CHAPTER 9

Maxwell's Model in Harmonic Regime

9.1 A CONCRETE PROBLEM: THE MICROWAVE OVEN

9.1.1 Modelling

Our last model, about the microwave oven, is typical of the class of time-harmonic problem *with* displacement currents taken into account, in bounded regions.

Such an oven is a cavity enclosed in metallic walls, containing an antenna and something that must be heated, called the "load" (Fig. 9.1). One may model the antenna by a current density j^g, periodic in time (the typical frequency is 2450 MHz), hence $j^g(t) = \mathrm{Re}[\mathrm{J}^g \exp(i\omega t)]$, the support of J^g being a part of the cavity. Note that this current has no reason to be divergence-free. The average power necessary to sustain it, which will be retrieved in part as thermal power in the load, is $-\frac{1}{2}\mathrm{Re}[\int \mathrm{J}^g \cdot \mathrm{E}^*]$. The load occupies a part of the cavity and is characterized by *complex*-valued coefficients ε and μ, for reasons we shall explain.

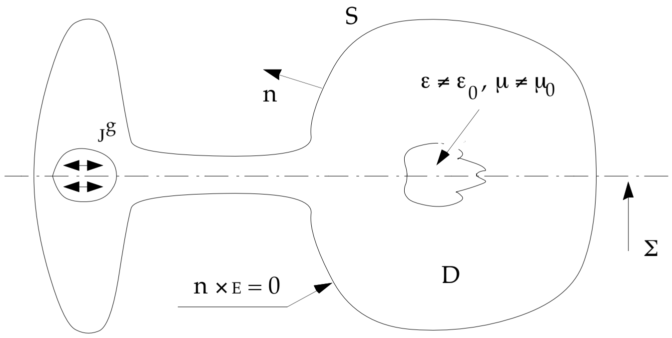


FIGURE 9.1. Notations for the microwave oven problem.

The conductivity of metallic walls is high enough to assume $\mathrm{E} = 0$ there, so the equations are

(1) $\quad - i\omega\, \mathrm{D} + \mathrm{rot}\, \mathrm{H} = \mathrm{J}^g + \sigma\, \mathrm{E}$ in D,

(2) $\quad i\omega\, \mathrm{B} + \mathrm{rot}\, \mathrm{E} = 0$ in D,

(3) $\quad n \times \mathrm{E} = 0$ on S.

The load, as a rule, is an aqueous material with high permittivity, hence a strong polarization in presence of an electric field. Moreover, because of the inertia of dipoles, the alignment of the polarization vector p on the electric field is not instantaneous, as we assumed in Chapter 1. If one sticks to the hypothesis of linearity of the constitutive law, then $p(t) = \int^t f(t - s)\, e(s)\, ds$, where the function f is a characteristic of the medium. After Fourier transformation, this becomes

$$\mathrm{P}(\omega) = \sqrt{2\pi}\, \mathrm{F}(\omega)\, \mathrm{E}(\omega),$$

(F is the *transfer function* of the polarized medium), hence $\mathrm{D}(\omega) = \varepsilon_0\, \mathrm{E}(\omega) + \mathrm{P}(\omega) \equiv \varepsilon\, \mathrm{E}(\omega)$, with $\varepsilon(\omega) = \varepsilon_0\, \mathrm{E} + \sqrt{2\pi}\, \mathrm{F}(\omega)$, a complex and frequency-dependent permittivity. It is customary to set $\varepsilon = \varepsilon' - i\, \varepsilon''$, with real ε' and ε''. It all goes (transfer $i\, \varepsilon''$ to the right-hand side in (1)) as if one had a real permittivity ε, a conductivity $\varepsilon''\omega$ (in addition to the normal ohmic conductivity—but the latter can always be accounted for by the ε'' term, by adding σ/ω to it), and a current density $j(t) = \mathrm{Re}[\mathrm{J} \exp(i\omega t)]$, where $\mathrm{J} = \omega\, \varepsilon'' \mathrm{E}$. The reason for the minus sign can be seen by doing the following computation, where $T = 2\pi/\omega$ is the period:

$$\frac{1}{T} \int_{t-T}^{t} ds \int_D j(s) \cdot e(s) = \frac{1}{T} \int_{t-T}^{t} ds \int_D \mathrm{Re}[\mathrm{J} \exp(i\omega t)] \cdot \mathrm{Re}[\mathrm{E} \exp(i\omega t)]$$

$$= \mathrm{Re}[\mathrm{J} \cdot \mathrm{E}^*]/2 \equiv \omega\, \varepsilon''\, |\mathrm{E}|^2 / 2,$$

since this quantity, which is the thermal power yielded to the EM compartment (cf. Chapter 1, Section 3), now agrees in sign with ε''.

Of course, f cannot directly be measured, just theorized about (cf. [Jo]). But ε' and ε'' can (cf. Fig. 1.4).[1]

[1]As real and imaginary parts of the Fourier transform of one and the same function, ε' and ε'' are not independent (they are "Hilbert transforms" of each other). One could thus, in theory, derive one from the other, provided one of them is known over *all* the spectrum, and with sufficient accuracy. This is of course impossible in practice, and ε' and ε'' are independently measured (as real and imaginary parts of the impedance of a sample) on an appropriate frequency range.

For the sake of symmetry and generality, let's also write $\mu = \mu' - i\,\mu''$, hence the definitive form of the equations:

(4) $$-\,i\omega\,\varepsilon\,E + \mathrm{rot}\,H = J^g,\quad i\omega\,\mu\,H + \mathrm{rot}\,E = 0 \text{ in } D,\quad n \times E = 0 \text{ on } S,$$

with $\varepsilon = \varepsilon' - i\,\varepsilon''$ and $\mu = \mu' - i\,\mu''$. They will normally be coupled with the heat equation, the source-term being the average thermic power $i\omega\,\varepsilon''|E|^2/2$ (plus $i\omega\,\mu''|H|^2/2$, if this term exists). The parameter ε, temperature-dependent, will therefore change during the heating.

9.1.2 Position of the problem

We want a variational formulation of (4), for J^g given in $\mathbb{L}^2_{\mathrm{div}}(D)$, where the unknown will be the field E, after elimination of H. Let $\mathbb{E}(D)$ denote the (complex) space $\mathbb{L}^2_{\mathrm{rot}}(D)$, and

$$\mathbb{E}^0(D) = \{E \in \mathbb{E}(D) : n \times E = 0 \text{ on } S\}.$$

The scalar product of two complex *vectors* U and V is as in Chapter 8 (*no* conjugation on the right), but we shall adopt a space-saving notational device, as follows: If U and V are two complex *fields*, we denote by $(U, V)_D$, or simply (U, V), the expression $\int_D U(x) \cdot V(x)\,dx$ (which, let's stress it again, is *not* the Hermitian scalar product).

