2 Rewriting the Maxwell equations

Deconstruction calls for reconstruction: We now resettle the Maxwell system in the environment just described, paying attention to what makes use of the metric structure and what doesn't. In the process, differential forms will displace vector fields as basic entities.

2.1 Integration: Circulation, flux, etc.

Differential forms are simply, among mathematical objects, those meant to be integrated. So let us revisit Integration.

In standard integration theory [46, 85, 113], one has a set X equipped with a measure dx. Then, to a pair $\{A, f\}$, where A is a part of X and f a function, integration associates a number, denoted $\int_A f(x)\, dx$ (or simply $\int_A f$, if there is no doubt on the underlying measure), with additivity and continuity with respect to both arguments, A and f. In what follows, where only nodding acquaintance with this theory is assumed, we operate a slight change of viewpoint: Instead of leaving the measure dx in background of a stage on which the two objects of interest would be A and f, we consider the whole integrand $f(x)\, dx$ as a single object (later to be given its proper name, "differential form"), and A as some piecewise smooth manifold of A_3. This liberates integration from its dependence on the metric structure: The integral becomes a map of type *MANIFOLD* × *DIFFERENTIAL_FORM* → *REAL* (by linearity, *CHAIN* will eventually replace *MANIFOLD* there), which we shall see is the right approach as far as Electromagnetics is concerned. The transition will be in two steps, one in which the Euclidean structure is used, one in which we get rid of it.

The dot product of E_n induces measures on its submanifolds: By definition, the Euclidean measure of the parallelotope built on p vectors $\{v_1, \ldots, v_p\}$ anchored at x, i.e., of the set $\{x + \sum_i \lambda^i v_i : 0 \le \lambda^i \le 1,\ i = 1, \ldots, p\}$, is the square-root of the so-called Gram determinant of the v_i's, whose entries are the dot products $v_i \cdot v_j$, for all i, j from 1 to p. One can build from this, by the methods of classical measure theory [46], the p-dimensional measures, i.e., the lineal, areal, volumic, etc., measures of a (smooth, bounded) curve, surface, volume, etc. For $p = 0$ not to stand out as an exception there, we attribute to an isolated point the measure 1. (This is the so-called *counting measure*, for which the measure of a set of points is the number of its elements.)

We shall consider, corresponding to the four dimensions $p = 0, \ldots, 3$ of manifolds in E_3, four kinds of integrals which are constantly encountered in Physics. Such integrals will be defined on cells first, then extended by linearity to chains, which covers the case of piecewise smooth manifolds.

First, $p = 0$, a point, x say. The integral of a smooth function φ is then[16] $\varphi(x)$. If the point is inner oriented, i.e., if it bears a sign $\epsilon(x) = \pm 1$, the integral is by convention $\epsilon(x)\varphi(x)$.

Next ($p = 1$), let c be a 1-cell. At point $x = c(t)$, define the *unit tangent vector* $\tau(x)$ as the vector at x equal to $\partial_t c(t)/|\partial_t c(t)|$, which inner-orients c. Given a smooth vector field u, the dot product $\tau \cdot u$ defines a real-valued function on the image of c. We call *circulation* of u, along c thus oriented, the integral $\int_c \tau \cdot u$ of this function with respect to the Euclidean measure of lengths.

Remark. Integrals are limits of Riemann sums. In the present case, such a sum can be obtained as suggested by Fig. 11, left: Chop the curve into a finite family $\mathcal{S}$ of adjacent curve segments s, pick a point x_s in each of them, and let $\vec{s}$ be the vector, oriented along c, that

[16] This is also its integral over the set $\{x\}$, with respect to the counting measure, in the sense of Integration theory. The integral over a *finite* set $\{x_1, \ldots, x_k\}$, in this sense, would be $\sum_i \varphi(x_i)$. Notice the difference between this and what we are busy defining right now, the integral on a 0-chain, which will turn out to be a weighted sum of the $\varphi(x_i)$s.

joins the extremities of s. The Riemann sum associated with $\mathcal{S}$ is then $\sum_{s \in \mathcal{S}} \vec{s} \cdot u(x_s)$, and converges towards $\int_c \tau \cdot u$ when $\mathcal{S}$ is properly refined. ◊

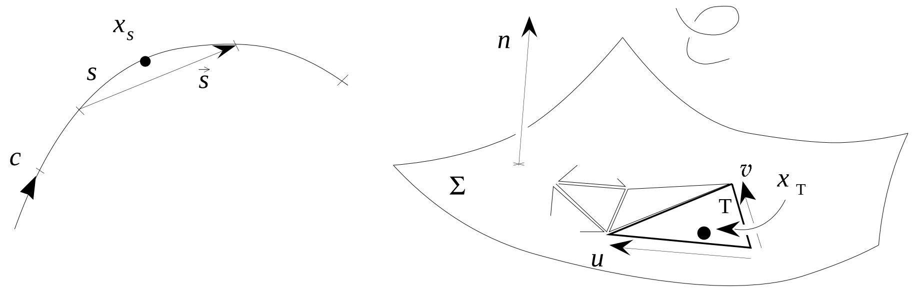

Figure 11. Forming the terms of Riemann sums. Left: generic "curve segment" s, with associated sampling point x_s and vector $\vec{s}$. Right: generic triangular small patch T, with sampling point x_T. Observe how, with the ambient orientation indicated by the icon, the vectorial area of T happens to be $1/2\, u \times v$.

Further up ($p = 2$), let Σ be a 2-cell, to which a crossing direction has been assigned, and choose the parameterization $\{s, t\} \to \Sigma(s, t)$ in such a way that vectors $\eta(s, t) = \partial_s \Sigma(s, t) \times \partial_t \Sigma(s, t)$ point in this direction. Then set $n(x) = \eta(s, t)/|\eta(s, t)|$, at point $x = \Sigma(s, t)$, to obtain the outer-orienting *unit normal field*. Given a smooth vector field u, we define the *flux* through Σ, thus outer oriented, as the integral $\int_\Sigma n \cdot u$ of the real-valued function $n \cdot u$ with respect to, this time, the Euclidean measure of *areas*. (No ambiguity on this point, since the status of Σ as a surface has been made clear.)

Remark. For Riemann sums, dissect Σ into a family $\mathcal{T}$ of small triangular patches T, whose vectorial areas are $\vec{\mathrm{T}}$, pick a point x_T in each of them, and consider $\sum_{\mathrm{T} \in \mathcal{T}} \vec{\mathrm{T}} \cdot u(x_T)$. ◊

Last, for $p = 3$, and a 3-cell V with outer orientation $+$, the integral of a function f is the standard $\int_V f$, integral of f over the image of V with respect to the Lebesgue measure. With outer orientation $-$, the integral is $-\int_V f$. The inner orientation of V is irrelevant here. This is consistent with the frequent physical interpretation of $\int_V f$ as the quantity, in V, of something (mass, charge, ...) present with density f in V. *Outer* orientation, on the other hand, helps fix bookkeeping conventions when f is a rate of variation, like for instance, heat production or absorption.

Now, let us extend the notion to chains based on oriented cells. In dimension 0, where an oriented point is a point-cum-sign pair $\{x, \epsilon\}$, a 0-chain m is a finite collection $\{\{x_i, \epsilon_i\} : i = 1, \ldots, k\}$ of such pairs, each with a weight μ^i. The integral $\int_m \varphi$ is then defined as $\sum_i \mu^i \epsilon_i \varphi(x_i)$.[17] In dimension 1, the circulation along the 1-chain $c = \sum_i \mu^i c_i$ is $\int_c \tau \cdot u = \sum_i \mu^i \int_{c_i} \tau \cdot u$. The flux $\int_\Sigma n \cdot u$ through the *twisted* (beware!) chain $\Sigma = \sum_i \mu^i \Sigma_i$ is defined as $\sum_i \mu^i \int_{\Sigma_i} n \cdot u$. As for dimension 3, a twisted chain manifold V is a finite collection[18] $\{\{V_i, \epsilon_i\} : i = 1, \ldots, k\}$ of 3D blobs-with-sign, with weights μ^i, and $\int_V f$ is, by definition, $\sum_i \mu^i \epsilon_i \int_{V_i} f$.

