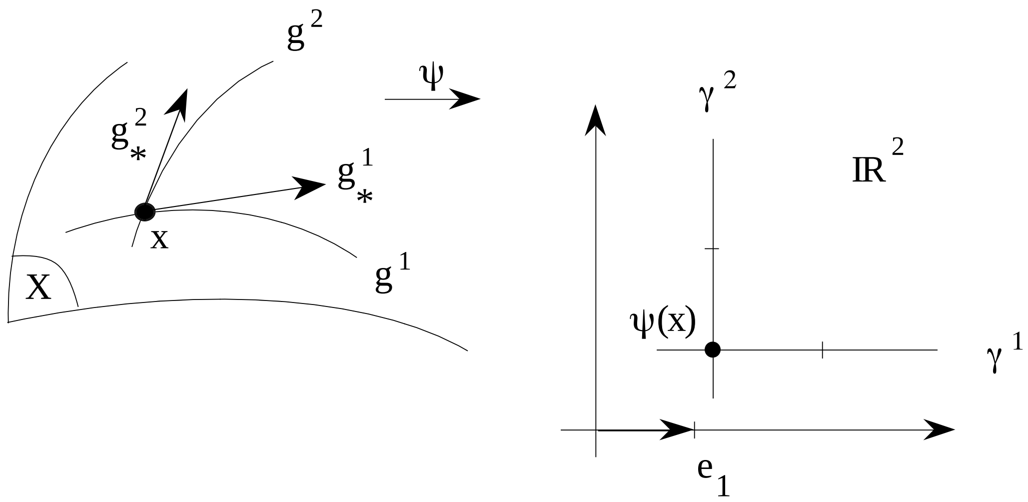


Figure 32. Basis vectors, for $n = 2$.

Remark 2. If x is a boundary point of X (Fig. 33), we obviously have a problem. For such a point, T_xX is not a vector space (but only a half-space, or more generally a cone), if Def. 4 is to be taken literally. Indeed, recall that X "looks like a half-space" in the neighborhood of a boundary point. So there are two kinds of trajectories through x: those "tangent to the boundary" (this makes sense in a chart, and is a chart-independent feature) and the "incoming" ones. If g is one of the latter, and g_* the associated tangent vector, $-g_*$ is not part of T_xX, according to Def. 4. This is too drastic, and one will rather define T_xX as the vector space *spanned* by the tangent vectors at x. Vectors such that g_*^2 or g_*^3 (Fig. 33) are said to be *incoming*, the sames with the opposite sign are said to be *outgoing*, and those like g_*^1 (or $-g_*^1$) are said to be *tangent to the boundary*. The latter form an $(n-1)$-dimensional subspace of T_xX, which will easily be seen to be isomorphic to $T_x\partial X$. ◊

Exercise 24: Let ψ_α and ψ_β be two charts about x, $u = \psi_\beta \circ (\psi_\alpha)^{-1}$, and e_i the basis vectors in $\mathbb{R}^n$. Show that, if $\xi = \psi_\alpha(x)$,

$$u_*(\xi)\, e_j = \Sigma_i\, \partial u^i/\partial x^j(\xi)\, e_i$$

and conclude to the C^{k-1}-compatibility of charts of TX if those of X are C^k-compatible.

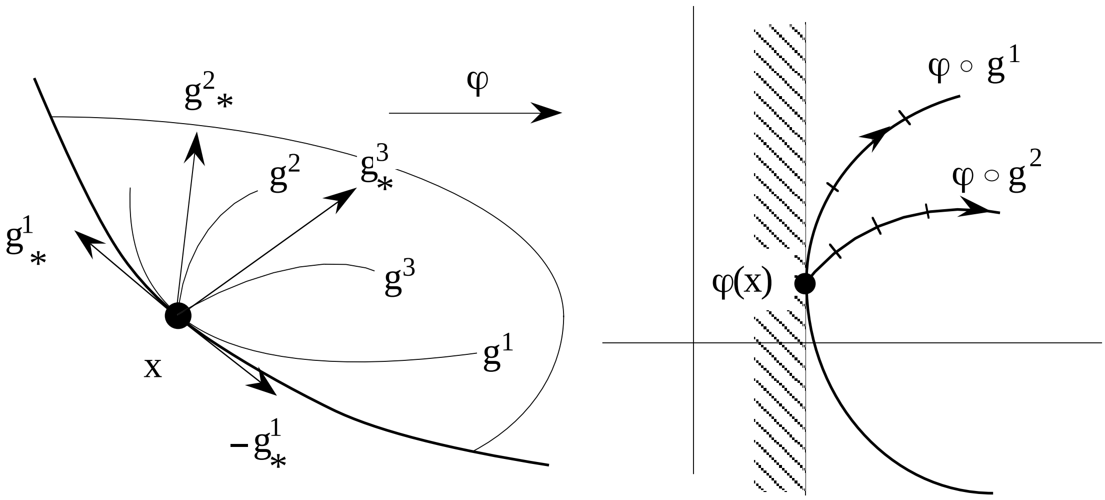


Figure 33. Do tangent vectors at x form a vector space or a cone? (The trajectory g^1 is tangent to the boundary, g^2 is incoming.)

The manifold TX is a bundle, the so-called *tangent bundle*. Its fibre is V_n, the fibre above x is T_xX, and the foregoing exercise has shown what transition functions are. The structural group is GL_n, the group of all invertible linear maps from V_n onto itself. (It *is* a Lie group: the representation of its elements by matrices, once chosen a frame base, is a chart, which is enough to endow the group with a manifold structure.) Sections of TX are called *vector fields.*

2.2.2 Cotangent space

Let us turn to the covectors. Let f* be a covector at x. If ψ is a chart about x, then $(\psi^{-1})^*\, f^*$ is a covector at ψ(x), and here again $T_x{}^*X$ can be identified with V_n. The basis covectors are the equivalence classes which contain the functions

$$f_i = y \to \psi_i(y) - \psi_i(x)$$

that is, $f_i = \varphi_i \circ \psi$, where φ_i, of type $\mathbb{R}^n \to \mathbb{R}$, is $\varphi_i = \eta \to \eta^i - \psi^i(x)$, and ψ^i is the i^{th} component of the chart ψ (Fig. 34).

By associating the n components of ψ(y) with the n components of $(\psi^{-1})^*(x)\, f^*(y)$ (a covector at ψ(y)), one gets a chart for the manifold of pairs {y, f*(y)}, i.e., the manifold of all covectors. The latter is thus of dimension 2n and will be denoted by T*X.

Exercise 25: Show that (just as for Exer. 24) these charts are compatible.

Remark 3: If $x \in \partial X$, same problem as in Remark 2, same solution. ◊

Like TX, T*X is a bundle, the so-called *cotangent bundle*, with the same structural group as TX. The sections of T*X, or fields of covectors, are called *differential forms of degree 1*, or *1-forms.*

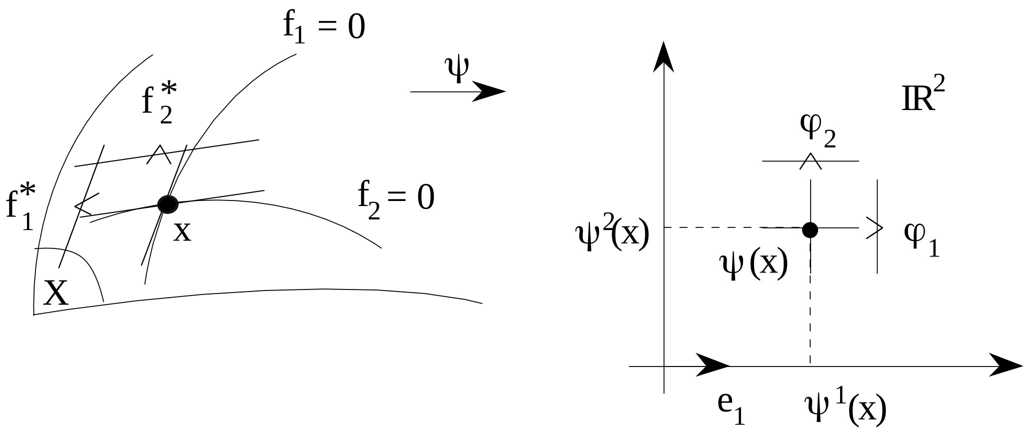


Figure 34. Basis covectors, for n = 2.

One often denotes by v the sections of TX: The value of v at a point x of X is thus a pair, consisting of x and of a tangent vector $v(x) \in T_xX$ at this point. A popular generic notation for 1-forms is ω.

If X and Y are two manifolds and $u \in X \to Y$ is a differentiable mapping between them, we now know (at last!) what the *derivative* of u is (Fig. 35). It's the bundle map $u_* \in TX \to TY$ that maps the pair $\{x, g_*\}$, where g is a trajectory through x, to $\{u(x), u_*(x)\, g_*\}$. This is sometimes called the *tangent mapping*. (Note that u may not be differentiable everywhere; in that case, $dom(u_*)$ is only a part of X.) As one sees, u gives birth to another map of different type, u_*. It also induces a map $u^* \in T^*Y \to T^*X$, of yet another type. What goes on here well illustrates the notion of *functor:* a mechanism which, given maps between objects of some category (here, the manifolds), builds other maps, which operate between objects of a different category (here, vector bundles). The words "objects", "functors", "categories", here, are used in an informal way, but they take on a precise meaning in the frame of the theory of categories, which was purportedly devised to study this kind of phenomena. (See [65] on this.) Its realm is the study of diagrams similar to the one in Fig. 36, the meaning of which should be obvious (the p's denote projections of the various bundles onto their bases).

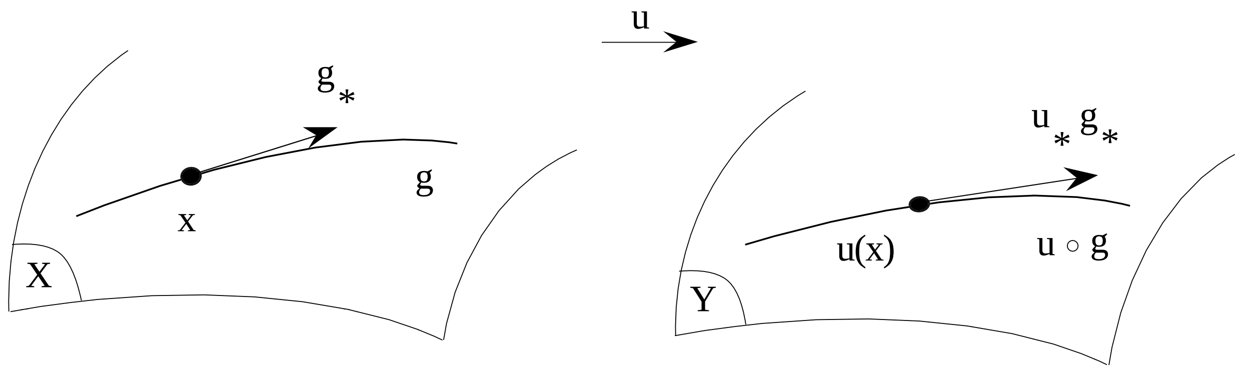


Figure 35. The "tangent mapping" $u_* \in TX \to TY$.

Remark 4. To $g \in \mathbb{R} \to X$ $(g(0) = x)$ and $f \in X \to \mathbb{R}$ $(f(x) = 0)$ correspond the mappings g_* and f^*, which send the unit vector e of $T_0\mathbb{R}$ and the unit covector ε of $T_0{}^*\mathbb{R}$ onto $g_*(0)e$ and $f^*(x)\varepsilon$ respectively. Identifying $T_0\mathbb{R}$ and $T_0{}^*\mathbb{R}$ with $\mathbb{R}$, and thus e and ε with 1, one easily sees that $g_*(0)e$ and $f^*(x)\varepsilon$ are the vector of T_xX and the covector of $T_x{}^*X$ that we called respectively g_* and f^* up to now. This is a posteriori justification for this notation, as promised p. 41. ◊

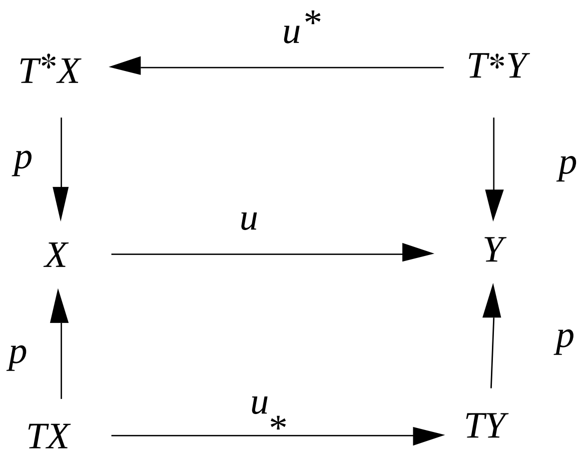


Figure 36. The functors $_*$ and *.

Exercise 26: Let v and v' be two sections of TX. Define $v + v'$ and αv, where $\alpha \in X \to \mathbb{R}$. (Same thing for ω and ω', sections of T^*X.) Conclude that TX and T^*X are *modules* on the ring of functions over X. (A module is to a ring what a vector space is to a field.) Verify that if $u \in X \to Y$, the operations u_* and u^* do distribute with respect to addition, and that $u_*(\alpha v) = \alpha\, u_* v$, as well as $u^*(\alpha\omega) = \alpha\, u^*\omega$.

2.2.3 Duality between vectors and covectors

We still have to deal with the vector-covector duality. Given $g \in \mathbb{R} \to X$, a trajectory through x, and $f \in X \to \mathbb{R}$, a function vanishing at x, both smooth, one may take the derivative at 0 of the composition product $f \circ g$ (of type

$\mathbb{R} \to \mathbb{R}$). Note this:

(14) $\langle g_*, f^* \rangle = d/dt\, (f \circ g)|_{t=0}$.

This number only depends on the classes of f and g (thanks to (11) and (12), as usual). We have thus obtained a bilinear mapping on $T_xX \times T_x{}^*X$. It is *non-degenerate*, i.e., it vanishes for all f^* only if $g_* = 0$, and vice-versa. This establishes a duality between T_xX and $T_x{}^*X$.

Let us now consider $u \in X \to Y$, a trajectory g through x, and a function f vanishing at $y = u(x)$, hence the following diagram:

$$\mathbb{R} \xrightarrow{g} X \xrightarrow{u} Y \xrightarrow{f} \mathbb{R}.$$

Then, after (14), one has

$$d/dt\, (f \circ u \circ g)_{t=0} = \langle u_*(x) g_*, f^* \rangle_{T_yY,\, T_y{}^*Y} = \langle g_*, u^*(y) f^* \rangle_{T_xX,\, T_x{}^*X},$$

showing that the linear maps $u_*(x)$ and $u^*(y)$ are mutually *transposed* (or "dual"), as we saw was the case when $X = V_m$ and $Y = V_n$.

2.3 Differential calculus on manifolds

We shall now see how a differential calculus, as powerful as the familiar one is in vector spaces, can be developed about manifold mappings.

On the face of formula (14), one would like to write

(15) $\langle g_*, f^* \rangle \equiv \partial f/\partial x\; dg/dt|_{t=0}$,

i.e., to chain the differentiations of f and of g. Even though such a chain rule has no validity, since the right-hand side of (15) has no meaning yet, it is quite suggestive: the action of covector f^* on vector g_* may be conceived as the differentiation of f "in the direction of g_*". In fact, this interpretation would be correct if X were V_n, $\partial f/\partial x$ then being the gradient of f at x. So we are entitled to define the *gradient of* f *at* x as the covector f^*, just as g_* was the velocity vector at x on the trajectory g.

This interpretation of (15) as the derivative of f is evidence that a vector *field* (i.e., a section of TX) can always be seen as a differential operator: the one that associates with function f the function $x \to \langle v_x, f^* \rangle$, that is, according to (15), the derivative of f at x in the direction of v_x. Conversely, if ∂ is a first-order differentiation operator, one may prove the existence of a unique vector field v such that

(16) $$\partial f = x \to \langle v_x, f^* \rangle.$$

In other words, *vector fields are first-order differential operators* on manifolds. Some text-books, like e.g., [96, 105], define tangent vectors this way. We did not, but we still can reflect this point of view at the notational level by denoting ∂_v both the vector field v and the differentiation operator, and $\partial_v f$ the function which appears on the right-hand side of (16). The basis vectors (relative to a chart about x) are often denoted as $\partial/\partial x_i$, but the plain (and more logical) notation ∂_i seems to be gaining favor. Let us adopt it. So, denoting by v^i the components of v in this basis, we have

$$\partial_v = \Sigma_{i=1,\ldots,n} v^i \, \partial_i$$

which legitimates the notation

(17) $$\partial_v f = \Sigma_{i=1,\ldots,n} v^i \, \partial_i f$$

(which, if $v = g_*$, is nothing else than (15)!).

Remark 5: Life would be hard if notation could not be abused. It is now quite natural to write, for a vector v_x at x,

$$v_x = \Sigma_{i=1,\ldots,n} v^i_x \, \partial_i \, ,$$

and for a vector field v,

$$v = x \to \Sigma_{i=1,\ldots,n} v^i(x) \, \partial_i,$$

the $v^i(x)$'s being the coordinates of v_x in some basis about x. This is an abuse, on two counts: First, though v(x) is a point of the fibre TX, i.e., a pair consisting of a point x and a vector at x, only the latter is made explicit; but this is only natural. On the other hand, a *section* of TX is denoted as if it was a *function* taking its values in the fibre, whereas we toiled to emphasize the difference between these two concepts. But again, this abuse is natural: for locally, in the domain of a chart about x,, sections are indeed functions taking their values in the fibre, by the very definition of a bundle. From here on, we shall indulge in the abuse without any further apologies. ◊

Consider, again, $u \in X \to Y$, with X and Y of dimensions m and n, and $f \in Y \to \mathbb{R}$. Let v be a vector field on V, and $w = u_* v$. One has a basis $\{\partial_J : J = 1, \ldots, m\}$ for $T_x X$, a basis $\{\partial_i : i = 1, \ldots, n\}$ for $T_y Y$, with $y = u(x)$. (Note again the use of small caps.) In these bases, induced by charts about x and y which need not explicitly be written, one has, following the pattern of (17),

(18) $$\partial_v = \Sigma_{J = 1, \ldots, m} v^J \partial_J, \quad \partial_w = \Sigma_{i = 1, \ldots, n} w^i \partial_i.$$

On the other hand, by way of definition of u_*,

$$\partial_w f = \partial_v (f \circ u),$$

and this suggests the following development, where we let the rules of differential calculus play freely:

$$\partial_v (f \circ u) = \Sigma_{J = 1, \ldots, m} v^J \partial_J (f \circ u) = \Sigma_J v^J \Sigma_{i = 1, \ldots, n} \partial_i f / \partial_J u^i,$$

that is,

(19) $$\partial_w = \Sigma_J v^J \Sigma_i \partial_J u^i \partial_i.$$

This is not formally valid, since $\partial_J u^i$ has no precise meaning for the time being, but let us persist. One also has $\partial_w = u_* \partial_v$, thus, after (18),

$$\partial_w = \Sigma_J v^J u_* \partial_J .$$

But $u_* \partial_J$ is a vector which can be written as follows, in the base of the ∂_is:

(20) $$u_* \partial_J = \Sigma_{i = 1, \ldots, n} \partial_J u^i \partial_i,$$

if we decide to call $\partial_J u^i$ its components (compare with (13), p. 44!). Thus we get back (19), and this *gives meaning* to the $\partial_J u^i$s of (19): these are the components of vector $u_* \partial_J$. Now (19) is legal, so we know how to apply the chain rule: this is all that was required to extend to manifolds the familiar rules of differential calculus.

After (13) (p. 44), one may as well denote by $\partial u^i / \partial x^J$ the $\partial_J u^i$s of (19) and (20). Let us stress that none of these expressions have intrinsic meaning: we just gave one to them, with (20). (To *compute* these numbers, if need be, one uses charts.) The notation is such that one may now apply the rules of differential calculus "as if" the manifolds X and Y were affine spaces. This combines with Einstein's convention of implied summation with respect to repeated indices (not adopted in this book) to make a powerful tool, which of course one should know, and this is why we insisted on its foundations. But to exclusively rely on its use would not be a good idea (no more, to suggest an analogy, than to systematically rely on analytical methods in matters of geometry).

What about covectors? The basis covectors, in a given chart, are in general denoted by dx^i, but the notation d^i seems, as for basis vectors, more logical. These basis covectors are the linear mappings which to $v \in T_xX$ assign the components v^i. Thus

$$d^i v = v^i,$$

hence the different possible expressions of the duality between a vector v and a covector f^*:

$$\langle v, f^* \rangle = \Sigma_i \, \partial_i f \, v^i = \Sigma_i \, \partial_i f \, d^i v = \partial_v f$$

$$= (\Sigma_i \, v^i \, \partial_i) \, f = \Sigma_i \, \partial_i f \, d^i v = (\Sigma_i \, \partial_i f \, d^i) \, v,$$

which suggests the following notation: df for the field of covectors generated by f, and

$$df = \Sigma_i \, \partial_i f \, d^i$$

for its expression in the covector basis. The operator d thus introduced is called *exterior derivative*. One is now entitled to write

$$df(v) = \partial_v f$$

as another version of (15) (when $g_* = v$).

This certainly leaves much to be desired. One should like more symmetric expressions, like e.g., $d^f v$ in lieu of $df(v)$. But can one go against tradition, which so firmly backs the use of df ? The object thus denoted, the so-called *gradient* of f, is a field of covectors, i.e., a 1-form, associated with f, whose effect on a vector field v consists in taking the derivative of f in direction v at each point. One is facing here a familiar notion, sometimes very difficult to grasp during the calculus curriculum, that of *differential*. The differential is a machinery whose purpose is to evaluate "the (first order) variation of a function f in the neighborhood of a point x". Answer: "this variation is a function of the displacement vector v; this function is called the differential of f; its expression is $df(v)$". Differentiation and derivation, as one can see, correspond to dual points of view, because one might as well answer as follows: "this variation is a function of f, the function under consideration, once given the displacement vector; this function consists in taking the derivative of f along v; its expression is $\partial_v f$".

Remark 6: The gradient is often defined as a vector field, instead of as a 1-form. We'll have more to say on this later. ◊

Exercise 27: Write down the counterpart of formula (20) for the covectors $u^* d^i$.

Exercise 28: In case u is a diffeomorphism (which implies dim(X) = dim(Y)), show that

$$(u^{-1})_* = (u_*)^{-1}$$

and the same about u^*.

At this stage, we may begin to see vectors and covectors as "geometric objects" which, so to speak, "live" on manifolds. When one goes via u from a manifold X to another manifold Y, vectors on X are "pushed forward" by u_*, while covectors on Y are "pulled back" to X by u^*. Vectors and covectors can be written as linear combinations of basis vectors and basis covectors. Basis vectors are akin to derivations along the coordinate lines (the g^i of Fig. 32). The basis covectors assign to a vector its components. The effect of a vector on a covector is a real number, invariant with respect to changes of charts. The bilinear mapping thus obtained is non-degenerate (cf. p. 49), hence a duality between vectors and covectors. Vectors are akin to derivation operations, and covectors to differentials of functions.

We shall now discover other objects which live on a manifold, those of the same family which stand at a given point forming the fibre of some bundle. These are the *tensors*. Among them, *differential forms* play a major rôle.

2.4 Differential forms

2.4.1 Multi-covectors

We already met with two "vectors" of classical physics which are in fact, from the geometrical viewpoint, covectors: *force* (whose effect on a displacement-velocity vector is a power) and the *electric field* e, which one can identify with the force it exerts on charged particles. There are "vectors", like for instance the magnetic induction b, which correspond to still different objects. One knows the rôle played, in several instances, by the flux of b across a surface, or a surface element. But a "surface element" is, in precise terms, a pair of vectors, v_1 and v_2, say, tangent at some point of the surface referred to. The flux across this element is obviously a linear function of v_1 and v_2. Moreover, it changes sign when the order of the vectors is changed, according to the intuitive idea that surface elements $\{v_2, v_1\}$ and $\{v_1, v_2\}$ are the same, but with opposite orientations. (We shall come back to the notion of orientation, with a precise definition, in a moment.) So, if we state the following definition:

Definition 6: *One calls a* bi-covector, *or* 2-covector *at* x, *any* $\mathbb{R}$-*valued mapping* ω *on* $T_xX \times T_xX$, *linear with respect to both arguments, and* antisymmetric *(or* skew-symmetric), *i.e.,*

(21) $$\omega(v_1, v_2) = -\omega(v_2, v_1) \ \forall \ v_1, v_2 \in T_xX,$$

we realize it is custom-made to fit b(x), the magnetic induction at point x: b(x) is indeed a 2-covector at x. The field b itself is a field of such objects, that is, a cross-section of the bundle of 2-covectors: this is what is called a *2-form*.

The way to generalization is straightforward: one will call *p-covector* at x a multilinear *alternating* map on T_xX, i.e., following up on (21), one that changes sign when two among the p vector factors are exchanged (one also says "skew-symmetric"). Hence the notion of a "p-form":

Definition 6 (continued): *A p-form, or differential form of degree p, is a field of p-covectors.*

The definition of a p-covector carries over to the case p = 0: it is then an argument-free function, i.e., a plain number, based at x. A 0-form is thus a function on X. (A smooth function, of course: recall this is understood for all sections of bundles we may be led to consider.)

So here is a new family of bundles (*vector* bundles, clearly) on X. What is the dimension of the fibre? Let us begin with the case p = 2. Let ω_x be a 2-covector at x and $\{\partial_i : i = 1, \ldots, m\}$ a basis for T_xX. If $v_j = \Sigma_i v^i_j \partial_i$, with j = 1 or 2, one has

$$\omega_x(v_1, v_2) = \omega_x(\Sigma_i v^i_1 \partial_i, \Sigma_j v^j_2 \partial_j) = \Sigma_{i,j} \omega_x(\partial_i, \partial_j) v^i_1 v^j_2.$$

By antisymmetry, knowing the n(n – 1)/2 numbers $\omega_x(\partial_i, \partial_j)$ for $i < j$ is enough to compute ω, so the dimension of the fibre is n(n – 1)/2. One may thus write

$$\omega_x(v_1, v_2) = \Sigma_{1 \leq i < j \leq n} \omega_x(\partial_i, \partial_j)(v^i_1 v^j_2 - v^i_2 v^j_1).$$

There are two kinds of factors in this expression: the $\omega_x(\partial_i, \partial_j)$, that can be denoted as $\omega_{ij}(x)$, which characterize ω_x, and the bilinear (with respect to v) expressions. Each of these is the result of applying to v_1 and v_2 a particular 2-covector, denoted $d^i \wedge d^j$:

(22) $$(d^i \wedge d^j)(v_1, v_2) = v^i_1 v^j_2 - v^i_2 v^j_1.$$

These $d^i \wedge d^j$ are the basis vectors in the fibre of 2-covectors above x.

Remark 7. People formed in some European traditions may be mistaken by the use of the symbol $\wedge$, and think they recognize in (22) an old acquaintance, in the case $n = 3$. But this is an illusion: the notion of vector product of two vectors (the "cross product", in Gibbsian tradition) has nothing to do here, and will not be met before long. ◊

If ω is a 2-form, it is thus only natural to write, in the neighborhood of a point x,

$$\omega = x \to \Sigma_{i<j}\, \omega_{ij}(x)\, d^i \wedge d^j.$$

The notational abuse is the same as the one (lambasted, then forgiven) in Remark 5.

When $p > 2$, the dimension of the fibre is given by the exercise that follows.

Exercise 29: Let ω be a p-form on X. Justify the notation

(23) $$\omega = x \to \Sigma_{\sigma \in C(n,\, p)}\, \omega_\sigma(x)\, d^{\sigma(1)} \wedge \ldots \wedge d^{\sigma(p)}$$

where $C(n, p)$ is the set of *increasing* injections from the segment [1, p] of IN into the segment [1, n]. What is the dimension of the fibre?

As one sees, the game stops when $p > n$, because a multilinear alternating mapping yields 0 when its vector factors are not linearly independent, and p vectors cannot be independent if $p > n$. So there are no non-trivial covectors for $p > n$.