A precise formulation of (4) is then: *find* $E \in \mathbb{E}^0(D)$ *such that*

(5) $$(i\omega\,\varepsilon\,E, E') + ((i\omega\,\mu)^{-1}\,\mathrm{rot}\,E, \mathrm{rot}\,E') = -\,(J^g, E') \quad \forall\, E' \in \mathbb{E}^0(D).$$

Unfortunately, the existence question is not trivial, because the bilinear form $a(E, E')$ on the left-hand side of (5) is not coercive. Indeed,

$$\mathrm{Re}[a(E, E^*)] = \omega \int_D \varepsilon''\,|E|^2 + \omega^{-1}\int_D \mu''/|\mu|^2\,|\mathrm{rot}\,E|^2,$$

which vanishes if the support of E does not overlap with those of ε'' and μ'', and

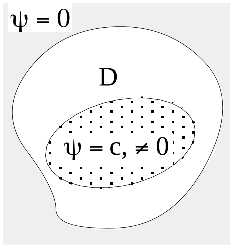
$$\mathrm{Im}[a(E, E^*)] = \omega \int_D \varepsilon'\,|E|^2 - \omega^{-1}\int_D \mu'/|\mu|^2\,|\mathrm{rot}\,E|^2$$

has no definite sign (and no premultiplication by a scalar will cure that).

But the restriction to a *bounded* domain (finite volume is enough, actually) introduces some compactness which makes up for this lack of coercivity, at least for non-singular values of ω, thanks to the Fredholm alternative.

9.2 THE "CONTINUOUS" PROBLEM

9.2.1 Existence



Let's first prove an auxiliary result. Let D be a regular bounded domain in E_3, with boundary S. For simplicity (but this is not essential), assume D simply connected. Let Ψ^0 be the space of restrictions to D of functions ψ of $L^2_{grad}(E_3)$ for which grad $\psi = 0$ outside D. (If S is connected, they belong to the Sobolev space $H^1_0(D)$, but otherwise, it's a slightly larger space, for ψ can be a nonzero constant on some parts of S, as shown in inset.) Call V the following closed subspace of $\mathbb{L}^2(D)$:

$$V = \{v \in \mathbb{L}^2(D) : (v, \operatorname{grad} \psi') = 0 \quad \forall\ \psi' \in \Psi^0\}.$$

Proposition 9.1. *Let* J *be given in* $\mathbb{L}^2(E_3)$, *with* div $J = 0$ *and* supp(J) $\subset$ D. *Suppose* $\mu' \geq \mu_0$ *in* D. *There exists a unique* $A \in \mathbb{E}^0(D)$ *such that*

(6) $\qquad (\mu^{-1} \operatorname{rot} A, \operatorname{rot} A') = (J, A') \quad \forall\ A' \in \mathbb{E}^0$

as well as $\varepsilon A \in V$, *and the map* $G = J \to \varepsilon A$ *is* compact *in* V.

Before giving the proof, note that such a field A verifies

(6') $\qquad \operatorname{rot}(\mu^{-1} \operatorname{rot} A) = J, \quad \operatorname{div} \varepsilon A = 0 \ \text{ in D}, \quad n \times A = 0 \ \text{ on S},$

but these conditions are not enough to determine it, unless S is connected. In that case, $V = \{v \in \mathbb{L}^2(D) : \operatorname{div} v = 0\}$. But otherwise, V is a strictly smaller subspace, characterized by $\int n \cdot v = 0$ on each connected component of S, hence as many similar conditions[2] on A, to be appended to the "strong formulation" (6). Note also that $J \in V$, under the hypotheses of the statement, so G does operate from V to V.

Proof of Prop. 1. Uniqueness holds, because the kernel of rot in $\mathbb{E}^0$ is precisely grad Ψ^0 (this is why Ψ^0 was defined this way). The proof will consist in showing that one passes from J to εA by composing continuous maps, one of which at least is compact.

Set $U = \chi * J$, with $\chi = x \to 1/(4\pi\ |x|)$, and take its restriction ${}^D U$ to D. The map $J \to {}^D U$ thus defined is compact in $\mathbb{L}^2(D)$ [Yo]. Therefore, the map $J \to \operatorname{rot} U \in \mathbb{L}^2(E_3)$ is compact, too, for if $\{J_n\}$ is a sequence of V

[2]This is one of the advantages of weak formulations: They foster thoroughness, by reminding one of conditions which one might have overlooked in the first place.

such that $J_n \rightharpoonup J$ (weak convergence—cf. A.4.3), then $U_n \rightharpoonup U$ by continuity, and hence (dot-multiply by a test field A' and integrate by parts) $\text{rot}\, U_n \rightharpoonup \text{rot}\, U$. Moreover, $\int |\text{rot}\, U_n|^2 = \int_D J_n \cdot U_n^*$, and ${}^D U_n$ tends to ${}^D U$, so the norm of $\text{rot}\, U_n$ converges towards that of $\text{rot}\, U$, therefore $\text{rot}\, U_n$ tends to $\text{rot}\, U$. Setting $H = \text{rot}\, {}^D U$, one has thus proved the compactness of the map $J \to H$.

Now let $\Phi \in L^2_{grad}(D)$ be such that

$$(\mu\,(H + \text{grad}\,\Phi), \text{grad}\,\Phi') = 0 \quad \forall\, \Phi' \in L^2_{grad}(D).$$

The solution of this problem is unique up to an additive constant only, but grad Φ is unique, and the map $H \to \text{grad}\,\Phi$ is continuous. Let us set $B = \mu(H + \text{grad}\,\Phi)$. Then $\text{div}\, B = 0$ and $n \cdot B = 0$, so the prolongation by 0 of B outside D is divergence-free. If one sets $A_0 = {}^D(\text{rot}(\chi * B))$ —again, the restriction to D —then $\text{rot}\, A_0 = B$, and the mapping $H \to A_0$ is continuous.