Note that we have implicitly defined integrals on piecewise smooth manifolds there, since these can be considered as cell-based chains with "orientation matching weights" (1 if the cell's orientation and the manifold's match, -1 if they don't).

Thus the most common ways[19] to integrate things in three-space lead to the definition of

[17] One might think, there, that orientation-signs and weights do double duty. Indeed, a convention could be made that all points are positively oriented, and this would dispose of the ϵ_is. We won't do this, for the sake of uniformity with respect to dimension.

[18] Again, one might outer-orient such elementary volumes by giving them all a $+$ sign, reducing the redundancy, and we refrain to do so for the same reason.

[19] Others reduce to one of these in some way. For instance, when using Cartesian coordinates x–y–z, $\int_c f(x, y, z)\, dx$ is simply the circulation along c, in the sense we have defined above, of the field of x-directed basis vectors magnified by the scalar factor f.

integrals over *inner* oriented manifolds or chains in cases $p = 0$ and 1 and *outer* oriented ones[20] in cases $p = 2$ and 3. An unpleasant asymmetry. But since we work in *oriented* Euclidean space, where one may, as we have seen, derive outer from inner orientation, and the other way round, this restores the balance, hence finally *eight* kinds of integrals, depending on the dimension and on the nature (internal or external) of the orientation of the underlying chain.

Thus we have obtained a series of maps of type $CHAIN \to REAL$, but in a pretty awkward way, one must admit. Could there be an underlying unifying concept that would make it all simpler?

2.2 Differential forms, and their physical relevance

Indeed, these maps belong to a category of objects that can be defined without recourse to the Euclidean structure, and have thus a purely affine nature:

Definition. *A straight [resp. twisted]* differential form *of degree p, or p-*form, *is a real-valued map ω over the space of straight [resp. twisted] p-chains, linear with respect to chain addition, and continuous in the sense of the above-defined topology of chains* (end of §1.5).

Differential forms, thus envisioned, are dual objects with respect to chains, which prompts us to mobilize the corresponding machinery of functional analysis [113]: Call $\mathcal{F}^p$ [resp. $\tilde{\mathcal{F}}^p$] the space of straight [resp. twisted] p-forms, as equipped with its so-called "strong" topology.[21] Then $\mathcal{C}_p$ and $\mathcal{F}^p$ [resp. $\tilde{\mathcal{C}}_p$ and $\tilde{\mathcal{F}}^p$] are *in duality* via the bilinear bicontinuous map $\{c, \omega\} \to \int_c \omega$, of type p-$CHAIN \times p$-$FORM \to REAL$. A common notation for such duality products being $\langle c\,;\omega\rangle$, we shall use that as a convenient alternative[22] to $\int_c \omega$. A duality product should be *non-degenerate*, i.e., $\langle c'\,;\omega\rangle = 0\ \forall\, c'$ implies $\omega = 0$, and $\langle c\,;\omega'\rangle = 0\ \forall\,\omega'$ forces $c = 0$. The former property holds true by definition, and the latter is satisfied because, if $c \neq 0$, one can construct an ad hoc smooth vector field or function with nonzero integral, hence a nonzero form ω such that $\langle c\,;\omega\rangle \neq 0$.

The above eight kinds of integrals, therefore, are instances of differential forms, which we shall denote (in their order of appearance) by ${}^0\varphi$, ${}^1u$ (circulation of u), ${}^2\tilde{u}$ (flux of u), ${}^3\tilde{\varphi}$, and ${}^0\tilde{\varphi}$, ${}^1\tilde{u}$, ${}^2u$, ${}^3\varphi$. This is of course ad hoc notation, to be abandoned as soon as the transition from fields to forms is completed. Note the use of the pre-superscript p, accompanied or not by the tilde as the case may be, as an *operator,* that transforms functions or vector fields into differential forms (twisted ones, if the tilde is there). This operator, being relative to a specific Euclidean structure is as a rule metric- and orientation-dependent. (We'll use $^{\mathbf{p}}$ and $\tilde{\ }$ versus ^{p} and $\tilde{\ }$ to distinguish[23] the $\{\bullet, \mathbf{Or}\}$ and the $\{\cdot, Or\}$ structure.) For instance, the 2 in ${}^2u$ means that, given the straight 2-chain S, one uses both the inner orientation of each of its components and the ambient orientation to define a crossing direction, then the metric in order to build a normal vector field n in this direction, over each component of the chain. Then, $\langle S\,;{}^2u\rangle = \int_S n \cdot u$ defines ${}^2u$, a straight 2-form indeed. (Notice that $\langle S\,;{}^2u\rangle$ does *not*

[20] A tradition initiated by Firestone [40] distinguishes between so-called "across" and "through" physical quantities [19, 20], expressible by circulations and fluxes, respectively. As we shall see, this classification is not totally satisfying.

[21] Differential forms converge, in this topology, if their integrals converge uniformly on bounded sets of chains. (A *bounded* set B is one that is *absorbed* by any neighborhood V of 0, i.e., such that $\lambda B \subset V$ for some $\lambda > 0$.) We won't have to invoke such technical notions in the sequel.

[22] In line with the convention of Note 4, we shall denote by ω the map $c \to \langle c\,;\omega\rangle$, and feel free to write $\omega = c \to \langle c\,;\omega\rangle$. Of course, the symmetric construct $c = \omega \to \langle c\,;\omega\rangle$ is just as valid. (Maps of the latter kind, from forms to reals, were called *currents* by De Rham [83]. See [82], p. 220, for the physical justification of the term.) There are, a priori, much more currents than chains, and one should not be fooled by the expression "in duality" into thinking that the dual of $\mathcal{F}^p$, i.e., the so-called *bidual* of $\mathcal{C}_p$, is $\mathcal{C}_p$ itself.

[23] This play on styles, needless to say, is just a temporary ad hoc device, not to be used beyond the present Section. Later we shall revert to the received "musical" notation, which assumes a single, definite metric structure in background, and cares little about ambiguity: $\sharp u$ denotes the vector proxy of form u, and $\flat$U is the form represented by the vector field U.

depend on the ambient orientation.)

Remark. In the foregoing example, it would be improper to describe $\langle S\,;{}^2u\rangle$ as the flux of u "through" S, since the components of S, a straight chain, didn't come equipped with crossing directions. These were derived from the ambient orientation, part of the Euclidean structure, instead of being given as an attribute of S's components. To acknowledge this difference, we shall refer to $\int_S n\cdot u$ as the flux "embraced by" S. This is not mere fussiness, as will be apparent when we discuss magnetic flux. ◇

One may wonder, at this point, whether substituting the single concept of differential form for those of point-value, circulation, flux, etc., has gained us any real generality, besides the obvious advantage of conceptual uniformity. Let us examine this point carefully, because it's an essential part of the deconstruction of Euclidean space we have undertaken.

On the one hand, the condition that differential forms should be continuous with respect to deformations of the underlying manifolds doesn't leave room, in dimension 3, for other kinds of differential forms than the above eight. First, it eliminates many obvious linear functionals from consideration. (For instance, γ being an outer-oriented curve, the *intersection number*, defined as the number of times γ crosses S, counted algebraically (i.e., with sign $-$ if orientations do not match), provides a linear map $S\to S\wedge\gamma$, which is not considered as a bona fide differential form. Indeed, it lacks continuity.) Second, it allows one, by using the Riesz representation theorem, to build vector fields or functions that reduce the given form to one of the eight types: For instance, given a 1-form ω, there is[24] a vector field Ω such that $\langle c\,;\omega\rangle=\int_c \tau\cdot\Omega$, which is our first example of what will later be referred to as a "proxy" field: A scalar or vector field that stands for a differential form. For other degrees, forms in 3D are representable by vector fields ($p=1$ and 2) or by functions ($p=0$ and 3).