The case $p = n$ is special. On V_n, there is a well-known n-covector, namely the *determinant* of n vectors, in a given basis. Changing the basis yields another n-covector (another n-linear alternating map), but proportional to the former, as one well knows, and as is easily seen by doing the computation in some basis. Conversely, every n-linear alternating map is a multiple of some determinant. So the fibre of n-covectors is of dimension 1. A field of n-covectors is called a *volume* if it does not vanish on X (a *local volume at* x if it does not vanish in some neighborhood of x). The word is well-chosen, for the determinant of n vectors is indeed, in elementary geometry, the volume of the parallelotope built on them. The sign is characteristic of the *orientation* of the set of n vectors, a concept we shall soon encounter again.

Exercise 30: Let $u \in X \to Y$ and ω_y be a 2-covector at $y = u(x)$. Set, on $T_xX \times T_xX$,

$$(u^*\omega)_x = \{v_1, v_2\} \to \omega_y(u_* v_1, u_* v_2).$$

Check this is a covector at x. Take a basis for T_xX. Consider the bases (induced by u) for T_yY and for the p-covectors at x and y. Write an expression for $u^*\omega$ in these bases.

2.4.2 The algebra of covectors: exterior product

As one may suspect, the notation $d^i \wedge d^j$, for one of the basis 2-covectors, is not a single block. It can be conceived as the result of an operation, denoted $\wedge$, that creates a 2-covector from covectors d^i and d^j. The operation can be iterated, according to (23), to yield p-covectors. In fact, it can be defined for forms of any degree. Let us first agree that if σ is an increasing injection from the integer segment [1, p] into [1, p + q], then ς is the *complementary* injection, of domain [1, q], whose image is the set of integers that do not belong to the codomain of the first one, and that $\text{sign}(\sigma, \varsigma)$ is the signature of the permutation of [1, p + q] thus obtained. Then:

Definition 7: *Let* ω *and* η *be a p-* and *a q-covector. Set, if* $p + q \leq n$,

$$(24) \qquad (\omega \wedge \eta)(v_1, \ldots, v_{p+q}) = \sum_{\sigma \in C(p,\, p+q)} \text{sign}(\sigma, \varsigma)\, \omega(v_{\sigma(1)}, \ldots, v_{\sigma(p)})\, \eta(v_{\varsigma(1)}, \ldots, v_{\varsigma(q)}).$$

If $p = 0$ *(i.e., if* ω *is a function),* $\omega \wedge \eta$ *is simply denoted* $\omega\, \eta$.

The reader will satisfy herself that $d^i \wedge d^j$ of (22) does correspond to this definition, and that $d^1 \wedge \ldots \wedge d^n$ is indeed the determinant of n vectors. Operation $\wedge$ is called the *exterior product,* or simply *wedge product*. It is associative (contrary to the cross product, also denoted $\wedge$ by some, although the Gibbsian notation with a cross is obviously preferable) and *anticommutative,* in the following sense:

$$(25) \qquad \omega \wedge \eta = (-1)^{pq}\, \eta \wedge \omega.$$

All this is easily verified. Note that $\omega \wedge \omega = 0$ if p is odd.

Exercise 31: Write $\omega \wedge \eta$ in the form (23).

Exercise 32: Show that $\text{sign}(\sigma, \varsigma) = (-1)^k$, where $k = \Sigma_{i = 1, \ldots, p} (\sigma(i) - 1)$, and that $\text{sign}(\varsigma, \sigma) = (-1)^{pq}\,\text{sign}(\sigma, \varsigma)$. Prove (25).

Thus, the fibres of covectors above a point are not foreign to each other. Actually, one should rather consider *all* p-covectors at x as elements of a single family, the *algebra* of covectors: the structure of algebra is conferred on it by the operation $\wedge$. It is named "Grassmann algebra".

Some dissymmetry has crept in with this proliferation of covectors, with the effect to spoil the simple vector-covector duality we had at the beginning. One regains balance by introducing objects dual to the p-covectors for $p > 1$. One thus calls *p-vector* an element of the dual of the vector space of p-covectors. (Beware that a p-vector is *not* a collection of p vectors!) A field of 0-vectors is a function. The Grassmann algebra of multi-vectors also exists, but is less popular and less often applied than the multi-covectors one. (A noteworthy exception is [52], an account of classical electrodynamics based on multi-vectors.)

Exercise 33: Derive the following coordinate expression for a field of p-vectors:

$$u = x \rightarrow \Sigma_{\sigma \in \mathcal{C}(n,\, p)}\, u_{\sigma(x)}\, \partial_{\sigma(1)} \wedge \ \ldots \wedge \partial_{\sigma(p)},$$

where the $\partial_i \wedge \ \ldots \wedge \partial_j$ are p-vectors that will be defined with reference to the basis p-covectors $d^i \wedge \ \ldots \wedge d^j$.

A word about tensors, to conclude. These are fields of multilinear mappings, but not necessarily alternating ones, which do not work exclusively on vectors of the tangent space (as p-covectors do) or on covectors (like p-vectors) but on both kinds. We'll encounter one later (the metric tensor).

Chapter 3

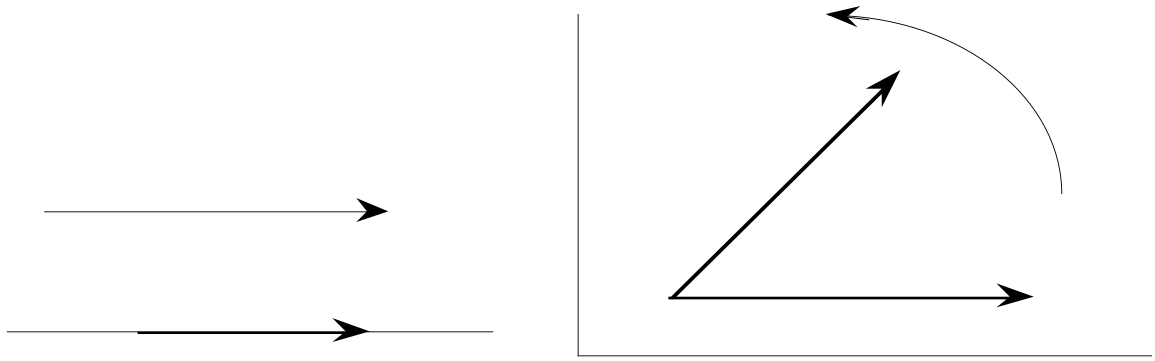
Orientation and integration

3.1 Orientability of a manifold

3.1.1 Volumes

Let X be of dimension n, x a point and Ω_x an n-covector, non-zero (i.e., $\Omega_x(v_1, \ldots, v_n) \neq 0$ if the v_i are independent). Suppose first n = 1 or n = 2. The presence of Ω_x then *orients* the tangent space (Fig. 37). This is clear if n = 1: a line is oriented if one can know left from right, rear from front, past from future, etc. All this amounts to be able to tell "positive" and "negative" vectors apart: a vector v will be positive if $\Omega_x(v) > 0$. Similarly, for n = 2, one has an orientation when one knows the meaning of "turning left", or "counter-clockwise": if v_1 and v_2 are two vectors, v_2 is "left to v_1", or else "v_1 and v_2 form a direct frame", if $\Omega_x(v_1, v_2) > 0$. For n = 3, space is oriented when one can know whether three vectors form a direct frame: so is the case when $\Omega_x(v_1, v_2, v_3) > 0$. As two different 3-forms, Ω_x and Ω'_x, yield numbers with either matching or opposite signs, there are only two possible orientations (and the "right-hand rule" is there to remind us of which of the two classes of 3-forms orients positively). In dimension n, a basis $v_1, \ldots, v_n$ will be *directly* oriented, or a *direct frame*, if $\Omega_x(v_1, \ldots, v_n) > 0$, a *retrograde* frame in the other case.

One may choose a consistent orientation system in a whole neighborhood U of x, provided one has a smooth field of n-forms $x \to \Omega_x$, whose domain includes U, and *non-vanishing* in U. Then, not only one may tell, at every point y of U, whether a system of n independent vectors at y is or is not positively oriented, but this orientation, this sign associated with the system of vectors, continuously depends on x: if, for n smooth vector fields v_i, one has $\Omega_x(v_1(x), \ldots, v_n(x)) > 0$ at point x, this stays valid by continuity if one substitutes a nearby y for x.



$\Omega_x(v) > 0$ $\Omega_x(v_1, v_2) > 0$

Figure 37. Notion of orientation for $n = 1$ and $n = 2$.

In particular, there is a system of n vector fields whose orientation is the same all over the domain of a chart about x: these are the basis vectors $x \to \partial_i(x)$, as defined p. 50. To orient the neighborhood of x amounts to deciding whether these n vectors form a direct or a retrograde frame.

Thus one may *locally* orient a manifold, and in two different ways. The question of orientability is whether one can do that, in a consistent manner, over the *whole* manifold.

The Möbius strip example (Fig. 38) suggests how to do it, and also how one can fail at this task. How to do it: Choose an atlas, orient the domain of each chart. When two such domains overlap, the orientations are either the same or opposite, but one may (at least if the intersection of domains is in one piece, which can always be arranged) change one of the orientations and proceed step by step, thus trying to make all orientations compatible. But this process can fail: it does fail with MS, because this particular manifold can be described by using three charts whose orientations, whatever the combination one chooses among the six possible ones, are inconsistent. Our intuition of an "orientable" manifold is one for which this process succeeds. But if so is the case, one may obtain, by smoothly patching the local n-forms together, an n-form which never vanishes on X, what we called earlier a volume. Hence the following definition:

Definition 8: *A manifold is* orientable *if one may endow it with a volume. Two volumes* Ω *and* Ω' *"define the same orientation" if* $\Omega' = \alpha\,\Omega$, *with* $\alpha > 0$.

An orientation is thus, in full rigor, an equivalence class of volumes, the equivalence relation being the one given in Def. 8 above. Thus, on a connected manifold, there are two possible orientations, or none.

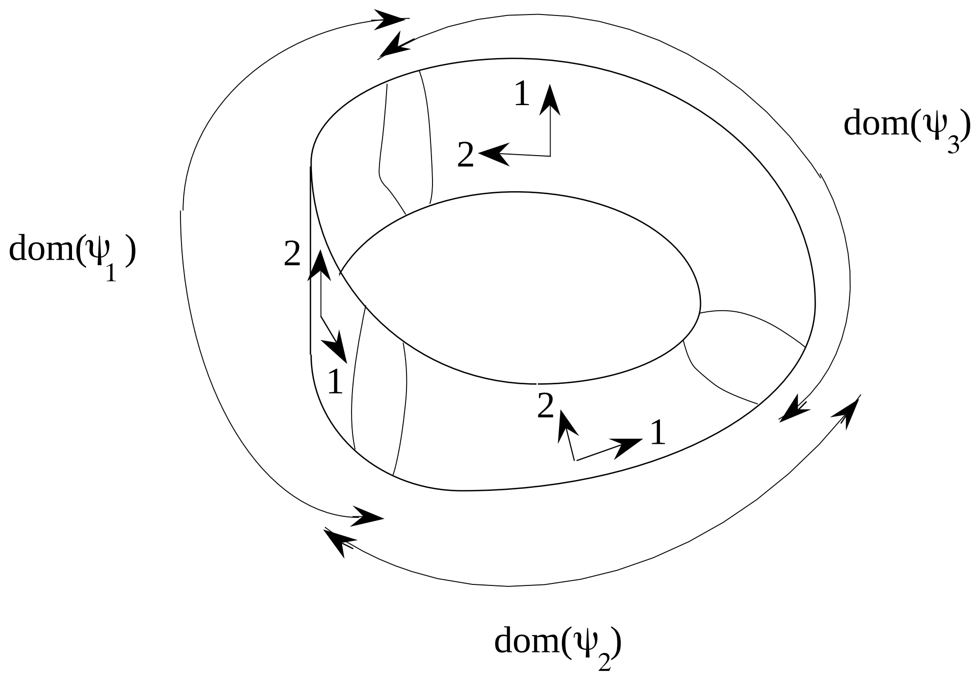


Figure 38. Orienting a Möbius strip, by continuation: the method, with here three charts, and (try something else, and see) its failure.

Whether we live in an orientable manifold has been much speculated about, and remains an unanswered question. Imagine the manifold obtained by removing from E_3 the interior of two spheres (Fig. 39) and by gluing the surfaces according to the indicated identification. It cannot be oriented, as one will convince oneself by looking at what happens to an oriented frame which slips along the trajectory γ, when it reaches point A and goes through. If we lived in such a universe, and assuming that it inherits from E_3 its metric (a concept on which we shall return), we could see, from Earth T, two images of the same galaxy G, sent along the two geodesics GT and GBT. An astronaut traveling along TG, then GBT, would come back with the heart on the right side. Maybe it's what happens to the heroes of the movie *The Black Hole*, the epilogue of which leaves us uncertain about what the post-mortem disclosed.

Exercise 34: How is determined point B on Fig. 39?

Exercise 35: Check that the manifolds of Figs. 15 and 16 (p. 23) are non-orientable.

Exercise 36 (Fig. 40): Describe a manifold of dimension 3, non orientable, without boundary, compact, obtained by identifying opposite faces of a cube in some specific way.

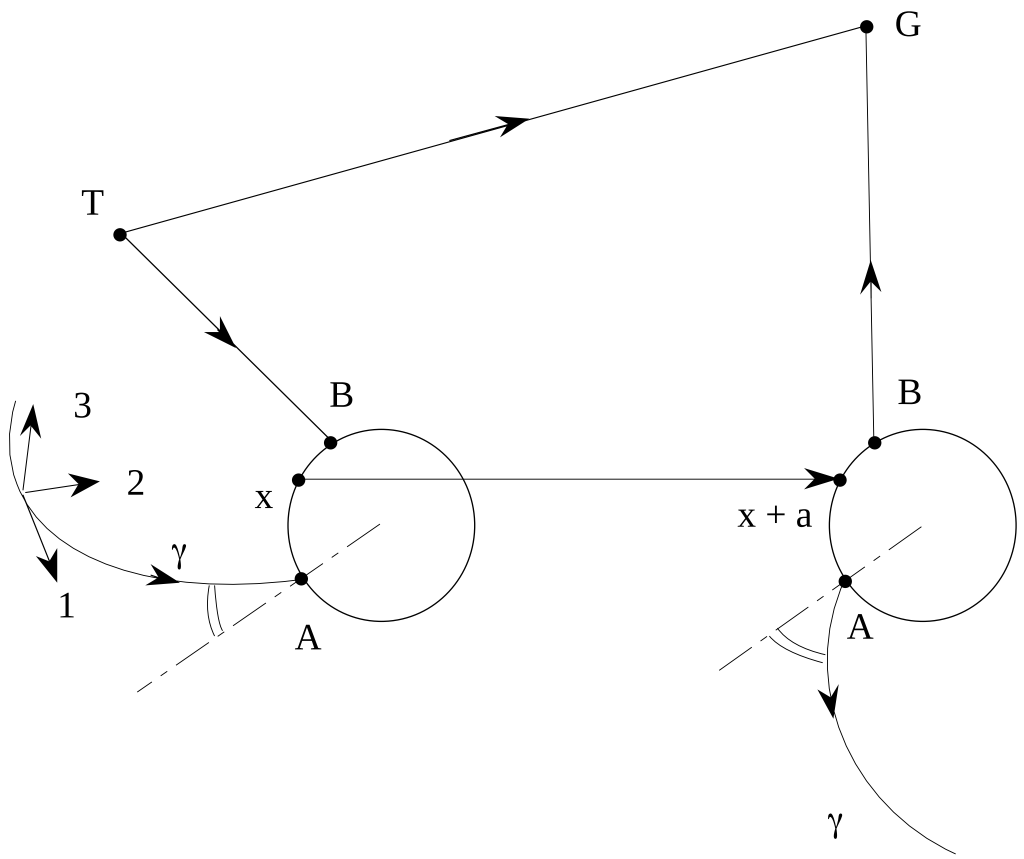


Figure 39. By identifying two spheres of identical radius, at distance a one from the other, according to the equivalence x ~ x + a, one turns the remaining space into a non-orientable manifold of dimension 3.

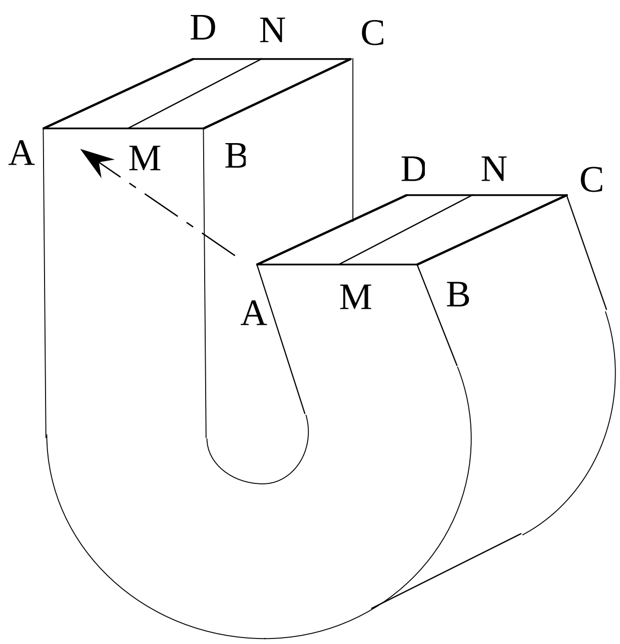


Figure 40. A suggestion for Exer. 36.

3.1.2 Transverse fields

The non-orientability of the Möbius strip is sometimes "proved" as follows: Choose a continuous field of normals in the neighborhood of some point, which can always be done. Then try to extend such a field to the whole ribbon. That fails. Therefore . . .

Such reasoning is incorrect, and it is worthwhile to understand why. Let us first give a long overdue definition:

Definition 9. *An* immersion *of a manifold* X *into a manifold* Y *is a mapping* $u \in X \rightarrow Y$, *of domain* X, *such that* $u_*(x)$ *is injective at all points of* X.

Note that an immersion is not necessarily itself injective: if it is, and *if* u *is a diffeomorphism* of X into u(X) (important! cf. Fig. 41), one calls it an *embedding*. (Cf., e.g., [62].)

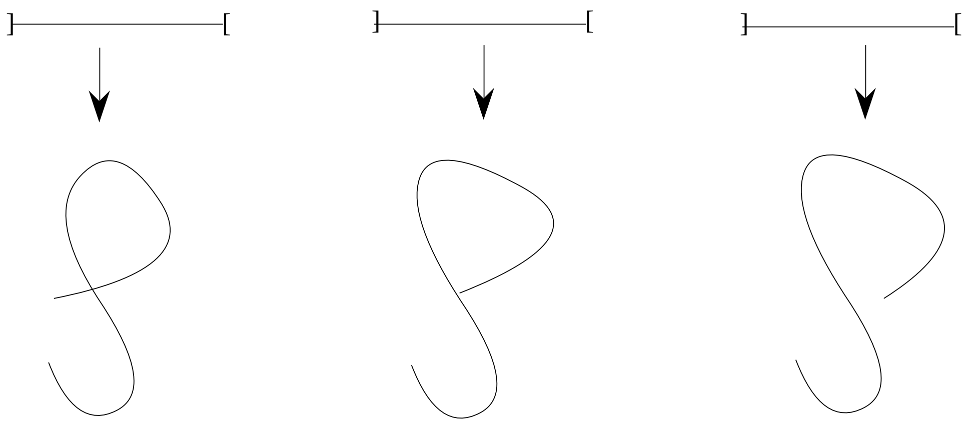


Figure 41. Three immersions of]0, 1[into $\mathbb{R}^2$. Only the last one is an embedding. The one in the middle is indeed injective (no double point), but its image in $\mathbb{R}^2$, with the topology induced by $\mathbb{R}^2$, is not a manifold.

Exercise 37. Find an injection of]0, 1[into $\mathbb{R}^2$ which is not an immersion.

Let thus u be an immersion, the dimensions of X and of Y being $m - 1$ and m respectively. So the image of $T_x X$ under u_* is a subspace of codimension 1 in the tangent space $T_y Y$ at $y = u(x)$. Let there be for each $x \in X$ a vector n(x) of $T_y Y$, non vanishing, not included in $u_*(T_x X)$. If now $x \rightarrow n(x)$ is continuous, one says that n is *transverse* with respect to X. (Example: the field of outward going unit normals to a closed surface of E_3.)

When a submanifold X is thus endowed with a continuous field of transverse vectors, it can inherit an orientation from the ambient manifold Y, to the extent that Y itself is oriented. For if Ω is a volume on Y, the $(m - 1)$-form

$$\{\xi_1, \ldots, \xi_{m-1}\} \to \Omega(n, u_*\xi_1, \ldots, u_*\xi_{m-1})$$

does constitute a volume for X. So if Y is orientable, the existence of a transverse field on X implies the orientability of X (and the other way round, too—but this is not easily proved).

But if Y is not orientable, it may contain orientable submanifolds deprived of any transverse field, or the other way round, as one will see by working out the following two exercises.

Exercise 38: Check: the midcircle of a Möbius strip has no transverse field.

Exercise 39: Find, on Fig. 40, an immersed Möbius strip, equipped with a continuous field of normals.

So the fact that MS, when immersed in E_3 the usual way, has no continuous field of normals does not prove anything about its orientability. The reasoning was wrong because the existence of such a field of normals is not a property of X or of the ambient manifold Y, but a property of the immersion u. What is involved is in fact the orientability of u itself (a concept we shall define in a moment).

Remark 8. Let ω be a p-form, v a vector field. The operation which consists in building the $(p - 1)$-form

$$\{\xi_2, \ldots, \xi_p\} \to \omega(v, \xi_2, \ldots, \xi_p)$$

(used above to give a volume to X) is called *contraction of* ω *by* v, or *inner product*. The result is often denoted $i_v\omega$. We'll have use for it later. ◊

Exercise 40: Is the manifold SO_3 (of all rotations about a fixed point) orientable? (Hint: first check that SO_3 can be obtained from a sphere of radius π by identifying antipodal surface points.)

3.1.3 Orientation covering

To any manifold X, orientable or not, one can associate an orientable manifold as follows (look at Fig. 26, p. 34). It will be a bundle on base X, with for fibre a two-points set, say $\{1, -1\}$, and for structural group the group S_2 of permutations of two objects, also denoted $\{1, -1\}$. To build it, consider an atlas on X, say $\{\psi_\alpha : \alpha \in \mathcal{A}\}$, and glue local Cartesian products (which consist in two copies of $\mathrm{dom}(\psi_\alpha)$, namely $\{-1\} \times \mathrm{dom}(\psi_\alpha)$ and $\{+1\} \times \mathrm{dom}(\psi_\alpha)$) with transition functions $g_{\alpha\beta} = 1$ or -1 depending on whether the orientations of the basis vectors do or do not coincide on the intersection $\mathrm{dom}(\psi_\alpha) \cap \mathrm{dom}(\psi_\beta)$ (which may always be taken connected, provided there are enough charts). The result is a two-sheeted covering of X, say $\tilde{X}$, which will easily be seen to be orientable: Indeed, the foregoing recipe is but a paraphrase of the definition of orientability given earlier.

Exercise 41: Satisfy yourself that $\tilde{X}$ does not depend on the chosen system of charts.

One calls $\tilde{X}$ the *orientation covering* of X. When X is connected and orientable, $\tilde{X}$ consists in two disjoint connected parts (two copies of X). When X is connected but not orientable, $\tilde{X}$ is connected.

The fibre above x, for $x \in X$, consists in two points, that we shall note x^+ and x^-. Thus, if p is the projection onto the base, $px^+ = px^- = x$, and $p^{-1}(x) = \{x^+, x^-\}$ (Fig. 42).

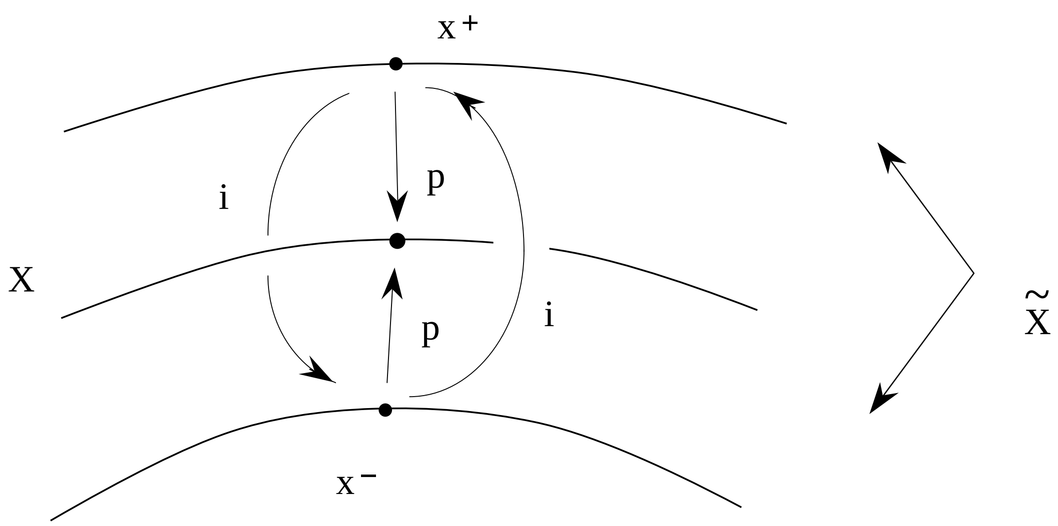


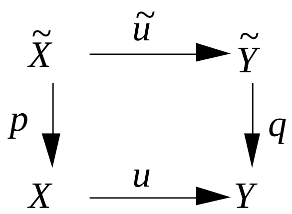
Figure 42. (The involution i will be used later.)

Exercise 42: Let an orientation of $\tilde{X}$ be given. Consider a positively oriented system of basis vectors at x^+, and an other one at x^-. Show their images by p_* are bases at x, with *opposite* orientations.

Exercise 43: Show that the continuous "lifts" $r^+ = x \to x^+$ (such that pr^+ be the identity) and r^-, that one can always define locally, cannot be continued all over X unless X is orientable.

A first application of these notions is the definition of the orientability of a function $u \in X \to Y$: it will be orientable if it can be lifted to a bundle map of $\tilde{X}$ into $\tilde{Y}$ with isomorphism from fibre to fibre. In precise terms,

Definition 10: *A smooth function* $u \in X \to Y$ *is* orientable *if there exists a smooth* $\tilde{u} \in \tilde{X} \to \tilde{Y}$ *which makes the following diagram commute:*



(where p and q are the projections), *and under which the two points of* $p^{-1}(x)$ *have distinct images*.

This amounts to saying that one may "associate, in a continuous way, the orientations of a neighborhood of x and of a neighborhood of $u(x)$" (a sentence to which, actually, only Def. 10 is able to give precise meaning!). Such an association mechanism was described above in the case when u is an immersion of codimension 1 endowed with a transverse field. Such a mapping is therefore orientable.

Note that if $\tilde{u}$ does exist, $\tilde{u} \circ i$ (where i is the involution of Fig. 42) satisfies the same requirements: There are thus two ways (or none) to orient a mapping when X is connected.

Exercise 44: If X and Y are orientable, $u \in X \to Y$ is.

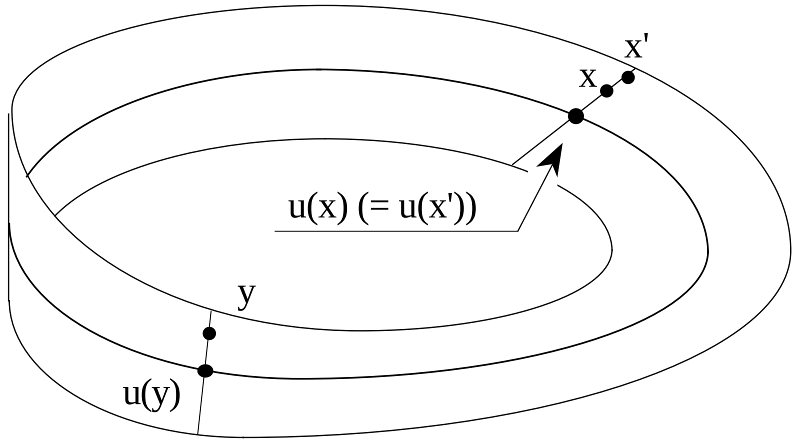


Figure 43. Retraction of MS onto its middle line

Exercise 45: The "retraction" of MS onto its midline (Fig. 43) is not orientable. More generally, if dom(u) = X, if Y is orientable, but not X, then u is not orientable. (Suggestion: continuous maps preserve connectedness.)

