CHAPTER 8

Eddy-current Problems

We now move beyond magnetostatics to tackle a non-stationary model. The starting point is again Maxwell's system with Ohm's law:

(1) $-\partial_t d + \text{rot}\, h = j,$ (2) $\partial_t b + \text{rot}\, e = 0,$

(3) $d = \varepsilon\, e,$ (4) $j = j^g + \sigma e,$ (5) $b = \mu\, h,$

where j^g is a given current density. In almost all of this chapter, we suppose j^g "harmonic", that is, of the form[1]

(6) $j^g(t) = \text{Re}[\text{J}^g \exp(i\omega t)],$

where J^g is a *complex*-valued vector field, and we'll look for all fields in similar form: $h(t) = \text{Re}[\text{H} \exp(i\omega t)]$, $e(t) = \text{Re}[\text{E} \exp(i\omega t)]$, etc. The functional spaces where these fields roam will still be denoted by $\mathbb{L}^2$, $\mathbb{L}^2_{\text{rot}}$, etc., but it should be clearly understood that *complexified* vector spaces are meant (see A.4.3). By convention, for two complex vectors $\text{U} = u_R + iu_I$ and $\text{V} = v_R + iv_I$, one has $\text{U} \cdot \text{V} = u_R \cdot v_R - i\, u_I \cdot v_I + i(u_I \cdot v_R + u_R \cdot v_I)$, the Hermitian scalar product being $\text{U} \cdot \text{V}^*$, where the star denotes complex conjugation, and the norm being given by $|\text{U}|^2 = \text{U} \cdot \text{U}^*$, not by $\text{U} \cdot \text{U}$. Note that an expression such as $(\text{rot}\, \text{U})^2$ should thus be understood as $\text{rot}\, \text{U} \cdot \text{rot}\, \text{U}$, not as $|\text{rot}\, \text{U}|^2$.

The form (6) of the given current is extremely common in electrotechnical applications, where one deals with alternating currents at a well-defined frequency f. The constant $\omega = 2\pi f$ is called *angular frequency*.

Under these conditions, system (1–5) becomes

(7) $-i\omega \text{D} + \text{rot}\, \text{H} = \text{J},$ (8) $i\omega\, \text{B} + \text{rot}\, \text{E} = 0,$

(9) $\text{D} = \varepsilon\, \text{E},$ (10) $\text{J} = \text{J}^g + \sigma\text{E},$ (11) $\text{B} = \mu\, \text{H}.$

[1] There are other possibilities, such as $\text{Im}[\text{J}^g \exp(i\omega t)]$, or $\text{Re}[\sqrt{2}\ \text{J}^g \exp(i\omega t)]$, etc. What matters is consistency in such uses.

8.1 THE MODEL IN H

The model under study in the present chapter is further characterized by a few simplifications, the most noteworthy being the neglect, for reasons we now indicate, of the term $-i\omega D$ in (7).

8.1.1 A typical problem

Figure 8.1 ([N a], pp. 209–247) shows a case study, typical of computations one may have to do when designing induction heating systems, among other examples: An induction coil, fed with alternative current, induces currents (called "eddy currents" or, in many national traditions, "Foucault currents") in an aluminum plate, and one wants to compute them.

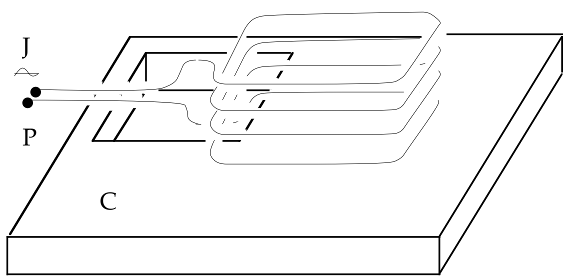


FIGURE 8.1. The real situation ("Problem 7" of the TEAM Workshop [Na]): compute eddy currents induced in the "passive conductor" C by an inductive coil, or "active conductor", which carries low-frequency alternating current. (The coil has many more loops than represented here, and occupies volume I of Fig. 8.2 below.) The problem is genuinely three-dimensional (no meaningful 2D modelling). Although the pieces are in minimal number and of simple shape, and the constitutive laws all linear, it's only during the 1980s that computations of similar complexity became commonplace.

Computing the field inside the coil, while taking its fine structure into account, is a pratical impossiblity, but is also unnecessary, so one can replace the situation of Fig. 8.1 by the following, idealized one, where the inducting current density is supposed to be given in some region I, of the same shape as the coil (Fig. 8.2).