Notice that the tangential trace of A_0 is a gradient, by the Stokes theorem, for the flux of B through a closed circuit drawn on S vanishes, since $n \cdot \text{rot}\, A_0 = n \cdot B = 0$. For this reason, the set of the $\Psi \in L^2_{grad}(D)$ for which $n \times (A_0 + \text{grad}\,\Psi) = 0$ is not empty, and there is one among them (unique up to an additive constant) for which

$$(\varepsilon\,(A_0 + \text{grad}\,\Psi), \text{grad}\,\Psi') = 0 \quad \forall\, \Psi' \in \Psi^0.$$

Then $A = A_0 + \text{grad}\,\Psi$ is the desired solution, and grad Ψ continuously depends on A_0, with respect to the norm of $\mathbb{L}^2(D)$. The map $J \to A$ is therefore compact, hence the compactness of the operator $G = J \to \varepsilon A$, whose domain is the subspace V. ◊

Let's call *singular* (or *resonating)* the nonzero values of ω for which the homogeneous problem associated with (5) has a nontrivial solution, i.e., $E \neq 0$ such that

$$(i\omega\,\varepsilon\, E, E') + ((i\omega\,\mu)^{-1}\,\text{rot}\, E, \text{rot}\, E') = 0 \quad \forall\, E' \in \mathbb{E}^0. \tag{7}$$

Such an E verifies $\varepsilon E \in V$ (take $E' \in \text{grad}\,\Psi^0$) as well as $\text{rot}(\mu^{-1}\,\text{rot}\, E) = \omega^2 \varepsilon\, E$ (integrate by parts). In other words, $\varepsilon E = \omega^2 G \varepsilon E$. Thus, εE is an eigenvector of G, corresponding to the eigenvalue ω^{-2}. (One says that the pair $\{E, H\}$, where $H = -(\text{rot}\, E)/i\omega\mu$, is an "eigenmode" of the cavity, for the angular frequency ω.) By Fredholm's theory, there exists a denumerable infinity of eigenvalues for G, each with finite multiplicity, and not clustering anywhere except at the origin in the complex plane.[3] The singular values are thus the square roots of the inverses of the eigenvalues

[3]Owing to uniqueness in (6), 0 is not an eigenvalue of G.

of G. (A priori, eigenvalues are complex, unless both ε and μ are real.)

Theorem 9.1. *For each non-singular value of* ω, *problem* (5) *is well posed, i.e., has a unique solution* E, *and the map* $J^g \to E$ *is continuous from* $\mathbb{L}^2(D)$ *into* $\mathbb{E}(D)$.

Proof. Since ω is not singular, uniqueness holds. Let's look for a solution of the form $E = -\,i\omega\, A - \text{grad}\,\Psi$, with $A \in \mathbb{E}^0$, $\varepsilon\, A \in V$, and $\Psi \in \Psi^0$. Set $E' = \text{grad}\,\Psi'$ in (5), with $\Psi' \in \Psi^0$. This yields

$$(i\omega\, \varepsilon\, (A + \text{grad}\,\Psi), \text{grad}\,\Psi') = (J^g, \text{grad}\,\Psi') \quad \forall\, \Psi' \in \Psi^0,$$

and hence, since εA is orthogonal to all $\text{grad}\,\Psi'$,

(8) $$(i\omega\, \varepsilon\, \text{grad}\,\Psi, \text{grad}\,\Psi') = (J^g, \text{grad}\,\Psi') \quad \forall\, \Psi' \in \Psi^0,$$

a well-posed problem in Ψ^0, hence the continuity of $J^g \to \text{grad}\,\Psi$ in $\mathbb{L}^2(D)$. This leaves A to be determined. After (5), one must have

$$(\mu^{-1}\,\text{rot}\,A, \text{rot}\,A') = (J^g + i\omega\, \varepsilon\, E, A') \quad \forall\, A' \in \mathbb{E}^0$$
$$= (J^g - i\omega\, \varepsilon\, \text{grad}\,\Psi, A') + \omega^2\, (\varepsilon A, A') \quad \forall\, A' \in \mathbb{E}^0.$$

But this is the Fredholm equation of the second kind,

$$(1 - \omega^2 G)\, \varepsilon\, A = G\, (J^g - i\omega\, \varepsilon\, \text{grad}\,\Psi),$$

hence A by the Fredholm alternative, if ω is not a singular value, and provided $J^g - i\omega\, \varepsilon\, \text{grad}\,\Psi \in V$ —which is what (8) asserts. ◊

9.2.2 Uniqueness

Hence the question: Are there singular values? For an empty cavity ($\mu = \mu_0$ and $\varepsilon = \varepsilon_0$), or with lossless materials (μ and ε real and positive), *yes*, since all eigenvalues of G are then real and positive. If $\omega \neq 0$ is one of them and $E = e_R$ a nonzero associated real solution of (7) (there *is* a *real* one), then $H = i\, h_I$, with real h_I. The existence of such a solution means that a time-periodic electromagnetic field, of the form $e(x, t) = \text{Re}[E(x) \exp(i\omega t)] \equiv e_R(x) \cos \omega t$ and $h(x, t) = -\, h_I(x) \sin \omega t$ can exist forever in the cavity, without any power expense, and also of course without loss.

To verify this point, let's start from the equations $i\omega\, \varepsilon E - \text{rot}\, H = 0$ and $i\omega\, \mu\, H + \text{rot}\, E = 0$, dot-multiply by E and H, add, and integrate over D: by the curl integration-by-parts formula, and because of $n \times E = 0$, this gives $\int_D \varepsilon\ \mathrm{E}^2 + \int_D \mu\ \mathrm{H}^2 = 0$, that is, since $\mathrm{E} = \mathrm{e_R}$ and $\mathrm{H} = \mathrm{i\ h_I}$,

$$\int_D \varepsilon\ |\mathrm{e_R}|^2 = \int_D \mu\ |\mathrm{h_I}|^2.$$

But the energy contained in D at time t, which is (cf. Chapter 1)

$$\begin{aligned} W(t) &= \tfrac{1}{2} \int_D (\varepsilon\ |\mathrm{e}(t)|^2 + \mu\ |\mathrm{h}(t)|^2) \\ &= \tfrac{1}{2} \int_D \varepsilon\ |\mathrm{e_R}|^2 \cos^2 \omega t + \tfrac{1}{2} \int_D \mu\ |\mathrm{h_I}|^2 \sin^2 \omega t \\ &\equiv \tfrac{1}{2} \int_D \varepsilon\ |\mathrm{e_R}|^2 \equiv \tfrac{1}{2} \int_D \mu\ |\mathrm{h_I}|^2, \end{aligned}$$

is indeed constant, and is the sum of two periodic terms of identical amplitudes, "electric energy" and "magnetic energy" in the cavity, the former vanishing at each half-period and the latter a quarter-period later.