Exercise 46: A diffeomorphism is always orientable.

3.2 "Twisted" objects

3.2.1 Twisted functions

We'll now indulge in an apparently gratuitous game, the point of which will only later become apparent. It's again a matter of building a non-trivial bundle, one which is, so to speak, "warped by the orientation", just like the above one, but the fibre this time will be $\mathbb{R}$, instead of being a pair of points. The structural group again contains two elements (which are mappings from $\mathbb{R}$ onto $\mathbb{R}$): the identity $\lambda \to \lambda$ and the inversion $\lambda \to -\lambda$.

Let thus X be a manifold, $\{\psi_\alpha : \alpha \in \mathcal{A}\}$ a system of charts, with domains so chosen that all their intersections be connected. Let us consider the Cartesian products $\mathbb{R} \times \mathrm{dom}(\psi_\alpha)$ and $\mathbb{R} \times \mathrm{dom}(\psi_\beta)$, and let us identify them according to the following rule: a pair $\{\lambda, x\}$ belonging to one is equivalent to a pair $\{\mu, y\}$ belonging to the other if $x = y$ to start with, and if $\lambda = +\mu$ or $-\mu$, depending on whether the orientations induced by the charts coincide or not in the common domain $\mathrm{dom}(\psi_\alpha) \cap \mathrm{dom}(\psi_\beta)$.

The fibered manifold produced by this operation (let us call it $\tilde{\mathcal{A}}(X)$) is independent of the chosen system of charts. If X is orientable, $\tilde{\mathcal{A}}(X)$ is simply the Cartesian product $\mathbb{R} \times X$, and sections of this bundle are nothing else than real-valued functions defined on X. But if X is not orientable, they are objects of a new kind. To better understand their nature, let us observe that if ψ_α and ψ_β have a common domain, but contrary orientations, the above procedure calls for the identification of $\{\lambda, x\}$ with $\{-\lambda, x\}$. So if one insists on considering a cross-section s of $\tilde{\mathcal{A}}$ as a function defined on X, its values are not real numbers, but *pairs* {real value, orientation}, or more accurately, equivalence classes of such pairs, the equivalence relation being

$$\{\lambda, \Omega\} \sim \{-\lambda, -\Omega\}$$

where Ω is a local volume giving the orientation.

Sections of $\tilde{\mathcal{A}}(X)$ are called *twisted functions*, which well fits such bizarre objects. Can it be that physics really needs them?

To make sure of that, let us consider the problem of Fig. 44, which consists in computing eddy-currents induced in a thin metallic conductor. Suppose one wants to apply the stream-function method. As described in the standard case when the surface is endowed with a field of normals n, this method consists in expressing the current density j as $j = -\, n \times \text{grad } a$, where a is a function on C to be determined. One meshes C, and unknowns are the nodal values of a.

A notorious problem with this method is the possibility that a be multivalued, which can be remedied by properly placing cuts (Remark 1, p. 35). This difficulty is not the one we want to discuss, so we make sure to avoid it by introducing the perfectly permeable magnetic circuit M of Fig. 44. The magnetic field vanishes there, so, by Ampère's Theorem, there is no global current, hence no grounds for multivaluedness.

The other difficulty, the one which does concern us, is the absence of a continuous field of normals. This, however, does not rule out the stream-function method, because we may define a *locally* (Fig. 45). Let's pick a point x_0, decide that $a(x_0) = 0$, and assign to $a(x)$ the *circulation* along some path joining x_0 to x of vector $j^\perp$ (that is, j rotated ninety degrees to the left). Since $\text{div } j = 0$, $\text{rot } j^\perp = 0$, so $a(x)$ is independent on the chosen path, and $j = -(\text{grad } a)^\perp$ by construction.

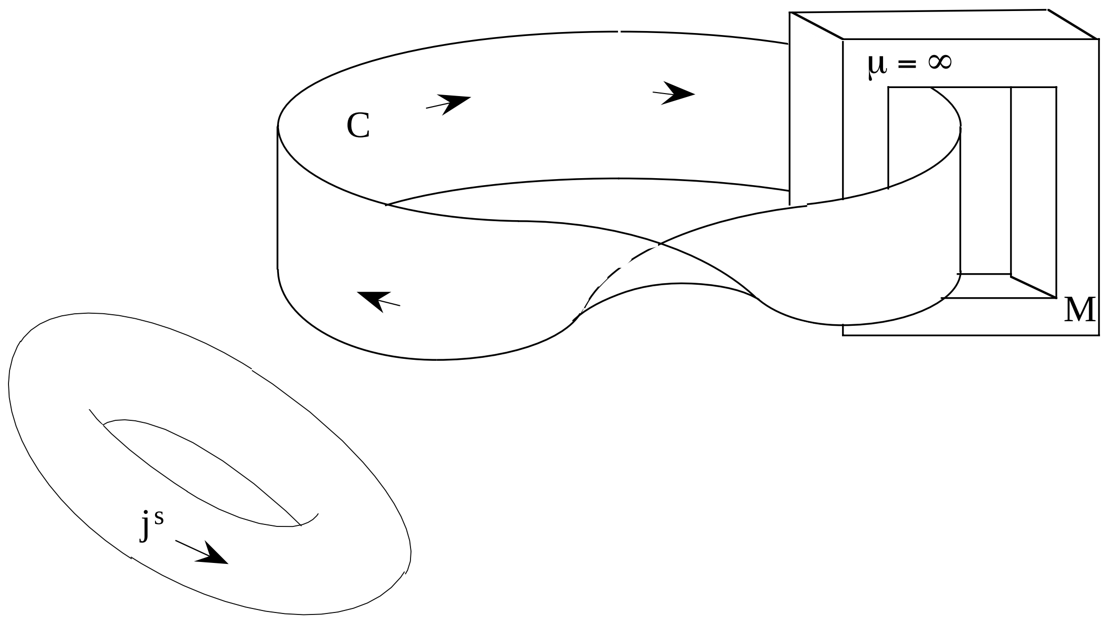


Figure 44. Induced eddy currents in an electrically conductive Möbius strip C. The presence of the perfect magnetic circuit M forces the total intensity in C to be 0.

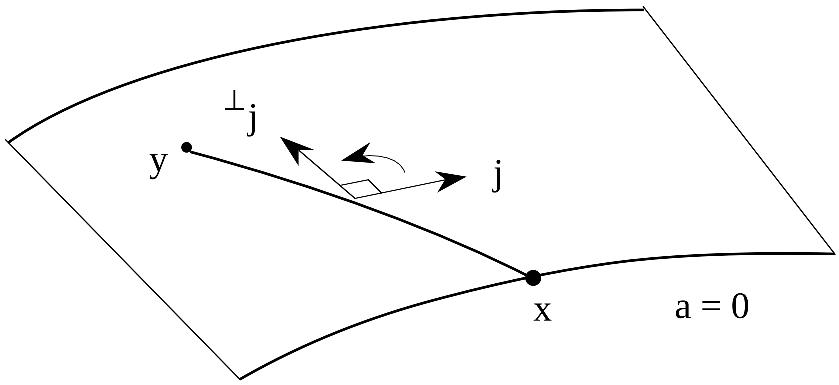


Figure 45. Construction of the stream-function a near x_0.

The above use of the words "to the left" testifies on the paramount rôle of orientation in this procedure: if right and left are permuted, this changes the sign of a (without changing j!). The physically relevant object at point x is thus not the real value a(x), but the pair formed by a(x) and the orientation about x (with the convention that the pair {–a(x), opposite orientation} represents the same object). Thus a is not a "genuine" function, but the local, and orientation-dependent representation of a geometric object in which one recognizes a "twisted function" as defined above.

Exercise 47: Having cut the strip, as in Fig. 46, one may choose an orientation. Check then that, for two points facing each other on opposite sides of the cut (like B and B'), one has a(B) = – a(B').

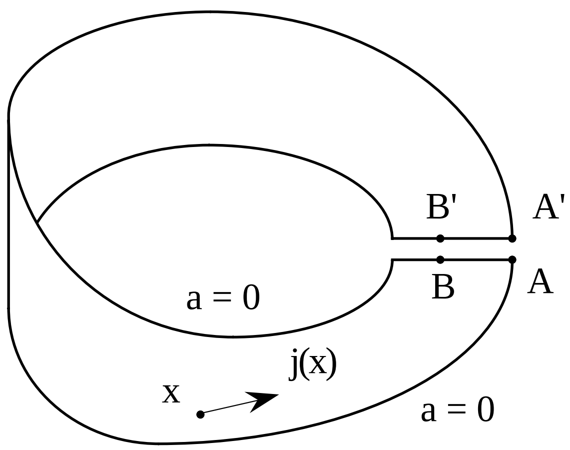


Figure 46. On the edge of the strip, a = 0 (no incoming nor outgoing current).

Exercise 48: Let j be a given current density on the non orientable surface of Fig. 16. Draw the cuts which are necessary to make the defintion of a stream-function possible, and tabulate all the relations between values of a on opposite sides of a cut. They take two distinct forms (hence two different kinds of cuts). Explain this.

3.2.2 Odd functions

As one will have guessed, there are some links between the above twisted functions and ordinary functions defined on the orientable covering. Let us describe them.

For this, let i be the mapping (from $\tilde{X}$ into itself) defined by $i(x^+) = x^-$ and $i(x^-) = x^+$. (The label + or – is assigned to both points of the fibre in an arbitrary way, since they play symmetrical rôles, but this does not prevent i from being well defined.) This map is a diffeomorphism (**Exercise 49**: check this) and, since $i \circ i$ is the identity, an *involution* of $\tilde{X}$ onto itself. Now, we'll say a function $f \in \tilde{X} \to \tilde{X}$ is *even* (resp. *odd*) if

$$f \circ i = f \qquad (\text{resp. } f \circ i = -f).$$

Of course, a function can be neither even nor odd.

Since an even function f assumes the same values at both points of $\tilde{X}$ above x, one may "pull down to x" this common value, thus associating with f a function living on X. The converse being possible, one sees that even functions on $\tilde{X}$ can be identified with functions on X.

Odd functions will prove more interesting. Choose an orientation on $\tilde{X}$. Sitting at x^+, above $x \in X$, let us take a basis at x^+, positively oriented. Take the image of these vectors by p_*, hence a basis at x. One thus has at point x a real value, $f(x^+)$, and an orientation. The same operation at point x^- yields the opposite value $f(x^-)$ and the opposite orientation, by the very definition of $\tilde{X}$. These two opposite pairs are but a single element of the fibre above x of the bundle $\tilde{\mathcal{A}}(X)$, according to the above-mentioned construction rules. In other words, to each odd function on $\tilde{X}$ corresponds a twisted function on X.

Conversely, one may lift any twisted function defined on X to an odd function on $\tilde{X}$, which is easier to conceive and to handle. But one will remark (**Exercise 50:** try it) that such a lift can be performed in two different ways, which yield functions of opposite signs, the sign depending on the chosen orientation of $\tilde{X}$. There is thus no canonical correspondence between twisted functions on X and odd functions on $\tilde{X}$. (A twisted function is actually a *pair* {odd function on $\tilde{X}$, orientation of $\tilde{X}$}, with the same quotient operation as above.) This slight difference motivates the contrasted use done here of two terms ("twisted" and "odd") which historically were applied to the same thing. (De Rham [83] calls "odd" the objects — functions, differential forms, tensors ... — which I call here "twisted", according to modern usage. One also says "oriented".)

Remark 9: How to *practically* represent twisted functions, for computational purposes? The manifold X is described by a limited number of charts, whose domains are orientable. So one selects (arbitrarily) an orientation for each of them. A twisted function is then represented, in each chart, by a function a and a sign, $\varepsilon = +1$ or -1. One will call this sign, with a slight abuse, "the local orientation of the twisted function a". (Of course, $\{-a, -\varepsilon\}$ represents the *same* twisted function in this chart.) If one has, for some reason, to change the orientation of a chart, one changes the sign of the ε (relative to this chart) in the data structure of each twisted function. ◊

3.2.3 Other twisted objects

Once understood, the process is easily generalized: thus there are fields of twisted vectors, twisted differential forms, etc. A twisted vector is a pair {vector, local orientation}, with the now standard proviso that the pair consisting of the *opposite* vector and the *other* orientation represents the same twisted vector. Here also, one may introduce the notion of *odd* vector field on $\tilde{X}$: it's a cross-section v of $T\tilde{X}$ that satisfies

$$i_* v = -v.$$

(Cf. Fig. 47. Note incidentally that i_* is an involution on $T\tilde{X}$.)

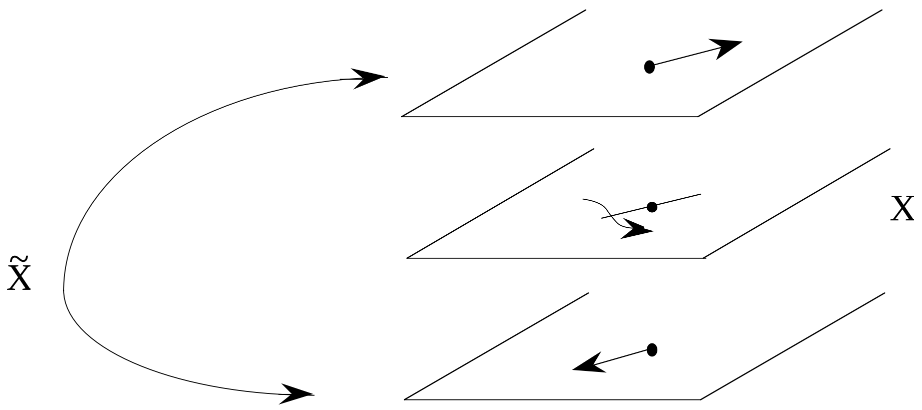


Figure 47. Odd vector field, above x.

The Möbius strip case (Fig. 48) proves that one may find on the orientable covering an odd field, continuous, which does not project on X (whichever definition of such a global projection one tries) as a continuous field. On the contrary, the corresponding twisted field is continuous. Burke, with his characteristic felicity in choosing graphical conventions, had found a way to visualise this continuity (Fig. 48). (Remark the use of an arrowed segment to represent a twisted vector. The length of the segment is the vector's modulus, the arrow is a kind of orientation, but "external", like the one conferred upon a surface by a field of normals, as seen earlier.)

In data structures, fields of twisted vectors are represented, for each chart, by a field of ordinary vectors and a sign (called "local orientation" of the field), in a similar way as functions.

Exercise 51: Let $u \in X \to Y$ and $\tilde{v}$ be a field of twisted vectors on X. How can one define $u_* \tilde{v}$? (Suggestion: cf. Def. 10.)

Let us finally define twisted forms. A *twisted p-covector* at x is an element of the *twisted* (by the orientation) *bundle* of p-covectors, that is, following the method we already used several times, a pair {p-covector, local orientation}, the pair formed of the opposite p-covector and of the other orientation representing the same object. A *twisted p-form* on X is a field of twisted p-covectors.

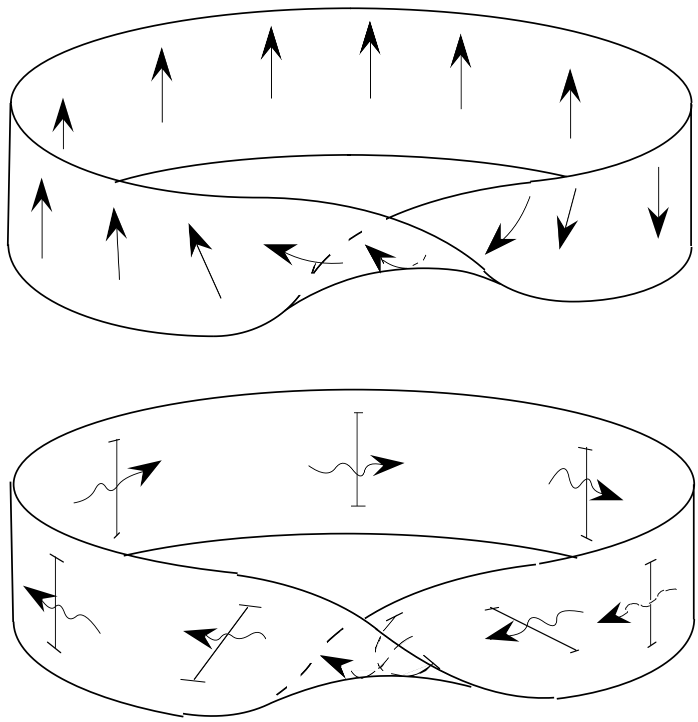


Figure 48. Field (regular and nowhere vanishing) of "twisted vectors" on a Möbius strip, and the impossibility of representing it by a (regular) field of "genuine" vectors.

Again, as was the case with twisted vectors, reasoning on ordinary forms living on $\tilde{X}$ may be easier. Let i be the involution which permutes the two points of $\tilde{X}$ above a given point of X, and ω a p-form on $\tilde{X}$. One can define, as already done above,

$$i^*\omega = \{\xi_1, \ldots, \xi_p\} \rightarrow \omega(i_*\xi_1, \ldots, i_*\xi_p),$$

where the ξ_is are vectors of $T\tilde{X}$. (An involution again.) A p-form ω on $\tilde{X}$ qualifies as *odd* if $i^*\omega = -\omega$.

Once chosen the orientation of $\tilde{X}$, there is a one-to-one correspondence between odd forms on $\tilde{X}$ and twisted forms on X.

Vectors and twisted covectors are in duality. Let $\tilde{v} = \{v, \varepsilon\}$ and $\tilde{\omega} = \{\omega, \varepsilon'\}$, in local representation (the chosen local orientations may not coincide). Set

$$<\tilde{\omega}, \tilde{v}> = \varepsilon' \, \varepsilon <\omega, v>.$$

This duality bracket is an orientation independent quantity, since changing the orientation of the representation of $\tilde{\omega}$, for instance, yields

$$<\tilde{\omega}, \tilde{v}> = \varepsilon' \, (-\varepsilon) <-\omega, v>,$$

i.e., the same value.

Similarly, if $\tilde{\omega}$ is a p-covector, represented by $\{\omega, \varepsilon\}$, and $\tilde{\xi}_i$, with $i = 1, \ldots, p$, a set of twisted vectors, each represented by $\{\xi_i, \varepsilon_i\}$, the effect of $\tilde{\omega}$ on them is

$$\tilde{\omega}(\tilde{\xi}_1, \ldots, \tilde{\xi}_p) = \varepsilon \, \varepsilon_1 \ldots \varepsilon_p \;\; \omega(\xi_1, \ldots, \xi_p).$$

This justifies a redefinition of twisted p-covectors as alternating multilinear mappings on the vector space of twisted vectors. The dual objects are *twisted p-vectors*.

We shall now examine the case p = n. Twisted p-forms are then called "densities" [27], because they well model, as one will see, the physical notion of density (of charge, of matter, of energy, etc.). This is so because *twisted n-forms can be integrated*, i.e., they may appear as integrands under summation signs.

3.3 Integration

Let X be a connected manifold of dimension n, not necessarily orientable, which will be assumed in all this Section to be *triangulable*, according to the definition to be given below. This hypothesis is done for technical reasons, to help define the integral in a sense similar to Riemann's. The object we wish to integrate is a twisted n-form, or density. Remarkably, this will prove feasible without any preexisting notion of measure, contrary to what happens in standard integration theory.

3.3.1 Triangulation

We shall call *reference p-simplex* the following closed set of $\mathbb{R}^p$:

(26) $$S^p = \{x \in \mathbb{R}^p : \; x_i \geq 0 \;\; \forall i, \;\; \Sigma_i \, x_i \leq 1\}.$$

Subsets of S^p obtained by replacing one or more of the inequalities in (26) by equalities are called *faces* of S^p. Thus, in particular, the vertices of S^p and the empty set are faces. One denotes by $e_1, \ldots, e_p$ the basis vectors of $\mathbb{R}^p$.

A (plain) *p-simplex* will be an embedding $s \in \mathbb{R}^p \to X$, with $\text{dom}(s) = S^p$. (One may occasionally call "simplex" the image $s(S^p)$ — then denoted $|s|$ — but it will be an abuse: a simplex is a *mapping*, for some p, of S^p into X (Fig. 49).)

For convenience in what follows, we introduce the following notion: a map of type $S^p \to S^q$ will qualify as *simplicial* if it is affine, injective, and transforms vertices into vertices. (It then transforms all faces of S^p into faces of S^q of same dimension.) By convention, a function with empty domain is also taken as simplicial.

Definition 11: *A* simplicial tessellation *of a manifold* X *of dimension* n *is a family* S *of* n-*simplices in* X *with the following properties*:

(a) *If* s *and* σ *are two simplices of* S, *the mapping* $s \circ \sigma^{-1}$ *is simplicial*,

(b) *If* $s \neq \sigma$, *then* $|s| \neq |\sigma|$,

(c) $\cup_{s \in S} |s| = X$,

(d) *Any compact part of* X *is covered by a finite union of images* $|s|$.

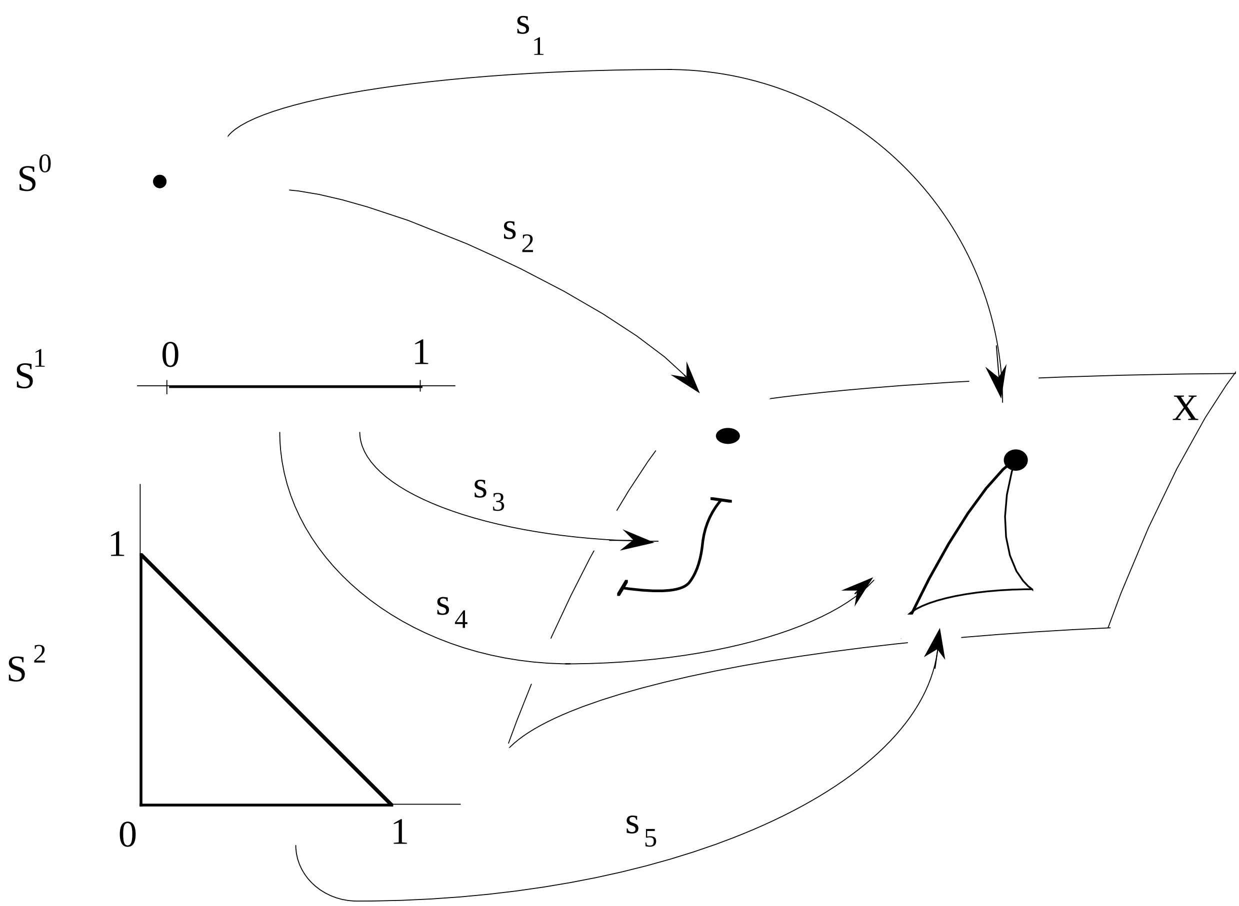


Figure 49. A few simplices, in dimension 2.

These axioms do correspond to the notion of mesh familiar to finite elements users. Note however that the shape of the elements is optional, and can thus be adapted to the curvature of the boundary ∂X.

Finite elements theory leads to introducing p-simplices, with $p < n$, relative to a tessellation, to account for the notions of nodes, edges, faces, etc. For our present needs, n-simplices will do.

By various *subdivision* procedures (which need not be detailed here, cf. e.g. [5], p. 125), one may associate with each simplex s of S a family of simplices forming a simplicial tessellation of $|s|$, and whose union for all s of S forms a simplicial tessellation of X, which is then called a *refinement* of the first one (Fig. 50).

A manifold is *triangulable* when it can be endowed with a simplicial tessellation. Smooth manifolds (p. 13) are always triangulable ([51], p. 1291, [28]). By subdivision, one may refine a triangulation at will.

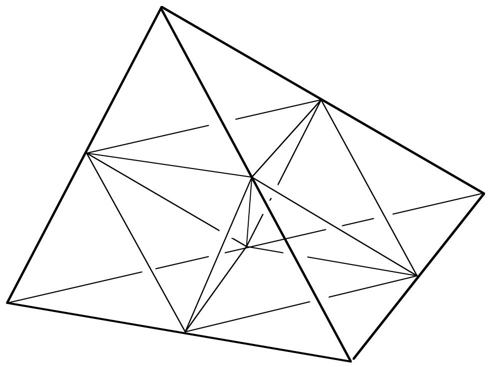


Figure 50. Subdivision of a 3-simplex. Note that the central octahedron can be divided in three different ways. (Beware, this is not a "barycentric subdivision" [2, 84], which would introduce other vertices at the centres of the faces and of the tetrahedron, for a total of 24 tetrahedra.)

3.3.2 The integral of an n-form: tentative definition

Let ω be an n-form on X. How could one define a linear mapping, that would be denoted $\omega \to \int_X \omega$, and would have the linearity properties one expects from an integral? By imitation of Riemann's procedure, one might think of assigning to each simplex s of a given triangulation a real number $\langle\omega, s\rangle$, to then take the sum

(27) $$I_S(\omega) = \Sigma_{s \in S} \langle\omega, s\rangle.$$

If this sum tends to a finite limit $I(\omega)$, when S is repeatedly subdivided, one will be entitled to say that ω is integrable and to call this limit the integral of ω.

The problem is thus to put forward a reasonable definition for $\langle\omega, s\rangle$. (All the rest, showing that the limit exists, is independent of the initial triangulation, etc., is far from being technically trivial, but the reader is assumed to have taken this kind of medicine at least once, and thus not to be in need of it any more.)

As we have nothing like a notion of length or a measure on X, there is not much leeway in the definition of $\langle\omega, s\rangle$ The only thing ω can do is to act on n vectors of TX to yield a number. So we need to associate n vectors with s in a natural way. How? the only candidates are the images $s_* e_i$ of basis vectors of $\mathbb{R}^n$. So let us try this:

(28) $$\langle\omega, s\rangle = 1/n!\ \omega(s_* e_1, \ldots, s_* e_n) \equiv 1/n!\ s^*\omega(e_1, \ldots, e_n).$$

Exercise 52: Before reading on, satisfy yourself that "it works" on the following example: $X = [0, 1]$, $\omega_x = f(x)\,dx$ (the 1-form which to vector ξ at x assigns the product of its unique component ξ by $f(x)$). Consider the simplicial tessellation

$$s_i = t \in [0, 1] \to x_{i-1} + t\,(x_i - x_{i-1})$$

where the x_is are points of X such that $0 = x_0 < x_1 < \ldots < x_m = 1$, and interpret (28).