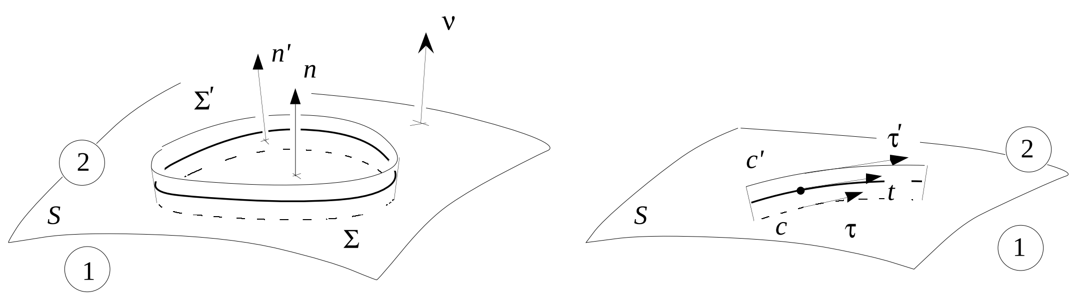

Figure 12. The interface S, equipped with the unit normal field ν, separates two regions where the vector field u is supposed to be smooth, except for a possible discontinuity across S. Suppose Σ or c, initially below S, is moved up a little, thus passing into region 2. Under such conditions, the flux of u through Σ (left) and circulation of u along c (right) can yet be *stable*, i.e., vary continuously with deformations of c and Σ, provided u has some partial regularity: As is well known, and easily proven thanks to the Stokes theorem, *normal* continuity (zero jump $[\nu\cdot u]$ of the normal component across the interface) ensures continuity of the flux $\int_\Sigma n\cdot u$ with respect to Σ (left), while *tangential* continuity of u (zero jump $[u_S]$ of the tangential component across the interface) is required for continuity of the circulation $\int_c \tau\cdot u$ (right) with respect to c. Forms ${}^0\varphi$ and ${}^0\tilde\varphi$ require a continuous φ, but piecewise continuity of the proxy function φ is enough for ${}^3\varphi$ and ${}^3\tilde\varphi$.

However, the continuity condition requires less regularity from the proxy fields than the smoothness we have assumed up to now. Not to the point of allowing them to be only piecewise smooth: What is required lies in between, and should be clear from Fig. 12, which revisits a well known topic from the present viewpoint. As one sees, the contrived "transmission conditions", about tangential continuity of this or normal continuity of that, are implied by

[24] The proof is involved. From a vector field v, build a 1-chain $\sum_i \mu_i s_i$, akin to the graphic representation of v by arrows, i.e., s_i is an oriented segment that approximates v in a region of volume μ_i. Apply ω to this chain, go to the limit. The real-valued linear map thus generated is then shown, thanks to the continuity of ω, to be continuous with respect to the L^2 norm on vector fields. Hence a Riesz vector field Ω, which turns out to be a proxy for ω.

the very definition of forms as continuous maps.

Last, the generalization is genuine in spatial dimensions higher than 3: A two-form in 4-space, for instance, has no vector proxy, as a rule.

So, although differential forms do extend a little the scope of integration, this is but a marginal improvement, at least in the 3D context. The real point lies elsewhere, and will now be argued: Which differential form is built from a given (scalar or vector) field depends on the Euclidean structure, *but the physical entity one purports to model via this field does not,* as a rule. Therefore, the entity of physical significance is the form, conceived as an affine object, and not the field. Two examples will suffice to settle this point.

Consider an electric charge, Q coulombs strong, which is made to move along an oriented smooth curve c, in the direction indicated by the tangent vector field τ. We mean a *test* charge, with Q small enough to leave the ambient electromagnetic field field $\{\mathrm{E}, \mathrm{B}\}$ undisturbed, and a *virtual* motion, which allows us to consider the field as frozen at its instant value. The work involved in this motion is Q times the quantity $\int_c \tau \cdot \mathrm{E}$, called the *electromotive force* (e.m.f.) *along* c, and expressed in volts (i.e., joules per coulomb). No unit of length is invoked in this description.

Then why is E expressed in volts *per meter* (or whatever unit one adopts)? Only because a vector v such that $|v| = 1$ is one meter long, which makes $\mathrm{E} \cdot v$, and the integral $\int_c \tau \cdot \mathrm{E}$ as well, a definite amount of *volts*, indeed. This physical data, of course, only depends on the field and the curve, not on the metric structure. Yet, change the dot product, from $\cdot$ to $\bullet$ (recall that $u \bullet v = Lu \cdot Lv$), which entails a change in the measure of lengths (hence a rescaling of the unitary vector, now $\boldsymbol{\tau}$ instead of τ), and the circulation of E is now[25] $\boldsymbol{\int}_c \boldsymbol{\tau} \bullet \mathrm{E}$ $= \int_c \tau \cdot L^a L\mathrm{E}$, a different (and physically meaningless) number. On the other hand, there is a field $\mathbf{E}$ such that $\boldsymbol{\int}_c \boldsymbol{\tau} \bullet \mathbf{E} = \int_c \tau \cdot \mathrm{E}$, namely $\mathbf{E} = (L^a L)^{-1}\mathrm{E}$. Conclusion: *Which vector field encodes the physical data* (here, e.m.f.'s along all curves) *depends on the chosen metric, although the data themselves do not.* This metric-dependence of E is the reason to call it a vector *proxy:* It merely *stands* for the real thing, which is the mapping $c \to$ <e.m.f. along c>, i.e., a differential form of degree 1, which we shall from now on denote by e.

Thus, summoning all the equivalent notations introduced so far,

$$(7) \qquad e = {}^1\mathrm{E} = {}^1\mathbf{E} = c \to \langle c\,; e\rangle, \text{ where } \langle c\,; e\rangle \equiv \int_c e = \int_c \tau \cdot \mathrm{E} = \boldsymbol{\int}_c \boldsymbol{\tau} \bullet \mathbf{E}.$$

This (straight) 1-form is the right mathematical object by which to represent the electric field, for it tells all about it: Electromotive forces along curves are, one may argue [98], all that can be observed as regards the electric field.[26] To the point that one can get rid of all the vector-field-and-metric scaffolding, and introduce e directly, by reasoning as follows: The *1-CHAIN* $\to$ *REAL* map we call e.m.f. depends linearly and continuously, *as can experimentally be established,* on the chain over which it is measured. But this is the very definition of a 1-form. Hence e is the minimal, necessary and sufficient, mathematical description of the (empirical) electric field.

Remark. The chain/form duality, thus, takes on a neat physical meaning: While the form

[25] The integral on the left, as hinted by the boldface summation sign, is with respect to the "bold" measure of lengths. The easiest way to verify this equality (and others like it to come) is to work on the above Riemann sums $\sum_s v_s \bullet \mathrm{E}(x_s)$ of the "bold" circulation of E: One has, for each term (omitting the subscript), $v \bullet \mathrm{E} = Lv \cdot L\mathrm{E} = v \cdot L^a L\mathrm{E}$, hence the result.

[26] Pointwise values cannot directly be measured, which is why they are somewhat downplayed here, but of course they do make sense, at points of regularity of the field: Taking for c the segment $[x, x + v]$, where v is a vector at x that one lets go to 0, generates at the limit a linear map $v \to \omega_x(v)$. This map, an element of the dual of T_x, is called a *covector* at x. A 1-form, therefore, can be conceived as a (smooth enough) field of covectors. In coordinates, covectors such as $v \to v^i$, where v^i is the i-th component of v at point x, form a basis for covectors at x. (They are what is usually denoted by $\mathrm{d}x^i$; but d^i makes better notation, that should be used instead, on a par with ∂_i for basis vectors.)

e models the field, chains are abstractions of the *probes*, more or less complex, that one may place here and there in order to measure it. ♢

The electric field is not the whole electromagnetic field: it only accounts for forces (and their virtual work) exerted on non-moving electric charges. We shall deal later with the other part, the magnetic field, and recognize it as a 2-form. But right now, an example involving a twisted 2-form will be more instructive.

So consider current density, classically a vector field J, whose purpose is to account for the quantity of electric charge, $\int_\Sigma n \cdot \mathrm{J}$, that traverses, per unit of time, a surface Σ, in the direction of the unit normal field n that outer-orients it. (Note again this quantity is in ampères, whereas the dimension of the proxy field J is A/m^2.) This map, $\Sigma \to <$ intensity through $\Sigma >$, a twisted 2-form (namely, ${}^2\tilde{\mathrm{J}}$), is what we can measure and know about the electric current, and the metric plays no role there. Yet, change $\cdot$ to $\bullet$, which affects the measure of areas, and the flux of J becomes[27] $\int_\Sigma \mathbf{n} \bullet \mathrm{J} = |\det(L)| \int_\Sigma n \cdot \mathrm{J}$. The "bold" vector proxy, therefore, should be $\mathbf{J} = |\det(L)|^{-1}\mathrm{J}$, and then ${}^2\tilde{\mathbf{J}} = {}^2\tilde{\mathrm{J}}$. Again, different vector proxies, but the same twisted 2-form, which thus appears as the invariant and physically meaningful object. It will be denoted by j.