This equivalent distribution of currents in I is easily computed, hence the source current J^g of (11), with I as its support. (One takes as given a mean current density, with small scale spatial variations averaged out.) If $\{E, H\}$ is the solution of (7–11), H is then a correct approximation to the actual magnetic field (inside I, as well), because the same currents are at stake (up to small variations near I) in both situations. However, the field E, as given by the same equations, has not much to do with the actual electric field, since in particular the way the coil is linked to the power supply is not considered. (There is, for instance, a high electric field between the connections, in the immediate vicinity of point P of Fig. 8.1, a fact which of course cannot be discovered by solving the problem of Fig. 8.2.)

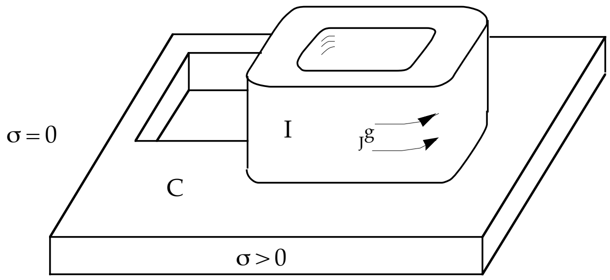


FIGURE 8.2. The modelled imaginary situation: Subregion I (for "inductor") is the support of a known alternative current, above a conductive plate C.

These considerations explain why emphasis will lie, in what follows, on the magnetic field H, and not on E (which we shall rapidly eliminate from the equations).

8.1.2 Dropping the displacement-currents term

Let us now introduce the main simplification, often described as the "low-frequency approximation", which consists in neglecting the term of "Maxwell displacement currents", that is $i\omega\, D$, in (7). By rewriting (7), (9), and (10) in the form $\text{rot } H = J^g + \sigma E + i\omega\, \varepsilon E$, one sees that this term is negligible in the conductor inasmuch as the ratio $\varepsilon\,\omega/\sigma$ can be considered small. In the air, where $\sigma = 0$, everything goes as if these displacement currents were added to the source-current J^g, and the approximation is justified if the ratio $\|i\omega D\|/\|J^g\|$ is small ($\|\ \|$ being some convenient norm). In many cases, the induced currents $J = \sigma E$ is of the same order of magnitude as the source current J^g, and the electric field is of the same order of magnitude outside and inside the conductor. If so, the ratio of $i\omega D$ to J^g is also on the order of $\varepsilon\,\omega/\sigma$. *The magnitude of the ratio* $\varepsilon\,\omega/\sigma$ *is thus often a good indicator of the validity of the low-frequency approximation.* In the case of induction heating at industrial frequencies, for instance $\omega = 100\,\pi$, the magnitude of σ being about 5×10^6, and $\varepsilon = \varepsilon_0 \cong 1/(36\pi \times 10^9)$, the ratio $\varepsilon\omega/\sigma$ drops to about 5×10^{-16}, and one cannot seriously object to the neglect of $-\,\mathrm{i}\omega \mathrm{D}$. Much higher in the spectrum, in simulations related to some medical practices such as hyperthermia [GF], where frequencies are on the order of 10 to 50 MHz, the conductivity of tissues on the order of 0.1 to 1, and with $\varepsilon \sim 10$ to $90\ \varepsilon_0$ [K&], one still gets a ratio $\varepsilon\omega/\sigma$ lower than 0.3, and the low-frequency approximation may still be acceptable, depending on the intended use. "Low frequency" is a very relative concept.

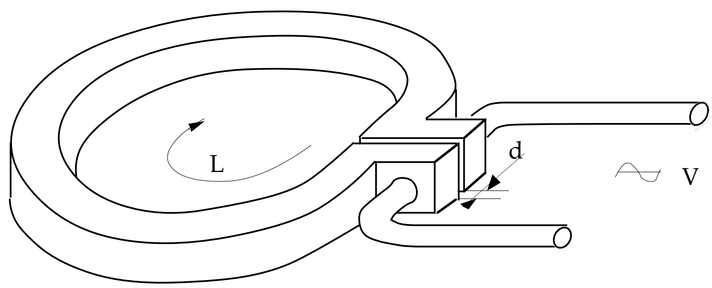


FIGURE 8.3. A case where capacitive effects may not be negligible (d << L).