At first glance, it looks as if we had grasped the wanted notion: Suppose ω represents the density of electric charge in a region X. One subdivides X into small (warped) tetrahedra, the $s_* e_i$ are tangent vectors which roughly match their (curved) edges, the values $\omega(s_*e_1, \ldots, s_*e_n)$ are, to a multiplicative factor (which is the charge density), the volumes of parallelepipeds built on these vectors, the factor $n!$ (here, $n = 3$) connects these with the volumes of tetrahedra, and one gets with (27) an approximation of the total charge.

Orientation problems, unfortunately, ruin this scenario. Let us substitute for some s another simplex s', with $|s'| = |s|$, such that the map $\varphi = s^{-1} \circ s'$ be simplicial (a condition imposed by point *a* of Def. 11). Then φ permutes the vertices of S^n. According to the parity of this permutation, even or odd, one will have $<\omega, s'> = \pm <\omega, s>$, thus all our construction breaks down.

A simple fix would consist in only considering orientable manifolds. For if X is endowed with an orientation, as given by a volume Ω, one may arrange for all simplices s to be "positively oriented", i.e., such that $\Omega(s_* e_1, \ldots, s_*e_n) > 0$. Or else, and this is equivalent, one may set[1]

(29) $$<\omega, s> = 1/n!\ \omega(s_*e_1, \ldots, s_*e_n)\ \mathrm{sgn}(\Omega(s_*e_1, \ldots, s_*e_n))$$

instead of (28). This time, the definition of $I_S(\omega)$ is indeed insensitive to the orientations of the s's.

However, with this new definition, the *sign* of the resulting integral depends on the orientation of X, and one can only integrate on orientable manifolds. This is very unpleasant, for why should global quantities like total mass, charge, etc., depend on the orientation — quite arbitrary — conferred on ambient space? Moreover, one may wish to integrate on *non*-orientable manifolds (for instance, to compute the mass of a Möbius strip of known density).

[1] sgn is the "sign" fonction: -1 or 1 depending on the sign of the argument, 0 when it is 0.

3.3.3 The integral of a twisted n-form

So we'll approach the problem in another way: give up on integrating ordinary n-forms, and concentrate on twisted n-forms, or *densities*, which bear with them, by their very definition, the necessary orientation. Let $\tilde{\omega}$ be a density, locally represented by $\{\omega, \Omega\}$, where Ω is a volume (arbitrarily selected) defined in a neighborhood of $|s|$. One defines $<\tilde{\omega}, s>$ as in (29), i.e.,

(30) $$<\tilde{\omega}, s> = 1/n!\ \omega(s_* e_1, \ldots, s_* e_n)\ \mathrm{sgn}(\Omega(s_* e_1, \ldots, s_* e_n)).$$

This number is independent of the choice of Ω, now. Similarly, the sum

$$I_S(\tilde{\omega}) = \Sigma_{s \in S} < \tilde{\omega}, s>$$

is left unchanged if one substitutes s' for s as above, for the possible change of sign in (30) is compensated by that of Ω. From this point, one carries on with the theory (subdivision of *S*, existence of a limit which does not depend on *S*, linearity and additivity of the integral, etc.) without any further problem.

The introduction of twisted forms finds a posteriori justification in this remarkable result: a density is (or is not) integrable on a manifold X, irrespective of its orientability, and without any preliminary construction of a measure.

The theory extends to ordinary n-forms, provided X is orientable: one just turns the given form into a density by adjoining an orientation to it. On the other hand, an n-form cannot be integrated on a non-orientable manifold.

Exercise 53: Did you ever worry about the fact that

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

according to an elementary approach to integration (the one which relies on the notion of primitive) whereas in more elaborated theories the number $\int_A f(x)\,dx$ (where A is a part of IR) can be defined without any reference to orientation? Show that in the former case $f(x)\,dx$ is a 1-form, and a density in the latter.

Exercise 54: Let $u \in X \to Y$ be a diffeomorphism. Show that

$$\int_X u^* \tilde{\omega} = \int_Y \tilde{\omega}\ .$$

(Remind that u is orientable, cf. Exer. 46, p. 67.)

Exercise 55: Let ρ be a density (in the common sense of the word) of charge in a region X of E_3. Define a twisted 3-form whose integral on X will be the total charge.

Exercise 56: What is the relationship between the notions of "measure" and of "density"?

3.3.4 Integrals of p-forms

If $\tilde{\omega}$ is a twisted p-form on a manifold Y of dimension n, with $p < n$, one will be able to integrate it on an immersed manifold of dimension p: If $u \in X \to Y$ is this immersion, one will define (cf. Exer. 54)

$$\int_{u(X)} \tilde{\omega} = \int_X u^* \tilde{\omega},$$

once a proper definition of $u^* \tilde{\omega}$ will be at hand. Let thus $\tilde{\omega} = \{\omega, \Omega\}$ be a local representation of $\tilde{\omega}$ in the neighborhood of $y = u(x)$. One knows how to pull-back ω to $u^*\omega$ at x. If one may adjoin to it a local orientation Ω' about x, naturally derived from Ω, the pair so obtained will be, by way of definition, $u^* \tilde{\omega}$.

As we saw above (Section 3.1.2), this is possible if u is orientable (Def. 10, p. 66). To any given local volume Ω one may then associate a local volume Ω' on X. One will check that the pair $\{u^*\omega, \Omega'\}$ so obtained does represent a twisted p-form on X. By definition, this form is $u^*\omega$.

Thus twisted p-forms can be integrated on some immersed manifolds of dimension p, those with an orientable immersion. (One also says that such manifolds have an "external orientation", a dubious terminology, since an external orientation is not an orientation, cf. Exer. 39.) The integral establishes a duality between the two kinds of objects.

One should not jump to the conclusion that ordinary differential forms cannot be integrated: provided the manifold is orientable, one may always turn them into twisted forms (just select an orientation) and the whole theory applies. The only difference is the dependence of the sign of the integral on orientation.

Physical entities for which integration makes sense are in general twisted p-forms. Here follows an especially important example.

Electric current density (let us denote this entity by $\tilde{j}$) is commonly regarded as a vector field. Actually, it's a twisted 2-form. To make this point, let us start from the idea that one should be able to associate with $\tilde{j}(x)$ some differential object, whose integral would have to be a flow of charge. More to the point, if S is a closed surface, an integration over S should yield the outgoing flow (or incoming, at will). But such a flow is independent, by its very nature, on orientations of both S and space. The to-be-defined 2-form (2, because the dimension of S is two) is thus not an ordinary 2-form, whose integral depends on orientation as we know, but a twisted form. Another compounding argument is: the words "incoming" or "outgoing" suggest that the surface through which one wants to compute a flow must be endowed with an external orientation. (Indeed, the idea of a flow of charge through a Möbius strip doesn't lend itself to any reasonable definition.) But, as we know, such externally oriented surfaces are precisely those on which 2-forms can be integrated.

All this concurs to suggest the proper mathematical object to model the notion of current density is some twisted 2-form. Knowing this, we necessarily arrive at the following definition: $\tilde{j}(x)$ is the twisted 2-covector for which a representation is the pair $\{j(x), \Omega\}$, where Ω is a local volume and $j(x)$ the 2-covector which assigns to a pair of vectors at x, say ξ and η, the flow of charge (through the parallelogram built on them) in the direction of a vector $n(x)$ such that $\{\xi, \eta, n(x)\}$ be a direct frame for the orientation Ω. In order to check the correctness of this definition, we must verify that the *other* representation of $\tilde{j}(x)$, to wit the pair $\{-j(x), -\Omega\}$, measures the *same* flow of charge. Indeed, the covector $-j(x)$ assigns to ξ and η a number which is this flow with a change of sign, thus the flow in the direction of $-n(x)$, and $\{\xi, \eta, -n(x)\}$ is effectively a direct frame for the orientation $-\Omega$.

Let now S be a surface, closed or not, endowed with a transverse field n (which defines the "crossing direction"). The pull-back of $\tilde{j}(x)$ on S is a twisted 2-form, whose integral over S is the flow crossing this surface along the direction indicated by n. Thus the twisted 2-form $\tilde{j}(x)$ well performs its intended function: it tells about the flow across externally oriented surfaces.

This long discussion may have been more irritating than convincing for some readers, who may have objected: "This is a lot of trouble for a rather modest result. If your purpose was to model the notion of current density (be it of electrical current or of any kind of 'fluid'), why not use a vector field? I call 'flux density' at x, on the surface S as oriented by n, the real number $j(x) \cdot n(x)$. To get the total flow, I integrate this function of x over S, and the result is indeed independent of any orientation. (I concede all this assumes an underlying integration theory, *including the definition of a measure borne by* S, but as you said, I did my homework about this in the past, so why not cash in on it?)"

The words I have emphasized are the weak point in this line of argument. The problem is not the technical difficulty of defining a measure on S, it roots in the absence of metric information on which to base such a definition: whatever the unit length and the unit of area on S, the flow through it (as expressed, for instance, in amperes) will be the same. This is the point of defining current density as a twisted 2-form: this way no metric, no previous notion of length, area, etc., is assumed.

Even if such notions have been introduced for other reasons, making use of them is not necessarily a good idea. Think for instance of a problem featuring the current flow through a *deformable* material surface. It will be simpler in such a case to think in terms of a 2-form, without any recourse to a measure of areas that would vary with time, with the easy to imagine complications this would bring in at the computational level.

What have been said about current density is valid for other kinds of flow: heat, fluids, etc. One may also think of adding to the list the magnetic flux, i.e., the induction field b. However, b is *not* a twisted 2-form, as we shall see later. (One may suspect this by noticing how the flux of b is linked with the circulation of the electric field by Faraday's law, for orientation plays a part in the matter.)

3.4 Stokes Theorem

Another famous topic, to which we won't pay as much attention as is customary, because the clanking technique involved hides a single and simple idea: one *defines* an operator, denoted d, in such a way that Stokes' Theorem, i.e.,

(31) $$\int_X d\omega = \int_{\partial X} \omega,$$

hold *locally*. One then easily finds it to hold *globally*. Operator d thus appears as a formal adjoint to ∂ in the duality between p-forms and p-submanifolds.

Consider first a manifold X of dimension n, and a (p−1)-form ω. Sitting at x, one considers p vectors $\xi_1, \ldots, \xi_p$. One may always define a simplex s such that the images $s_* e_i$ of basis edges of the reference p-simplex S^p coincide with the ξ_is. One of the vertices is x (Fig. 51). Let us orient |s| so that the ξ_is form a direct frame. This induces an orientation on $\partial|s|$ (for which a transverse field is at hand, the one obtained by mapping an *outgoing* vector field on the boundary of S^p to one on |s|, via s). One then integrates ω on $\partial|s|$ with this orientation, hence a number, denoted $\alpha(1)$.

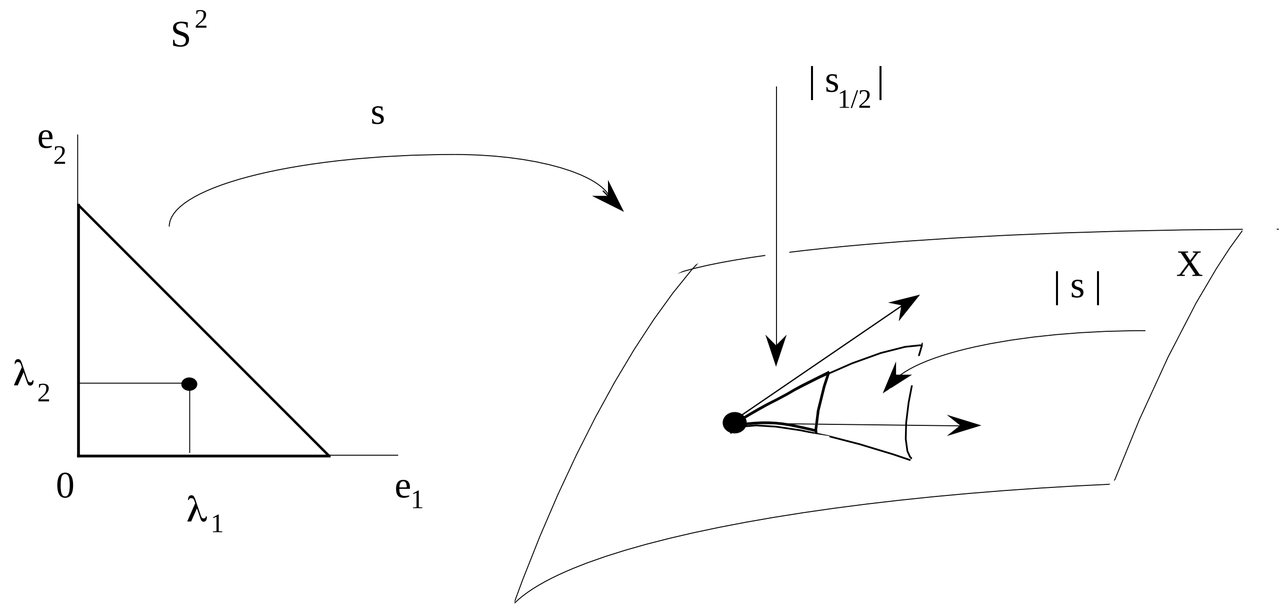


Figure 51. Definition of d. The λ_i are the coordinates of a generic point in S^p (here, p = 2).

Let now s_ε be the simplex built from s by applying the transform

$$s_\varepsilon(\lambda) = s(\varepsilon\, \lambda),$$

where $\lambda \in S^p$ (Fig. 51). Integrating on s_ε yields a number $\alpha(\varepsilon)$. As easily shown by working in a chart about x, the quantity $\alpha(\varepsilon)/\varepsilon$ tends to a limit when $\varepsilon \to 0$, and this limit is multilinear and alternating with respect to the ξ_is. We now set

$$\text{(32)} \qquad \eta(\xi_1, \ldots, \xi_p) = \lim \alpha(\varepsilon)/\varepsilon$$

hence a covector at x. Then,

Definition 12: dω *is the field of the covectors in* (32).

The definition can be extended to twisted forms, by setting $d\{\omega, \Omega\} = \{d\omega, \Omega\}$, where Ω is a local volume. The operator d thus obtained is called *exterior derivative.*

Exercise 57: Consider $u \in X \to Y$. Show that $d\, u^*\omega = u^*\, d\omega$. (Hint: Exer. 54, a simplex s at x, and the simplex $u \circ s$ at $y = u(x)$.)

One then proves (31), in the case $p = n$, by working on a simplicial tessellation of X, and by taking into account the cancellation of contributions of most $(n - 1)$-simplices to the second integral (this, because both opposite induced orientations appear, for all simplices but those belonging to ∂X).

Exercise 58: Note that $\partial \partial X$ is always empty, and derive $d^2 = 0$ from this.

Last, thanks to (31) and Exer. 57, one tackles the case of an immersed manifold X of dimension p. (The immersion has to be orientable if ω is a twisted form, whereas X has to if ω is an ordinary form.)

Remark 10: One says that a p-form ω is *closed* if $d\omega = 0$, is *exact* if $\omega = d\alpha$ for some $(p-1)$-form α. Since $d^2 = 0$, an exact form is closed. On the same pattern, a manifold X is "closed" if $\partial X = 0$ (but one will rather say that it is "a *cycle*"), it is "a *boundary*" if there exists some Y such that $X = \partial Y$. The question of the converse statement then arises. When is a cycle a boundary? When is a closed form exact? Such questions make the subject matter of the Chapters "homology" and "cohomology" of algebraic topology [2, 5, 44, 53, 67, . . .]. ◊

In spite of the simplicity of the definition of d, the explicit *formula*, due to Palais [77], which expresses $d\omega(\xi_1, \ldots, \xi_p)$ in terms of intrinsic quantities like $\omega(\xi_2, \ldots, \xi_p)$, etc., is not simple (cf. [68], p. 107). Better here to use a chart. If

$$\omega(x) = \Sigma_{\sigma}\, \omega_{\sigma}(x)\ d^{\sigma(1)} \wedge \ldots \wedge d^{\sigma(p)}$$

(for the meaning of this notation, cf. (23), p. 55), one has

$$\text{(33)} \qquad d\omega(x) = \Sigma_{i = 1, \ldots, p}\ \Sigma_{\sigma}\, \partial_i\, \omega_{\sigma}(x)\ d^i \wedge d^{\sigma(1)} \wedge \ldots \wedge d^{\sigma(p)}.$$

This can be taken as an analytical definition of d. Indeed, d is often introduced this way.

Exercise 59: With the help of (33), verify that the basis covector d^i of Section 2.3 (p. 52) is actually the d of the function "i^{th} coordinate", $x \rightarrow x^i$.

Exercise 60: Show (by first putting ω and η in the form (23)), that

$$\text{(34)} \qquad d(\omega \wedge \eta) = d\,\omega \wedge \eta + (-1)^{\deg(\omega)}\, \omega \wedge d\,\eta.$$

Exercise 61: A twisted 0-form (say $\tilde{a}$) is a twisted function, i.e., at each point, a value $a(x)$ and a sign $\varepsilon(x)$, with $\{a, \varepsilon\} = \{-a, -\varepsilon\}$. Show that the integral of $\tilde{a}$ over a finite set of points A is

$$\Sigma_{x \in A}\, \varepsilon(x)\, a(x).$$

Apply Stokes theorem on a path joining x_1 to x_2 and recover the notion of gradient.

Exercise 62: One heats up a heat-conducting Möbius strip from its boundary. Define on MS an appropriate twisted 1-form, such that Stokes theorem, when applied to all MS, expresses heat conservation. (Note this should be an intrinsic 1-form, one defined on MS directly, and not as the pull-back of some 1-form on E_3.)

Exercise 63: Discuss the relationship between current density (a twisted 2-form) and electric charge (a twisted 3-form); between heat flux (a twisted 2-form) and thermal power.

Chapter 4

Additional structures on a manifold

The structure of differentiable manifold by itself, as provided by charts, has proved very rich, allowing the definition of vectors, forms, the d, the integral, etc. However, the time has come to add something to it.

What is to follow will more easily be understood by way of analogy. As one knows, vectors of V_n and covectors of V_n^* are in duality. (This simply means that to any pair $\{\omega, v\}$ one can assign a number $<\omega, v>$, this correspondence being bilinear and non-degenerate (cf. p. 49).) Thus V_n and V_n^* are isomorphic to each other, but there is no *canonical* isomorphism, i.e., no natural way to match a vector with a given covector, and the other way round. On the other hand, as soon as V_n is endowed with a scalar product (which turns it into the Euclidean vector space E_n), such associations become possible: for the mapping $v \to u \cdot v$, where u is a fixed vector, defines a covector ω_u, hence a canonical isomorphism. The same phenomenon happens in Hilbert space (it's the Riesz theorem). The scalar product, in both cases, is the additional element of structure which makes the definition of such an isomorphism possible.

Something analogous will happen here: the additional element of structure will first be a *density*, then a *metric*.

4.1 Measurable manifolds

Let X be a manifold and $\tilde{\Omega}$ a density, or twisted n-form on X, fixed, nowhere vanishing on X. We shall call such a structure a *measurable manifold*.

By way of definition, $\tilde{\Omega}(x)$ is represented in the domain of a chart by a pair {n-covector, orientation}, and the orientation in turn is represented by a local volume, which can be the n-covector itself (Ω, say), since it does not vanish anywhere. So $\tilde{\Omega}$ is, locally, the pair $\{\Omega, \Omega\}$ (or $\{-\Omega, -\Omega\}$). Integration of $\tilde{\Omega}$ on a part A (of dimension n) will thus yield (cf. (30)) something which is *positive* and *additive* with respect to A, from which one may define a measure on X, in the sense of Lebesgue measure theory (hence the name of "measurable manifold" we tentatively use here).

Examples where such a structure can provide a good model are: For X, the continuum of material points of a deformable solid, and for $\tilde{\Omega}$, the mass; For X, a given territory, and for $\tilde{\Omega}$, the population density.

Exercise 64: Find other similar examples, i.e., with a natural *density* but no natural *metric* on manifold X.

4.1.1 Duality between densities and functions

Consider now on $\{X, \tilde{\Omega}\}$ *another* density $\tilde{\omega}$, locally represented by $\{\omega, \Omega\}$. Then the real number

$$\rho(x) = \omega(\xi_1, \ldots, \xi_n)/\Omega(\xi_1, \ldots, \xi_n)$$

is obviously independent of the ξ_is (by linearity) and insensitive to orientation. Therefore $x \to \rho(x)$ is a function (a genuine one, not a twisted one) associated with the density ω, and one may legitimately write

(35) $$\tilde{\omega} = \rho\,\tilde{\Omega}.$$

(Note that ρ is not necessarily positive.) One says that ρ and $\tilde{\omega}$ are *dual* to each other.

For instance, if X is a deformable solid and $\tilde{\Omega}$ the mass, $\tilde{\omega}$ can be the heat content, or the charge, or the volume of space occupied, or etc. Then $\rho(x)$ is what is commonly called the "density" of this substance: quantity of heat per unit of mass (i.e., specific enthalpy), charge per unit of mass, specific volume, etc. (This vindicates, a posteriori, the use of the name "density" for twisted n-forms.)

One could wonder about the choice of sophisticated mathematical objects like densities to model the physical notion known by this name. Why not simply the scalar ρ? Because ρ alone is not enough: one needs a measure with respect to which integrate it (the density of charge, for instance, is understood "with respect to" mass, or volume, etc.). The density $\tilde{\omega}$, after (35), incorporates both notions: scalar density and measure.

The distinction we are doing there is often obscured by the "Eulerian" setting one usually favors, which consists in considering physical space E_3 as the ambient manifold. Once a unit of length has been chosen, there is a natural volume (the one which is usually called volume, precisely), a conventional orientation, and thus a natural density $\tilde{\Omega}$, to which all other densities can be compared. But when field computation in deformable bodies is in order, it is very profitable to outgrow this point of view and to shift to the "Lagrangian" one, where the ambient manifold is the body itself. The geometric notions introduced here then take all their interest (cf. [20]).

In short, there is, on a measurable manifold, a canonical isomorphism between twisted n-forms and functions.

This works the same way as regards ordinary n-forms and twisted functions: If ω is such a form, it can be matched with the twisted function $\{\rho, \Omega\}$, with $\rho\Omega = \omega$. It happens that Electromagnetism features a natural 3-form: the *magnetic charge* (div b, in ordinary language, and db if one considers b, as one should, as a 2-form). The corresponding charge density function is thus actually a twisted function, or as Treatises have it, sometimes a bit esoterically, a "pseudo-scalar". (Fortunately, free magnetic charges do not exist in nature, up to now, which makes this dependence of the sign of charge on orientation rather irrelevant. The absence of magnetic charge, on the other hand, may have something to do with its geometric nature. Cf. [92].)

4.1.2 Duality in general

Let now j be a field of (genuine) vectors. Since the mapping

$$\{\xi_2, \ldots, \xi_n\} \rightarrow \Omega(j, \xi_2, \ldots, \xi_n), \qquad (36)$$

considered at point x, is an $(n - 1)$-covector, one obtains, by pairing it with the orientation Ω, a twisted $(n - 1)$-form $\tilde{j}$ (said *dual* to j). Conversely, there corresponds to a given $(n - 1)$-covector a unique vector, after (36) (just check uniqueness, which implies existence, in a finite dimensional space), so things go both ways there again: to the twisted $(n - 1)$-form $\tilde{j}$ corresponds a dual vector field, j.

For $n = 3$, this corresponds to the already discussed case of electric current.

As one may have anticipated, the notion of divergence of a vector field now comes in a natural way (but only now!). The *divergence* of the vector field j, dual to the twisted $(n - 1)$-form $\tilde{j}$, is the function div j such that

(37) $$d\tilde{j} = (\text{div } j)\ \tilde{\Omega}.$$

Exercise 65: Let $u \in X \to Y$ be an immersion, with $\dim(X) = m - 1$ and $\dim(Y) = m$. One assumes the existence of a transverse field n on X. Let v be a vector field on Y. Show that the expression "component of v with respect to n" can be given a precise meaning. (Suggestion: Fig. 52. One denotes this component by v_n for the rest of this exercise.)

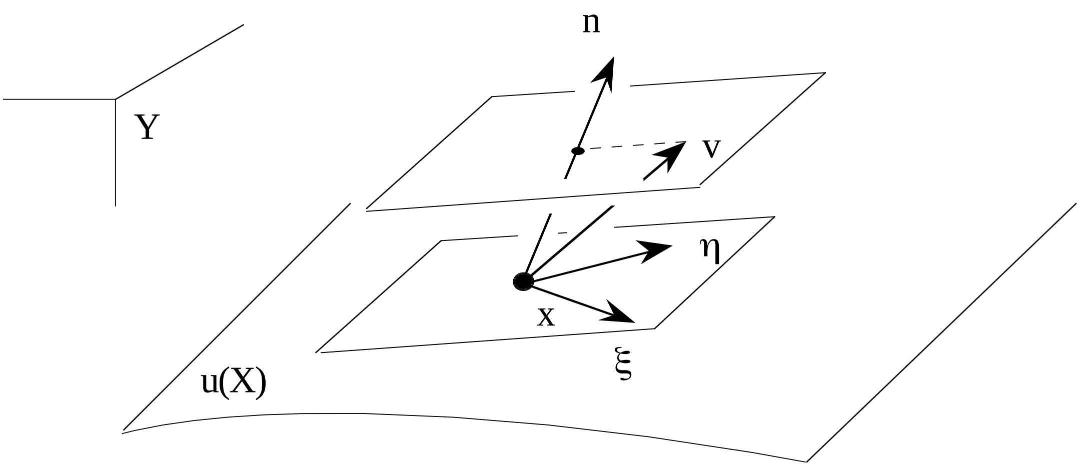


Figure 52.

Exercise 65 (continued): Let now $\tilde{\Omega}$ be a standard density on Y. Build from it and from n a density on X, denoted $n\tilde{\Omega}$. Find back v_n by comparing with $n\tilde{\Omega}$ the $(m-1)$-form which is dual to v.

Exercise 65 (end): From (37), Stokes theorem, and what precedes, derive Ostrogradskii's theorem:

$$\int_{\partial X} j_n = \int_X \text{div } j,$$

and explain the notation. (Beware, n is a field of outgoing vectors, but *not* the field of normals, for lack of any metric structure on which to base the notion of orthogonality!)

Thus, the presence of a standard density allows one to pair objects of different types which otherwise would be unrelated: functions and densities, vector fields and twisted $(n - 1)$-forms. More generally, $\tilde{\Omega}$ associates an ordinary (resp. twisted) p-form to a field of twisted (resp. ordinary) $(n - p)$-vectors, as displayed in Fig. 53, by a transformation which is called the "dual map". (Its geometric definition is only palatable if $p = 0$, 1, $n - 1$ or n. See [89], p. 25, for the analytical definition.)

There does not seem to exist a standard symbol to denote this dual map with. Let us adopt $\diamond$ for this purpose (not to be used beyond this Section). Thus $\tilde{j} =$

$\diamond$ j. In the other direction, one prefers to set $\diamond \tilde{j} = (-1)^n j$, instead of $\diamond \tilde{j} = j$, to get rid of a few minus signs in some formulas, so the transformation $\diamond$ is not quite an involution. Actually,

$$\diamond\diamond = (-1)^{p(n-p)}$$

when applied to a p-form or to a p-vector.