Such a behavior seems unlikely in the case of a loaded cavity, since the energy of the field decreases in the process of yielding heat to the region $C = \mathrm{supp}(\varepsilon'') \cup \mathrm{supp}(\mu'')$. How can this physical intuition be translated into a proof? It's easy to see that E and H vanish in C: Setting $\mathrm{E}' = \mathrm{E}^*$ in (7), one gets

$$\begin{aligned} \omega \int_D \varepsilon''\ |\mathrm{E}|^2 + \mathrm{i}\omega \int_D \varepsilon'\ |\mathrm{E}|^2 + \int_D (\omega\ |\mu|^2)^{-1} \mu''\ |\mathrm{rot}\ \mathrm{E}|^2 \\ + \int_D (\mathrm{i}\omega\ |\mu|^2)^{-1} \mu'\ |\mathrm{rot}\ \mathrm{E}|^2 = 0, \end{aligned}$$

hence, taking the real part, $\mathrm{E} = 0$ or $\mathrm{rot}\ \mathrm{E} = 0$ on C, hence $\mathrm{H} = 0$, too, and therefore $\mathrm{E} = 0$ after (2). But what opposes the existence of a nonzero mode E that would be supported in the complementary region D – C, what one may call an "air mode" ?

If such an air mode existed, both $\mathrm{n} \times \mathrm{E}$ and $\mathrm{n} \times \mathrm{H}$ would vanish on ∂C, which is impossible if E and H are to satisfy Maxwell's equations in region D – C. We shall prove this by way of a mathematical argument, here encapsulated as a context-independent statement, cast in non-dimensional form:

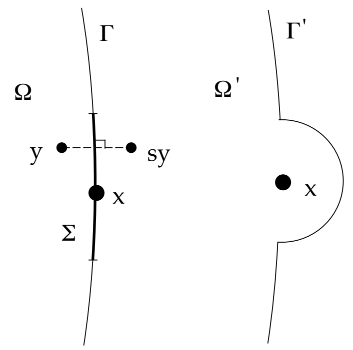
Proposition 9.2. *Let* Ω *be a regular domain of* E_3. *Let* u *and* v *satisfy*

(13) $\mathrm{i\ rot\ u} = \mathrm{v}, \qquad - \mathrm{i\ rot\ v} = \mathrm{u}$ in Ω,

(14) $\mathrm{n} \times \mathrm{u} = 0, \qquad \mathrm{n} \times \mathrm{v} = 0$ on Σ,

where Σ *is a part of* Γ *with a smooth boundary and a non-empty interior (relative to* Σ*). Then* u *and* v *vanish in all* Ω.

Proof. (Though reduced to the bare bones by many oversimplifications, the proof will be long.) Both u and v satisfy $\operatorname{div} u = 0$ and $-\Delta u = u$ in Ω, after (13). It is known (cf., e.g., [Yo]), that every u which satisfies $(\Delta + 1)u = 0$ in some open set is *analytic* there. This is akin to the Weyl lemma discussed in 2.2.1, and as mentioned there, this result of "analytic ellipticity" is valid also for $\operatorname{div}(a \operatorname{grad}) + b$, where a and b are smooth. If we could prove that all derivatives of all components of u and v vanish at some point, u and v would then have to be 0 in all Ω, by analyticity.



The problem is, we can prove this fact, but only for boundary points such as x (inset), which is in the relative interior of Σ, not inside Ω. So we need to expand Ω in the vicinity of x, as suggested by the inset, and to make some continuation of the equations to this expanded domain Ω' in a way which preserves analytic ellipticity. To do this, first straighten out Σ around x by an appropriate diffeomorphism, then consider the mirror symmetry s with respect to the plane where Σ now lies. Pulling back this operation to Ω gives a kind of warped reflexion with respect to Σ, still denoted by s. Let us define continuations of u and v to the enlarged domain Ω' by setting $\tilde{u}(sy) = -\, s u(y)$ and $\tilde{v}(sy) = s_* v(y)$, where s_* is the mapping induced by s on vectors (cf. A.3.4, p. 301). Now the extensions $\tilde{u}$ and $\tilde{v}$ satisfy in Ω' a system similar to (13–14), with some smooth coefficients added, the solutions of which are similarly analytic.

The point about vanishing derivatives remains to be proven. This is done by working in an appropriate coordinate system $y \to \{y^1, y^2, y^3\}$ with the above point x at the origin, y^1 and y^2 charting Σ, and y^3 along the normal. In this system, $u = \{u^1, u^2, u^3\}$, and $u^j(x^1, x^2, 0\} = 0$, for $j = 1, 2$, and the same for v. Derivatives $\partial_i u^j$ vanish for all j and $i = 1$ or 2 at point x by hypothesis. One has also $\partial_1 u^3 = -\, i\, \partial_1(\partial_1 v^2 - \partial_2 v^1) = 0$ on Σ, and in the same way, $\partial_2 v^3 = 0$, $\partial_1 u^3 = 0$, $\partial_2 u^3 = 0$. Since $\operatorname{rot} u = \{\partial_2 u^3 - \partial_3 u^2, \partial_3 u^1 - \partial_1 u^3, \partial_1 u^2 - \partial_2 u^1\} \equiv \{-\, \partial_3 u^2, \partial_3 u^1, 0\}$ is also 0 at x, we have $\partial_3 u^j(x) = 0$ for $j = 1$ and 2. The last derivative to consider is $\partial_3 u^3 \equiv \operatorname{div} u - (\partial_1 u^1 + \partial_2 u^2) = 0$, since $\operatorname{div} u = \operatorname{div} \tilde{u} = 0$ at point x. This disposes of first-order derivatives.

What has just been proved for a generic point of Σ implies that all first-order derivatives of u and v vanish on Σ. Thus, differentiating (13) with respect to the ith coordinate, we see that $\partial_i u$ and $\partial_i v$ satisfy

(13–14), so the previous reasoning can be applied to derivatives of u and v of all order: They all vanish at x, and now the analyticity argument works, and yields the announced conclusion. ◊

The anti-air-mode result now comes by setting $\Omega = D - C$ and $\Sigma = \partial C$, and by choosing a system of units in which $\varepsilon_0 \mu_0 \omega^2 = 1$. (Another corollary is that one cannot simultaneously impose $n \times H$ and $n \times E$ on a part of nonzero area of the cavity boundary.)