But there are circumstances in which the electric field outside conductors is far larger than inside, and these are as many special cases. Consider Fig. 8.3, for instance, where L is the length of the loop and d the width of the gap. A simple computation (based on the relation $V \sim d\,|\mathrm{E}|$) shows that the ratio of $\varepsilon\omega\mathrm{E}$ *in the gap* to the current density *in the conductor* is on the order of $\varepsilon\omega L/d\sigma$, and thus may cease to be negligible when the ratio L/d gets large. This simply amounts to saying that the *capacitance* C of this gap, which is in ε/d, cannot be ignored in the computation when its product by the *resistance* R, which is in L/σ, reaches the order of ω^{-1} (recall that RC, whose dimension is that of a time interval, is precisely the time constant of a circuit of resistance R and capacitance C). One can assert in general that dropping $\mathrm{i}\omega\mathrm{D}$ from the equations amounts to neglecting *capacitive effects.* This is a legitimate approximation when the energy of the electromagnetic field is mainly stored in the "magnetic" compartment, as opposed to the "electric" one, in the language of Chapter 1.

One then has, instead of (7), $\mathrm{rot}\ \mathrm{H} = \mathrm{J}$. Consequently, $\mathrm{div}\ \mathrm{J} = 0$, and if the supports of J^g and σ are disjoint, one must assume $\mathrm{div}\ \mathrm{J}^g = 0$, after (11).

To sum up, we are interested in the family of problems that Fig. 8.4 depicts: a bounded conductor, connected, a given harmonic current, with bounded support, not encountering the conductor. (The last hypothesis is sensible in view of Fig. 8.1, but is not valid in all conceivable situations.) The objective is to determine the field H, from which the eddy currents of density $J = \text{rot}\, H$ in the conductor will be derived.

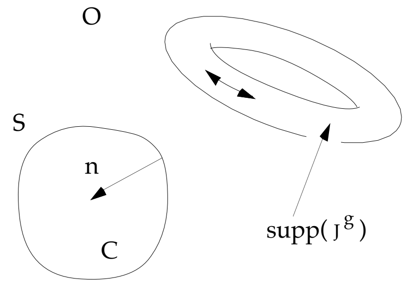


FIGURE 8.4. The theoretical situation. Note the convention about the normal unitary field on S, here taken as outgoing with respect to the "outer domain" $O = E_3 - C$.

From the mathematical point of view now, let thus C be a bounded domain of space, S its surface (Fig. 8.4), and J^g a given complex-valued field, such that $\text{supp}(J^g) \cap C = \varnothing$, and $\text{div}\, J^g = 0$. Again, we denote by O the domain which complements $C \cup S$, that is to say, the topological interior of $E_3 - C$ (take notice that O contains $\text{supp}(J^g)$). Domains C and O have S as common boundary, and the field of normals to S is taken as outgoing with respect to O. Conductivity σ and permeability μ being given, with $\text{supp}(\sigma) = C$, and $\sigma_1 \geq \sigma(x) \geq \sigma_0 > 0$ on C, where σ_0 and σ_1 are two constants, as well as $\mu(x) \geq \mu_0$ on C and $\mu(x) = \mu_0$ in O, one looks for $H \in \mathbb{L}^2_{rot}(E_3)$, complex-valued, such that

(12) $$i\omega\mu H + \text{rot}\, E = 0, \quad J = J^g + \sigma E, \quad \text{rot}\, H = J.$$

8.1.3 The problem in H, in the harmonic regime

Let us set, as we did up to now, $\mathbb{H} = \mathbb{L}^2_{rot}(E_3)$ (complex), and

$$\mathbb{H}^g = \{H \in \mathbb{H} : \text{rot}\, H = J^g \text{ in } O\},$$

$$\mathbb{H}^0 = \{H \in \mathbb{H} : \text{rot}\, H = 0 \text{ in } O\}.$$

We shall look for H in $\mathbb{H}^g$. One has $\mathbb{H}^g = H^g + \mathbb{H}^0$, with, as in magnetostatics (cf. the h^j of Chapter 7), $H^g = \text{rot}\, A^g$, where

$$A^g(x) = \frac{1}{4\pi}\int_{E_3} \frac{J^d(y)}{|x-y|}\,dy.$$

It all goes as if the source of the field was H^g, that is, the magnetic field that would settle in the presence of the inductor alone in space. The difference $\tilde{H} = H - H^g$ between effective field and source field is called *reaction field*.