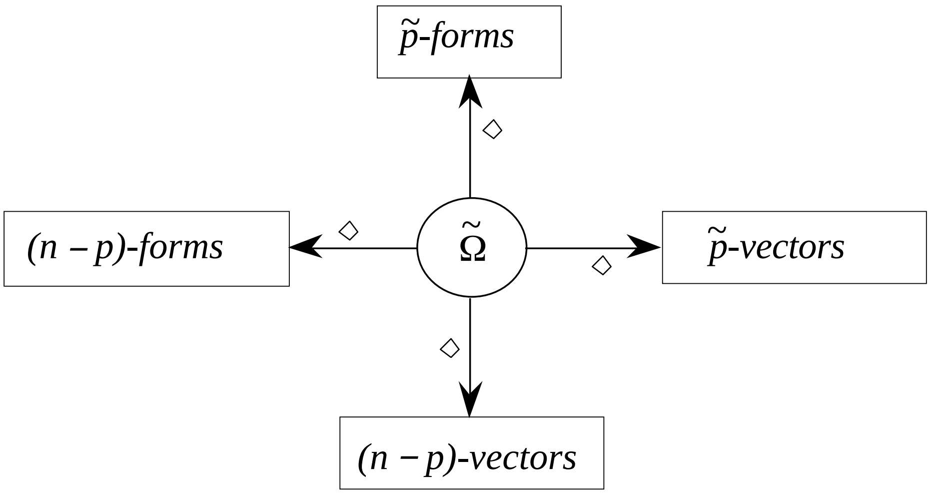


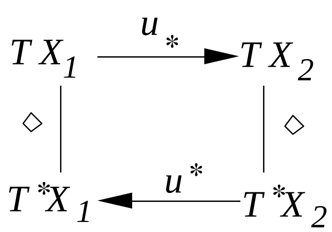
Figure 53. Correspondences under the dual map induced by a standard density $\tilde{\Omega}$. (The sign ~ is an abbreviation for "twisted".)

Exercise 66 ("covariance" of the flux, and more generally of the integral of an (n − 1)-form): Show that, with proper hypotheses on $u \in X \to Y$ and S,

$$\int_S u^* \tilde{j} = \int_{u(S)} \tilde{j} \,,$$

where $\tilde{j}$ is a twisted (n − 1)-form (dim(X) = dim(Y) = n).

Exercise 67: Let $(X_1, \tilde{\Omega}_1)$ and $(X_2, \tilde{\Omega}_2)$ be two measurable manifolds, and $u \in X_1 \to X_2$ an orientable map. One will say that u is a *volume-preserving map* if $u^* \tilde{\Omega}_2 = \tilde{\Omega}_1$ (or $-\tilde{\Omega}_1$). Study in that case the commutativity of the diagram:



4.2 Riemannian manifolds

4.2.1 Metrics

Definition 13. *One has a* metric g *on a manifold* X *when there exists, at each point* x, *a bilinear map*

$$g_x \in T_xX \times T_xX \to \mathbb{R},$$

symmetric with respect to both arguments, positive definite, i.e.,:

(38) $$g_x(v, v) > 0 \Leftrightarrow v \neq 0,$$

with smooth dependence on x.

In coordinates,

$$g_x(v,v) = \Sigma_{i,j}\, g_{ij}(x)\, v^i\, v^j,$$

the g_{ij} (or "coefficients of the metric tensor") being smooth functions of x, with $g_{ij} = g_{ji}$.

A manifold endowed with a metric is a *Riemannian* manifold. Note that g_x is a scalar product on T_xX, thus a metric gives each tangent space a Euclidean structure. So one will abbreviate, if there is only one metric in sight, as follows:

$$g_x(v, v) = v \cdot v\ .$$

Given such a structure, things like the norm of a vector, the angle of two vectors, orthogonality, etc., make sense. (But beware: only in the tangent space at a point. There is no way to take the scalar product of tangent vectors at two distinct points.)

Distance between two points also makes sense, as follows. The map $v \to [g_x(v, v)]^{1/2}$, of type $T_xX \to \mathbb{R}$, is not a covector, since it lacks linearity. But by restriction to one-dimensional submanifolds, it yields a density, called "the length element", which can be integrated, for instance along an arc connecting x with y. The result is the length of this arc. By taking the infimum of lengths of all arcs from x to y, one gets the *distance* between x and y. Axioms for a distance are easily checked.

One also gets new correspondences. Let v be a vector field. Then

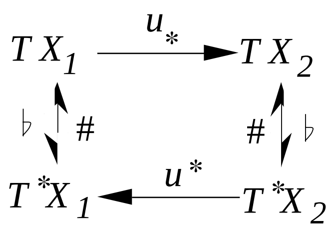
(39) $x \to (\xi \to v(x) \cdot \xi)$

defines a 1-form, often denoted with a flat sign: $\flat v$. Conversely, to a 1-form ω corresponds (according to the Riesz theorem) a vector field denoted with a sharp, $\# v$, such that $\omega(\xi) = (\# v) \cdot \xi$. Of course,

$$\flat\# = \#\flat = 1.$$

The same operators can be defined in an obvious way for twisted vectors and forms.

Exercise 68: Let (X_1, g_1) and (X_2, g_2) be two Riemannian manifolds. One says that $u \in X_1 \to X_2$ is an *isometry* if $g_2(u_* v, u_* w) = g_1(v, w)$ for any pair of vectors v and w at x. Study in that case the commutativity of the diagram:



(Note that u has to be a diffeomorphism and $\dim(X) = \dim(Y)$.)

Remark 11: If $f \in X \to \mathbb{R}$ is a function, $\# df$ is a vector field, denoted grad f (cf. Remark 6). If $v \in X \to TX$ is a vector field, $\# d \flat v$ is vector field, denoted rot v. ◊

Exercise 69: Study the commutativity of u_* with the operators grad and rot. (Cf. Exer. 68.)

One may also sharpen a p-form into a field of p-vectors, or flatten such a field into a p-form. (This is more easily done in coordinates, and the exercise is left to the reader.) One thus obtains the diagram of Fig. 54.

4.2.2 Hodge operator

But this diagram doesn't tell the whole story. For the existence of a metric entails that of a (local) volume, thanks to (38). To get it, one first selects a local orientation. Then, for a given set of vectors $\xi_1, \ldots, \xi_n$, one builds the Gram matrix G with the dot-products $\xi_i \cdot \xi_j$ as entries, and one sets

(40) $\Omega(\xi_1, \ldots, \xi_n) = \pm\, [\det(G)]^{1/2}$

(where det is the determinant), with the sign + or – according to the orientation of the ξ_is. Then $\tilde{\Omega} = \{\Omega, \Omega\}$ (also equal to $\{-\Omega, -\Omega\}$) constitutes a standard density.

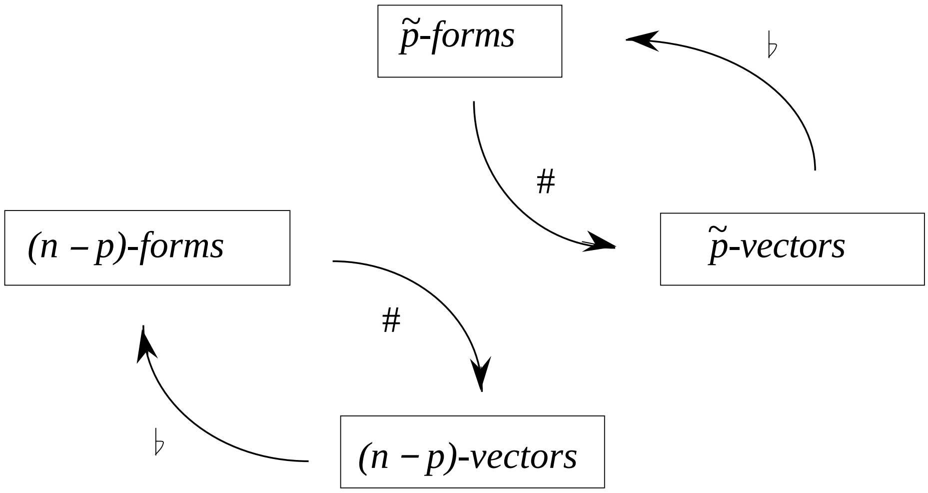


Figure 54. Correspondences set by a metric.

So the Riemannian structure encompasses that of measurable manifold, so #, ♭, and the dual map are available. One calls *Hodge operator* (denoted $*$) the composition of ♭ and of the dual map:

$$* = \flat \diamond .$$

One shows (**Exercise 70**: do it in the case of a 1-form) this is equal to $\diamond$ #. This "star operator" thus takes ordinary (resp. twisted) p-forms to twisted (resp. ordinary) (n – p)-forms. Hence the scheme of Fig. 55, obtained by superposition of the two previous ones.

Exercise 71: Show that

(41) $$* * = (-1)^{p(n-p)}.$$

Remark 12: It would be natural to call Hodge operator as well the one indicated on Fig. 55, which turns p-vectors into (n – p)-vectors (twisted or not as the case may be). Common usage, however, seems to reserve the name for the operator which works on forms. ◊

Contrary to the dual map, seldom used, the Hodge operator is a major tool, so the sketchy definitions we just suggested are not enough. Here follows a direct one. First remark that a p-covector ω in Euclidean space is known if one knows how it acts on an *orthonormal* system of p vectors: for, given any p vectors, one may first orthogonalize them without changing the value of ω, then scale them to length one and take scaling factors into account thanks to the linearity of ω.

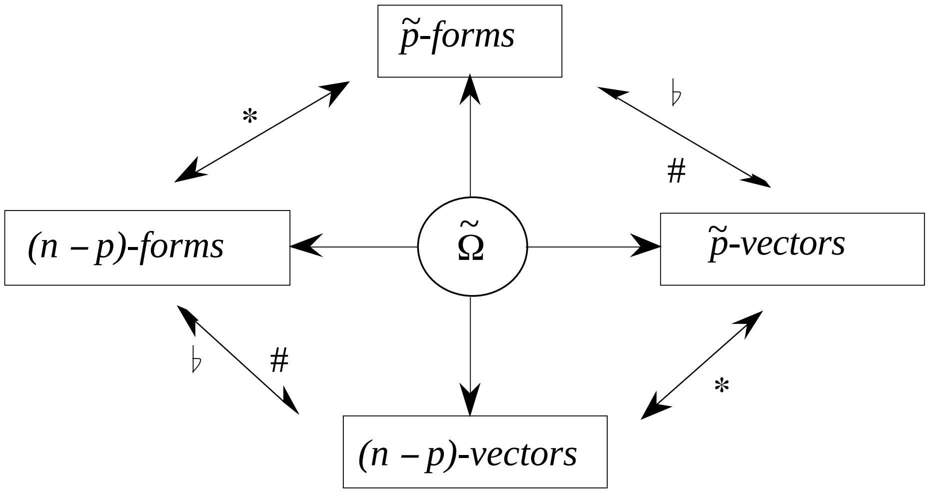


Figure 55. Canonical correspondences for a Riemannian manifold.

Let thus $\tilde{\omega}$ be a twisted p-covector represented by $\{\omega, \Omega\}$, where Ω is the volume (40). Let $e_{p+1}, \ldots, e_n$ be a system of $n - p$ orthonormal vectors. To this incomplete basis, one may append p normalised vectors $e_1, \ldots, e_p$, orthogonal between them and to the previous ones, and such that $\Omega(e_1, \ldots, e_n) > 0$. Then,

Definition 14: $*\tilde{\omega}$ *is the* (n – p)-*covector*

(42) $$\{e_{p+1}, \ldots, e_n\} \to \omega(e_1, \ldots, e_p).$$

This is unambiguous, because $\omega(e_1, \ldots, e_p)$ is the same for any eligible system of e_i's.

Exercise 72: Justify the foregoing assertion. (Hint: begin with $p = n$; then there exists a constant λ such that $\omega(e_1, \ldots, e_p) = \lambda\Omega(e_1, \ldots, e_p)$, and this latter quantity is indeed invariant.)

For a p-covector ω, $*\omega$ is the twisted covector obtained by pairing the covector defined by (42) with the orientation Ω.

There is a remarkably simple coordinate expression of the Hodge operator when the chosen basis is *orthonormal*:

(43) $\quad * (d^1 \wedge \ldots \wedge d^p) = d^{p+1} \wedge \ldots \wedge d^n.$

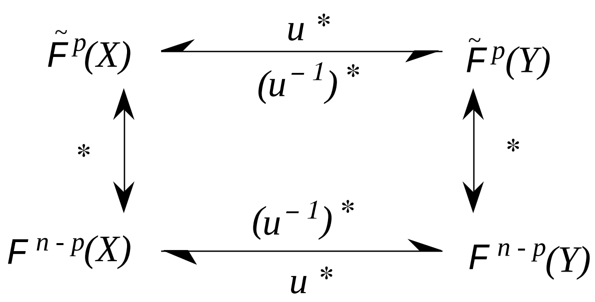
(In particular, in dimension 3, $* dx = dy \wedge dz$, $* dy = dz \wedge dx$, etc.) From (43), one gets $* (d^{\sigma(1)} \wedge \ldots \wedge d^{\sigma(p)})$ for any injection σ of the segment $[1, p]$ of $\mathbb{N}$ into $[1, n]$. The only problem is to find the right sign, a simple and dull exercise in combinatorics, but a prerequisite for the one which follows.

Exercise 73: Write down $*\omega$, where ω is the covector

$$\omega = \Sigma_{\sigma \in C(n,\, p)}\, \omega_\sigma\, d^{\sigma(1)} \wedge \ldots \wedge d^{\sigma(p)}.$$

(Cf. (23) for the notation.)

Exercise 74: Let $u \in X \rightarrow Y$ be an isometry. Investigate the commutativity of the diagram:



(where $\mathcal{F}^p(X)$ denotes the space of p-forms on X, and ~ the twisted forms).

4.2.3 Scalar product

If one could take the scalar product of two p-covectors at x, this would yield by integration over all X a bilinear form on $\mathcal{F}^p(X)$ with all the properties of a scalar product, like the one defined on the functional space L^2 (which would then correspond to the special case $p = 0$).

The presence of a metric should make this program feasible: for if $u \cdot v$ makes sense, u and v being vectors, setting $\omega \cdot \eta = (\#\omega) \cdot (\#\eta)$ transfers this scalar product to covectors, which covers the case $p = 1$. Can this be generalized?

Yes, thanks to the Hodge operator. Let $\tilde{\Omega}$ be the standard density, and ω and η two p-covectors at x. Since $*\eta$ is a twisted $(n - p)$-covector and $(n - p) + p = n$, the wedge product of ω by $*\eta$ is a twisted n-covector, i.e., a multiple of $\tilde{\Omega}$. The multiplicative factor (a true function) is the wanted scalar product. One thus defines $\omega \cdot \eta$, at point x, by

$$\omega(x) \wedge * \eta(x) = \omega(x) \cdot \eta(x)\, \tilde{\Omega}_x,$$

and one sets

(44) $$(\omega, \eta) = \int_X \omega \wedge * \eta,$$

which is but the integral of $x \to \omega(x) \cdot \eta(x)$ with respect to the measure induced by $\tilde{\Omega}$.

Exercise 75: Going back to Def. 7 (p. 56), check that (ω, η) is symmetric and that $(\omega, \omega) \geq 0$.

Exercise 76: Show that if ω and η are 1-forms, $(\omega, \eta) = (\#\omega, \#\eta)$, as expected.

4.3 Hilbertian structures on spaces of forms

Starting from (44), we now establish an *integration par parts formula*, that will generalize the familiar ones involving the divergence,

$$\int_X \varphi \operatorname{div} b + \int_X b \cdot \operatorname{grad} \varphi = \int_{\partial X} n \cdot b \;\; \varphi$$

(cf. (4)), and the curl, (5).

4.3.1 Traces (tangential and normal) of a form

We begin with the notion of "outgoing (unit) normal field".

Let's recall (cf. Remark 2, p. 45) that if $x \in \partial X$, there are three kinds of vectors at x: "tangent to the boundary" (these span in $T_x X$ a subspace $T_x \partial X$, of codimension one), "incoming", and "outgoing". Among the latter, a unique one is orthogonal to $T_x \partial X$ and of length 1 (with respect, of course, to the metric g_x): this is the "outgoing unit normal vector", denoted $n(x)$. One easily checks, within a chart, that $x \to n(x)$ is continuous. (Hence a transverse field.)

If $\tilde{\Omega}$ is the standard density associated with g, $n \tilde{\Omega}$ (cf. Exer. 65) is a density on ∂X. Since g, by restriction to $T\partial X$, defines a metric on it, there is a naturally defined surfacic Hodge operator, also denoted $*$ (the distinction with the one on X will always be clear in context) and a standard density on ∂X, which is nothing else than $n \tilde{\Omega}$ (take a direct orthonormal basis in $T_x \partial X$, and add $n(x)$ to it).

Exercise 77: A priori, two standard densities (with opposite signs) can be constructed from the metric induced on ∂X. Selecting $n \tilde{\Omega}$ amounts to orienting the map i, and also to deciding, among two possibilities, how the local orientation of X induces a local orientation on ∂X. Verify that the choice thus done conforms to standard conventions (Fig. 56).

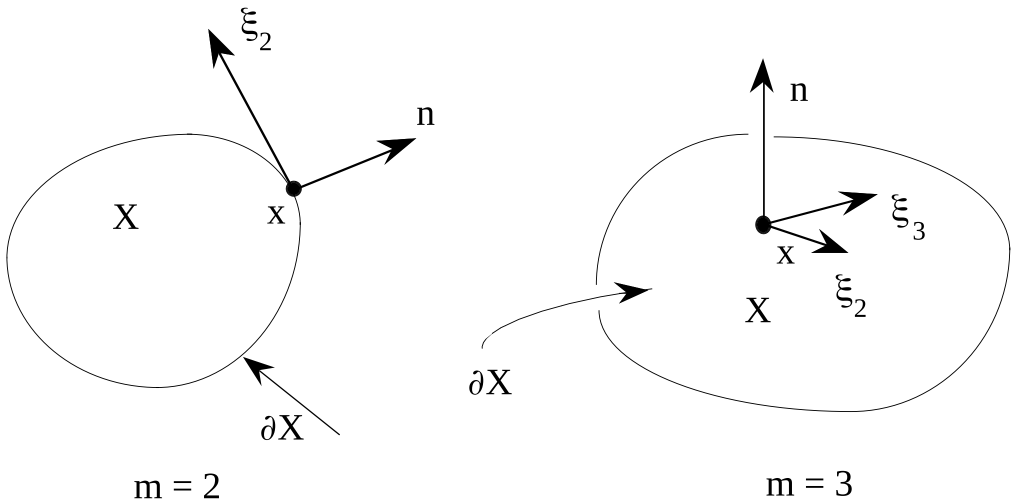


Figure 56. Induced orientation in two and three dimensions. The frames $\{n, \xi_2\}$ and $\{n, \xi_2, \xi_3\}$ are direct orthonormal.

We now define *traces*, normal and tangential, on the manifold's boundary, for a p-form (twisted or not). The *tangential trace* of $\omega \in \mathcal{F}^p(X)$ is its pull-back $i^*\omega$, where $i \in \partial X \to X$ is the canonical embedding of ∂X into X. (Since i is oriented, thanks to the transverse field n, twisted forms can be pulled-back, so $\tilde{\omega} \in \tilde{\mathcal{F}}^p(X)$ also has a trace.)

To avoid overloading the symbol $*$, we shall denote this p-form on ∂X by $t\omega$. So,

$$t\omega(\xi_1, \ldots, \xi_p) = \omega(\xi_1, \ldots, \xi_p)$$

when the ξ_is are tangent to the boundary at $x \in \partial X$.

Remark 13: The Stokes theorem can thus be written

$$\int_X d\omega = \int_{\partial X} t\omega,$$

which corrects the slight notational abuse in (31). ◊

Exercise 78: Verify (cf. Def. 7, p. 56) that t distributes with respect to $\wedge$:

$$t(u \wedge v) = t\,u \wedge t\,v.$$

As for the normal trace, it is not only a matter of notation, but a new notion, that could not be defined before having introduced a metric. One calls *normal trace* of a p-covector ω at $x \in \partial X$ the $(p-1)$-covector

(45) $$n\omega(x) = \{\xi_2, \ldots, \xi_p\} \to \omega(n(x), \xi_2, \ldots, \xi_p).$$

The definition extends to p-forms and also (if orientations are associated according to the above-mentioned rule, Exer. 77) to twisted p-forms. The reader will check that n, as an operator from $\mathcal{F}^p(X)$ into $\mathcal{F}^{p-1}(\partial X)$, or from $\tilde{\mathcal{F}}^p(X)$ into $\tilde{\mathcal{F}}^{p-1}(\partial X)$, satisfies

(46) $$n = (-1)^{(p-1)\dim(X)} * t *$$

(which could be used as a definition).

Exercise 79: Prove the equivalence of (45) and (46), and check that

(47) $$*\, n = t\, *, \quad n\, * = (-1)^p * t\,.$$

4.3.2 Green's formula

Now things start to fly. Let u be a $(p-1)$-form and v a p-form on X, with $\dim(X) = m$. Denote $(\ ,\ )$ the scalar product defined in (44) and $<\ ,\ >$ the analogous scalar product on ∂X. Since $u \wedge * v$ is a twisted $(m-1)$-form, one may invoke Stokes theorem, hence

$$\int_X d(u \wedge * v) = \int_{\partial X} t(u \wedge * v).$$

Expanding the left-hand side with the help of (34) and the right-hand side thanks to (47) and Exer. 78 , one gets

$$\int_X du \wedge * v - (-1)^p \int_X u \wedge d * v = \int_{\partial X} t\,u \wedge * n\, v$$

i.e., (46) and (44) being taken into account,

$$(du, v) - (-1)^{p + (p-1)(m-p+1)} (u, * d * v) = <tu, nv>.$$

One then defines δ (the *codifferential*), as applied to a form of degree p, by

(48) $\delta = (-1)^{m(p-1)+1} * d *$

hence the integration by parts formula

(49) $(du, v) - (u, \delta v) = \langle tu, nv \rangle,$

which can be called *Green's formula* as rightly as (4)(5), since all formulas named after Green stem from it.

Exercise 80: Show that, if u and u' are p-forms,

$$(du, du') + (\delta u, \delta u') = (-\Delta u, u') + \langle n\, du, t\, u' \rangle - \langle t\, \delta u, nu' \rangle$$

where $\Delta = -(d\,\delta + \delta\, d)$. (This is what is called "Green's formula" in calculus textbooks.)

Exercise 81: Prove that, if u and u' are p-forms,

$$(\delta\, d\, u, u') - (u, \delta\, d\, u') = \langle tu, n\, du' \rangle - \langle tu', n\, du \rangle$$

("second Green's formula").

Exercise 82: Check $d\,t = t\,d$. Show that $\delta * = \pm * d$, and $d * = \pm * \delta$, and watch for the dependence of the sign on the degree of the form to which these operators are applied. Conclude that $n\delta = -\delta n$. Then work out the following formulary:

$$* td = n\delta *, \quad * \delta t = dn *, \quad * t\delta = -nd *.$$

4.3.3 Extensions of the theory

From this stems a theory of the Laplace operator on a manifold, quite similar to the standard one. The essentials are in [43] and [76] (cf. also [1, 34]). Let's take a glance at it.

The corner stone is the scalar product (44). One completes the vector space of square integrable p-forms with respect to this scalar product, hence a Hilbert space, denoted $F^p(X)$ (or $\tilde{F}^p$, in the case of twisted forms). Thanks to the Hodge operator, there is an isometry between $F^p(X)$ and $\tilde{F}^{n-p}(X)$. A theory similar to that of Sobolev spaces develops, by considering the scalar product

$$((\omega, \eta)) = (\omega, \eta) + (d\omega, d\eta)$$

and by completing, hence a space $F_d^p(X)$, the topology of which is such that

$$d \in F_d^{\,p}(X) \to F^{p+1}(X)$$

(of domain $F_d^{\,p}(X)$) is continuous. The codifferential δ of (48) then appears as the adjoint of d. Setting

$$\Delta = -\,(d\delta + \delta d),$$

one gets an unbounded operator of $F^p(X)$, the *Laplace operator*. Differential forms such that $\Delta\omega = 0$ are called *harmonic*. Last, any form of $F^p(X)$ can be written as a sum

$$\omega = d\alpha + \delta\beta + \gamma$$

with $\alpha \in F^{p-1}_d$, $\beta \in F^{p+1}_\delta$ and γ harmonic. This is "Hodge decomposition" [48].

Doing this requires the same kind of technical results as used in the elementary theory: trace theorems, Poincaré-like inequalities, etc. The method consists in working within a chart, where a p-form ω is represented by a family of functions, whose traces on ∂X are distributions belonging to miscellaneous Sobolev spaces. In particular, one may call $H_p^{-1/2}(\partial X)$ the space of traces (of p-forms) on ∂X such that these functions be in $H^{-1/2}(\partial X)$. Then the following result holds [78]: traces on ∂X of forms belonging to $F_d^{\,p}(X)$ span the space

$$\{\alpha \in H_p^{-1/2}(\partial X) : \ d\alpha \in H_{p+1}^{-1/2}(\partial X)\}.$$

The minus sign may come as a surprise: for if $p = 0$, we are used to find the trace in $H^{1/2}(\partial X)$, not merely in $H^{-1/2}(\partial X)$. But this is indeed what this general result says in that case: $d\alpha \in H_1^{-1/2}$ means that the $n - 1$ components of the gradient of α (taken in ∂X) are in $H^{-1/2}$, so $\alpha \in H^{1/2}$. The case $n = 3$ and $p = 1$ is especially interesting and one will come back to it in Section 5.1.

4.4 Back to dimension 3: the cross product

To prepare for this transition, here follows a new viewpoint on an old subject. Let X be a Riemannian manifold of dimension three and $\tilde{\Omega}$ the associated standard density. Let u and v be two vector fields. Select an orientation in the vicinity of x, and call Ω the local volume. Then

$$\xi \to \Omega(u, v, \xi)$$

is a covector. Paired with the orientation, it forms a twisted covector, whose sharp is a twisted vector: this is the one that is denoted $u \times v$, the cross product of u and v. As a twisted vector, it can be represented by a vector, once the orientation has been fixed, but its sign will change with the orientation. This explains the oddities of the cross product, the reason why it only exists in three dimensions, and the rationale behind the subtle distinction done by the Treatises ([80], p. 200, . . .) between "polar" vectors (the true ones) and "axial vectors" (the twisted ones).

Exercise 83: If u is twisted and v ordinary, show that $u \times v$ is an ordinary vector. What happens in the case of two twisted vectors u and v?

Careless use of the cross product may lead to confusion. Consider, for instance, the formula which gives Lorentz force,

$$f = j \times b.$$

Force is, by its very definition, a covector, since it operates linearly on virtual displacement vectors, yielding the virtual work. But if we were right in treating b as a 2-form and j as a twisted 2-form, how can such a formula make sense? What kind of "product" is it that would yield a covector from two 2-covectors, one of them twisted? A step forward consists in defining f as the covector $v \rightarrow b(v, j)$ where j is the current density vector field. (The argument v is but the field of virtual displacements.) This shows that the metric was irrelevant, but since going from $\tilde{j}$ to j involves the operator $\diamond$, some density has to intervene. Which density? Clearly the one that measures volumes, since f is a density of force per volume unit ("volume" and "density" being taken with their common meaning in this sentence).

Can one go further and get rid of even this standard density? For this, one should combine b (a 2-form) and $\tilde{j}$ (a twisted 2-form) in order to find something like a *covector-valued*, not real-valued, density (twisted 3-form), so that by integration over some region, one could find the total force. There is a geometric object which fits this description: $v \rightarrow i_v b \wedge j$, where i denotes the inner product of Remark 8, p. 64. This is the correct representation of the field of Lorentz forces. One sees the concept of "vector-valued differential form " emerging here, and this opens new avenues. We shall refrain from walking them (not without some regret), to concentrate on dimensions two and three, and on structures specific to these dimensions.

Chapter 5

Differential forms in E_3 and the structure of Maxwell equations

5.1 Differential forms in dimension 3

In this Chapter, we take for granted the notions of *function* $\varphi \in E_3 \to \mathbb{R}$ and of *vector field* $v \in E_3 \to E_3$ (being understood that E_3 means the *affine* space on the left of the arrow, the *vector* space on the right). The scalar product of two vectors u and v is denoted $u \cdot v$, and the mixed product of three vectors is $\text{vol}(u, v, w)$. Recall that $\text{vol}(u, v, w) = u \cdot (v \times w)$. The frame formed by three independent vectors is said to have a "direct orientation" if their mixed product is positive.

