APPENDIX C

A Cheaper Way to Complementarity

For reference, let us state again the basic problem of Chapter 6: In domain D, the surface S of which is partitioned as $S^h \cup S^b$, *find among pairs* {h, b} *which satisfy*

(1) $\operatorname{rot} h = 0$ in D, (3) $\operatorname{div} b = 0$ in D,

(2) $n \times h = 0$ on S^h (4) $n \cdot b = 0$ on S^b,

(6) $\int_c \tau \cdot h = I$, (7) $\int_C n \cdot b = F$,

a minimizer for the error in constitutive law

$$E(b, h) = \int_D \mu^{-1} |b - \mu h|^2.$$

(Cf. Fig. 6.1 for the definition of the "link" c and the "cut" C.) As we saw, there is a minimizer whatever I and F, with h and b weakly solenoidal and irrotational, respectively, and a unique value of the ratio R = I/F (the reluctance) for which the constitutive law

(5) $b = \mu h$ in D,

is satisfied, i.e., E(b, h) = 0.

Complementarity consists in simultaneously solving for $h = \operatorname{grad} \varphi$ by nodal elements for φ and for $b = \operatorname{rot} a$ by edge elements for a, hence (assuming one uses the same mesh m for both, which is not mandatory), a rot-conformal h_m satisfying (1–2)(6) and a div-conformal b_m satisfying (3–4)(7), "m-weakly" solenoidal and irrotational respectively, but not linked by (5). This gave us bilateral bounds for R and (by computing $E(b_m, h_m)$) upper bounds for both approximation errors $\int_D \mu |h - h_m|^2$ and $\int_D \mu^{-1} |b - b_m|^2$.

Alas, this nice approach has a serious drawback: As we saw in Section 6.3, the part of the computation that yields a is much more expensive[1] than the determination of φ.

Therefore, it would be interesting to be able to save on this effort, in the case of the a-method, by making good use of the information one has, once h_m has been determined. In quite fuzzy terms for the moment—but this will become more and more precise—can the solution in terms of φ somewhat be corrected to yield a truly solenoidal (not only m-solenoidal) approximation of b?

C.1 LOCAL CORRECTIONS

So let's suppose we have computed h_m, satisfying Eqs. (1), (2), and (6), and such that

$$(8) \qquad \int_D \mu\, h_m \cdot \operatorname{grad} \lambda^n = 0 \quad \forall n \in \mathcal{N}_h,$$

where λ^n (preferred in this Appendix to w_n, for notational uniformity) is the barycentric function of node n, and $\mathcal{N}_h = \mathcal{N} - \mathcal{N}(S^h)$ the set of nodes not included in S^h. We want some $b \in W^2{}_m$, divergence-free, and—in order to make use of the knowledge of the solution we have already acquired—close to h_m.

What we have done in Chapter 6 seems to give an obvious solution: Look for a minimizer of the error in constitutive law,

$$(9) \qquad b = \operatorname{arginf}\{\int_D \mu^{-1} \, |b' - \mu h_m|^2 : \, b' \in \mathbb{B}^F{}_m\},$$

where $\mathbb{B}^F{}_m = \{b \in W^2{}_m(D) : \operatorname{div} b = 0, \; n \cdot b = 0 \text{ on } S^b, \; \int_C n \cdot b = F\}$. Vector fields in this space are linear combinations of face elements,

$$(10) \qquad b = \sum_{f \in \mathcal{F}_b} \mathbf{b}_f w_f,$$

where $\mathcal{F}_b$ abbreviates $\mathcal{F}(S - S^b)$, the set of faces not included in S^b. (This way, (10) implicitly takes the no-flux condition (4) into account.) But the remaining nonzero face-DoFs $\mathbf{b}_f$ are not independent for $b \in \mathbb{B}^F{}_m$. They are constrained by linear relations:

[1]Just for practice, let's do it again, this time with the ratio T/N equal to 6. Thanks to the Euler–Poincaré formula, one has $E \sim 7N$ and $F \sim 12\,N$. The average number of faces that contain a given edge is $3F/E$, so each edge has $9F/E$ "neighbors", if one defines as neighbors two edges that belong to a common tetrahedron. The number of off-diagonal entries of the edge-element matrix is thus $9F$, that is, $108\,N$, against $14\,N$ for the matrix created by the φ-method.