This notational scheme will be systematized: Below, we shall call e, h, d, b, j, a, etc., the differential forms that the traditional vector fields $\mathrm{E}, \mathrm{H}, \mathrm{D}, \mathrm{B}, \mathrm{J}, \mathrm{A}$, etc., represent.

2.3 The Stokes theorem

The Stokes "theorem" hardly deserves such a status in the present approach, for it reduces to a mere

Definition. *The* exterior derivative $\mathrm{d}\omega$ *of the* $(p-1)$*-form* ω *is the* p*-form* $c \to \int_{\partial c} \omega$.

In plain words: To integrate $\mathrm{d}\omega$ over the p-chain c, integrate ω over its boundary ∂c. (This applies to straight or twisted chains and forms equally. Note that d is well defined, thanks to the continuity of ∂ from $\mathcal{C}_{p-1}$ to $\mathcal{C}_p$.) In symbols: $\int_{\partial c} \omega = \int_c \mathrm{d}\omega$, which is the common form of the theorem, or equivalently,

$$(8) \qquad \langle \partial c\,; \omega \rangle = \langle c\,; \mathrm{d}\omega \rangle \;\; \forall\, c \in \mathcal{C}_p \text{ and } \omega \in \mathcal{F}^{p-1}$$

(put tildes over $\mathcal{C}$ and $\mathcal{F}$ for twisted chains and forms), which better reveals what is going on: d is the *dual* of ∂ [113]. As a corollary of (2),

$$(9) \qquad \mathrm{d} \circ \mathrm{d} = 0.$$

A form ω is *closed* if $\mathrm{d}\omega = 0$, and *exact* if $\omega = \mathrm{d}\alpha$ for some form α. (Synonyms, perhaps more mnemonic, are *cocycle* and *coboundary*. The integral of a cocycle over a boundary, or of a coboundary over a cycle, vanishes.)

Remark. In A_n, all closed forms are exact: this is known as the *Poincaré Lemma* (see, e.g., [88], p. 140). But closed forms need not be exact in general manifolds: this is the dual aspect of the "not all cycles bound" issue we discussed earlier. Studying forms, consequently, is another way, dual to homology, to investigate topology. The corresponding theory is called *cohomology* [55, 64]. ♢

In three dimensions, the d is the affine version of the classical differential operators, grad, rot, and div, which belong to the Euclidean structure. Let's review this.

First, the gradient: Given a smooth function φ, we define $\operatorname{grad}\varphi$ as the vector field such that, for any 1-cell c with unit tangent field τ,

$$(10) \qquad \int_c \tau \cdot (\operatorname{grad}\varphi) = \int_{\partial c} \varphi,$$

[27] Same trick, with Riemann sums of the form $\sum_{\mathrm{T}} \vec{\mathbf{T}} \bullet \mathrm{J}(x_{\mathrm{T}})$. After (4) and (6), $\vec{\mathbf{T}} \bullet \mathrm{J} = L\vec{\mathbf{T}} \cdot L\mathrm{J} = L^a L\vec{\mathbf{T}} \cdot \mathrm{J} = |\det(L)|\, \vec{\mathrm{T}} \cdot \mathrm{J}$. Hence $\int_\Sigma \mathbf{n} \bullet \mathrm{J} = |\det(L)| \int_\Sigma n \cdot \mathrm{J}$.

the latter quantity being of course $\varphi(c(1)) - \varphi(c(0))$. By linearity, this extends to any 1-chain. One recognizes (8) there. The relation between gradient and d, therefore, is ${}^1(\text{grad}\,\varphi) = \text{d}\,{}^0\varphi \equiv \text{d}\varphi$, the third term being what is called the *differential* of φ. (The zero superscript can be dropped, because there is only one way to turn a function into a 1-form, whatever the metric.) The vector field $\text{grad}\,\varphi$ is a proxy for the 1-form $\text{d}\varphi$.

Thus defined, $\text{grad}\,\varphi$ depends on the metric. If the dot product is changed from "$\cdot$" to "$\bullet$", the vector field whose circulation equals the right-hand side of (10) is a different proxy, $\mathbf{grad}\varphi$, which relates to the first one, as one will see using (4), by $\text{grad}\,\varphi = L^a L\,\mathbf{grad}\varphi$.

Up in degree, rot and div are defined in similar fashion. So, all in all,

$$(11)\qquad {}^1(\text{grad}\,\varphi) = \text{d}\,{}^0\varphi, \quad {}^2(\text{rot}\,u) = \text{d}\,{}^1u, \quad {}^3(\text{div}\,v) = \text{d}\,{}^2v.$$

Be well aware that all forms here are *straight*. Yet their proxies may behave in confusing ways with respect to orientation, as we shall presently see.

About curl, (11) says that the curl of a smooth field u, denoted $\text{rot}\,u$, is the vector field such that, for any inner oriented surface S,

$$(12)\qquad \int_S n \cdot \text{rot}\,u = \int_{\partial S} \tau \cdot u.$$

Here, τ corresponds to the induced orientation of ∂S, and n is obtained by the Ampère rule. So the ambient orientation is explicitly used. Changing it reverses the sign of $\text{rot}\,u$. The curl behaves like the cross product, in this respect. If, moreover, the dot product is changed, the bold curl and the meager one relate as follows:

Proposition 1. *With* $u \bullet v = Lu \cdot Lv$ and $\mathbf{Or} = \text{sign}(\det(L))Or$, *one has*

$$(13)\qquad \mathbf{rot}u = (\det(L))^{-1}\,\text{rot}(L^a L u).$$

Proof. Because of the hybrid character of (12), with integration over an outer oriented surface on the left, and over an inner oriented line on the right, the computation is error prone, so let's be careful. On the one hand (Note 25), $\boldsymbol{\int}_{\partial S} \boldsymbol{\tau} \bullet u = \int_{\partial S} \tau \cdot L^a L u = \int_S n \cdot \text{rot}(L^a L u)$. On the other hand (Note 27), setting $\mathbf{J} = \mathbf{rot}u$, we know that $\boldsymbol{\int}_S \mathbf{n} \bullet \mathbf{J} = |\det(L)| \int_S n \cdot \mathbf{J}$, hence ... but wait! In Note 27, we had both normals n and $\mathbf{n}$ on the same side of the surface, but here (see Fig. 4, left), they may point to opposite directions if $\mathbf{Or} \neq Or$. The correct formula is thus $\boldsymbol{\int}_S \mathbf{n} \bullet \mathbf{rot}u = \det(L) \int_S n \cdot \mathbf{rot}u \equiv \int_S n \cdot \text{rot}(L^a L u)$, hence (13). $\diamondsuit$

As for the divergence, (11) defines $\text{div}\,v$ as the function such that, for any volume V with outgoing normal n on ∂V,

$$(14)\qquad \int_V \text{div}\,v = \int_{\partial V} n \cdot v.$$

No vagaries due to orientation this time, because both integrals represent the same kind of form (twisted). Moreover, $\mathbf{div}v = \text{div}\,v$, because the same factor $|\det(L)|$ pops up on both sides of $\boldsymbol{\int}_V \mathbf{div}v = \boldsymbol{\int}_{\partial V} \mathbf{n} \bullet v$. (The integrals, with boldface summation sign, are with respect to the "bold" measure. For the one on the left, it's the 3D measure $|\mathbf{vol}|$, and $\mathbf{vol} = \det(L)\,\text{vol}$ after (4).)