5.1.1 Vector fields and differential forms

Definition 15: *A* p-covector ω of E_3 *is a function of type* $E_3 \times \ldots \times E_3 \to \mathbb{R}$ (p factor spaces), *linear with respect to all its arguments, and* alternating, *i.e., changing sign when one permutes two of the arguments:*

$$\omega(\xi_1, \xi_2, \ldots) = -\omega(\xi_2, \xi_1, \ldots),$$

etc. (One also says "skew-symmetric".)

As a direct consequence of the definition, $\omega = 0$ for $p > 3$, and for $p = 0$, ω is a real constant.

A vector u generates a 1-covector, that will be denoted ${}^1u$:

(50) $${}^1u = \xi \to u \cdot \xi,$$

and a 2-covector, denoted ${}^2u$:

(51) $${}^2u = \{\xi, \eta\} \to \text{vol}(u, \xi, \eta).$$

Conversely, if ω is a p-covector, with $p = 1$ or 2, there exists a unique vector u such that $\omega = {}^{p}u$. (This would not happen in dimension higher than 3.) Similarly, a real number φ generates a 0-covector, which is just the constant φ, and a 3-covector, denoted ${}^{3}\varphi$, which is the product of φ by $\mathrm{vol}(\xi, \eta, \zeta)$:

$$ {}^{3}\varphi = \{\xi, \eta, \zeta\} \to \varphi\ \mathrm{vol}(\xi, \eta, \zeta) $$

Definition 16: *A* differential form ω of degree p, *or* p-form, *on* E_3, *is a smooth field of p-covectors.*

The notion is only interesting when $0 \le p \le 3$. To any smooth function φ, there corresponds a 0-form ${}^{0}\varphi$ and a 3-form ${}^{3}\varphi$, and to any smooth vector field u, a 1-form ${}^{1}u$ and a 2-form ${}^{2}u$, thanks to the above correspondences. One will denote by $\mathcal{F}^{p}$ the vector space of smooth p-forms, with compact support[1], on E_3.

Exercise 84: One defines the operation $\wedge$ by ${}^{0}\varphi \wedge {}^{p}\varphi' = {}^{p}(\varphi\,\varphi')$ for $p = 0$ or 3, by ${}^{0}\varphi \wedge {}^{p}u = {}^{p}(\varphi\, u)$ for $p = 1$ or 2, and by ${}^{1}u \wedge {}^{1}v = {}^{2}(u \times v)$, thence ${}^{1}u \wedge {}^{2}v = {}^{3}(u \cdot v)$. Show this is indeed the wedge product of Def. 7, p. 56.

Remark 14: After (50), the correspondence $u \to {}^{1}u$ does not depend on the orientation of E_3. To the contrary, after (51), the form associated with u is $-{}^{2}u$ if one reverses the orientation. The geometric object associated with u via (51) is thus not really a 2-form but a pair {2-form, orientation}, the kind of thing we called a "twisted 2-form" in 3.2.3, and the correspondence defined by (51) is the "dual map" of Fig. 53. On the other hand, ${}^{1}u$ is a genuine 1-form, and the correspondence (50) is the "flat" of (39). Similarly, ${}^{3}\varphi$ is a twisted form. Since in all this chapter we assume a fixed, once and for all, orientation, these distinctions will not be done (but the reader who has already tackled the subject matter of Chap. 3 is invited to do it on his or her or its own). ◊

5.1.2 Operators d and ∗

Definition 17 (cf. Def. 14, p. 93): *One calls* Hodge operator, *denoted* $*$, *one or the other correspondence defined by the equalities*

$$ *\,{}^{0}\varphi = {}^{3}\varphi, \quad *\,{}^{3}\varphi = {}^{0}\varphi, $$

$$ *\,{}^{1}u = {}^{2}u, \quad *\,{}^{2}u = {}^{1}u, $$

Remark 15: In (41), p. 92, we had $** = (-1)^{(n-1)p}$, where n was the spatial dimension. Here, $n = 3$, hence the absence of any sign change. ◊

[1] Recall that the *support* of a field is the closure of the set of points where it does not vanish.

Definition 18: *The* scalar product *of two* p-*forms is, for* p = 0 *or* 3,

$$({}^p\varphi, {}^p\psi) = \int_{E_3} \varphi(x)\, \psi(x)\, dx$$

and for p = 1 *or* 2,

$$({}^p u, {}^p v) = \int_{E_3} u(x) \cdot v(x)\, dx$$

One calls F^p *the space obtained from* $\mathcal{F}^p$ *by completion with respect to the distance induced by this scalar product.*

Last, one takes for granted the "naive" definitions, in Cartesian coordinates, of grad, rot and div. Then,

Definition 19: *One defines the operator* $d \in \mathcal{F}^p \to \mathcal{F}^{p+1}$ ("exterior derivative") *by*

$$d({}^0\varphi) = {}^1(\text{grad}\ \varphi),\ \ d({}^1u) = {}^2(\text{rot}\ u),\ \ d({}^2u) = {}^3(\text{div}\ u),\ \ d({}^3\varphi) = 0,$$

for p = 0, 1, 2 *and* 3 *respectively, and* $\delta \in \mathcal{F}^p \to \mathcal{F}^{p-1}$ *for* $1 \le p \le 3$, *by*

$$\delta = (-1)^p * d *$$

and $\delta({}^0\varphi) = 0$ *for* p = 0.

Exercise 85: Show that $\delta({}^1u) = -\,{}^0(\text{div}\ u)$, $\delta({}^2u) = {}^1(\text{rot}\ u)$, $\delta({}^3\varphi) = -\,{}^2(\text{grad}\ \varphi)$.

Figs. 57 and 58 are two possible graphical displays of the structures we have just set out. We shall make use of the former in the sequel.

Exercise 86: Place the relation ${}^2h = -\,\delta({}^3\varphi)$ on Figs. 57 and 58.

Note that $d^2 = 0$ and $\delta^2 = 0$. As one knows, rot u = 0 ⇒ u = grad φ, and div u = 0 ⇒ u = rot a. The collection of all these results, which is "Poincaré's Lemma", is thus expressed in the language of differential forms: *A* closed *p-form* ω *on* E_3 *(i.e., such that* dω = 0*) is* exact *(i.e., there exists* α *such that* ω = dα).

5.1.3 Forms on a surface, traces

Let S be a surface embedded in E, endowed with a field of unit normals n. One will make use of the following notation (Fig. 59): φ_S for the restriction of a function, $u_S(x)$ for the projection of u(x) onto the tangent plane at point x, u_S for the function $x \to u_S(x)$ with domain S. One has $u_S = -\, n \times (n \times u)$. Last, tω, the *tangential trace* on S of a p-form ω, is

$$t\omega = \{\xi_1, \ldots, \xi_p\} \to \omega(\xi_1, \ldots, \xi_p),$$

(cf. Section 4.3.1), and nω, its *normal trace*, is

$$n\omega = \{\xi_2, \ldots, \xi_p\} \to \omega(n, \xi_2, \ldots, \xi_p).$$

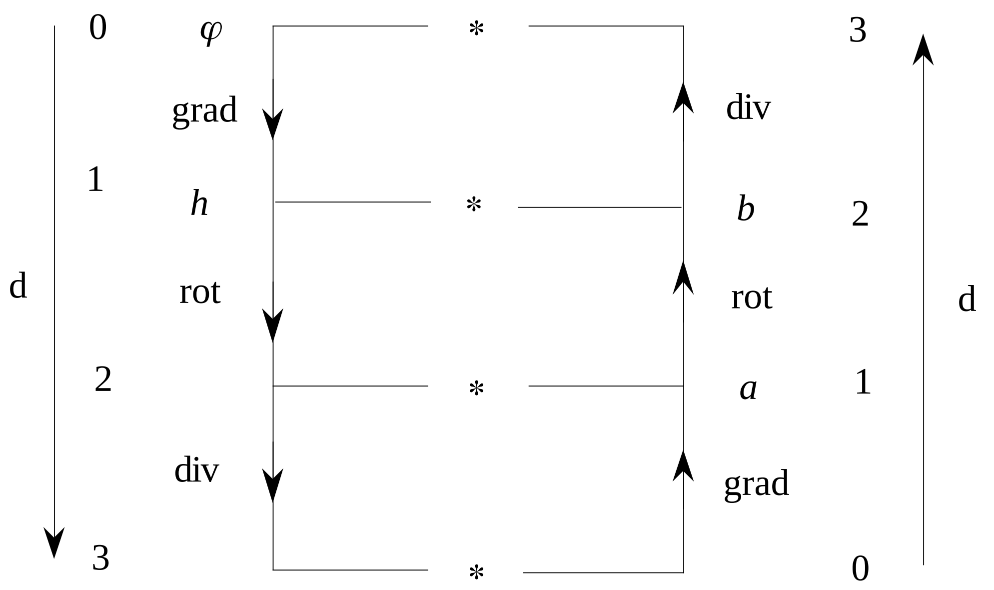


Figure 57. A graphical convention for the visualization of spaces $\mathcal{F}^p$ and of their relationships. Here, for instance, one has $^2b = *^1h = d(^1a) = {}^2(\text{rot } a)$ and $^1h = d^0\varphi = {}^1(\text{grad } \varphi)$.

The metric of E_3 and its orientation descend to S as follows. If ξ and η are two tangent vectors at $x \in S$, then $\xi \cdot \eta$ is naturally defined. Thanks to n, one may select an orientation on S by deciding that if $\text{vol}(n, \xi, \eta) > 0$ (i.e., when η is "to the left" of ξ with respect to the normal, cf. Fig. 59), the frame {ξ, η} is direct. One will easily see that the 2-volume (or "area") of the parallelogram built on ξ and η is vol(n, ξ, η), which therefore is the standard volume 2-form on S.

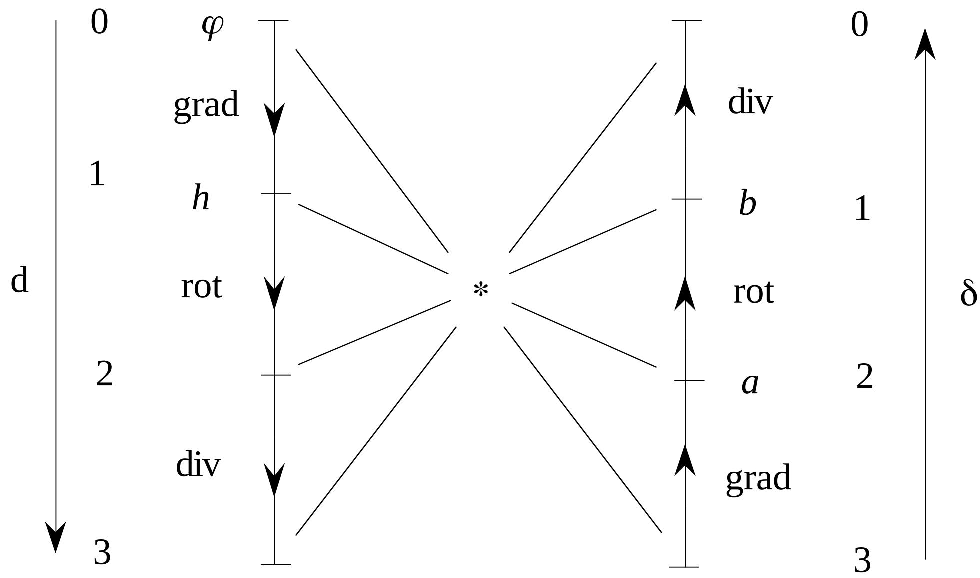


Figure 58. Another possible graphical convention.

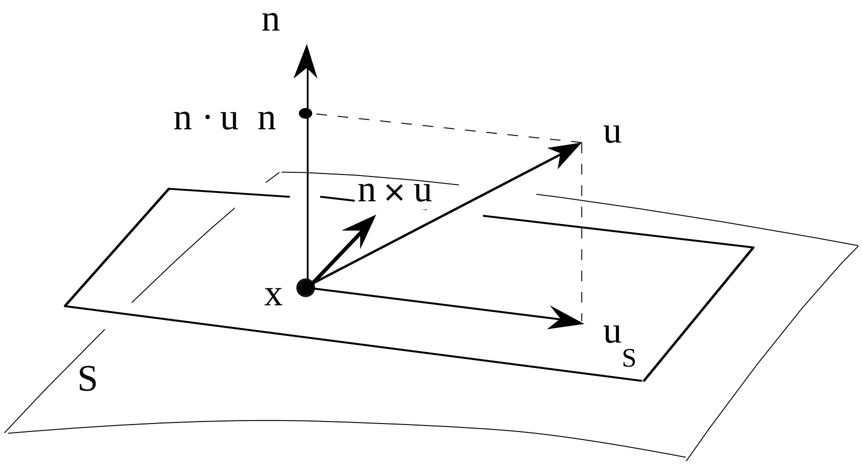


Figure 59. Notations.

Thanks to these metric elements, one may associate functions or vector fields defined on S and p-forms, exactly as above. To the function φ of domain S, corresponds the 0-form $^0\varphi$, and also the 2-form

$$^2\varphi = x \to (\{\xi, \eta\} \to \mathrm{vol}(n(x), \xi, \eta)).$$

To the field of tangent vectors u corresponds the 1-form

(52) $$^1u = x \to (\xi \to u(x) \cdot \xi).$$

But now there is *another* way to get a 1-form, that we'll denote by ${}^1\tilde{u}$ (because it is actually a twisted form, cf. Section 3.3.3):

(53) $${}^1\tilde{u} = x \to (\xi \to \mathrm{vol}(n(x), u(x), \xi)).$$

Remark that ${}^1\tilde{u} = {}^1(n \times u)$.

The one-to-one correspondence between these two 1-forms is achieved by the Hodge operator (still denoted $*$; the context will suffice to distinguish it from the $*$ of dimension three). One has (by way of definition, but the reader is invited to *justify* this definition by referring to (42)):

$$*u = {}^1\tilde{u} = {}^1(n \times u),\ *\ {}^1\tilde{u} = -{}^1u.$$

As for other values of p, one has of course

$$*\ {}^0\varphi = {}^2\varphi,\ *\ {}^2\varphi = {}^0\varphi.$$

It is now natural to study the relationship between traces on S of a function or vector field, on the one hand, and traces of the associated differential forms, on the other hand. It's an exercise, whose solution is given by the following table:

p	ω	t ω	n ω
0	${}^0\varphi$	${}^0\varphi_S$	X
1	${}^1u$	${}^1u_S$	${}^0(n \cdot u)$
2	${}^2u$	${}^2(n \cdot u)$	$-\,{}^1(n \times u)$
3	${}^3\varphi$	X	${}^2\varphi_S$

Remark 16: That $n\,{}^2u = -{}^1(n \times u)$, and not ${}^1(n \times u)$, is a bit unaesthetic, but unwelcome minus signs will pop up somewhere in the theory, whatever the sign conventions one starts with. ◊

From (52), (53) and the previous table, one gets the formulas

$$t\,* = *\,n,\ *\,t = (-1)^p\, n\,*,$$

obtained above in the general case (Exer. 79).

5.1.4 Integration

We know (cf. Section 3.3.4) that a p-form can be integrated on an *oriented* manifold of dimension p (Fig. 60). A point x (dimension 0) is oriented by giving it a sign, + or –. The integral of ${}^0\varphi$ is then defined as $\pm\,\varphi(x)$, the sign being consistent with the orientation. An arc connecting x_0 with x_1 is oriented by giving a field of unit tangent vectors, which one can take as being

$$s \to \tau(s) = \partial_s\gamma(s) \,/\, \|\partial_s\gamma(s)\|,$$

where $\gamma \in [0,1] \to \mathbb{R}$ is a parametric representation of the arc. (Note there are two possible such fields, τ and – τ, which depend on the parameterization by their signs only.) The integral of ${}^1u$ is, by definition,

$$\int_\gamma {}^1u = \int_\gamma \tau \cdot u = \int_{[0,1]} \tau(s) \cdot u(s)\, ds.$$

That of ${}^1\tilde{u}$ is

$$\int_\gamma {}^1\tilde{u} = \int_\gamma \tau \cdot (n \times u),.$$

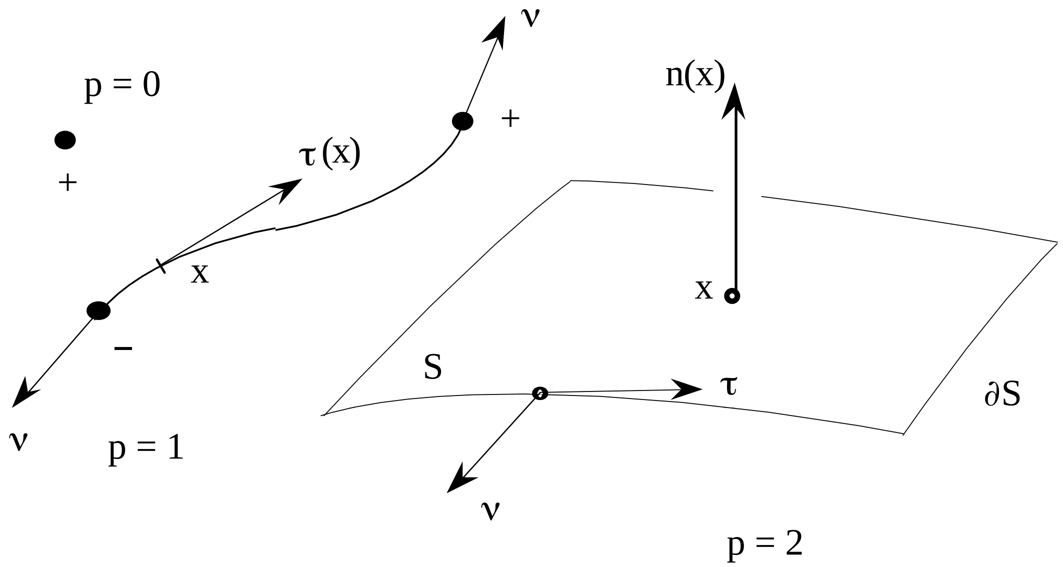


Figure 60. Orientation of p-manifolds, p = 0, 1, 2, and induced orientations on their boundaries. One is reminded (Exer. 77, p. 96, and Fig. 56) that the boundary of an orientable manifold inherits from it an orientation, thanks to the outgoing vector field (here ν), which can always be defined if the boundary is smooth enough.

A surface S is oriented by giving a field of normals n. If the 2-form is defined by a function φ on S, like ${}^2\varphi$, its integral is, still by definition,

$$\int_S {}^2\varphi = \int_S \varphi = \int_S \varphi(x)\, dx$$

where dx is the surfacic measure. If it comes from a vector field by pulling back $\omega = {}^2u$, one has

$$\int_S t\omega = \int_S {}^2n \cdot u = \int_S (n \cdot u)(x)\, dx.$$

(This is the *flux* of u through S.)

5.1.5 The surfacic d

The operator d_S is defined, according to the principles set in Section 3.4, in order to express the local form of Stokes theorem. First, let φ be a function on S and v a tangent vector at x. There exists a trajectory $g \in \mathbb{R} \rightarrow S$, with $0 \in dom(g)$, with v as its tangent vector at the origin (i.e., $v = g_*$, with the notation of 2.2.1). The map

$$v \rightarrow d/dt\ \varphi(g(t))|_{t=0} \equiv \partial_v\varphi,$$

being linear in v, defines a 1-covector at x, that is denoted ${}^1grad_S\varphi$. The gradient itself is thus the vector $grad_S\,\varphi$ such that

$$(grad_S\,\varphi) \cdot v = \partial_v\varphi.$$

The expected relation

$$\varphi(\gamma(1)) - \varphi(\gamma(0)) \equiv \int_\gamma {}^1(grad\ \varphi),$$

which is Stokes theorem, does hold.

Remark 17: So, nothing new with this definition, which does correspond to the intuitive notion of surfacic gradient. We went into details in order to stress two points: *1°*- φ need not be defined outside S, *2°*- The metric on S, inherited from E_3, only plays a rôle if one insists on $grad_S\varphi$ being a field of (surfacic) *vectors*. The associated 1-form ${}^1(grad_S\varphi)$ does not depend on it (thus the d_S we are about to define will be metric independent as well). ◊

Let now u be a vector field on S, and O an open set, with smooth boundary ∂O, around x (Fig. 61). The boundary ∂O admits of an outgoing unitary field of tangent vectors to S, called ν. One sets, as a definition,

$$\mathrm{div}_S u = \lim[(\textstyle\int_{\partial O} \nu \cdot u)/\mathrm{area}(O)],$$

the limit being taken by letting the area of O tend to 0. One finally sets

(54) $$\mathrm{rot}_S\, u = -\,\mathrm{div}_S(n \times u).$$

Then, calling $\mathcal{F}^p(S)$ the set of p-forms on S,

Definition 20: *One defines* $d_S \in \mathcal{F}^p(S\,) \to \mathcal{F}^{p+1}(S)$ *via*

$$d_S\, {}^0\varphi = {}^1(\mathrm{grad}_S \varphi),\ \ d_S\, {}^1u = {}^2(\mathrm{rot}_S\, u),\ \ d_S\, {}^2\varphi = 0,$$

and $\delta_S \in \mathcal{F}^{p+1}(S) \to \mathcal{F}^p(S)$ *via*

$$\delta_S = (-1)^p * d_S *.$$

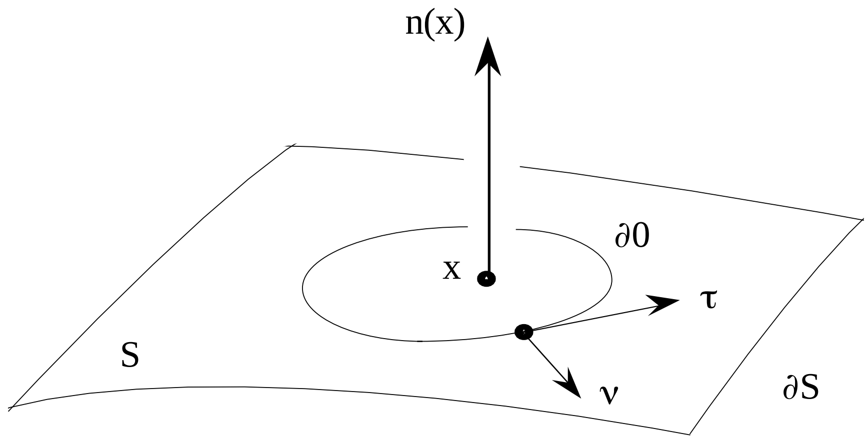


Figure 61.

This is an "ad-hoc" definition, just like Def. 19: one introduces d_S, starting from naive definitions of grad_S, rot_S, etc., in order to retrieve the operator d of differential geometry (Def. 12, and (33), p. 83). The virtue of this procedure is to quickly get to the point. Its drawback is to blur the distinction between different structural levels.

Exercise 87: Show that ${}^2(\mathrm{div}_S\, u) = d_S\, {}^1\tilde{u}$, and $\delta_S\, {}^1u = -\,{}^0(\mathrm{div}_S\, u)$.

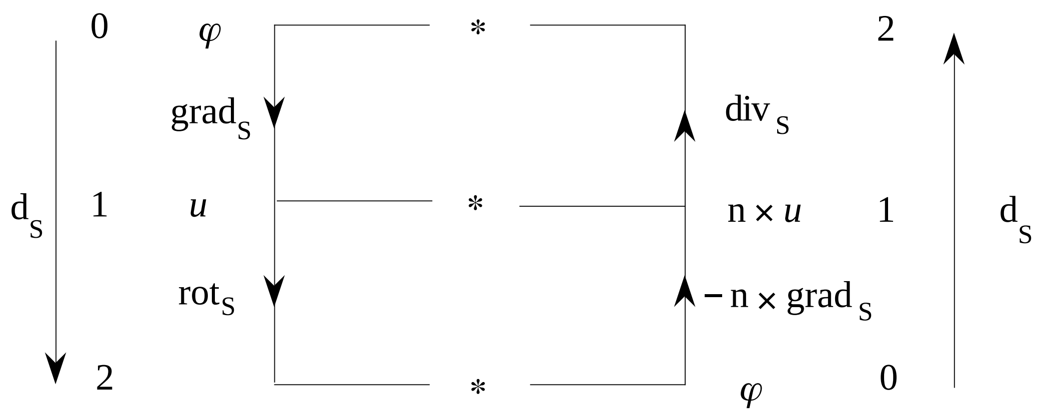


Figure 62. The structure made of the spaces $\mathcal{F}^p(S)$, the different realizations of d, and the Hodge operator $*$.

As above (Fig. 57, p. 104), we have a graphic representation of the foregoing structures (Fig. 62). Beware however of sign changes. One will notice the intervention, which is necessary in order to make this diagram complete, of operator $- n \times \text{grad}_S$, often itself denoted rot_S, just like the one in (54) (the distinction being brought to attention by various typographical tricks), because of this: if $u = \{0, 0, \varphi\}$ in a Cartesian system, and if φ does not depend on x^3, then $\text{rot}\, u = \{\partial_2\varphi, -\partial_1\varphi, 0\}$, i.e., precisely $-n \times \text{grad}\, \varphi$, where $n = \{0, 0, 1\}$. This is perhaps not reason enough to adopt such a notation, which is prone to confusion. Some advocate "grot_S" for $- n \times \text{grad}_S$. This could be a fine substitute.

Exercise 88: Let u be a vector field, whose domain contains S, such that $n \times u = 0$. Under which conditions is the equality $(\text{rot}\, u)_S = - n \times \text{grad}_S(n \cdot u)$ valid?

Exercise 89: Show that $\delta_S{}^2\varphi = - {}^1(n \times \text{grad}\, \varphi)$, and draw a diagram analogous to that of Fig. 62, but featuring δ_S.

Remark 18: Even if S is not orientable, one may erect a structure similar to that of Fig. 62. But the right side of the diagram is then occupied by "twisted forms", and $n \times u$ must be replaced by a field of twisted vectors, like the one of Fig. 48. ◊

5.1.6 Stokes Theorem

One will not be surprised to meet at this stage the different versions of Stokes' theorem, in particular

$$\int_\gamma \tau \cdot \text{grad}\, \varphi = \varphi(\gamma(1)) - \varphi(\gamma(0))$$

or else

$$\int_S \ n \cdot \text{rot } u = \int_{\partial S} \tau \cdot u = \int_S \text{rot}_S\, u.$$

What we did was actually a preparation for that. One will check in particular the relation dt = td (cf. Exer. 82), whose realizations fill the following table:

p	ω	t ω	t dω	d t ω
0	$^0\varphi$	$^0\varphi_S$	$^1(\text{grad } \varphi)_S$	$^1(\text{grad}_S\, \varphi_S)$
1	^1u	^1u_S	$^2(n \cdot \text{rot } u)$	$^2(\text{rot}_S\ u_S)$

The row p = 2 is missing, since a 3-form on S is necessarily zero.

Exercise 90: Complete the table, by adding to it columns for $n\,\omega$, $*\,\omega$, $n\,\delta\,\omega$, $\delta\,n\,\omega$, $*\,t\,\omega$, etc., and read off the relations $*\,t\,d = n\,\delta\,*$, $*\,\delta\,t = d\,n\,*$, $*\,t\,\delta = -n\,d\,*$, of the "formulary" of Exer. 82.

The foregoing exercise featured the forms

$$n\ d\,{}^0\varphi = n\,({}^1\text{grad } \varphi) = {}^0(n \cdot \text{grad } \varphi) = {}^0(\partial\varphi/\partial n),$$

$$n\ d\,{}^1u = n\,({}^2\text{rot } u\ ) = -\ {}^1(n \times \text{rot } u),$$

$$n\ d\,{}^2u = n\,{}^3(\text{div } u) = {}^2(\text{div } u).$$

The first two will be found again in the renderings of the classical Green formulas in the language of differential forms.

Let S be a closed surface, bounding region D, and n the outer normal[1]. Then, as one knows (4), (5),

(55) $$\int_D u \cdot \text{grad } \varphi + \int_D \varphi \text{ div } u = \int_S \varphi\ n \cdot u,$$

(56) $$\int_D v \cdot \text{rot } u - \int_D u \cdot \text{rot } v = \int_S (n \times u) \cdot v.$$

[1] D for "domain", with its technical meaning of "connected open set", which we avoided elsewhere, the word domain being here reserved for another notion. Remark however the two acceptions are very near to each other: D, or its closure, are indeed the domains of the various fields we consider.