$$\sum_{f \in \mathcal{F}(T)} \mathbf{D}_{Tf}\, \mathbf{b}_f = 0, \qquad \sum_{f \in \mathcal{F}(C)} \mathbf{D}_{C\,f}\, \mathbf{b}_f = F,$$

where $\mathcal{F}(C)$ is the set of faces that pave the cut C, and $\mathbf{D}_{C\,f} = \pm 1$ according to relative orientation. As $\mathbb{B}^F_m = \text{rot}\, A^F_m$, (9) is equivalent to finding a minimizer

$$a_m \in \text{arginf}\{\textstyle\int_D \mu^{-1} \,|\, \text{rot}\, a - \mu h_m \,|^2 : \; a \in A^F_m\}, \tag{11}$$

and there is no difference between doing that and directly solving for a by edge elements. The $b = \text{rot}\, a$ thus obtained, which is the approximation b_m of Chapter 6, is indeed the closest to h_m in energy. But no use is made of the knowledge of h_m this way.

Remark C.1. Problem (11) is the same as problem (6.21): Since $\int_D \mu^{-1} \,|\, \text{rot}\, a - \mu h_m \,|^2 = \int_D \mu^{-1} \,|\, \text{rot}\, a \,|^2 - 2\int_D h_m \cdot \text{rot}\, a + \int_D \mu \,|\, h_m \,|^2$ and (by Lemma 6.1) $\int_D h_m \cdot \text{rot}\, a = 0$, the two functionals in (11) and (6.21) differ by a constant, and minimization is performed on the same subspace. ◊

We now introduce the *localization* heuristics. Let's have a partition of unity over D, i.e., a family of piecewise-smooth functions χ^i, indexed over some finite set $\mathcal{I}$, and satisfying $\sum_{i \in \mathcal{I}} \chi^i = 1$. Any b can be written as a sum $b = \sum_{i \in \mathcal{I}} \chi^i b \equiv \sum_{i \in \mathcal{I}} b^i$. For one that suits our needs (divergence free, and close to h_m), each b^i should satisfy $\text{div}\, b^i = \text{div}(\chi^i b) = b \cdot \text{grad}\, \chi^i$, and should be close to $\chi^i \mu h_m$. Not knowing b, we replace $b \cdot \text{grad}\, \chi^i$ by the next best thing, which is $\mu h_m \cdot \text{grad}\, \chi^i$, and try to achieve $\text{div}\, b^i = \mu h_m \cdot \text{grad}\, \chi^i$ as best we can, while looking for b^i in W^2. Since then $\text{div}\, b^i$ belongs to W^3, and is thus mesh-wise constant, the best we can do is to request

$$\textstyle\int_T \text{div}\, b^i = \int_T \mu\, h_m \cdot \text{grad}\, \chi^i \quad \forall\, T \in \mathcal{T}, \tag{12}$$

for all indices i. Besides that, we also want b^i as close as possible, in energy, to $\chi^i \mu h_m$, hence

$$b^i = \text{arginf}\{\textstyle\int_D \mu^{-1} \,|\, b' - \mu \chi^i h_m \,|^2 : \; b' \in \mathbb{B}^i(h_m)\}, \tag{13}$$

where $\mathbb{B}^i(h_m)$ is an ad-hoc and provisional notation for the set of b^is in $\mathbb{B}^F_m$ that satisfy the constraints (12). Intuitively (and we'll soon confirm this), computing b^i is a *local* procedure. (Notice that $\text{div}\, b = \text{div}(\sum_i b^i) = 0$, by summing (12) over i.) This is the principle.

For its formal application, now, let us call $\mathcal{T}^i$ the set of tetrahedra whose intersection with $\text{supp}(\chi^i)$ has a nonzero measure, D^i their set union, and $\mathcal{F}^i$ the collection of all faces of such tetrahedra *except* those contained in the boundary ∂D^i (but not in S^h, cf. Fig. C.1). In (13),

$\mu\,\chi^i h_m = 0$ outside D^i, so we may search b^i among the restricted set of fields that vanish outside D^i, which means (since by normal continuity of b^i, its normal component on ∂D^i must be null) those of the form $\sum_{f\in\mathcal{F}^i} \mathbf{b}_f w_f$. Let us therefore introduce the notation

$$W^2{}_m(D^i) = \{b \in W^2{}_m(D) :\ b = \textstyle\sum_{f\in\mathcal{F}^i} \mathbf{b}_f w_f\},$$

and redefine $\mathbb{B}^i(h_m)$ as

$$\mathbb{B}^i(h_m) = \{b \in \mathbb{B}^F{}_m \cap W^2{}_m(D^i) :\ \textstyle\int_T \operatorname{div} b^i = \int_T \mu h_m \cdot \operatorname{grad}\chi^i \ \ \forall T \in \mathcal{T}'^i\}.$$

The b^is are given by

$$b^i = \operatorname{arginf}\{\textstyle\int_{D^i} \mu^{-1}\,|b' - \mu\chi^i h_m|^2 :\ b' \in \mathbb{B}^i(h_m)\}, \tag{14}$$

which differs from (13) only by the integration domain being D^i instead of D.