We may thus conclude that no resonant modes exist if there is a charge in a microwave oven, and that Maxwell equations have a unique solution in that case, assuming the tangential part of one of the fields H or E is specified at every point of the boundary.

9.2.3 More general weak formulations

Thus satisfied that (4), or better, its weak form (5), has a unique solution, we shall try to get an approximation of it by finite elements. But first, it's a good idea to generalize (5) a little, as regards source terms and boundary conditions, without bothering too much about the physical meaning of such generalizations.

First, let's exploit the geometrical symmetry, by assuming the source current J^g is symmetrically placed with respect to plane Σ. Then, for $x \in \Sigma$, one has $H(x) = - s_* H(x)$, where s is the mirror reflection with respect to Σ, and s_* the induced mapping on vectors. The boundary condition to apply is therefore, if n denotes the normal as usual,

(9) $\qquad n \times H = 0$ on Σ.

Therefore, our customary splitting of the boundary into complementary parts is in order: $S = S^e \cup S^h$, with $n \times E = 0$ and $n \times H = 0$, respectively, on S^e and S^h.

Second generalization: Introduce a right-hand side K^g in the second equation (4). This term does not correspond to anything physical (it would be a magnetic charge current, if such a thing existed), but still it pops up in a natural way in some modellings. For instance, if the field is decomposed as $H^g + \tilde{H}$, where H^g is a known field, a term $K^g = -\,i\omega\,\mu\,H^g$ appears on the right-hand side of the equation relative to the reaction field $\tilde{H}$.

This suggests also preserving the possibility of *non-homogeneous* boundary conditions: $n \times E = n \times E^g$ in (4) and $n \times H = n \times H^g$ in (9), where E^g and H^g are given fields, of which only the tangential part on S will matter. For instance, in the case of Fig. 9.1, one might wish to limit the computation to the oven proper, that is, the rightmost part of the cavity. Indeed, the middle part is a waveguide, in which the *shape* of the electric field, if not its amplitude, is known in advance. Hence the source–term E^g, up to a multiplicative factor.

All this considered, we shall treat a general situation characterized by the following elements: a regular bounded domain D limited by S, the latter being partitioned as $S = S^e \cup S^h$, and given fields J^g, K^g (in $\mathbb{L}^2_{div}(D)$), H^g, E^g (in $\mathbb{L}^2_{rot}(D)$). We denote, with the dependence on D understood from now on, $\mathbb{E} = \mathbb{L}^2_{rot}(D)$ and also $\mathbb{H} = \mathbb{L}^2_{rot}(D)$, then

$$\mathbb{E}^g = \{E \in \mathbb{E} : n \times E = n \times E^g \text{ on } S^e\},$$

$$\mathbb{E}^0 = \{E \in \mathbb{E} : n \times E = 0 \text{ on } S^e\},$$

$$\mathbb{H}^g = \{H \in \mathbb{H} : n \times H = n \times H^g \text{ on } S^h\},$$

$$\mathbb{H}^0 = \{H \in \mathbb{H} : n \times H = 0 \text{ on } S^h\},$$

and we set the following two problems:

$$\textit{find } E \in \mathbb{E}^g \textit{ such that } (i\omega\, \varepsilon\, E, E') + ((i\omega\, \mu)^{-1} \operatorname{rot} E, \operatorname{rot} E') =$$

$$- (J^g, E') + ((i\omega\, \mu)^{-1} K^g, \operatorname{rot} E') + \int_S n \times H^g \cdot E' \quad \forall\, E' \in \mathbb{E}^0, \tag{10}$$

$$\textit{find } H \in \mathbb{H}^g \textit{ such that } (i\omega\, \mu\, H, H') + ((i\omega\, \varepsilon)^{-1} \operatorname{rot} H, \operatorname{rot} H') =$$

$$(K^g, H') + ((i\omega\, \varepsilon)^{-1} J^g, \operatorname{rot} H') - \int_S n \times E^g \cdot H' \quad \forall\, H' \in \mathbb{H}^0. \tag{11}$$

(Beware, there is no relation, a priori, between K^g and H^g, or between J^g and E^g.)

Integrating by parts, one easily sees that *each* weak formulation (10) or (11) solves the following "strong" problem (compare with (4)):

$$\left|\begin{array}{l} -\, i\omega\, \varepsilon\, E + \operatorname{rot} H = J^g \text{ in } D, \quad n \times E = n \times E^g \text{ on } S^e, \\ i\omega\, \mu\, H + \operatorname{rot} E = K^g \text{ in } D, \quad n \times H = n \times H^g \text{ on } S^h. \end{array}\right. \tag{12}$$

One may therefore solve (12), approximately, by discretizing either (10) or (11). But (and this is *complementarity*, again!) one obtains solutions which slightly differ, and yield complementary information about the exact solution.

9.3 THE "DISCRETE" PROBLEM

9.3.1 Finite elements for (10) or (11)

Let $m = \{\mathcal{N}, \mathcal{E}, \mathcal{F}, \mathcal{T}\}$ be a mesh of D, and $W^1_m(D)$ (or, for shortness, W^1_m) the edge-element subspace of $\mathbb{L}^2_{rot}(D)$ of Chapter 5. The idea is to *restrict* the formulations (10) and (11) to this subspace.

For this, let us denote by $\mathcal{E}^e$ [resp. $\mathcal{E}^h$] the subset of $\mathcal{E}$ formed by edges that belong to S^e [resp. to S^h], and let $\mathbb{E}_m$ and $\mathbb{H}_m$ be two copies of W^1_m. Set

$$\mathbb{E}^g_m = \{E \in \mathbb{E}_m : \int_e \tau \,.\, E = \int_e \tau \,.\, E^g \ \ \forall\, e \in \mathcal{E}^e\},$$

$$\mathbb{E}^0_m = \{E \in \mathbb{E}_m : \int_e \tau \,.\, E = 0 \ \ \forall\, e \in \mathcal{E}^e\},$$

$$\mathbb{H}^g_m = \{H \in \mathbb{H}_m : \int_e \tau \,.\, H = \int_e \tau \,.\, H^g \ \ \forall\, e \in \mathcal{E}^h\},$$

$$\mathbb{H}^0_m = \{H \in \mathbb{H}_m : \int_e \tau \,.\, H = 0 \ \ \forall\, e \in \mathcal{E}^h\}.$$