Remark. The invariance of div is consistent with its physical interpretation: if v is the vector field of a fluid mass, its divergence is the rate of change of the volume occupied by this mass, and though volumes depend on the metric, volume *ratios* do not, again after (4). $\diamondsuit$

For reference, Fig. 13 gathers and displays the previous results. This is a commutative diagram, from which transformation formulas about the differential operators can be read off.[28]

As an illustration of how such a diagram can be used, let us prove something the reader has probably anticipated: the invariance of Faraday's law with respect to a change of metric and

[28] It should be clear that L could depend on the spatial position x, so this diagram is more general than what we contracted for. It gives the correspondence between differential operators relative to different Riemannian structures on the same 3D manifold.

orientation. Let two vector fields E and B be such that $\partial_t \mathrm{B} + \mathrm{rot}\, \mathrm{E} = 0$, and set $\mathbf{B} = \mathrm{B}/\det(L)$, $\mathbf{E} = (L^a L)^{-1}\mathrm{E}$, which represent the same differential forms (call them b and e) in the $\{\bullet\ , \mathbf{Or}\}$ framework, as B and E in the $\{\cdot, Or\}$ one. Then $\partial_t \mathbf{B} + \mathbf{rot} \mathbf{E} = 0$. We now turn to the significance of the single physical law underlying these two relations.

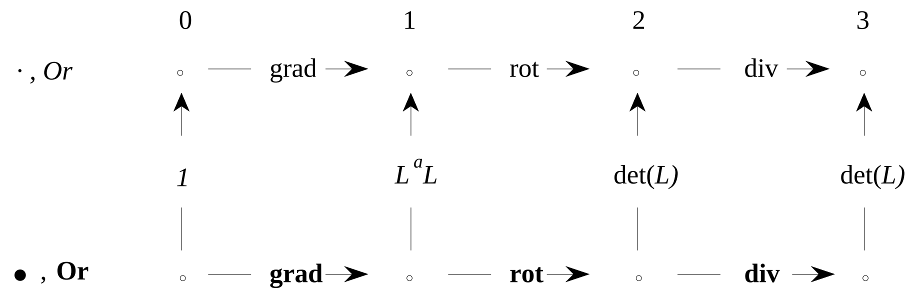

Figure 13. Vertical arrows show how to relate vector or scalar proxies that correspond to the *same* straight form, of degree 0 to 3, in two different Euclidean structures. For *twisted* forms, use the same diagram, but with $|\det(L)|$ substituted for $\det(L)$.

2.4 The magnetic field, as a 2-form

Electromagnetic forces on moving charges, i.e., currents, will now motivate the introduction of the magnetic field. Consider a current loop, I ampères strong, which is made to move—virtual move, again—so as to span a surface S (Fig. 14). The virtual work involved is then I times $\int_S n \cdot \mathrm{B}$ (the "cut flux rule"), as explained in the caption. Experience establishes the linearity and continuity of the factor $\int_S n \cdot \mathrm{B}$, called the *induction flux,* as a function of S. Hence a 2-form, again the minimal description of the (empirical) magnetic field, which we denote by b and call *magnetic induction*.

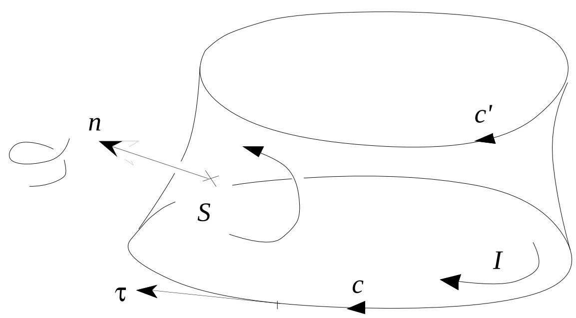

Figure 14. Conventions for the virtual work due to B on a current loop, in a virtual move from position c to position c'. The normal n is the one associated, by Ampère's rule, with the inner orientation of S, a surface such that $\partial S = c' - c$. The virtual work of the $\mathrm{J} \times \mathrm{B}$ force, with $\mathrm{J} = I\tau$, is then I times the flux $\int_S n \cdot \mathrm{B}$.

In spite of the presence of n in the formula, b is not a twisted but a straight 2-form, as it should, since ambient orientation cannot influence the sign of the virtual work in any way. Indeed, what is relevant is the direction of the current along the loop, which inner-orients c, and the inner orientation of S is the one that matches the orientation of the chain $c' - c$ ("final position minus initial position" in the virtual move). The intervention of a normal field, therefore, appears as the result of the will to represent b with help of a vector, the traditional B such that $b = {}^2\mathrm{B}$. No surprise, then, if this vector proxy "changes sign" with ambient orientation! Actually, it cannot do its job, that is, represent b, without an ambient orientation.

If one insists on a proxy that can act to this effect in autonomy, this object has to carry on its back, so to speak, an orientation of ambient space, i.e., it must be a field of *axial* vectors. Even so, the dependence on metric is still there, so the benefit of using such objects is tiny. Yet, why not, if one is aware that (polar) vector field and axial vector field are just mathematical *tools*,[29] which may be more or less appropriate, depending on the background structures, to represent a given physical entity. In this respect, it may be useful to have a synoptic guide (Fig. 15).

Nature of the proxy	*for a*	straight	*or*	twisted	*DF of degree*
	function	polar		axial	*0*
	vector field	polar		axial	*1*
	vector field	axial		polar	*2*
	function	axial		polar	*3*

Figure 15. Nature of the proxies in *non*-oriented 3D space with dot product.

We can fully appreciate, now, the difference between j and b, between current flow and magnetic flux. Current density, the twisted 2-form j, is meant to be integrated over surfaces Σ with crossing direction: its proxy J is independent of the ambient orientation. Magnetic induction, the straight 2-form b, is meant to be integrated over surfaces S with inner orientation: its proxy B changes sign if ambient orientation is changed. Current, clearly, flows through a surface, so intensity is one of these "through variables" of Note 20. But thinking of the magnetic flux as going *through* S is misleading. Hence the expression used here, flux *embraced by* a surface.[30]

2.5 Faraday and Ampère

We are now ready to address Faraday's famous experiment: variations of the flux embraced by a conducting loop create an electromotive force. A mathematical statement meant to express this law with maximal economy will therefore establish a link between the integral of b over a fixed surface S and the integral of e over its boundary ∂S. Here it is: one has

$$\partial_t \int_S b + \int_{\partial S} e = 0 \quad \forall\, S \in \mathcal{C}_2, \tag{15}$$

i.e., for any straight 2-chain, and in particular, any inner oriented surface S. Numbers in (15) have dimension: webers for the first integral, and volts (i.e., Wb/s) for the second one. *Inner* orientation of ∂S (and hence, of S itself) makes much physical sense: it corresponds to selecting one of the two ways a galvanometer can be inserted in the circuit of which ∂S is an idealization. Applying the Stokes theorem—or should we say, the definition of d—, we find the local, infinitesimal version of the global, integral law (15), as this:

$$\partial_t b + \mathrm{d}e = 0, \tag{16}$$

the metric- and orientation-free version of $\partial_t \mathrm{B} + \mathrm{rot}\,\mathrm{E} = 0$.

As for Ampère's theorem, the expression is similar, except that twisted forms are now involved:

$$-\partial_t \int_\Sigma d + \int_{\partial\Sigma} h = \int_\Sigma j \quad \forall\, \Sigma \in \tilde{\mathcal{C}}_2, \tag{17}$$

i.e., for any twisted 2-chain, and in particular, any outer oriented surface Σ. Its infinitesimal form is

$$-\partial_t d + \mathrm{d}h = j, \tag{18}$$

[29] Thus axiality or polarity is by no means a property of the physical objects. But the way physicists write about it doesn't help clarify this. For instance [5, p. 61]: "In physics, the electric field E is called a vector, while the magnetic field B is called an axial vector, because E changes sign under parity transformation, while B does not." Or else [84]: "It is well known that under the space inversion transformation, $P : (x, y, z) \to (-x, -y, -z)$, the electric field transforms as a polar vector, while the magnetic field transforms as an axial vector, $P : \{E \to -E, B \to B\}$." This may foster confusion, as the blunders in [6] demonstrate.