Let us seek a weak formulation. From the first Eq. (12), and using the curl integration by parts formula, one has

$$0 = \int_{E_3} (i\omega\,\mu\, H + \text{rot } E)\cdot H' = i\omega \int_{E_3} \mu\, H\cdot H' + \int_{E_3} E\cdot \text{rot } H' \quad \forall\, H' \in \mathbb{H}^0.$$

As $\text{rot } H' = 0$ *outside* C (this is the key point), one may eliminate E by using the other two equations (12): for $E = \sigma^{-1}(J - J^g) \equiv \sigma^{-1} J$ in C, and thus $E = \sigma^{-1}\,\text{rot } H$. We finally arrive at the following prescription: *find* $H \in \mathbb{H}^g$ *such that*

$$\text{(13)} \qquad \int_{E_3} i\,\omega\,\mu\, H\cdot H' + \int_C \sigma^{-1}\,\text{rot } H\cdot \text{rot } H' = 0 \quad \forall\, H' \in \mathbb{H}^0.$$

Proposition 8.1. *If* $H^g \in \mathbb{L}^2_{rot}(E_3)$, *problem* (13) *has a unique solution* H, *and the mapping* $H^g \to H$ *is continuous from* $\mathbb{L}^2(E_3)$ *into* $\mathbb{H}$.

Proof. Let us look for H in the form $\tilde{H} + H^g$. After multiplication of both sides by $1 - i$, the problem (13) takes the form $a(\tilde{H}, H') = L(H')$, where L is continuous on $\mathbb{H}$, and

$$a(\tilde{H}, H') = \int_{E_3} \omega\,\mu\,\tilde{H}\cdot H' + \int_C \sigma^{-1}\,\text{rot } \tilde{H}\cdot \text{rot } H'$$
$$+\, i\left(\int_{E_3} \omega\,\mu\,\tilde{H}\cdot H' - \int_C \sigma^{-1}\,\text{rot } \tilde{H}\cdot \text{rot } H'\right).$$

As one sees, $\text{Re}[a(\tilde{H}, \tilde{H}^*) \geq C\,(\int_{E_3} |\tilde{H}|^2 + \int_C |\text{rot } \tilde{H}|^2)$ for some positive constant C. This is the property of *coercivity* on $\mathbb{H}^0$ under which Lax–Milgram's lemma of A.4.3 applies, in the complex case, hence the result. ◊

Remark 8.1. Problem (13) amounts to looking for the point of stationarity ("critical point") of the complex quantity

$$\mathcal{Z}(H) = i\omega \int_{E_3} \mu\, H^2 + \int_C \sigma^{-1}\,(\text{rot } H)^2,$$

when H spans $\mathbb{H}^g$. (There is a tight relationship between $\mathcal{Z}$ and what is called the *impedance* of the system.) So it's not a variational problem in the strict sense, and the minimization approach of former chapters is no longer available. By contrast, this emphasizes the importance of Lax–Milgram's lemma. ◊

The electric field has thus disappeared from this formulation. One easily retrieves it in C, where $E = \sigma^{-1}$ rot H. One may find it outside C by solving a static problem[2], formally similar to magnetostatics in region O, but this is rarely called for. (And anyway, this outside field would be fictitious, as already pointed out.)

To simplify, and to better emphasize the basic ideas, we first consider, in Section 8.2 below, the case when the passive conductor is contractible (i.e., simply connected with a connected boundary, cf. A.2.3), as in Fig. 8.4. It's obviously too strong a hypothesis (it doesn't hold in the above case), but the purpose is to focus on the treatement of the outer region, by the same method as for "open space" magnetostatics in Chapter 7. In Section 8.3, we'll reintroduce loops, but forfeit infinite domains, thus separately treating the two main difficulties in eddy-current computation.

8.2 INFINITE DOMAINS: "TRIFOU"

The key idea of the method to be presented now, already largely unveiled by the treatment of open-space magnetostatics of Chapter 7, is to reduce the computational domain to the conductor, in order not to discretize the air region[3] around. The method, implemented under the code name "Trifou", was promoted by J.C. Vérité and the author from 1980 onwards [B1, BV, BV'], and provided at the time the first solution of general applicability to the three-dimensional eddy-currents problem.

8.2.1 Reduction to a problem on C

We tackle problem (13), assuming C contractible (no current loops, no non-conductive hole inside C). In that case, the outside region O also is contractible.