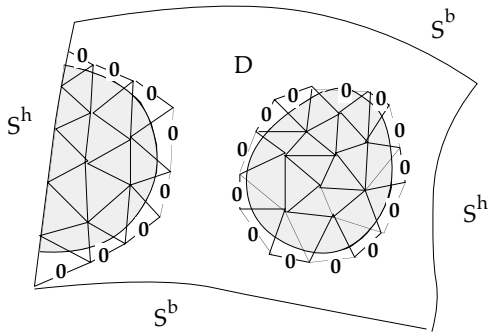

FIGURE C.1. Two examples showing the relation between $\operatorname{supp}(\chi^i)$ (shaded) and $\mathcal{T}'^i$. Faces of $\mathcal{F}^i$ (appearing as edges in this 2D drawing) are those not marked with a 0.

Before going further, let us point to an easily overlooked difficulty: If D^i does not encounter S^h (we call $\mathcal{J}_h$ the subset of $\mathcal{J}$ for which this happens), then $\int_{D^i} \operatorname{div} b^i = 0$. So unless

$$\textstyle\int_{D^i} \mu h_m \cdot \operatorname{grad}\chi^i = 0 \quad \forall\, i \in \mathcal{J}_h, \tag{15}$$

some of the affine sets $\mathbb{B}^i(h_m)$ may well be empty! Fortunately, there are easy ways to enforce condition (15). One is to use the barycentric functions as partition of unity, and then $\mathcal{J}_h$ coincides with $\mathcal{N}_h$, so (15) is equivalent to (8), which is indeed satisfied if h_m was computed by the φ-method. More generally, if all the χ^is are linear combinations of the λ^ns, which we shall assume from now on, (15) holds, and we are clear.

C.2 SOLVING PROBLEM (14)

The previous remark sheds some light on the algebraic structure of Problem (14), which determines b^i. The number of unknowns is the number of faces in $\mathcal{F}^i$, say F^i. There are T^i tetrahedra in $\mathcal{T}^i$, hence T^i constraints, but only $T^i - 1$ of those are independent, owing to (15), in the general case where $i \in \mathcal{I}_h$. This leaves $F^i - T^i + 1$ face DoFs with respect to which to minimize the energy error in (14). In the case of connected simply connected regions D^i, one has

$$N^i - E^i + F^i - T^i = -1,$$

where N^i and E^i are the numbers of *inner* nodes and edges in D^i (those not on ∂D^i), so the number of independent DoFs is $E^i - N^i$.

To go further in the identification of these DoFs, let $b^i(h_m)$ be some member of $\mathbb{B}^i(h_m)$, constructed by a procedure, the description of which we defer for an instant. Then $\operatorname{div} b^i = \operatorname{div} b^i(h_m)$ in D^i, hence $b^i = b^i(h_m) + \operatorname{rot} a^i$, with

$$a^i = \sum_{e \in \mathcal{E}^i} \mathbf{a}^i{}_e w_e, \tag{16}$$

a linear combination of edge elements, indexed over the E^i inner edges of D^i. The a^is which are gradients yield $\operatorname{rot} a^i = 0$, and the dimension of the subspace they span is N^i, so we have indeed $E^i - N^i$ independent degrees of freedom, as far as b^i is concerned. (One might use the N^i loose ones to "gauge" a^i. But this is not necessary, as stressed in Chapter 6.)