[30] This exposes the relative inadequacy of the "across vs. through" concept, notions which roughly match those of straight 1-form and twisted 2-form [20]. Actually, between lines and surfaces on the one hand, and inner or outer orientation on the other hand, it's *four* different "vectorial" entities one may have to deal with, and the vocabulary may not be rich enough to cope.

again the purely affine version of $-\partial_t \mathrm{D} + \mathrm{rot}\,\mathrm{H} = \mathrm{J}$. Since j is a twisted form, d must be one, and h as well,[31] which suggests that its proxy H will not behave like E under a change of the background Euclidean structure. Indeed, one has $\mathbf{H} = |\det(L)|\,(L^a L)^{-1}\mathrm{H}$ in the now familiar notation. In non-oriented space with metric, the proxy H would be an axial vector field, on a par with B. Vector proxies D and J would be polar, like E.

At this stage, we may announce the strategy that will lead to a discretized form of (15) and (17): Instead of requesting their validity for *all* chains S or Σ, we shall be content with enforcing them for a *finite* family of chains, those generated by the 2-cells of an appropriate finite element mesh, hence a system of differential equations. But first, we must deal with the constitutive laws linking b and d to h and e.

2.6 The Hodge operator

For it seems a serious difficulty exists there: Since b and h, or d and e, are objects of different types, simple proportionality relations between them, such as $b = \mu h$ and $d = \epsilon e$, won't make sense if μ and ϵ are mere scalar factors. To save this way of writing, as it is of course desirable, we must properly redefine μ and ϵ as *operators*, of type 1-*FORM* $\rightarrow$ 2-*FORM*, one of the forms twisted, the other one straight.

So let's try to see what it takes to go from e to d. It consists in being able to determine $\int_\Sigma d$ over any given outer oriented surface Σ, knowing two things: the form e on the one hand, i.e., the value $\int_c e$ for any inner oriented curve c, and the relation $\mathrm{D} = \epsilon\mathrm{E}$ between the proxies, on the other hand. (Note that ϵ can depend on position. We shall assume it's piecewise smooth.) How can that be done?

The answer is almost obvious if Σ is a small[32] piece of plane. Build, then, a small segment c meeting Σ orthogonally at a point x where ϵ is smooth. Associate with c the vector $\vec{c}$ of same length that points along the crossing direction through Σ, and let this vector also serve to inner-orient c. Let $\vec{\Sigma}$ stand for the vectorial area of Σ, and take note that $\vec{\Sigma}/\mathrm{area}(\Sigma) = \vec{c}/\mathrm{length}(c)$. Now dot-multiply this equality by D on the left, ϵE on the right. The result is

$$\int_\Sigma d = \epsilon(x)\,\frac{\mathrm{area}(\Sigma)}{\mathrm{length}(c)}\int_c e, \tag{19}$$

which does answer the question.

How to lift the restrictive hypothesis that Σ be small? Riemann sums, again, are the key. Divide Σ into small patches T, as above (Fig. 11, right), equip each of them with a small orthogonal segment c_{T}, meeting it at x_{T}, and such that $\vec{c}_{\mathrm{T}} = \vec{\mathrm{T}}$. Next, define $\int_\Sigma d$ as the limit of the Riemann sums[33] $\sum_{\mathrm{T}} \epsilon(x_{\mathrm{T}}) \int_{c_{\mathrm{T}}} e$. One may then define the *operator* ϵ, with reuse of the symbol, as the map $e \rightarrow d$ just constructed, from $\mathcal{F}^1$ to $\tilde{\mathcal{F}}^2$. A similar definition holds for μ, of type $\tilde{\mathcal{F}}^1 \rightarrow \mathcal{F}^2$, and for the operators ϵ^{-1} and μ^{-1} going in the other direction. (Later, we shall substitute ν for μ^{-1}.)

Remark. We leave aside the anisotropic case, with a (symmetric) tensor ϵ^{ij} instead of the scalar ϵ. In short: Among the variant "bold" metrics, there is one in which ϵ^{ij} reduces to unity. Then apply what precedes, with "orthogonality", "length", and "area" understood in the sense of this modified metric. (The latter may depend on position, however, so this stands a bit outside our present framework. See details in [17].) ◊

[31] A *magnetomotive force* (m.m.f.), therefore, is a real value (in ampères) attached to an *outer* oriented line γ, namely the integral $\int_\gamma h$.

[32] To make up for the lack of rigor which this word betrays, one should treat c and Σ as "p-vectors" ($p = 1$ and 2 respectively), which are the infinitesimal avatars of p-chains. See [13] for this approach.

[33] Singular points of ϵ, at which $\epsilon(x_{\mathrm{T}})$ is not well defined, can always be avoided in such a process, unless Σ coincides with a surface of singularities, like a material interface. But then, move Σ a little, and extend d to such surfaces by continuity.

Remark. When the scalar ϵ or μ equals 1, what has been defined is the classical *Hodge operator* of differential geometry [21, 88], usually denoted by $*$, which maps p-forms, straight or twisted, to $(n-p)$-forms of the other kind, with $** = \pm 1$, depending on n and p. In dimension $n = 3$, it's a simple exercise to show that the above construction then reduces to $*^1u = {}^2\tilde{u}$, which prompts the following definition: $*^0\varphi = {}^3\tilde{\varphi} * {}^1u = {}^2\tilde{u}, *^2u = {}^1\tilde{u}, {}^3\varphi = *^0\tilde{\varphi}$. Note that $** = 1$ for all p. ◇

Note the essential role of the metric structure in this definition: areas, lengths, and orthogonality depend on it. So we now distinguish, in the Maxwell equations, the two metric-free main ones,

(16) $\partial_t b + \mathrm{d}e = 0,$ (18) $-\partial_t d + \mathrm{d}h = j,$

and the metric-dependent constitutive laws

(20) $b = \mu h,$ (21) $d = \epsilon e,$

where μ and ϵ are operators of the kind just described. To the extent that no metric element is present in these equations, except for the operators μ and ϵ, from which one can show the metric can be inferred [17], one may even adopt the radical point of view [22] that μ and ϵ *encode* the metric information.

2.7 The Maxwell equations: Discussion

With initial conditions on e and h at time $t = 0$, and conditions about the "energy" of the fields to which we soon return, the above system makes a well-posed problem. Yet a few loose ends must be tied.

First, recall that j is supposed to be known. But reintroducing Ohm's law at this stage would be no problem: replace j in (18) by $j^s + \sigma e$, where j^s is a given twisted 2-form (the source current), and σ a third Hodge-like operator on the model of ϵ and μ.

2.7.1 Boundary conditions, transmission conditions

Second, boundary conditions, if any. Leaving aside artificial "absorbing" boundary conditions [69], not addressed here, there are essentially four basic ones, as follows.

Let's begin with "electric walls", i.e., boundaries of perfect conductors, inside which $\mathrm{E} = 0$, hence the standard $n \times \mathrm{E} = 0$ on the boundary. In terms of the form e, it means that $\int_c e = 0$ for all curves c contained in such a surface. This motivates the following definition, stated in dimension n for generality: S being an $(n-1)$-manifold, call $\mathcal{C}_p(S)$ the space of p-chains whose components are all supported in S; then,

Definition. *The* trace $\mathrm{t}_S\omega$ *of the p-form ω is the restriction of ω to $\mathcal{C}_p(S)$,*

i.e., the map $c \to \int_c \omega$ restricted to p-chains based on components which are contained in S. Of course this requires $p < n$. So the boundary condition at an electric wall S^e is $\mathrm{t}_{S^e} e = 0$, which we shall rather write, for the sake of clarity, as "$\mathrm{t}e = 0$ on S^e." Symmetrically, the condition $\mathrm{t}h = 0$ on S^h corresponds to a magnetic wall S^h.