[2]Namely the following problem: rot $E = -i\omega\mu H$, $D = \varepsilon_0 E$, div $D = Q$ in O, with $n \times E$ known on the boundary S, where Q is the density of electric charge outside C. The difficulty is that the latter is not known in region I, for lack of information on the fine structure of the inductor. One may assume $Q = 0$ with acceptable accuracy if the objective is to obtain E near C (hence in particular the surface charge on S, which is $\varepsilon_0\, n \cdot E$). Such information may be of interest in order to appraise the magnitude of capacitive effects.

[3]It's not always advisable thus to reduce the computational domain D to the passive conductor C. It's done here for the sake of maximum simplicity. But "leaving some air" around C may be a good idea, for instance, in the presence of small air gaps, conductors of complex geometry, and so forth. Methods for such cases will be examined in Section 8.3.

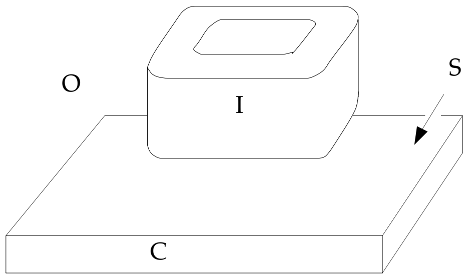


FIGURE 8.5. Model problem for the study of the hybrid approach in "Trifou", finite elements in the conductor C, and integral method over its boundary S to take the far field into account. The support of the given current density J^g is the inductor I. Contrary to Fig. 8.2, C here is loop-free, and we restrict consideration to this case to separate the difficulties. Section 8.3 will address loops (but shun the far-field effect).

Let's keep the notations of Chapter 7: Φ is the space of magnetic potentials (the Beppo Levi space of 7.2.1), Φ_O is composed of the restrictions to O of elements of Φ, and the set of elements of Φ_O that have in common the trace ϕ_S is denoted $\Phi_O(\phi_S)$. Let Φ^{00} stand[4] for the subspace $\{\phi \in \Phi : \phi = 0 \text{ on } D\}$. Set

$$\mathbb{K}^g = \{H \in \mathbb{H}^g : \int_{E_3} H \cdot \text{grad } \phi' = 0 \quad \forall \phi' \in \Phi^{00}\}$$

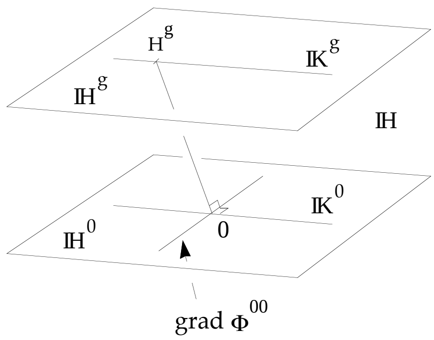
(the support of the integrand reduces to O, in fact), and

$$\mathbb{K}^0 = \{H \in \mathbb{H}^0 : \int_{E_3} H \cdot \text{grad } \phi' = 0 \quad \forall \phi' \in \Phi^{00}\}.$$

We note that

$$(14) \qquad \mathbb{H}^0 = \mathbb{K}^0 \oplus \text{grad } \Phi^{00},$$

by construction, and that $\mathbb{K}^g = H^g + \mathbb{K}^0$, for H^g is orthogonal to grad Φ^{00}, since div $H^g = 0$. (Cf. the inset drawing.)



By their very definition, the elements of $\mathbb{K}^g$ and of the parallel subspace $\mathbb{K}^0$ satisfy div $H = 0$ in O. This property is shared by the required solution, since div $H = \mu_0^{-1}$ div $B = 0$ in O. One may therefore expect to find this solution in $\mathbb{K}^g$. Which is indeed the case:

[4]The notation Φ^0 is reserved for an analogous, but slightly larger space (see below).

Proposition 8.2. *The solution* H *of Problem* (13) *lies in* $\mathbb{K}^g$.