The first formula is nothing else than

$$(d\ {}^0\varphi,\ {}^1u) - ({}^0\varphi,\ \delta\ {}^1u) = <t\ {}^0\varphi,\ n\ {}^1u>,$$

i.e. (as in (49), only with different notations):

(57) $$(d\omega,\ \eta) - (\omega,\ \delta\eta) = <t\omega,\ n\eta>,$$

with $\eta = {}^0\varphi$ and $\omega = {}^1u$ on the manifold with boundary formed by the closure of D. The second formula is

$$(d\ {}^1u,\ {}^2v) - ({}^1u,\ \delta\ {}^2v) = <t\ {}^1u,\ n\ {}^2v>$$

for $n\ {}^2v = -\ {}^1(n \times v)$, and the scalar product of ${}^1u$ and $-\ {}^1(n \times v)$ is equal to $-\ u \cdot (n \times v) = (n \times u) \cdot v$. But this is not the only possible interpretation: the reader will see that (55) can also be understood as

$$(d\ {}^2u, {}^3\varphi) - ({}^2u,\ \delta\ {}^3\varphi) = <t\ {}^2u,\ n\ {}^3\varphi>$$

and (56) as

$$-\ (d\ {}^1v,\ {}^2u) + ({}^1v,\ \delta\ {}^2u) = -\ <t\ {}^1v,\ n\ {}^2u>.$$

Could one derive from (57) other interesting formulas? Not so, obviously, since all possible cases have been considered: $\omega = {}^0\varphi$, ${}^1u$ and ${}^2u$. The fact that only two formulas exist in the present case stems from the symmetry of (57) with respect to the Hodge operator: if the dimension is $2q$ or $2q - 1$, there are only q different Green formulas.

Thus, in dimension 2, there is only one, corresponding to $\omega = {}^0\varphi$ and $\eta = {}^1u$:

$$\int_S u \cdot \mathrm{grad}_S\varphi + \int_S \varphi\ \mathrm{div}_S u = \int_{\partial S} \varphi\ \ \nu \cdot u,$$

(ν is the outgoing normal, with respect to ∂S, in the tangent plane to S). The other one ($\omega = {}^1u$ and $\eta = {}^2\varphi$) only *looks* different, because of (54) (**Exercise 91**):

$$\int_S \varphi\ \mathrm{rot}_S u + \int_S (n \times \mathrm{grad}_S\varphi) \cdot u = \int_{\partial S} \varphi\ \ \tau \cdot u,$$

where τ is the tangent vector to ∂S of Fig. 61.

By setting $\eta = d\omega'$ in (57), one gets a second group of Green formulas, which are of frequent use (cf. Exer. 80):

$$\int_D \text{grad } \varphi \cdot \text{grad } \varphi' = \int_D - \Delta\varphi\ \varphi' + \int_S \partial_n\varphi\ \varphi'$$

(where $\partial_n\varphi$ is the normal derivative), and

$$\int_D \text{rot } u \cdot \text{rot } u' = \int_D (\text{rot rot } u) \cdot u' - \int_S \text{vol}(n, \text{rot } u, u').$$

From this would stem a third group, basic to boundary integral methods, whose geometric structure is discussed in [21].

5.2 Maxwell's equations

We shall end with a study of Maxwell equations, with the help of the geometric tools introduced in this course. First, a few words on the nature of the intellectual exercise we shall thus indulge in.

5.2.1 Modelling

It's a *modelling* process, that is to say, the construction of a mathematical structure which is supposed to represent a definite compartment of the real world (in our case, "classical" electromagnetic phenomena, to the exclusion of quantal ones). The use of a word like "model", so rich in connotations, may wrongly suggest that the outcome of such a work could be a kind of coarse image, or perhaps a mock-up, of reality. This is only partially correct. The physicist's ambition goes beyond a mere description of the world, it aims at gaining predictive and operative power. Models are thus meant to be *interrogated*, to produce new information, or more to the point, they should make explicit the implicit information built into them. This is requiring a strong, almost paradoxical property: how could mind constructs, a priori totally transparent to us, their makers, tell us something new about the world? This tiny miracle is commonplace, however. It is performed by these mathematical objects, *equations:*[1] to solve an equation consists in producing an object — its solution — endowed with specified properties, but which happens to have also other, unpredicted, properties, which reveal themselves to us as we look at it. This is why physical models reduce, when all is said and done, to equations: we formalize our knowledge of reality by setting them, we enrich it by solving them.

[1] Provided, of course, the word is taken in a broad sense. For instance, sending queries to a data-retrieval system by using a combination of key-words, or submitting a predicate to the evaluation of an expert system, consists from the present point of view in setting up an equation.

Does this mean that in every modelling, there would exist at the onset a solid, objective, unquestionable corpus of knowledge, and then a completely free choice of the building blocks of a new mathematical structure to be appended to it? Such a view would be too drastic, for this corpus is itself nothing but a system of models, whose constitutive parts and organization principles guide and restrain our choice. Indeed, all really innovative new modellings (like the one Einstein did to account for gravitation) turn the whole edifice upside down before settling in.

From the pedagogical point of view, however, such a presentation is convenient. So we shall suppose known and familiar to us, besides classical mechanics, a part of electromagnetism: the one that deals with the existence and empirical properties of *electric charge*. Consider the latter as a substance, the existence of which is an experimental fact (cat skin, electrolysis, Millikan's experiment, whatever): the question "how much charge is there in that region of space" thus makes sense. Moreover, we record the existence of what will be called the *field*, a time- and location-dependent physical reality which makes itself be perceived through the behavior of these charges. Our objective is to set up an *electrodynamics*, that is to say a theory (with some predictive power) of this behavior.

At the onset, we thus have a rather scanty[1] mathematical structure: space E_3, time (a real variable spanning $\mathbb{R}$), and a 3-form, the *charge density* ${}^3\tilde{\rho}$. Up to first infinitesimal order[2], the charge contained in a parallelepiped built on vectors ξ_1, ξ_2, ξ_3 at point x is[3] $\rho(x)\,\mathrm{vol}(\xi_1, \xi_2, \xi_3)$, the total charge in a region D is the integral $\int_D {}^3\tilde{\rho}$, i.e., $\int_D \rho(x)\,dx$. On this basis, we shall model what we know (from experimental evidence) of the effects of the field, while following an Occam-like (or Strunk-and-White-like . . . [94]) golden rule: omit unnecessary mathematical structures.

5.2.2 Electrical phenomena: first equation

Let us begin with the observed effects of the ambient field on *non-moving* charged particles: they sum up to this observation that to move a charge[4] some distance

[1] Up to a point. After all, Newtonian space-time $E_3 \times \mathbb{R}$ is a formidable edifice, the achievement of a protracted modelling process, which is clearly perceived as such now that physics has led us beyond Newtonian conceptions. We shall come back to this in the Conclusion.

[2] with respect to the norms of the three vectors.

[3] If the frame $\{\xi_1, \xi_2, \xi_3\}$ is direct. Otherwise, the sign has to be reversed. So we are indeed dealing with a *twisted* 3-form.

[4] A virtual movement, that one may conceive as a limit case for an arbitrarily slow real movement (think to reversible transformations in thermodynamics).

implies a work proportional, to first order, to this distance. We shall call "electric field" the physical entity which is responsible for these effects. (Of course this field is only a facet of the electromagnetic field: experience shows that moving charged particles are subject to other effects (deflection of the trajectories), which will later be ascribed to another facet of the field, the "magnetic field".) Which mathematical object shall we select to model the electric field with?

Consider a charge unit, concentrated at point x. To mathematically model what we mean by its "displacement", we have the right object at hand: it's a vector at x, say v(x). The work involved being experimentally found to be proportional to the displacement, we model it as a linear function of type $T_x E_3 \to \mathbb{R}$, i.e., as a covector at x. Specifying such a covector at each point thus suffices, by definition, to describe the electric field (cf. Fig. 3). The latter will thus be represented by a field of such covectors, i.e., *by a 1-form*, that we shall denote ${}^1e$.

In this composite symbol, one may rightly distinguish the vector field e and the tag 1, standing for an operator which transforms e into a 1-form, provided one is well aware that the metric of E_3 has been summoned in order to make this separation possible. If the metric was changed, the vector field e would be different, whereas the electric field, as a physical entity, would of course stay the same. So the 1-form ${}^1e$ better represents the electric field than the vector field e, and the form, from now on, not the vector, will be for us "the" electric field[1].

One knows (Sections 3.3, 5.1.4) that a 1-form can be integrated along an oriented path, yielding a number. Because of our interpretation of the field, this integral $\int_\gamma {}^1e$ is the work received when one pushes a unit charge along the trajectory γ. (The sign convention we are doing at this stage, work received rather than given, is unimportant for our purpose.) We shall call it "electromotive force (e.m.f.) along γ".

Considering now a charge distribution of density ${}^3\rho$, instead of a point charge at x, one will easily see (**Exercise 92**) that the work received during a movement described by the vector field v in the electric field e is, to first order, $\int_{E_3} \rho(x) \langle e(x), v(x)\rangle \, dx$, where $\langle \, , \, \rangle$ denotes the duality covector-vector. This quantity is of course invariant with respect to changes of metric, in spite of the presence of the volume element dx under the integral.

[1] We won't go so far as saying the electric field "is" a 1-form. This would amount to identify some "elements of physical reality" (if this makes sense!) with some mathematical elements of the modelling one makes, and this would go against our objective. Moreover, this would verge on dogmatism, since there is no reason for this representation of the field by a 1-form to be in all circumstances and for all purposes *the best one*. This being said, we shall not deny to ourselves the convenience of saying that, for instance, "charge is a twisted 3-form", etc. But it will be just an indication about the rôle held by the mathematical object (here ${}^3\rho$) in the structure one is building, not an ontological statement.

Exercise 93. Prove this by showing the integral can be written $\int_{E_3} i_v{}^3\rho \wedge {}^1e$, with the notation of Remark 8, p. 64.

Let us now look for the right object with which to model the electric *current*, i.e., the flux of charge. It must bear with it the information needed to answer the question: "What is the quantity of charge which crosses a given surface in a prescribed direction (per unit of time)?". So it has to be (cf. p. 80) a 2-form, say ${}^2\tilde{j}$, represented (in a metric-dependent way, just as ${}^1e$ above was represented by e) by a vector field j, the one usually called *current density*. Given two vectors at x, say ξ_1 and ξ_2, the flux of charge across the parallelogram defined by the two vectors ξ_1 and ξ_2, in the direction defined by some vector n at x, is to first order the quantity $\mathrm{vol}(j(x), \xi_1, \xi_2)\ \mathrm{sgn}(\mathrm{vol}(n, \xi_1, \xi_2))$. The latter is indeed independent of the orientation of ambient space, as it should, since if the orientation is reversed, the sign of $\mathrm{vol}(j(x), \xi_1, \xi_2)$ is reversed, but the sign of $\mathrm{vol}(n, \xi_1, \xi_2)$ is reversed too. We are led to the conclusion that the information on the flux is borne by the *pair* consisting in the form $x \to (\{\xi_1, \xi_2\} \to \mathrm{vol}(j(x), \xi_1, \xi_2))$ and the orientation, i.e., the twisted form associated with j (p. 153), hence the tilda in ${}^2\tilde{j}$.

Moreover, one may change not only the sign of the volume form, but the metric as well: the vector field j will be totally different, but the associated twisted 2-form will still be ${}^2\tilde{j}$. So the twisted 2-form ${}^2\tilde{j}$ legitimately represents the current density. The integral $\int_S {}^2\tilde{j}$ on a surface S endowed with an external orientation (cf. Section 3.3.4), for instance by a normal field, or a transverse field (cf. p. 64), is the flux of charge, per unit of time, through S, in the crossing direction thus defined. This indifference to orientation is specific to integrals of twisted forms, as we saw in 3.3.3.

At this stage, we may enrich the modelling with a first physical property (that one may view as coming from experience): *charge conservation*. From the Stokes theorem and Def. 19 (or the definition of divergence in (37)), we have

$$(58) \qquad \partial_t\left(\int_D {}^3\tilde{\rho}\right) + \int_S {}^2\tilde{j} = \int_D [\partial_t({}^3\tilde{\rho}) + d({}^2\tilde{j})]$$

$$\equiv \int_D [\partial_t({}^3\tilde{\rho}) + {}^{3\sim}(\mathrm{div}\, j)] \equiv \int_D {}^{3\sim}(\partial_t\rho + \mathrm{div}\, j)$$

for a region D of surface S. The outer orientation of S being from inside to outside (according to the convention adopted Fig. 56, p. 96), the left-hand side of (58) is only 0 if charge cannot be destroyed nor created, only displaced. So the principle of *charge conservation* can be expressed by the inequality

$$(59) \qquad \partial_t\, {}^3\tilde{\rho} + d\, {}^2\tilde{j} = 0,$$

i.e., $\partial_t \rho + \mathrm{div}\, j = 0$, in familiar notation.

Let's proceed. Like for all 3-forms in dimension 3, one has $d\, {}^3\tilde{\rho} = 0$. By Poincaré's Lemma (p. 153), there exists a 2-form ${}^2\tilde{\delta}$ (twisted, just as ${}^3\tilde{\rho}$ was) such that ${}^3\tilde{\rho} = d\, {}^2\tilde{\delta}$, therefore, after (59), $d[\partial_t\, {}^2\tilde{\delta} + {}^2\tilde{j}\,] = 0$. (Of course, ${}^2\tilde{\delta}$ is not unique, and we'll wait till this indetermination is lifted before giving it its proper name and symbol.) Again by Poincaré's Lemma, there exists a 1-twisted form ${}^1\tilde{\eta}$ (non unique) such that

$$(60) \qquad - \partial_t\, {}^2\tilde{\delta} + d\, {}^1\tilde{\eta} = {}^2\tilde{j}\,.$$

Exercise 94. Check the vector field δ is only defined up to a curl, and η to a gradient, and that only the transformations of the form[1] $\delta \leftarrow \delta + \mathrm{rot}\, u$, $\eta \leftarrow \eta - \partial_t u + \mathrm{grad}\, \varphi$ leave eq. (60) satisfied. Show that these so-called "gauge" transformations form a group.

Remark 19. The reader may have decided to get rid of symbols ^{1}, ^{2}, ~, etc., in order to solve Exer. 94, and why not, for all this Section. One of course wishes to promote this transition towards the "differential forms" viewpoint (without imposing it, however, for the reasons given in the Introduction).

5.2.3 Magnetic phenomena: second equation

Let us now turn to the "magnetic" facet of the field. It could be perceived through the effect of the electromagnetic field on moving charged particles, as suggested above, but one will rather invoke induction phenomena and Faraday's experiment, historically much more significant. Just as we perceive the electric field by the force it exerts on electrically charged particles, we test for the presence of a magnetic field with the help of specific experiments. But what is perceived this way is in general a *variation* of the field: in space, when one looks at a compass, or in time, when one measures the e.m.f. induced in a closed circuit by the movement of a magnet (Fig. 63). In the latter experiment, one notices that the e.m.f. V is an additive function of the surface S which bounds the circuit (I do say "surface", not "area"), and is proportional to the rate of change (speed of the magnet, etc.). So the empirical law of induction, as indeed Faraday put it forward, has to be $\partial_t \Phi(S) + V = 0$, where Φ has the linearity properties which characterize a flux through S, i.e., is the integral over S of some 2-form. So the right mathematical object to stand for the magnetic field is a 2-form, that we shall denote by ${}^2b$, whose integral over S is the above flux $\Phi(S)$. One calls it *magnetic induction*. Knowing that $V = \int_{\partial S} {}^1e$, one gets

[1] The expression $x \leftarrow f(x)$ should be understood as in programming practice (evaluate $f(x)$, then assign its value to variable x), and read: "x takes the value $f(x)$".

$$\partial_t \int_S {}^2b + \int_{\partial S} {}^1e = 0,$$

which, after Stokes' theorem, is equivalent to

(61) $$\partial_t {}^2b + d\, {}^1e = 0.$$

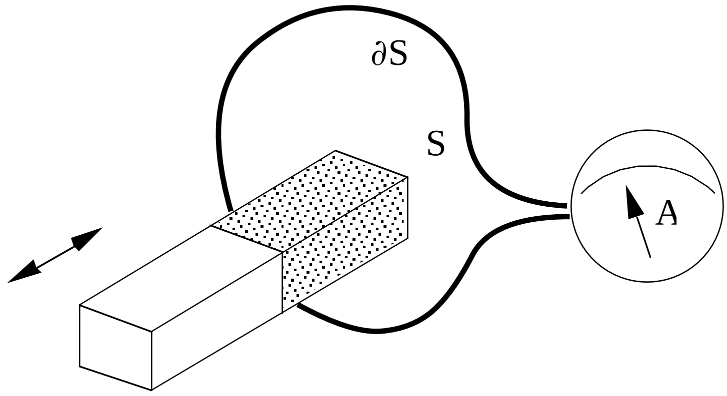


Figure 63. Demonstration of the induction phenomenon: moving the magnet evokes an e.m.f. in the circuit, acknowledged by the displacement of the pointer.

Remark 20. The name "magnetic field" would fit ${}^2b$ better, and some eminent authors use it in that acception [42]. But it is more traditionally reserved for another entity, namely one of the gauge-equivalent 1-forms ${}^1\tilde{\eta}$ of (60), whose connection with ${}^2b$ will soon be discussed. ◊

Remark 21. Readers who have been through Section 3.3.4 may have reacted this way: "Why a 2-form and not a twisted 2-form, to stand for something which has to be integrated over a surface in order to yield a flux? Why should the above argument about current density stop being valid here?" Because here the orientation does plays a rôle. If one reverses the (inner) orientation of S, the flux $\int_S {}^2b$ changes sign. If, as one *must* do to apply Stokes theorem, one simultaneously reverses the orientation of ∂S, the e.m.f. V changes sign, since ${}^1e$ is an ordinary 1-form. (This amounts to saying that there are two ways of plugging the ammeter, resulting in opposite values for V.) The choice of an *ordinary* 2-form for the magnetic induction b is thus consistent with the electric field itself being an *ordinary* 1-form. The same argument could be more quickly presented as follows: a form and its d are of the same kind, both ordinary or both twisted, so ${}^2b$ in (61) is of the same kind as ${}^1e$. ◊

So far, we twice appealed to experimental evidence, first when introducing charge and acknowledging its conservative character, then with Faraday's law. From this point, purely mathematical considerations led us to the following protomodel (written in vectorial notation):

(62) $\partial_t b + \text{rot } e = 0, \quad -\partial_t \delta + \text{rot } \eta = j, \;\; \text{div } \delta = \rho,$

where the last two equations imply the conservation of charge:

(63) $\partial_t \rho + \text{div } j = 0.$

We reached this point by modelling the *effects* of the field, but without accounting for the way it is *generated* by charges and currents. This, which is the essential part, we still have to do.

Experimental facts in this respect show that *currents create a magnetic field, charges create an electric field*. One may solicit them a little further, to have them suggest a *principle of superposition*, whose mathematical translation will be, as in other areas of physics, the postulated *linearity* of equations: the superposition of two distributions of charge (resp. of currents) has the same effects as the sum of the two corresponding electric (resp. magnetic) fields.[1]

The point is therefore to link b and e to j and ρ, or at least to objects already associated with them. We have that: the δ of (62), associated with ρ. The easiest way to achieve our goal is to postulate that *one* of these δ, say d, is proportional to e: $d = \varepsilon e$. This leaves, in a way which is almost forced on us, a relation to establish between b and one of the η (which are linked with j), denoted h: so, $b = \mu h$.

We thus obtain the model of *Maxwell's equations:*

(64) $\partial_t b + \text{rot } e = 0, \quad -\partial_t d + \text{rot } h = j, \quad \text{div } d = \rho,$

(65) $b = \mu h, \qquad\qquad d = \varepsilon \; e.$

When the charge distribution ρ and the current density j are given as functions of time, and satisfy the conservation relation (63), this model determines (as the mathematical analysis, now free to go in full gear, will show) the four constituents of the field, b, e, h, d. The coefficients μ and ε are functions of position, and their numerical values at a point can thus depend on the nature of the material about this point. This gives the model enough flexibility to account for phenomena

[1] Of course, this principle can fail to apply, for instance in presence of ferromagnetic materials. But as elsewhere in physics, we'll manage to treat such non-linearities at the level of "behavior laws", non-linear, specific to these materials. It suffices for this to avoid any premature identification between objects (such that, as we shall see, b and h, or d and e) which are linked by the linear relations suggested by the superposition principle.

encountered with some dielectrics (where $\varepsilon > \varepsilon_0$, its vacuum value) and with some so-called para- or diamagnetic materials (for which $\mu > \mu_0$ and $\mu < \mu_0$ respectively, μ_0 being the value in the vacuum). By allowing μ to be a function of the local state of the field, one may even account for some aspects of ferromagnetism.[1]

5.2.4 Maxwell's model, in terms of differential forms

One may very well feel unconvinced by the foregoing justification, in standard vectorial language, of the Maxwell model. We shall recast the argument in the language of differential forms, which helps make it stronger.

For this, let us first construct a diagram analogous to that of Fig. 57. Notice the way the latter diagram is doubled, in order to represent a p-form and its time-derivative in two parallel vertical planes (Fig. 64). As in Fig. 57, ordinary forms are on the left, and twisted forms on the right.

Next, let us place on this diagram the mathematical entities introduced up to now, beginning with charge and current density (Fig. 65). (To avoid overloading the diagram, we have denoted them ρ, j, etc., but we do mean the *forms*, not the functions or vector fields that stand for them.) Due to the conservation relation (59), there is only one way to place ρ and j. The reasoning based on Poincaré's Lemma by which we introduced d and h then simply consists in walking down the right part of the diagram while giving names to the entities encountered at each node along the way. As one will realize, there is not much of a choice in doing that: once j and ρ have been placed upstairs on the right, d and h will be located one floor below thanks to Poincaré Lemma, and the elements of the "gauge transformation" of Exer. 94 another level below (**Exercise 95**: place them). As for relation (61), i.e., $\partial_t b + \text{rot } e = 0$, its location is also forced.

Exercise 96. On Fig. 65, place a and ψ (respectively a 1-form and a 0-form, named "vector potential" and "electric potential"), such that $b = \text{rot } a$ and (thus) $e = -\partial_t a + \text{grad } \psi$. Study the "gauge transformations" from a pair $\{a, \psi\}$ into another.

So now, all the mathematically implied consequences of the existence of charge and its conservative character appear on the right side of the diagram, all what has to do with the effects of the field is on the left side. Knowing that charges and currents create the field, and having the Hodge operator as a vehicle from one side of the diagram to the other, what else can one do than assess the proportionality of b and e and of $*h$ and $*d$, hence (65)? Thus is model

[1] Within limits. Let us, incidentally, recall the MKSA values: $\mu_0 = 4\pi\ 10^{-7}$, and $\varepsilon_0 = 1/(c^2\ \mu_0)$, where c is the speed of light, about $3\ 10^8$.

(64)(65) found back, up to notations:

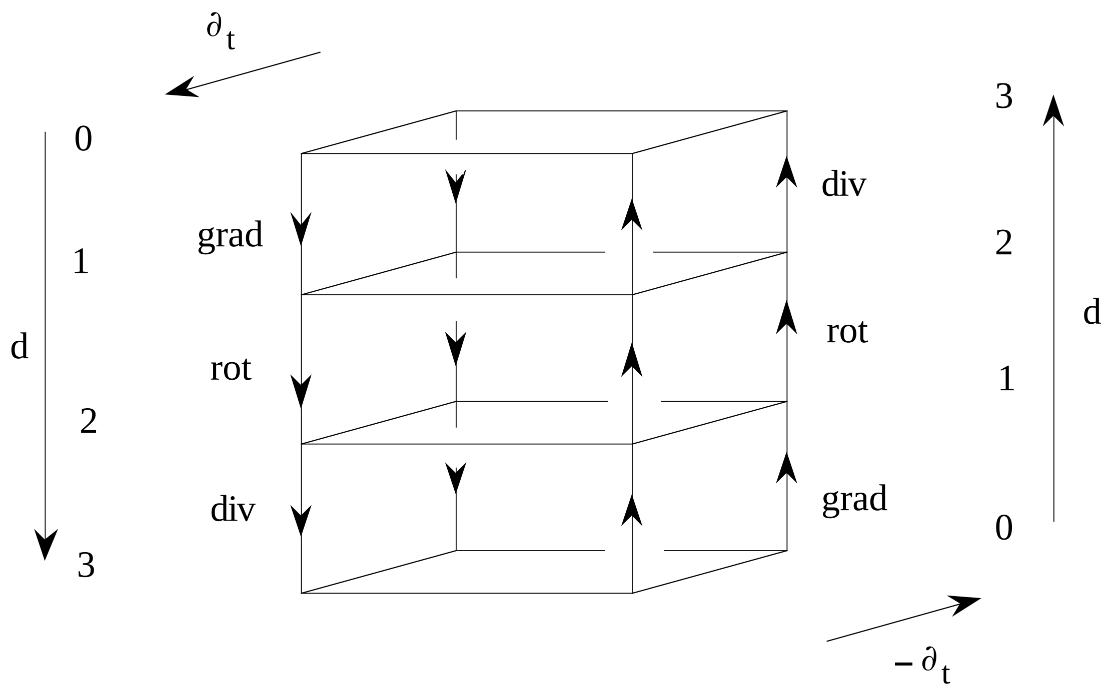


Figure 64. Combination of two copies of the diagram of Fig. 57, linked by the time-differentiation operator. This algebraic-differential structure is "home" to Maxwell equations. The horizontal bars on the back and front walls correspond to the Hodge operator.

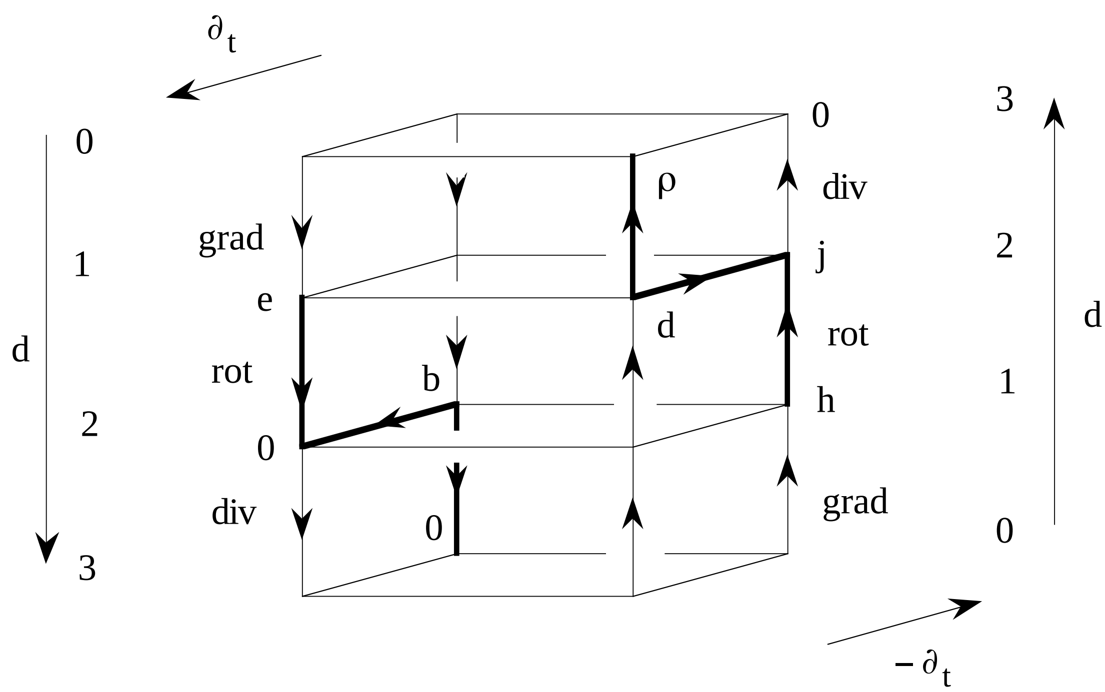


Figure 65. The "Tonti diagram" [98, 99] of Maxwell equations.