All we have to do now is to index by m all the spaces that appear in (10) and (11) in order to obtain *approximate* weak formulations for both problems:

find $E \in \mathbb{E}^g_m$ *such that* $(i\omega\, \varepsilon\, E, E') + ((i\omega\, \mu)^{-1} \operatorname{rot} E, \operatorname{rot} E') =$

(13) $$-\,(J^g, E') + ((i\omega\, \mu)^{-1} K^g, \operatorname{rot} E') + \int_S n \times H^g \cdot E' \ \ \forall\ E' \in \mathbb{E}^0_m,$$

find $H \in \mathbb{H}^g_m$ *such that* $(i\omega\, \mu\, H, H') + ((i\omega\, \varepsilon)^{-1} \operatorname{rot} H, \operatorname{rot} H') =$

(14) $$(K^g, H') + ((i\omega\, \varepsilon)^{-1} J^g, \operatorname{rot} H') - \int_S n \times E^g \cdot H' \ \ \forall\ H' \in \mathbb{H}^0_m.$$

These are linear systems, with a finite number of equations: The choice $E' = w_e$, at the right-hand side, for all edges e not contained in $\mathcal{E}^e$ [resp. $H' = w_e$, for all e not in $\mathcal{E}^h$] does give one equation for each unknown edge circulation.

By the usual notational shift, we shall cast these equations in matrix form. Let $E = \sum_{e \in \mathcal{E}} \mathbf{E}_e w_e$ and $H = \sum_{e \in \mathcal{E}} \mathbf{H}_e w_e$ be the required fields of $\mathbb{E}^g_m$ and $\mathbb{H}^g_m$, and denote $\mathbf{E} = \{\mathbf{E}_e : e \in \mathcal{E}\}$[4] and $\mathbf{H} = \{\mathbf{H}_e : e \in \mathcal{E}\}$ the DoF vectors. They span vector spaces $\mathbf{IE}_m$ and $\mathbf{IH}_m$, isomorphic to $\mathbb{C}^E$, where E is the number of edges in m. Remind that, for $\mathbf{U}$ and $\mathbf{U}'$ both in $\mathbb{C}^E$, one denotes

$$(\mathbf{U}, \mathbf{U}') = \sum_{e \in \mathcal{E}} \mathbf{U}_e \cdot \mathbf{U}'_e \equiv \sum_{e \in \mathcal{E}} (\mathrm{Re}[\mathbf{U}_e] + i\, \mathrm{Im}[\mathbf{U}_e]) \cdot (\mathrm{Re}[\mathbf{U}'_e] + i\, \mathrm{Im}[\mathbf{U}'_e]).$$

[4]In memoriam G.P.

By sheer imitation of what precedes, let us set

$$\mathbb{E}^g_m = \{\mathrm{E} \in \mathbb{E}_m : \mathrm{E}_e = \int_e \tau \cdot E^g \ \ \forall\, e \in \mathcal{E}^e\},$$

$$\mathbb{E}^0_m = \{\mathrm{E} \in \mathbb{E}_m : \mathrm{E}_e = 0 \ \ \forall\, e \in \mathcal{E}^e\},$$

$$\mathbb{H}^g_m = \{\mathrm{H} \in \mathbb{H}_m : \mathrm{H}_e = \int_e \tau \cdot H^g \ \ \forall\, e \in \mathcal{E}^h\},$$

$$\mathbb{H}^0_m = \{\mathrm{H} \in \mathbb{H}_m : \mathrm{H}_e = 0 \ \ \forall\, e \in \mathcal{E}^h\},$$

and also, for ease in the expression of the right-hand sides,

$$\mathrm{F}^g_e = -\,(J^g, w_e) + ((i\omega\,\mu)^{-1} K^g, \operatorname{rot} w_e) + \int_S n \times H^g \cdot w_e,$$

$$\mathrm{G}^g_e = (K^g, w_e) + ((i\omega\,\varepsilon)^{-1} J^g, \operatorname{rot} w_e) - \int_S n \times E^g \cdot w_e,$$

and $\mathrm{F}^g = \{\mathrm{F}^g_e : e \in \mathcal{E}\}$, as well as $\mathrm{G}^g = \{\mathrm{G}^g_e : e \in \mathcal{E}\}$.

Using the matrices $\mathbf{M}_1(\mu)$, $\mathbf{R}$, etc., of Chapter 5, one may restate the two problems as follows:

find $\mathrm{E} \in \mathbb{E}^g_m$ *such that*

(15) $$(i\omega\, \mathbf{M}_1(\varepsilon)\, \mathrm{E}, \mathrm{E}') + (i\omega)^{-1} (\mathbf{R}^t\, \mathbf{M}_2(\mu^{-1})\, \mathbf{R}\, \mathrm{E}, \mathrm{E}') = (\mathrm{F}^g, \mathrm{E}') \ \ \forall\, \mathrm{E}' \in \mathbb{E}^0_m,$$

find $\mathrm{H} \in \mathbb{H}^g_m$ *such that*

(16) $$(i\omega\, \mathbf{M}_1(\mu)\, \mathrm{H}, \mathrm{H}') + (i\omega)^{-1} (\mathbf{R}^t\, \mathbf{M}_2(\varepsilon^{-1})\, \mathbf{R}\, \mathrm{H}, \mathrm{H}') = (\mathrm{G}^g, \mathrm{H}') \ \ \forall\, \mathrm{H}' \in \mathbb{H}^0_m.$$

Since $\mathbb{E}^g_m$ and $\mathbb{E}^0_m$ [resp. $\mathbb{H}^g_m$ and $\mathbb{H}^0_m$] are parallel by their very definition, they have same dimension, so there are *as many equations as unknowns* in (15) [resp. in (16)]. All that is left to do is to assess the regularity, in the algebraic sense, of the corresponding matrices.