Let us now rewrite problem (14) in terms of a^i. We have

$$a^i \in \operatorname{arginf}\{\textstyle\int_{D^i} \mu^{-1} \,|\operatorname{rot} a' + b^i(h_m) - \mu\chi^i h_m|^2 : a' \in A^i\}, \tag{17}$$

where A^i is the set of fields of the form (16). When $\chi^i = \lambda^i$, the barycentric function of node number i, this simplifies a little, for

$$\textstyle\int_{D^i} \mu^{-1} \,|\operatorname{rot} a' + b^i(h_m) - \mu\lambda^i h_m|^2 = \int_{D^i} \mu^{-1} \,|\operatorname{rot} a' + b^i(h_m)|^2$$
$$\textstyle + \int_{D^i} \mu \,|\lambda^i h_m|^2 - 2\int_{D^i} \lambda^i \operatorname{rot} a' \cdot h_m,$$

and the latter term is $\frac{1}{2}\int_{D^i} \operatorname{rot} a' \,.\, h_m \equiv 0$, because both $\operatorname{rot} a'$ and h_m are piecewise constant, and the average value of λ^i is $1/4$ over all tetrahedra. The term to be minimized is then[2] $\int_{D^i} \mu^{-1} \,|\operatorname{rot} a' + b^i(h_m)|^2$.

[2]This is actually true in all cases when the χ^is are linear combinations of the λ^ns.

This only leaves the problem of determining $b^i(h_m)$. Refer back to (12), which says, equivalently, that

$$\int_{\partial T} n \cdot b^i = \int_{\partial T} \mu \chi^i n \cdot h_m \quad \forall T \in \mathcal{T}'^i.$$

Select $T^i - 1$ faces in such a way that no more than three of them belong to the same tetrahedron, and attribute to them the DoF

$$\mathbf{b}^i_f = \tfrac{1}{2} \sum_{T \in \mathcal{T}'^i} \int_f \mu \chi^i n \cdot {}^T h_m, \tag{18}$$

where ${}^T h_m$ is the value of h_m over T. (Only two tetrahedra give nonzero contributions to the sum, those sharing f, so $\mathbf{b}^i_f$ is the average flux through f.) Other DoFs will be determined from the linear relations (12), now in the right number.

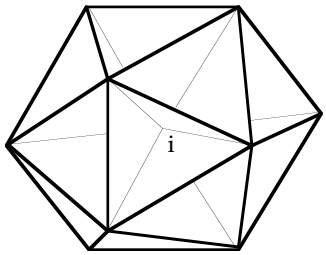

FIGURE C.2. A cluster of 20 tetrahedra around node i, with opaque inner faces and transparent surface faces. There are 12 inner edges, as many surface nodes, and 30 inner faces, in one-to-one correspondence with surface edges ($E^i = 12$, $F^i = 30$, $T^i = 20$).

As for the selection of faces, we shall describe this process only in the case where $\mathcal{J} \equiv \mathcal{N}$ and $\chi^i \equiv \lambda^i$. Then $N^i = 1$, and D^i is the cluster[3] of tetrahedra around node number i (cf. Fig. C.2). Its surface is a polyhedron with E^i nodes (the number of inner edges), F^i edges (for each inner face corresponds to a boundary edge) and T^i faces (the number of tetrahedra in the cluster). By Euler–Poincaré, $E^i - F^i + T^i = 2$ (don't be confused by the notational shift), and since each boundary face has three edges, $F^i = 3T^i/2$. One has thus $T^i = 2E^i - 4$ (typically 20, on the average, if we assume 5 tetrahedra per node) and $F^i = 3E^i - 6$ (typically, 30). (This fits: $T^i - 1$ independent constraints, $E^i - 1$ "genuine" edge-DoFs, and 19 + 11 = 30.) We must select $T^i - 1$ edges (out of F^i), leaving out $E^i - 1$. This is done by extracting a spanning tree from the graph, the nodes of which are the *faces* of ∂D^i (*not* the surface nodes!) with the surface edges as edges

[3]French readers, please use a French name, "grappe" or "agrégat", to translate cluster.

of the graph. I suppose a drawing could help: cf. Fig. C.3. Co-edges of the spanning tree point to the faces for which the computation (18) will be done (those which cut the surface along the thick lines of Fig. C.3).