(64) $\partial_t\, {}^2b + d\, {}^1e = 0, \quad -\, \partial_t\, {}^2\tilde{d} + d\, {}^1\tilde{h} = {}^2\tilde{j}, \quad d\, {}^2\tilde{d} = {}^3\tilde{\rho},$

(65) ${}^2b = \mu * {}^1\tilde{h}, \qquad {}^2\tilde{d} = \varepsilon * {}^1e.$

Coefficients ε and μ appear now as dependent on the choice of units, and their numerical values thus account for physical properties of space. One may thus distinguish in system (64)(65) the "vertical" equations (64), which are the geometric translation of fundamental principles (Faraday's law, charge conservation) and the "horizontal" equations (65), which express physical properties of space and (since ε and μ can assume other values than ε_0 and μ_0, as already pointed out) how they are modified by the presence of matter.

The "Tonti diagram" of Fig. 65 summarizes and condenses all these considerations into a single structure: it explains how (to draw on the metaphor) Maxwell equations "live" in the structure of Fig. 64. Tonti seems to have been among the first to point at the universality of diagrams of this kind in physics. (See also [85].)

Remark 22. One now perceives the rôle played by the Hodge operator, and thus by the metric structure of space $E_3 \times \mathbb{R}$, in modelling: Whereas the structure of differentiable manifold, operator d included, had been enough to geometrize the *separate* description of cause and effect, one needs a metric structure to geometrize the behavior laws, which are *relations* between cause and effect. This point of view suggests that the respective rôles of the constants like ε and μ and of the metric structure proper are not so strictly distributed. One might very well include the constants in the Hodge operator, and have the same Hodge operator intervene at both levels of Fig. 65: it's a matter of choice of units, of time and length units in particular (so that $c = 1$). In this spirit, putting an appropriate metric on the manifold $E_3 \times \mathbb{R}$ helps ironing out the distinction between anisotropic behavior laws (the case where ε and μ are tensors) and isotropic ones (scalar ε and μ, possibly dependent on position). This relativizes the "fundamental" character of some "fundamental constants" of physics (as remarked, e.g., in [49] or [64] ; cf. also [86]). One might push the geometrization of behavior laws even further, to the point where it would take some non-linearities in charge: one should for this introduce a metric not only on the base $E_3 \times \mathbb{R}$ but on some bundle on this base, whose fibre would consist in the set of possible states of the field. ◊

Remark 23. The discussion could have been shortened (though perhaps to the detriment of clarity) by working directly on a four-dimensional manifold M, space-time. Then b and e [resp. j and ρ] appear as the two descriptive elements of one and the same 2-form F [resp. of a twisted 1-form α], and Maxwell equations reduce to $dF = 0$ (this is the reduced form of (64)), $dG = \alpha$ (consequence, as above, of $d\alpha = 0$, charge conservation, and reduced form of (65)), and $G = *F$, where this time $*$ is the Hodge corresponding to an "indefinite" metric, the Minkowski metric, on M. In this presentation, which is quite standard [27, 32, 69, 73, 89, 103, . . .], one well distinguishes the three panels of the modelling triptych: Faraday's law translates as $dF = 0$, charge conservation as $dG = \alpha$, and the principle of superposition, or of linear dependence of cause on effect, as $G = *F$. Since $*$ here intervenes only to yield a linear map from the vector space of 2-forms onto that of twisted 2-forms, one may wonder whether the Minkowskian metric underlying $*$ is not a redundant element of structure, which could be done without. A result by di Carlo [35] seems to suggest otherwise: giving the map $G \rightarrow F$ (endowed with reasonable properties) would suffice to determine the metric. If so is the case, the metric of space-time is determined by the very nature of electromagnetic phenomena, and the remarkable "simplicity" of Maxwell equations is no more surprising.

Exercise 97. In a famous method of eddy-currents computation, known as "T-Ω" [29], one represents the field h in the form $h = T + \text{grad}\ \Omega$, where T is subject to some restrictions (consisting, for instance, in forcing to 0 one of its components). Place T and Ω on the diagram of Fig. 65. (One will edit the notation a little: for instance τ and ω, for the sake of consistency with the style which is prevalent in these notes.)

Exercise 98. Compare the diagram of Fig. 65 with the one that appears in [79], p. 59.

5.2.5 Quasi-static and static models

In electrotechnical applications, linear dimensions and time-constants are such that, in any appropriate system of units, the speed of light c assumes a very high numerical value. One then very naturally wishes to consider it as infinite, and to go to the limit in Maxwell's system. As $c = (\varepsilon\,\mu)^{-1/2}$, this amounts to letting *one* of the parameters ε and μ tend to 0. Which one, this depends on the nature of sources: when there are *high* densities of *slowly* moving charges ("weak currents"), one lets μ go to zero. In the opposite case (small or null charge densities, strong currents), one cancels ε instead.

Thus, in the weak currents model, there is an *uncoupling* into a one-parameter family of electrostatic problems:

(66) $$\text{div}\ d = \rho(t), \quad d = \varepsilon\, e, \quad \text{rot}\ e = 0,$$

to be solved first, followed by the solution of an analogous family of magnetostatic problems:

(67) $$\text{div}\,(\mu_r\, h) = 0, \quad \text{rot}\ h = j(t) + \partial_t d,$$

where μ_r is the (finite) ratio of the two infinitesimals μ and μ_0.

In the strong currents model, the situation is reversed: one first solves

(68) $\operatorname{div} b = 0, \ b = \mu h, \ \operatorname{rot} h = j(t),$

j being given, then

(69) $\operatorname{div}(\varepsilon_r e) = \rho, \ \operatorname{rot} e = -\partial_t b,$

where ε_r is the ratio of the two infinitesimals ε and ε_0. In both cases (Fig. 66), one has to successively solve two problems which obviously have *the same structure.*

The uncoupling is total with steady sources, since then (66) and (67) [resp. (68) and (69)] are two independent problems: one in *electrostatics* (at the front of the diagrams) one in *magnetostatics* (at the rear), and we then neatly see the structure in question: it always consists in finding a p-form ω and a $(3 - p)$-form η, Hodge conjugate to each other (up to a choice of metric), their d's being known.

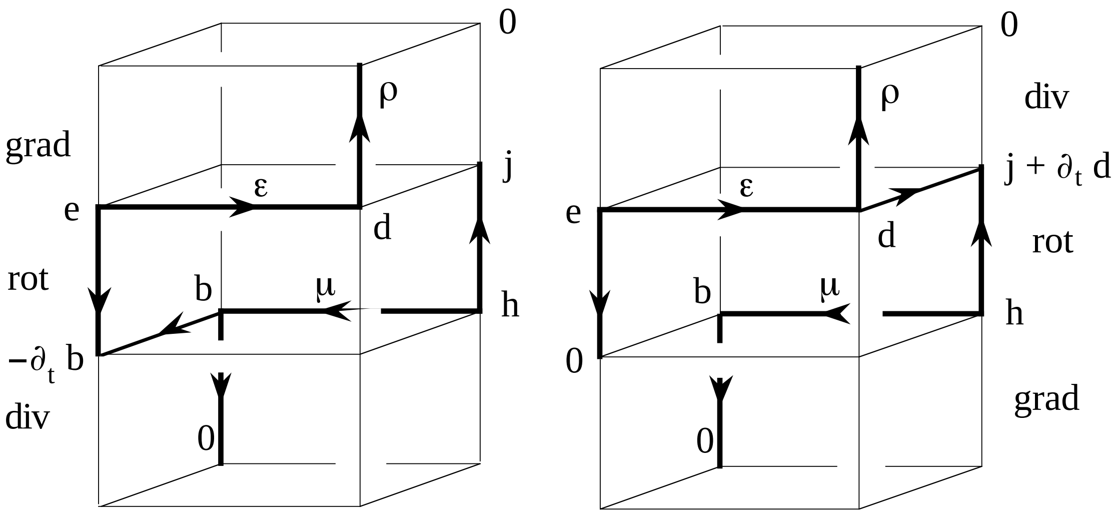


Figure 66. Tonti diagrams of the models with infinite c: Left, strong currents, right, weak currents. The side arrow disappears in the case of steady (i.e., time independent) sources, hence the uncoupling between electrostatics (at the front of the diagrams) and magnetostatics (at the rear).

This "paradigm" (as Kotiuga says [56], but we shall prefer to speak here of the "canonical problem"), as illustrated by Fig. 67, is not special to Maxwell equations: it forms the building block for most Tonti diagrams. It was early identified in Electromagnetism (cf. [102, 72]), but how to *discretize* it (by use of mixed finite elements) was only recently understood. See [17, 22] on this point.

Let us just recall that this model can be treated by introducing *potentials* φ and α such that $\omega = \omega^s + d\varphi$ and $\eta = \eta^s + d\alpha$, where ω^s and η^s are forms satisfying $d\omega^s = f$ and $d\eta^s = g$ (s for "sources", since these forms can be considered as the sources of the field, in lieu of f and g). Since, by elimination and substitution, one may always cast any of the entities ω, η, α, φ in the rôle of unknown (and even, in so-called mixed formulations, two of them together), one has a large array of possible, equivalent formulations of the canonical problem. They result, according to the choice of finite elements, in various numerical schemes, an attempted classification of which can be found in [22].

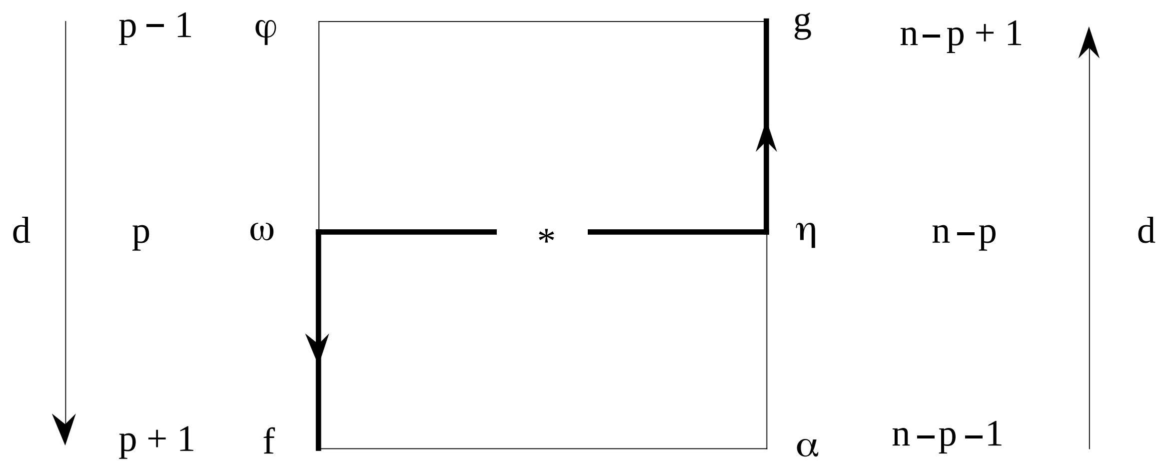


Figure 67. Tonti diagram of the "canonical problem": to find a p-form and an (n − p)-form, Hodge dual one to the other, knowing their exterior derivatives. One has placed in the diagram the "potentials" φ and α that may play a rôle in solving the problem.

To see how source-forms and potentials are introduced, consider (66) first. Let d^s be the "source-field" as defined by

$$d^s = x \to \text{grad}(x \to (4\pi)^{-1}\int_E \rho(y)\,|x - y|^{-1}\,dy))$$

(so $\text{div}\, d^s = \rho$). One then sets $d = d^s + \text{rot}\, u$, which turns (66) into

$$\text{rot}(\varepsilon^{-1}\,\text{rot}\, u) = -\,\text{rot}(\varepsilon^{-1}\, d^s).$$

But one might as well set $e = -\,\text{grad}\,\psi$ (the source-field is 0, in that case) and arrive at

$$-\,\text{div}(\varepsilon\,\text{grad}\,\psi) = \rho.$$

Symmetrically, one may solve (68) with help of the source field $h^s(t)$ given by the Biot and Savart formula:

$$h^s = x \rightarrow \mathrm{rot}(x \rightarrow (4\pi)^{-1} \int_E j(y)\, |x - y|^{-1}\, dy)),$$

by setting $h = h^s(t) + \mathrm{grad}\ \varphi$, hence

$$-\ \mathrm{div}(\mu\ \mathrm{grad}\ \varphi) = \mathrm{div}(\mu\ h^s(t)),$$

or by introducing the vector potential a such that $b = \mathrm{rot}\ a$ (again, zero source-field), hence

$$\mathrm{rot}(\mu^{-1}\ \mathrm{rot}\ a) = j.$$

Exercise 99. Apply the same methods to (67) and (69).

Thus, electrostatics as well as magnetostatics lead to "div-grad like" or "rot-rot like" problems, at leisure. *Electroquasistatics* and *magnetoquasistatics* (the weakly coupled models (66)(67) and (68)(69) respectively) call for the successive solution of such problems. The remarkable symmetries and analogies between them find their explanation in Fig. 67, a paradigm coming from differential geometry. This is our justification for having attempted to present the bases of this discipline in this course.

All this is far from being exhaustive, since we did not even mention mixed formulations, nor problems in bounded domains, nor discretization methods. See [24] for some complements.

5.2.6 The eddy-currents model

We must now get rid of the fiction according to which currents and charges would be given and known beforehand. For, assume a given material configuration (possibly as a function of time: let us call it "trajectory" for shortness) and also a given smooth function $t \rightarrow \{j, \rho\}$, arbitrary (call it "the current"). One may deduce the evolution of the field from this information, with help of the previous models, and thus obtain the forces acting on charged particles. (The force acting on a particle of charge q moving at speed v is $q(e + v \times b)$, cf. e.g. [61].) But then, these electromagnetic forces have no reason to be balanced by forces due to other causes. So neither this trajectory nor this current are the ones that will actually develop, and these can only be found by solving a *coupled* problem. The nature of this problem depends on how charges are linked to matter, and convey these forces to it. Depending on whether one deals with gases, liquids, plasmas, etc., the theory of these coupled problems may assume widely different forms.

There is however one kind of materials for which this theory stays simple (so simple that one often overlooks the fact that it refers to a coupled problem): solid conductors[1]. In such media, there is a simple proportionality relationship between the current density and the electric field:

(70) $$j = \sigma\, e,$$

where σ is the *conductivity* of the metal. This is *Ohm's law*. One may account for it by imagining that charges, so loosely linked with the crystal lattice that they are free to move, and practically inertialess, acquire in the local electric field some limit speed, for which the "friction" force, proportional to the speed, balances the force due to the electric field. (Reality is of course a bit more complex than this, but never mind: (70) agrees very well with observations.)

Again, as above with the first version of (65), p. 119, we have there a relation between two differential forms of different orders, so it only *looks* like a proportionality relationship. Actually, one has

(71) $${}^2j = \sigma * {}^1e,$$

as with the second version of (65), p. 122. This can be shown by direct reasoning. For, consider a metallic cube of resistivity σ^{-1}, of side-length one, built on three orthogonal vectors v_1, v_2, v_3. Let us apply a uniform electric field ${}^1e$ parallel to v_3. The potential difference between the two faces parallel to v_1 and v_2 is $V = {}^1e(v_3) \equiv e \cdot v_3$. A current density ${}^2j$ sets in. The corresponding intensity is $J = {}^2j(v_2, v_3) \equiv \mathrm{vol}(j, v_2, v_3)$. But then $J = V/R$, where the resistance R is σ^{-1}, so ${}^2j(v_1, v_2) = \sigma\, {}^1e(v_3)$, hence (71) by the very definition of the Hodge operator (Def. 17, p. 102).

Remark 24. The same reasoning would apply to (65), p. 122, a reluctance or a capacitance playing the rôle here devoted to the resistance. ◊

One must however modify (71) to account for the presence of *generators*. A generator is a region of space where charges, instead of being free to move (and thus to behave according to the law (71)) are in some way forced to follow definite

[1] What follows also holds for *liquid* conductors (liquid metals, salted water...) as far as velocities stay moderate.

trajectories. This involves some work expenditure (to counter the electromagnetic forces which act on these charges). Generators are thus regions of space where power is injected into the electric system. In most modellings, the current density in generators, j^s (again s for "source"), is thus a data, and one must amend (71) as follows:

$$^2\tilde{j} = \sigma * {}^1e + {}^2\tilde{j}^s, \tag{72}$$

with disjoint supports for j^s and σ, in general (but not always).

After (70), the equation $-\partial_t d + \text{rot } h = j$ takes the form $-\partial_t d + \text{rot } h = \sigma e$, i.e.,

$$\text{rot } h = \sigma e + \partial_t(\varepsilon e),$$

which suggests to compare the orders of magnitude of the two terms on the right, respectively called *conduction current* and (since Maxwell) *displacement current*. For this, let T be a characteristic span of time for the phenomenon under study, or as one says, a "time constant": orders of magnitude are in the ratio $\sigma/T\varepsilon$. This dimensionless number is *very large* in most electrotechnical applications. This is why, save a few exceptions, one adopts the "strong currents" model ($\varepsilon = 0$, and thus $\text{rot } h = j$, with $j = \sigma e + j^s$) when Ohm's law intervenes. One then obtains the eddy-currents model, that is, in vector notation:

$$\left| \begin{array}{l} \partial_t b + \text{rot } e = 0, \quad \text{rot } h = j, \\ \\ b = \mu h, \quad j = \sigma e + j^s, \end{array} \right. \tag{73}$$

and in terms of differential forms:

$$\left| \begin{array}{l} \partial_t {}^2b + d\,{}^1e = 0, \quad d\,{}^1\tilde{h} = {}^2\tilde{j}, \\ \\ {}^2b = \mu * {}^1\tilde{h}, \quad {}^2\tilde{j} = \sigma * {}^1e + {}^2\tilde{j}^s. \end{array} \right. \tag{74}$$

Remark 25. Hence, $\text{div } j = 0$, which is the form of the law of electricity conservation in this model. One also has $\rho = \text{div}(\varepsilon e) = 0$, so the 3-form $^3\rho$, which is the mathematical representation of charge, does not feature in the model any more. One should not from there conclude too fast that charges are physically negligible... Anyway, (73) does not determine a unique electric field (one may add to it the gradient of an electric potential ψ, provided $\text{grad } \psi = 0$ in regions where $\sigma = 0$). For this one should specify the charge ρ outside the support of σ, i.e., outside conductors. Let thus for instance[1] $\rho = 0$ in $E_3 - \text{supp}(\sigma)$. One solves

$$\text{rot } e = -\, \partial_t b, \quad \text{div } e = 0$$

in $E_3 - \text{supp}(\sigma)$. (The necessary boundary values are those of the tangential component of e (the "trace" of ^1e), which is known once j is, by Ohm's law.) One may *then* compute $\rho = \text{div}(\varepsilon e)$: it's a distribution, concentrated on interfaces between regions with different conductivities, and it is not *zero* (far from it . . .). Paradox? No. The parameter ε being small, one may consider the Taylor expansion of the field in terms of ε, in the neighborhood of $\varepsilon = 0$. Model (73) only gives the term of order 0 in this development, a term for which indeed $\rho = 0$. The procedure just suggested yields the *next* term, of order 1 in ε, or at least the part of this term relevant to e, and thus to ρ. This term is in $O(\varepsilon)$, which does not mean it is physically negligible. *Don't* put your hand on a naked conductor. ◊

Let us finally draw the Tonti diagram of this new model (Fig. 68). This consists in taking the "strong currents" diagram (Fig. 66, right), deleting all references to d and ρ, and to add Ohm's law, hence Fig. 68, left. At the cost of a small abuse of representation, one may flatten the diagram by not representing the differentiation with respect to time (Fig. 68, right). The "coupled" character of this problem is graphically obvious, and even more so if one reads the constitutive laws backwards ($h = \mu^{-1} b$, $e = \sigma^{-1} j$). One may consider this diagram as resulting from a merger of two canonical problems, the already met one of magnetostatics, at the bottom, and at the top, one that characterises a new model, the "conduction", or "electrokinetics" model: to find ^2j and ^1e, linked by an affine constitutive law, their respective d being given. The diagram suggests that e [resp. h] can serve as a vector potential for the model downstairs [resp. upstairs], so there are essentially two ways to solve the eddy-currents problem: with respect to the unknown h, or to the unknown e. There are of course many possible variations, since one may represent h and e in terms of other entities ($h = \tau + \text{grad } \omega$, $e = -\, \partial_t a + \text{grad } \psi$, etc.), and thus there is actually a "magnetic" family and an "electric" family of methods, the latter looking for h (hence for j), the former looking for e (hence b).

[1] but not necessarily. There may be space charges, this is not precluded by having replaced ε by 0. Please read on . . .

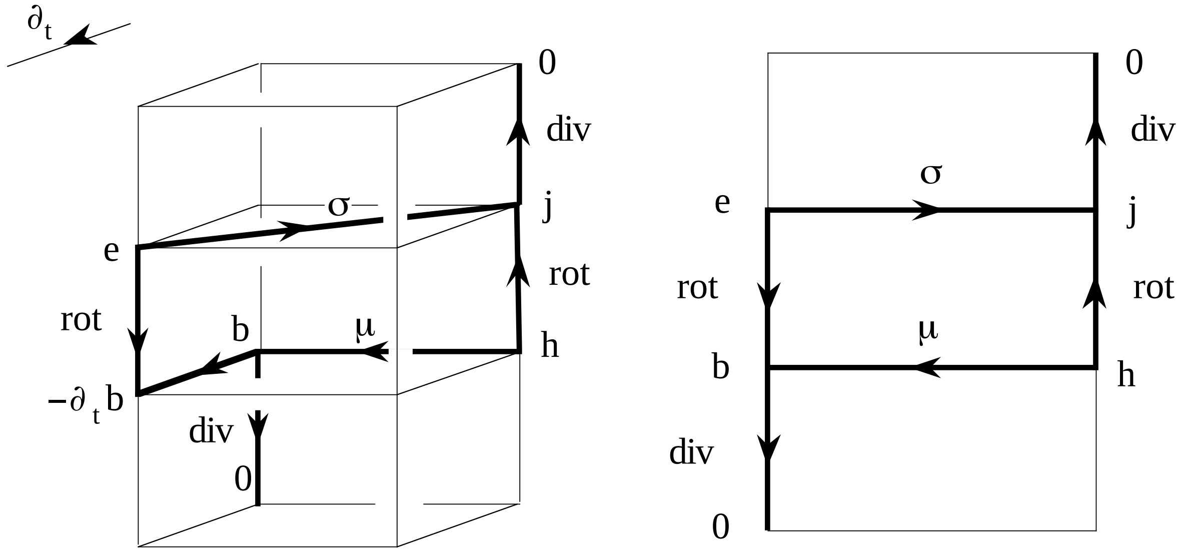


Figure 68. Tonti diagram for the eddy-currents equation

5.3 Epilogue: towards numerical schemes

How should one pursue? By taking the concept of Tonti diagrams in earnest. Each of them describes a particular way of housing the protagonists of the various models (e, b, h, etc.) in the mathematical structure first met, still empty, at Fig. 64. The idea is to discretize *the structure*, once and for all, and not each model on a piecemeal basis. This can be done, because this structure is nothing else than the cohomology of E_3, and mathematicians have developed methods of cohomological analysis which closely resemble what numerical analysts call discretization: one has in particular Whitney's complex [104, 36, 37], a structure associated with the simplicial tessellation of a manifold, analogous to the structure in Fig. 64, where each "vacant room" is a vector space of *finite* dimension. It suffices (if I dare say…— see [17, 19, 23] for details) to "accommodate" each of the "tenants" (h, b, e, etc.) in the "room" which corresponds to its nature to obtain numerical schemes for all these models.

But the time has come to stop.

Conclusion

To model physical space by the mathematical object E_3, and time by a real variable spanning $\mathbb{R}$, as we did all along this course, constitutes an intellectual decision: no "natural laws", no "a priori categories of human understanding" force this choice on us. The wisdom of such a choice can therefore be questioned, as in all modellings: why E_3? why $\mathbb{R}$? Today's scholars, coming after Einstein and Poincaré, have the benefit of hindsight about this, but let us replace ourselves in the situation as it was at the beginning of the 20th century. The laws of electromagnetism were expressed by the system of equations known since Maxwell [70, Chap. 9], and rewritten by his followers under the now classical form:

(75) $$-\partial_t d + \text{rot } h = j,$$

(76) $$\partial_t b + \text{rot } e = 0$$

(plus some relations between b and h, d and e, j and e or h — Ohm's law, Hall effect ...— depending on the medium). It was only natural to see them as describing a *dynamics:* a mathematical rule (here a system of partial differential equations) which governs the evolution of some objects—vector fields—living in E_3.

The modern point of view, acquired throughout a well known historical process, is different. It does not consider the geometric structure (E_3 and $\mathbb{R}$) as antedating equations (75)(76) (which would thus be, in a way, less essential, subordinate). It envisions this structure and these equations as a whole, "the model" (a mathematical one) of a definite compartment of reality (namely, "classical", i.e., non-quantal electromagnetic phenomena). It then wonders about the *necessity* of this model: hasn't it unnecessary structure? Is there not a more economical, hence "simpler" model (which does not mean more easily grasped by the layman, rather the contrary), that could assume the same function?

To bring this point home, let us consider the term rot e in (76). At first sight, it's the curl of a vector field, i.e., assuming a direct orthogonal basis $\{v_1, v_2, v_3\}$ on E_3, the vector field whose components are

$$\text{rot } e = \{\partial_2 e^3 - \partial_3 e^2, \partial_3 e^1 - \partial_1 e^3, \partial_1 e^2 - \partial_2 e^1\}.$$

Let's do this with all the terms of (76), hence three (unwieldy) partial differential equations. Sure, they "say the same thing" as (76), but by marshalling extra structure—the three basis vectors—which can be dispensed with. Indeed, the historical evolution has been, precisely, to do without them, thanks to the invention of vector analysis [31], hence (75)(76).

But then, why stop there? Is there not in (76) some unnecessary structure left? The vector space structure of E_3, for instance, is not really called for: to confer sense on rot e, which stems from e by an obviously local operation, it is enough to have the E_3 structure present locally. A three-dimensional manifold with a metric is all what is needed. Even metric is redundant, as we observed, since (76) rewrites as $\partial_t b + de = 0$, i.e., as a relation between the time-derivative of a 2-form (the object here denoted by b) with the d of a 1-form (the object denoted e), and all this makes sense on a "naked" manifold (even the dimension of the latter appears to be incidental). Same thing with eq. (75). Does that mean the metric is contingent and can be ignored? Not at all, because it played the leading rôle when we had to express the *constitutive laws:*

$$b = \mu\, h, \quad d = \varepsilon\, e$$

(and also when, not considering j as given any more, we introduced Ohm's law). But by dissecting the model in this fastidious way, we realize this: eqs. (75) and (76), the most fundamental, are those which require the less structure. A contrario, the quest for minimal structures, when one models a class of phenomena, helps one to recognize what in a model is fundamental, not to be tampered with, and what is inessential, thus modifiable. This much helps in enriching the model and in broadening its scope. This also helps understand analogies between different models, by revealing their common structure, and exposing their differences.

This analysis, as far as the above equations are concerned, goes even further, as one knows, to the point of unifying time and space into a single structure. It's the whole story of Relativity.

The approach thus suggested can be characterized in one word: geometrization. Indeed, it consists in understanding the equations of physics as necessary relations between some *geometric objects*, elements of sets endowed with a peculiar kind of structure (that some mathematicians have tried to characterize, cf. [93]), those which are called "spaces": vector spaces, fibered spaces, etc. All the manifold denizens we have met are in this sense geometric objects. To geometrize thus consists in identifying these objects, as well as the minimal structures necessary to account for their relationships, and to specify these relationships, all of this not in succession, but in a single sweep.

A practicing programmer cannot fail to see the analogy between geometrization thus conceived and "object oriented programming" [71], a modern development in the art of computer programming that one could characterize in terms almost identical to those used in the previous sentence. The concomitance of these two trends is perhaps no accident. As far as I am concerned anyway, their connection is strong: the long-term aim being the numerical solution of Maxwell's equations, which implies the writing, according to the rules of the craft, of specialized software, the "objects" in this programming cannot be without relation with the geometric objects whose behavior is ruled by these equations. Geometrizing the equations of electrodynamics is a prerequisite to the rational construction of computing software systems able to solve them.

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