The Stokes theorem shows that d and t commute: $\mathrm{dt}\omega = \mathrm{td}\omega$ for any ω of degree no higher than $n-2$. Therefore $\mathrm{t}e = 0$ implies $\mathrm{td}e = 0$, hence $\partial_t(\mathrm{t}b) = 0$ by (16), that is, $\mathrm{t}b = 0$ if one starts from null fields at time 0. For the physical interpretation of this, observe that $\mathrm{t}b = 0$ on S^b means $\int_S b = 0$ for any surface piece S belonging to S^b, or else, in terms of the vector proxy, $\int_S n \cdot \mathrm{B} = 0$, which implies $n \cdot B = 0$ on all S^b: a "no-flux" surface, called a "magnetic barrier" by some. We just proved anew, in the present language, that electric walls are impervious to magnetic flux. One will see in the same manner that $\mathrm{t}j = 0$ corresponds to "insulating boundaries" ($n \cdot \mathrm{J} = 0$) and $\mathrm{t}d = 0$ to "dielectric barriers" ($n \cdot \mathrm{D} = 0$). If j is given with $\mathrm{t}j = 0$ at the boundary of the domain of interest (which is most often the case) then $\mathrm{t}h = 0$ on S^h implies $\mathrm{t}d = 0$ there. (In eddy current problems, where d is neglected, but j is only partially given, $\mathrm{t}h = 0$ on S^h implies $\mathrm{t}j = 0$, i.e., no current through the surface.)

Conditions $\mathrm{t}b = 0$ or $\mathrm{t}d = 0$ being thus weaker than $\mathrm{t}e = 0$ or $\mathrm{t}h = 0$, one may well want to enforce them independently. Many combinations are thereby possible. As a rule (but there are exceptions in non-trivial topologies, see [15]), well-posedness in a domain D bounded by surface S obtains if S can be subdivided as $S = S^e \cup S^h \cup S^{eh}$, with $\mathrm{t}e = 0$ on S^e (electric wall), $\mathrm{t}h = 0$ on S^h (magnetic wall), and *both* conditions $\mathrm{td}e = 0$ and $\mathrm{td}h = 0$ on S^{eh}, which corresponds to $\mathrm{t}b = 0$ and $\mathrm{t}d = 0$ taken together (boundary which is both a magnetic and a dielectric barrier, or, in the case of eddy-current problems, an insulating interface).

Remark. It may come as a surprise that the standard Dirichlet/Neumann opposition is not relevant here. It's because a Neumann condition is just a Dirichlet condition composed with the Hodge and the trace operators [18]: Take for instance the standard $n \times \mu^{-1} \operatorname{rot} \mathrm{E} = 0$, which holds on magnetic walls in the E formulation. This is (up to an integration with respect to time) the proxy form of $\mathrm{t}h = 0$, i.e., of the *Dirichlet* condition $n \times \mathrm{H} = 0$. In short, Neumann conditions on e are Dirichlet conditions on h, and the other way round. They only become relevant when one eliminates either e or h in order to formulate the problem in terms of the other field exclusively, thus breaking the symmetry inherent in Maxwell's equations (which we have no intention to do unless forced to!). $\diamondsuit$

Third point, what about the apparently missing equations, $\operatorname{div} \mathrm{D} = \mathrm{Q}$ and $\operatorname{div} \mathrm{B} = 0$ in their classical form (Q is the density of electric charge)? These are not equations, actually, but relations implied by the Maxwell equations, or at best, constraints that initial conditions should satisfy, as we now show.

Let's first define q, the electric charge, of which the above Q is the proxy scalar field. Since j accounts for its flow, charge conservation implies $d_t \int_V q + \int_{\partial V} j = 0$ for all volumes V, an integral law the infinitesimal form of which is

$$\partial_t q + \mathrm{d}j = 0. \tag{22}$$

Suppose both q and j were null before time $t = 0$. Later, then, $q(t) = -\int_0^t (\mathrm{d}j)(s)\, ds$. Note that q, like $\mathrm{d}j$, is a *twisted* 3-form, as should be the case for something that accounts for the density of a substance. (Twisted forms are often called "densities", by the way [21].)

Now, if one accepts the physical premise that no electromagnetic field exists until its sources (charges and their flow, i.e., q and j) depart from zero, all fields are null at $t = 0$, and in particular, after (18), $d(t) = d(0) + \int_0^t [(\mathrm{d}h)(s) - j(s)]\, ds$, hence, by using (9), $\mathrm{d}d(t) = -\int_0^t (\mathrm{d}j)(s)\, ds \equiv q(t)$, at all times, hence the derived relation $\mathrm{d}d = q$. As for b, the same computation shows that $\mathrm{d}b = 0$.

So-called "transmission conditions", classically $[n \times \mathrm{E}] = 0$, $[n \cdot \mathrm{B}] = 0$, etc., at material interfaces, can be evoked at this juncture, for these too are not equations, in the sense of additional constraints that the unknowns e, b, etc., would have to satisfy. They *are* satisfied from the outset, being a consequence of the very definition of differential forms (cf. Fig. 12).

2.7.2 Wedge product, energy

Fourth point, the notion of energy. The physical significance of such integrals as $\int \mathrm{B} \cdot \mathrm{H}$ or $\int \mathrm{J} \cdot \mathrm{E}$ is well known, and it's easy to show, using the relations displayed on Fig. 13, that both are metric-independent. So they should be expressible in non-metric terms. This is so, thanks to the notion of *wedge product*, an operation which creates a $(p+q)$-form $\omega \wedge \eta$ (straight when both factors are of the same kind, twisted otherwise) out of a p-form ω and a q-form η. We shall only describe this in detail in the case of a 2-form b and a 1-form h, respectively straight and twisted.

The result, a twisted 3-form $b \wedge h$, is known if integrals $\int_V b \wedge h$ are known for all volumes V. In quite the same way as with the Hodge map, the thing is easy when V is a small parallelepiped, as shown in Fig. 16. Observe that, if $b = {}^2\mathrm{B}$ and $h = {}^1\tilde{\mathrm{H}}$, then $\int_V b \wedge h = \mathrm{B} \cdot \mathrm{H}\ \mathrm{vol}(V)$, if one follows the recipe of Fig. 16, confirming the soundness of the latter. The extension to finite-size volumes is made by constructing Riemann sums, as usual.

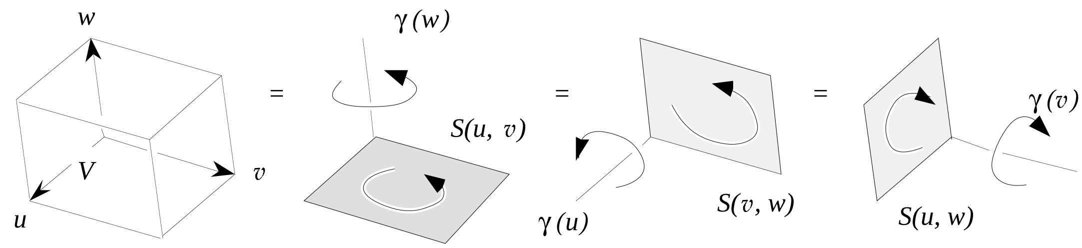

Figure 16. There are three ways, as shown, to see volume V, built on u, v, w, as the extrusion of a surface S along a line segment γ. A natural definition of the integral of $b \wedge h$ is then $\int_V b \wedge h = (\int_{S(u,v)} b)(\int_{\gamma(w)} h) + (\int_{S(v,w)} b)(\int_{\gamma(u)} h) + (\int_{S(u,w)} b)(\int_{\gamma(v)} h)$. Note the simultaneous inner and outer orientations of S and γ, which should match (if the outer orientation of V is $+$, as assumed), but are otherwise arbitrary.