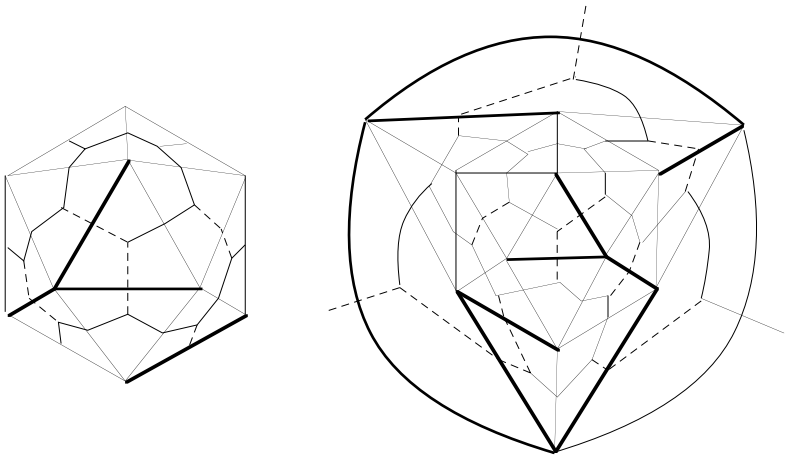

FIGURE C.3. Complementary spanning trees on the surface of an icosahedron, as seen in perspective view (left) and in stereographic projection from the center of the back face (right). The latter view shows only 19 faces, three of them with some distortion, as if one was peering inside the icosahedron from a point near the center of face 20 (the rear one on the left view), which thus corresponds to the outer region of the plane in the stereographic projection. The spanning tree of the face-to-edges graph of the text is in thin lines. Co-edges of this tree are in dotted lines; they intersect the thick-drawn edges of the polyhedron. Note that these edges themselves constitute a spanning tree for the node-to-edges tree on the surface. (This "complementarity" of the trees of both kinds is a general fact on closed simply connected surfaces; cf. Remark C.3.)

After this, Problem (17) amounts to solving a linear system of order E^i, with twice as many off-diagonal terms as there are edges on the surface of the cluster, that is, $2F^i = 2(3E^i - 6)$, typically 72. There are about N such problems (more precisely, N^h, the number of nodes in $\mathcal{N}^h$), so if we regroup them in a single large matrix, the latter will be block-diagonal (about N blocks), and contain about 60 N off-diagonal terms. This compares favorably with the 90 N we found in Chapter 6 for this typical mesh,[4] to say nothing of the intrinsic parallelism of the method.

[4]With $T/N \sim 6$, as in Note 1, typical figures are $E^i = 12$, $F^i = 36$, $T^i = 24$. One has then 72 N off-diagonal terms vs the 108 N computed in Note 1—the same ratio.

Remark C.2. All this cries out for some symmetrization: Suppose b_m has been obtained by the a-method. Could a local correction to $\mu^{-1}h_m$ be built, quite similarly, which would yield a curl-free companion to b_m? Indeed, this is straightforward, and the reader will easily do it by imitating all we have been doing up to now, systematically transposing h for b, rot for grad, etc. Economy is no more a factor there, since the global φ-method is likely to be cheaper than all alternatives, but the need for local corrections may arise in mesh-refinement procedures the same way. And in this respect, even though the a-method is not the preferred one in magnetostatics, due to its intrinsically high cost, we may hope a method elaborated in this context will transpose to the more general one of *eddy-current* problems, in which edge-elements and the curl-curl equation are natural ingredients, and the scalar potential method is not an option. A mesh-refinement procedure based on the present ideas would then take all its value. ◊

Remark C.3. Let's explain the intriguing "tree-complementarity" of Fig. C.3. (This will illustrate what was said in 5.3.3 about the relevance of graph-theoretical concepts.) Start from a spanning tree of the so-called "primal" graph, nodes to edges. Across each co-edge, there is a line joining the two nearby triangles. These lines form a subgraph of the "dual" graph (the faces-to-edges one). This "co-edge subgraph" visits all triangles, because no triangle can have all its edges in the spanning tree, since that would make a loop. It's connected, because the complement of the spanning tree is. It has no loops, because a loop would disconnect the primal spanning tree. (On other surfaces than spheres, the situation is completely different, as we observed in Section 5.3, Fig. 5.9.) ◊

C.3 CONCLUSION AND SPECULATIONS

We achieved our objective, which was to find some b, divergence-free, and close to μh, with moderate computational effort. In practice, one will thus solve for the scalar magnetic potential φ, hence $h = \text{grad}\,\varphi$, and compute b by the above procedure. Then everything one may wish to know about the error relative to this mesh is told by the estimator

$$E(b, h) = \sum_{T \in \mathcal{T}} \int_T \mu^{-1} |b - \mu h|^2, \tag{19}$$

which can be used to get bilateral bounds on the reluctance, or to map the local error in constitutive law. The approximation error with *this* mesh is therefore well documented. If it is found too large, a mesh refinement,

by appropriate subdivision of the "guilty" elements and their neighbors, cannot fail to improve the result.