Proof. By letting $H' = \text{grad}\,\Phi'$ in (13), where Φ' roams in Φ^{00}, one gets

$$0 = i\omega \int_{E_3} \mu\, H \cdot \text{grad}\,\Phi' = i\omega\,\mu_0 \int_{E_3} H \cdot \text{grad}\,\Phi' \quad \forall\, \Phi' \in \Phi^{00},$$

and hence $H \in \mathbb{K}^g$. ◊

To find the point of stationarity of $H \to Z(H)$ in $\mathbb{H}^g$, it is thus enough to look for it in $\mathbb{K}^g$, and to check that no other, spurious critical point is in the way. Indeed,

Corollary of Prop. 8.2. *Problem* (13) *is equivalent to* find $H \in \mathbb{K}^g$ such that

(15) $$i\omega \int_{E_3} \mu\, H \cdot H' + \int_C \sigma^{-1}\, \text{rot}\, H \cdot \text{rot}\, H' = 0 \quad \forall\, H' \in \mathbb{K}^0,$$

since this is the Euler equation for the search of critical points of Z in the affine subspace $\mathbb{K}^g$, and it has at most one solution.

Our effort, now, will concentrate on showing that Problem (15) is in fact "posed on C", meaning that a field in $\mathbb{K}^g$ (or in $\mathbb{K}^0$) is entirely determined by its restriction to C. I expect this to be obvious "on physical grounds", but this doesn't make the proofs any shorter. We are embarked on a long journey, till the end of 8.2.3. The operator P of Chapter 7 will play a prominent part in these developments.

Remark 8.2. Set $\Phi^0 = \{\Phi \in \Phi : \text{grad}\,\Phi = 0 \text{ on } C\}$. By introducing as before the orthogonal subspaces $\mathbb{K}^{g0}$ and $\mathbb{K}^{00}$, one would have $\mathbb{H}^0 = \mathbb{K}^{00} \oplus \text{grad}\,\Phi^0$, and one could proceed with the same kind of reduction, with $\mathbb{K}^{g0}$ strictly contained in $\mathbb{K}^g$. This may look like an advantage, but in practice, it makes little difference. ◊

Exercise 8.1. Show that $\int_S n \cdot H = 0$, and prove the analogue of Prop. 8.2 in the context suggested by Remark 8.2.

8.2.2 The space HΦ, isomorphic to $\mathbb{K}^g$

Let now HΦ (treated as a single symbol) stand for the vector space of pairs $\{H, \Phi_S\}$, where H is a field supported on C and Φ_S a surface potential "associated with" H, in the precise following sense[5]:

(16) $$H\Phi = \{\{H, \Phi_S\} \in \mathbb{L}^2_{rot}(C) \times H^{1/2}(S) : H_S = \text{grad}_S\, \Phi_S\},$$

where grad_S denotes the surface gradient. Note that the projection of HΦ on the first factor of the Cartesian product $\mathbb{L}^2_{rot}(C) \times H^{1/2}(S)$ is not

[5]Refer to Fig. 2.5 for H_S, the tangential part of H.

$\mathbb{L}^2_{rot}(C)$ in its entirety, for there are constraints that H must satisfy, in particular $n \cdot \text{rot}\, H = 0$ on S. On the other hand, Φ_S may be any function in $H^{1/2}(S)$. One provides $H\Phi$ with its natural Hilbertian norm, induced by the norm of the encompassing Cartesian product. Then,

Proposition 8.3. $H\Phi$ *is isomorphic to* $\mathbb{K}^g$ *and* $\mathbb{K}^0$.

Proof. Since C is simply connected, and $\text{rot}\, H^g = 0$ in C, there exists $\Phi^g \in L^2_{grad}(C)$ such that $H^g = \text{grad}\, \Phi^g$ in C. Let us still denote by Φ^g the harmonic continuation of this function outside C. Now, take $H \in \mathbb{K}^g$. One has $\text{rot}\, H = J^g = \text{rot}(H^g - \text{grad}\, \Phi^g)$ in O. Since O is simply connected, there exists a unique Φ in BL(O) such that the equality $H = H^g + \text{grad}(\Phi - \Phi^g)$ hold in O. By restricting H and Φ to C and S, a map from H to the pair $\{H, \Phi_S\}$ of $H\Phi$ is therefore defined. Conversely, such a pair $\{H(C), \Phi_S\}$ being given, let Φ be the exterior harmonic continuation of Φ_S. Set H equal to $H(C)$ in C and to $H^g + \text{grad}(\Phi - \Phi^g)$ outside C. This enforces $H \in \mathbb{L}^2_{rot}(E_3)$, because both tangential traces $H(C)_S$ and $H^g_S + \text{grad}_S(\Phi_S - \Phi^g_S) \equiv \text{grad}_S \Phi_S$ coincide, after (16). In O, $\text{rot}\, H = J^g$ and $\text{div}\, H = 0$ by construction, whence $H \in \mathbb{K}^g$. Moreover, the one-to-one correspondence thus established (in a way that would apply as well to $\mathbb{K}^0$, just consider the special case $J^g = 0$) is an isometry. Hence the announced isomorphisms: For if H is the difference between two elements of $\mathbb{K}^g$, then $H = \text{grad}\, \Phi$ outside C, and