9.3.2 Discrete models

To this effect, let's write the unknown vector E in the form $\mathrm{E} = {}^0\mathrm{E} + {}^1\mathrm{E}$, the 0's corresponding to edges of $\mathcal{E} - \mathcal{E}^e$ and the 1's to those of $\mathcal{E}^e$, so ${}^1\mathrm{E}$ is a known item. If $\mathbf{K}$ is some $E \times E$ matrix, the identity

$$(\mathbf{K}\,({}^0\mathrm{E} + {}^1\mathrm{E}), {}^0\mathrm{E} + {}^1\mathrm{E}) = ({}^{00}\mathbf{K}\,{}^0\mathrm{E}, {}^0\mathrm{E}) + ({}^{01}\mathbf{K}\,{}^1\mathrm{E}, {}^0\mathrm{E})$$
$$\ldots + ({}^{10}\mathbf{K}\,{}^0\mathrm{E}, {}^1\mathrm{E}) + ({}^{11}\mathbf{K}\,{}^1\mathrm{E}, {}^1\mathrm{E})$$

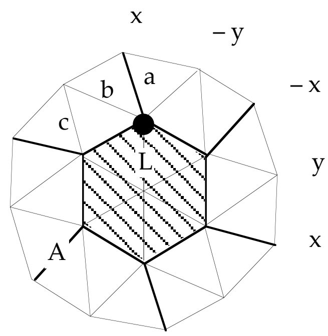
defines a partition of $\mathbf{K}$ in blocks of dimensions $E - E^e$ and E^e, where E^e is the number of edges in $\mathcal{E}^e$. Set, for simplicity, $\mathbf{K}_{\varepsilon\mu}(\omega) = i\omega\, \mathbf{M}_1(\varepsilon) + (i\omega)^{-1}\, \mathbf{R}^t\, \mathbf{M}_2(\mu^{-1})\mathbf{R}$, and let ${}^{00}\mathbf{K}_{\varepsilon\mu}(\omega)$, ${}^{01}\mathbf{K}_{\varepsilon\mu}(\omega)$, etc., be the corresponding blocks. Same thing for the matrix $\mathbf{K}_{\mu\varepsilon}(\omega) = i\omega\, \mathbf{M}_1(\mu) + (i\omega)^{-1}\, \mathbf{R}^t\, \mathbf{M}_2(\varepsilon^{-1})\, \mathbf{R}$, partitioned as ${}^{00}\mathbf{K}_{\mu\varepsilon}(\omega)$, etc. Then (15) and (16) can be rewritten as

(17) $${}^{00}\mathbf{K}_{\varepsilon\mu}(\omega)\, {}^{0}\mathrm{E} = {}^{0}\mathrm{F}^{g} - {}^{01}\mathbf{K}_{\varepsilon\mu}(\omega)\, {}^{1}\mathrm{E},$$

(18) $${}^{00}\mathbf{K}_{\mu\varepsilon}(\omega)\, {}^{0}\mathrm{H} = {}^{0}\mathrm{G}^{g} - {}^{01}\mathbf{K}_{\mu\varepsilon}(\omega)\, {}^{1}\mathrm{H},$$

two systems of orders $E - E^e$ and $E - E^h$, respectively. The values of ω (not the same for both) for which they are singular are approximations of the above singular values.

The question arises again: In the case of a loaded cavity, can there be *real* such values, i.e., discrete air modes? Unfortunately, no proof similar to the previous one seems possible for the discretized version of the problem, and the following counter-example shakes hopes of finding one without some qualifying assumptions, which remain to be found. Consider the stiffness matrix of the air region, including its boundary, and denote by $\mathbf{u}$ the vector of degrees of freedom for all edges except those on the boundary of the load, which form vector $\mathbf{v}$. Write the discrete eigenvalue problem as $\mathbf{Au} + \mathbf{B}^t\,\mathbf{v} = \lambda\mathbf{u}$, $\mathbf{Bu} + \mathbf{Cv} = \lambda\mathbf{v}$. The question is: Can one exclude solutions with $\lambda \neq 0$ but $\mathbf{v} = 0$, i.e., such that $\mathbf{Au} = \lambda\mathbf{u}$ and $\mathbf{Bu} = 0$? For the mesh in inset, and in 2D, where the equation reduces to $-\Delta\varphi = \lambda\varphi$, one cannot. Let a, b, c, off-diagonal coefficients, repeat by six-fold symmetry. Let $\mathbf{u}$ be such that $\mathbf{Au} = \lambda\mathbf{u}$, with $\lambda \neq 0$, and $\mathbf{u}$ antisymmetric, that is, such that the degrees of freedom x, y, etc., alternate as suggested. (There *is* such an eigenvector.) Now, the row of $\mathbf{Au} = \lambda\mathbf{u}$ corresponding to the marked node yields $(b - c)y + ax = 0$, hence $\mathbf{Bu} = 0$.



Discrete air modes thus cannot so easily be dismissed. Do they appear? This would destroy well-posedness. But even their existence for meshes “close”, in some sense, to the actual one would be enough to create difficulties, when solving $(-\omega^2\,\mathbf{A} + \mathbf{B})\,\mathbf{u} = \mathbf{f}$, that is, in the frequential approach. Note that, fortunately, alternative “time domain” approaches exist (Remark 8.4), not prone to such difficulties [DM].

Note finally that problems (10) and (11) were equivalent, but (17) and (18) are not. They yield *complementary* views of the solution: (17) gives E (approximately, of course), hence $\mathrm{B} = (\mathrm{K}^g - \mathrm{rot}\,\mathrm{E})/i\omega$, whereas (18) gives H, whence $\mathrm{D} = (\mathrm{rot}\,\mathrm{H} - \mathrm{J}^g)/i\omega$. These four fields satisfy Maxwell equations *exactly*. But the constitutive relations $B = \mu H$ and $D = \varepsilon E$ are not rigorously satisfied, and the magnitude of the discrepancy is a good gauge of the accuracy achieved in this dual calculation. See [PB] for a development of this idea, with applications to adaptive mesh refinement in particular.

9.3.3 The question of spurious modes

Let's end with a topic that much intrigued the microwave community during the past 20 years (cf. [KT] for a review).

Let's go back to (12), but for an *empty* cavity ($\varepsilon = \varepsilon_0$ and $\mu = \mu_0$ all over) and *not* excited by outside sources (so E^g and H^g are zero). The equations reduce to

(19) $$- i\omega\, \varepsilon_0\, E + \mathrm{rot}\, H = 0 \text{ in } D, \quad n \times E = 0 \text{ on } S^e,$$

(20) $$i\omega\, \mu_0\, H + \mathrm{rot}\, E = 0 \text{ in } D, \quad n \times H = 0 \text{ on } S^h,$$

and they have nonzero solutions for resonating values of ω, which in this case correspond to *all* the eigenvalues (since all are real). So, if $\omega \neq 0$ is such a value, one has $\mathrm{div}\, B = 0$ and $\mathrm{div}\, D = 0$, where $B = \mu_0 H$ and $D = \varepsilon_0 E$: The electric and magnetic induction fields are solenoidal. (Note this is not the case when $\omega = 0$: There are solutions of the form $H = \mathrm{grad}\, \Phi$ and $E = \mathrm{grad}\, \Psi$, with Φ and Ψ non-harmonic.) Of course, whatever the method, one does not solve (19) and (20) with absolute accuracy, and one does not expect the relations $\mathrm{div}(\mu_0 H) = 0$ and $\mathrm{div}(\varepsilon_0 E) = 0$ to hold true, but at least one may hope for the magnitudes[5] of these divergences to be small, and to get smaller and smaller when the mesh grain tends to zero under the usual anti-flattening restrictions. Yet, before the advent of edge elements, such was not the case; all meshes showed modes with sizable divergence, which had to be rejected as "non-physical". The discussion in Chapter 6 helps understand why the emergence of such "spurious modes" is a defect inherent in the use of classical node-based vector-valued elements, and indeed, using Whitney elements is a sufficient condition for such spurious modes not to appear, as we now show.