But the estimator (19) *could*, conceivably, fail to register this fact when recomputed on the refined mesh. Let's call m' the new mesh, and denote by $m' < m$ the fact that m' is a refinement of m. That $h_{m'}$ is "better" than h_m is no proof that $E(b_{m'}, h_{m'}) < E(b_m, h_m)$, if b_m is computed by local correction, since the partition of unity has changed with the mesh. On the other hand, this equality would hold if b_m was computed by the "expensive", nonlocal edge-element procedure. We shall denote by b^L_m and b^G_m (for "local" and "global") these two approximations.

Hence the following question: Does the approximation b^L_m converge, like b^G_m, towards the true b? The answer is likely to be *yes*, but the issue is not yet settled.

To make progress in this direction, let us establish two lemmas (useful by themselves):

Lemma C.1. *One has* $\sum_{i \in \mathcal{N}} \mathrm{vol}(D^i) = 4\ \mathrm{vol}(D)$, *where* vol *denotes the volume.*

Proof.
$$\sum_{i \in \mathcal{N}} \mathrm{vol}(D^i) = \sum_{i \in \mathcal{N}} \sum_{T \in \mathcal{T}(i)} \mathrm{vol}(T) = \sum_{T \in \mathcal{T}(i)} \sum_{i \in \mathcal{N}} \mathrm{vol}(T)$$
$$= 4 \sum_{T \in \mathcal{T}(i)} \mathrm{vol}(T) = 4\ \mathrm{vol}(D),$$

since each tetrahedron has four nodes. ◊

Lemma C.2. *Let* k *be the maximum number of edges adjacent to a node. Then, for a given vector field* u,

$$\int_D |u|^2 \le (k+1) \sum_{i \in \mathcal{N}} \int_D |\lambda^i u|^2. \tag{20}$$

Proof. Let us set $u^i = \lambda^i u$. Then, for any definite node ordering,

$$\int_D |u|^2 = \int_D |\sum_{i \in \mathcal{N}} \lambda^i u|^2 = \sum_{i \in \mathcal{N}} \int_D |u^i|^2 + \sum_{i \neq j} \int_D u^i . u^j$$
$$\le \sum_{i \in \mathcal{N}} \int_D |u^i|^2 + \sum_{i < j} \int_D (|u^i|^2 + |u^j|^2)$$
$$\le (k+1) \sum_{i \in \mathcal{N}} \int_D |u^i|^2,$$

hence (20). ◊

Now call $\gamma(m)$ the grain of the mesh. Recall the standard result of Chapter 4,

$$E(b^G_m, h_m) \le C_1\, \gamma(m)^2,$$

where the constant C_1 (like all similar ones to come) depends on D and the data, but not on the mesh. If one can prove that

(21) $\int_{D^i} \mu^{-1} \, | b^i - \mu \chi^i h_m |^2 \le C_2 \ \mathrm{vol}(D^i) \, \gamma(m)^2,$

it will follow from this and the lemmas that

$$E(b^L_m, h_m) \le C_3 \, \gamma(m)^2,$$

a quite nice prospect, since the convergence speed would be the same, with less computation, than with the global complementarity method, only of course with a larger multiplicative factor in the error estimate.

So all depends on (21) being true, which seems likely, but I have no proof.

Another legitimate question is, could the local correction be obtained in a more straightforward way? After all, we know μh in D^i, so we have a fair estimate of the fluxes of b through faces of ∂D^i. Why not just look for the b in $W^2(D^i)$, with these fluxes as Dirichlet data, that is closest to μh in energy? What we get this way is a local b directly, not something like the earlier b^i, which was only the local contribution to $b = \sum_i b^i$. It's much better, in a way, since if we are interested in b around specified isolated nodes, *one* computation, instead of about 13 to 15, will be required for each of these.

But while this may be all right if all we want is some local divergence-free correction of μh, in order to make good-looking, flux-preserving displays, for instance, such a procedure is not able to yield a *globally* valid b: Even if we implement it on a family of D^is which pave D, and try patching the results together, the b thus obtained will not be div-conformal (and hence, not divergence-free over D), because the boundary fluxes taken as data on any given face differ for the two computations in the domains D^i and D^j adjacent to that face. (The variant in which the fluxes are used as Neumann data instead has the same drawback, and more degrees of freedom.) This procedure, therefore, is of no value as regards error bounds.