Remark. Starting from the equality $\int b \wedge h' = \int \mathrm{B} \cdot \mathrm{H}'$, setting $b = \mu h$ yields $\int \mu\, h \wedge h' = \int \mu \mathrm{H} \cdot \mathrm{H}' = \int \mu \mathrm{H}' \cdot \mathrm{H} = \int \mu\, h' \wedge h$, a *symmetry* property of the Hodge operator to which we didn't pay attention so far. Note also that $\int \mu\, h \wedge h = \int \mu |\mathrm{H}|^2 > 0$, unless $h = 0$. Integrals such as $\int \mu h \wedge h'$, or $\int \nu b \wedge b'$, etc., can thus be understood as *scalar products* on spaces of forms, which can thereby be turned (after due completion) into Hilbert spaces. The corresponding norms, i.e., the square roots of $\int \mu h \wedge h$, of $\int \nu\, b \wedge b$, and other similar constructs on e or d, will be denoted by $|h|_\mu$, $|b|_\nu$, etc. ◊

Other possible wedge products are ${}^0\varphi \wedge \omega = {}^0(\varphi\omega)$ (whatever the degree of ω), ${}^1u \wedge {}^1v = {}^2(u \times v)$, ${}^2u \wedge {}^1v = {}^3(u \cdot v)$. (If none or both factors are straight forms, the product is straight.) It's an instructive exercise to work out the exterior derivative of such products, using the Stokes theorem, and to look for the equivalents of the standard integration by parts formulas, such as

$$\int_\Omega (\mathrm{H} \cdot \operatorname{rot} \mathrm{E} - \mathrm{E} \cdot \operatorname{rot} \mathrm{H}) = \int_{\partial\Omega} n \cdot (\mathrm{E} \times \mathrm{H}), \quad \int_\Omega (\mathrm{D} \cdot \operatorname{grad} \Psi + \Psi \operatorname{div} \mathrm{D}) = \int_{\partial\Omega} \Psi\, n \cdot \mathrm{D}.$$

They are, respectively,

$$(23) \qquad \int_\Omega (\mathrm{d}e \wedge h - e \wedge \mathrm{d}h) = \int_{\partial\Omega} e \wedge h, \qquad (24) \qquad \int_\Omega (\mathrm{d}\psi \wedge d + \psi\, \mathrm{d}d) = \int_{\partial\Omega} \psi\, d.$$

Now, let us consider a physically admissible field, that is, a quartet of forms b, h, e, d, which may or may not satisfy Maxwell's equations when taken together, but are each of the right degree and kind in this respect.

Definition. *The following quantities:*

$$(25) \qquad {}^1\!/_2 \int \mu^{-1} b \wedge b, \ \ {}^1\!/_2 \int \mu\, h \wedge h, \ \ {}^1\!/_2 \int \epsilon\, e \wedge e, \ \ {}^1\!/_2 \int \epsilon^{-1} d \wedge d,$$

are called, respectively, magnetic energy, magnetic coenergy, electric energy, *and* electric coenergy *of the field. The integral $\int j \wedge e$ is the* power *released by the field.*

The latter definition, easily derived from the expression of the Lorentz force, is a statement about field–matter energy exchanges from which the use of the word "energy" could rigorously be justified, although we shall not attempt that here (cf. [10]). The definition entails the following relations:

$${}^1\!/_2 \int \mu^{-1} b \wedge b + {}^1\!/_2 \int \mu\, h \wedge h \geq \int b \wedge h, \qquad {}^1\!/_2 \int \epsilon^{-1} d \wedge d + {}^1\!/_2 \int \epsilon\, e \wedge e \geq \int d \wedge e,$$

with equality if and only if $b = \mu h$ and $d = \epsilon e$. One may use this as a way to set up the constitutive laws.

Remark. The well-posedness evoked earlier holds if one restricts the search to fields with finite energy. Otherwise, of course, nonzero solutions to (16)(18)(20)(21) with $j = 0$ do exist (such as, for instance, plane waves). ◊

The integrals in (25) concern the whole space, or at least, the whole region of existence of the field. One may wish to integrate on some domain Ω only, and to account for the energy balance. This is again an easy exercise:

Proposition 2 (Poynting's theorem). *If the field* $\{b, h, e, d\}$ *does satisfy the Maxwell equations* (16)(18)(20)(21), *one has*

$$d_t \, [{}^1\!/_2 \int_\Omega \mu^{-1} b \wedge b + {}^1\!/_2 \int_\Omega \epsilon \, e \wedge e] + \int_{\partial\Omega} e \wedge h = - \int_\Omega j \wedge e$$

for any fixed domain Ω.

Proof. "Wedge multiply" (16) and (18), from the right, by e and $-h$, add, use (23) and Stokes. ◊

As one sees, all equalities and inequalities on which a variational approach to Maxwell's theory can be based do have their counterparts with differential forms. We shall not follow this thread any further, since what comes ahead is not essentially based on variational methods. Let's rather close this Section with a quick review of various differential forms in Maxwell's theory and how they relate.

2.7.3 "Maxwell house"

To the field quartet and the source pair $\{q, j\}$, one may add the *electric potential* ψ and the *vector potential* a, a straight 0-form and 1-form respectively, such that $b = \mathrm{d}a$ and $e = -\partial_t a + \mathrm{d}\psi$. Also, the *magnetic potential* φ (twisted 0-form) and the twisted 1-form τ such that $h = \tau + \mathrm{d}\varphi$, whose proxy is the T of Carpenter's "T–Ω" method [23]. None of them is as fundamental as those in (16)(18), but each can be a useful auxiliary at times. The *magnetic current* k and *magnetic charge* m can be added to the list for the sake of symmetry (Fig. 17), although they don't seem to represent any real thing [44].

For easier reference, Fig. 17 displays all these entities as an organized whole, each one "lodged" according to its degree and nature as a differential form. Since primitives in time may have to be considered, we can group the differential forms of electromagnetism in four similar categories, shown as vertical pillars on the figure. Each pillar symbolizes the structure made by spaces of forms of all degrees, linked together by the d operator. Straight forms are on the left and twisted forms on the right. Differentiation or integration with respect to time links each pair of pillars (the front one and the rear one) forming the sides of the structure. Horizontal beams symbolize constitutive laws.

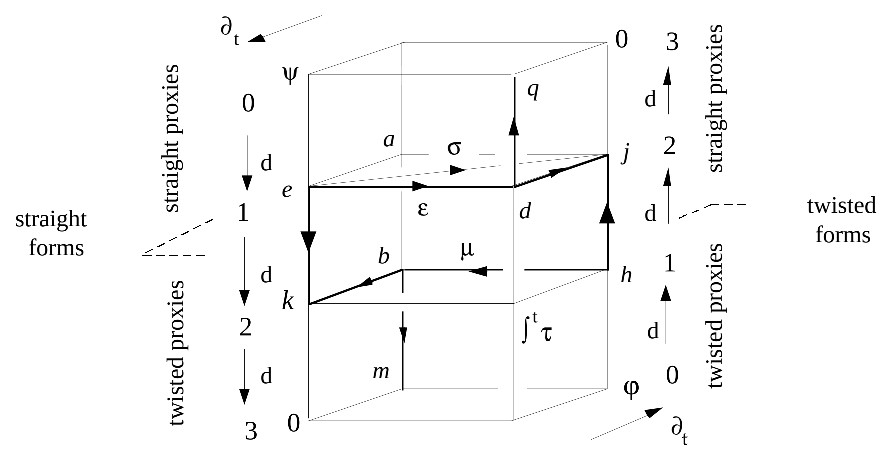

Figure 17. Structures underlying the Maxwell system of equations. For more emphasis on their symmetry, Faraday's law is here taken to be $\partial_t b + \mathrm{d}e = -k$, with $k = 0$. (The straight 2-form k would stand for the flow of magnetic charge, if such a thing existed. Then, one would have $\mathrm{d}b = m$, where the straight 3-form m represents magnetic charge, linked with its current by the conservation law $\partial_t m + \mathrm{d}k = 0$.)

As one can see, each object has its own room in the building: b, a 2-form, at level 2 of the "straight" side, the 1-form a such that $b = \mathrm{d}a$ just above it, etc. Occasional asymmetries (e.g., the necessity to time-integrate τ before lodging it, the bizarre layout of Ohm's law ...) point to weaknesses which are less those of the diagram than those of the received nomenclature or (more ominously) to some hitch about Ohm's law. Relations mentioned up to now can be directly read off from the diagram, up to sporadic sign inversions. An equation such as $\partial_t b + \mathrm{d}e = -k$, for instance, is obtained by gathering at the location of k the contributions of all adjacent niches, including k's, in the direction of the arrows. Note how the rules of Fig. 15, about which scalar- or vector-proxies must be twisted or straight, are in force.

But the most important thing is probably the neat separation, in the diagram, between "vertical" relations, of purely affine nature, and "horizontal" ones, which depend on metric. If this was not drawing too much on the metaphor, one could say that a change of metric, as encoded in ϵ and μ (due for instance to a change in their local values, because of a temperature modification or whatever) would shake the building horizontally but leave the vertical panels unscathed.

This suggests a method for *discretizing* the Maxwell equations: The orderly structure of Fig. 16 should be preserved, if at all possible, in numerical simulations. Hence in particular the search for finite elements *which fit differential forms*, which will be among our concerns in the sequel.