[5]Since μH is *not* normally continuous across faces, there is a problem of definition here, for the divergence of μH is a distribution, not a function. In order to assess the "magnitude of the divergence" of μH, one should evaluate the norm of the mapping $\varphi' \to \int_D \mu\, H \cdot \mathrm{grad}\, \varphi'$, that is

$$\sup\{\, |\int_D \mu H \cdot \mathrm{grad}\, \varphi'| \,/\, [\int_D |\varphi'|^2]^{1/2} : \ \varphi' \in L^2_{\mathrm{grad}}(D), \ \varphi' \neq 0\}.$$

In practice, a weighted sum of the "flux losses" at faces makes a good indicator.

The "continuous" spectral problem consists in finding the values of ω for which, all source data being zero, and μ and ε real positive, Problem (11), that is, *find* $\mathrm{H} \in \mathbb{H}^0$ *such that*

$$(\mathrm{i}\omega\, \mu\, \mathrm{H}, \mathrm{H}') + ((\mathrm{i}\omega\, \varepsilon)^{-1} \operatorname{rot} \mathrm{H}, \operatorname{rot} \mathrm{H}') = 0 \quad \forall\, \mathrm{H}' \in \mathbb{H}^0,$$

has a nonzero solution. (The situation with respect to (10) is symmetrical, as we have seen.) Let's consider some Galerkin approximation to this problem, by which one wants to *find* $\mathrm{H} \in \mathcal{H}_m$ *such that*

(21) $$(\mathrm{i}\omega\, \mu\, \mathrm{H}, \mathrm{H}') + ((\mathrm{i}\omega\, \varepsilon)^{-1} \operatorname{rot} \mathrm{H}, \operatorname{rot} \mathrm{H}') = 0 \quad \forall\, \mathrm{H}' \in \mathcal{H}_m,$$

where $\mathcal{H}_m$ is a *finite*-dimensional subspace of $\mathbb{H}^0$. (This is $\mathbb{H}^0_m$, for the same mesh m, if edge elements are used.) This problem has no nonzero solution, except for a finite number of values of ω, corresponding to the eigenvalues of the matrix that represents, in some basis of $\mathcal{H}_m$, the bilinear form of the left-hand side in (21).

Now, consider the kernel of rot in the space $\mathcal{H}_m$. It's some subspace $\mathcal{K}_m$ which is, if one assumes a simply connected D, the image by grad of some finite-dimensional space $\mathcal{F}_m$, composed of functions which belong to L^2_{grad}. If, for some $\omega \neq 0$, (21) has a nonzero solution H, the latter verifies, a fortiori,

(22) $$(\mu\, \mathrm{H}, \operatorname{grad} \Phi') = 0 \quad \forall\, \Phi' \in \mathcal{F}_m\,.$$

This is a familiar relation: We spent most of Chapter 4 studying its consequences, where we saw to which extent $\mu\,\mathrm{H}$ is satisfactory, as an "m-weakly solenoidal" field. This happens when $\mathcal{F}_m$ is a *good* approximation space for L^2_{grad}, that is, "big enough", in an intuitively clear sense.

When $\mathcal{H}_m$ is $\mathbb{H}^0_m$, that is, with edge elements, the subspace $\mathcal{F}_m$ is indeed big enough, as we saw in Chapter 5; thereby, spurious modes are effectively eliminated [Bo, PR, WP]. In contrast, the use of nodal elements entails spaces $\mathcal{F}_m$ of very small dimensions, possibly 0, as we saw in Chapter 6. In such a case, nothing warrants any kind of weak solenoidality of $\mu\mathrm{H}$, hence the occurrence of spurious modes, so often observed and deplored [KT] before the advent of edge elements.

You stop, but that does not mean you have come to the end.
P. AUSTER, "*In the Country of Last Things*"

REFERENCES

[Bo] A. Bossavit: "Solving Maxwell's Equations in a Closed Cavity, and the Question of Spurious Modes", **IEEE Trans., MAG-26,** 2 (1990), pp. 702–705.

[DM] D.C. Dibben, A.C. Metaxas: "Finite Element Time Domain Analysis of Multimode Applicators Using Edge Elements", **J. Microwave Power & Electromagn. Energy, 29,** 4 (1994), pp. 242–251.

[Jo] A.K. Jonscher: "The 'universal' dielectric response", **Nature**, 267 (23 June 1977), pp. 673–679.

[KT] A. Konrad, I.A. Tsukerman: "Comparison of high- and low-frequency electromagnetic field analysis", **J. Phys. III France**, 3 (1993), pp. 363–371.

[PR] L. Pichon, A. Razek: "Analysis of Three-Dimensional Dielectric Loaded Cavities with Edge Elements", **ACES Journal, 66,** 2 (1991), pp. 133–142.

[PB] L. Pichon, A. Bossavit: "A new variational formulation, free of spurious modes, for the problem of loaded cavities", **IEEE Trans., MAG-29,** 2 (1993), pp. 1595–1600.

[St] M.A. Stuchly, S.S. Stuchly: "Dielectric Properties of Biological Substances —Tabulated", **J. Microwave Power, 15**, 1 (1980), pp. 19–26.

[FS] K.P. Foster, H.P. Schwan: "Dielectric properties of tissues and biological materials: a critical review", **Critical Reviews in Biomedical Engineering, 17**, 1 (1989), pp. 25–104.

[WP] M.-F. Wong, O. Picon, V. Fouad-Hanna: "Résolution par éléments finis d'arête des équations de Maxwell dans les problèmes de jonctions et cavités micro-ondes", **J. Phys. III France** (1992), pp. 2083–2099.

[Yo] K. Yosida: **Functional Analysis**, Springer-Verlag (Berlin), 1965.