$$\int_{E_3} |H|^2 + \int_C |\text{rot}\, H|^2 = \int_C |H|^2 + \int_C |\text{rot}\, H|^2 + \int_O |\text{grad}\, \Phi|^2$$
$$= \int_C |H|^2 + \int_C |\text{rot}\, H|^2 + \int_S P\Phi_S\, \Phi_S,$$

which is indeed the square of the norm of the pair $\{H, \Phi_S\}$ in the product $\mathbb{L}^2_{rot}(C) \times H^{1/2}(S)$. ◊

Remark 8.3. The isomorphism depends of course on Φ^g, which in turn depends on J^g, up to an additive constant. ◊

8.2.3 Reformulation of the problem in $H\Phi$

So, let Φ^g_S be the function on S associated with J^g that specifies the above isomorphism. One has

$$\mathcal{Z}(H) = i\omega \int_{E_3} \mu\, H^2 + \int_C \sigma^{-1} (\text{rot}\, H)^2$$
$$= i\omega \int_C \mu\, H^2 + \int_C \sigma^{-1} (\text{rot}\, H)^2 + i\omega\, \mu_0 \int_O (H^g + \text{grad}(\Phi - \Phi^g))^2,$$

and we are looking for the critical point of this function in $H\Phi$. Since, thanks to the properties of P,

$$\int_O (\mathrm{H}^g + \mathrm{grad}(\Phi - \Phi^g))^2 = \int_O (\mathrm{H}^g - \mathrm{grad}\, \Phi^g)^2 + 2\int_S \mathrm{n} \cdot \mathrm{H}^g\, \Phi_S$$
$$+ \int_S \mathrm{P}\Phi_S\, \Phi_S - 2\int_S \mathrm{P}\Phi^g{}_S\, \Phi_S\,,$$

Problem (15), equivalent to the initial problem (13) under our assumptions, amounts to finding the critical point of the function $\tilde{\mathcal{Z}}$ (equal to $\mathcal{Z}$ up to a constant) thus defined:

$$\tilde{\mathcal{Z}}(\{\mathrm{H}, \Phi_S\}) = i\omega\, [\int_C \mu\, \mathrm{H}^2 + \mu_0 \int_S \mathrm{P}\Phi_S\, \Phi_S] + \int_C \sigma^{-1} (\mathrm{rot}\, \mathrm{H})^2$$
$$+ 2\, i\omega\, \mu_0 \int_S (\mathrm{n} \cdot \mathrm{H}^g - \mathrm{P}\Phi^g{}_S)\, \Phi_S,$$

whence, by taking the Euler equation, the following result (index S is understood in Φ_S, Φ'_S and $\Phi^g{}_S$):

Proposition 8.4. *When* C *is contractible,* (13) *is equivalent to* find $\{\mathrm{H}, \Phi\}$ in $\mathbf{H\Phi}$ such that

(17) $$i\omega\, [\int_C \mu\, \mathrm{H} \cdot \mathrm{H}' + \mu_0 \int_S \mathrm{P}\Phi\, \Phi'] + \int_C \sigma^{-1}\, \mathrm{rot}\, \mathrm{H} \cdot \mathrm{rot}\, \mathrm{H}'$$
$$= i\omega\, \mu_0 \int_S (\mathrm{P}\Phi^g - \mathrm{n} \cdot \mathrm{H}^g)\, \Phi' \quad \forall\, \{\mathrm{H}', \Phi'\} \in \mathbf{H\Phi}.$$

This is the final weak formulation, on which "Trifou" was based. The pending issue is how to discretize it. Clearly, H and H' in (17) will be represented by edge elements, and Φ and Φ' by surface nodal elements (these are compatible representations, thanks to the structural properties of the Whitney complex). The discretization of terms $\int_S \mathrm{P}\Phi\, \Phi'$ and $\int_S \mathrm{P}\Phi^g\, \Phi'$ by Galerkin's method will make use of the matrix $\mathbf{P}$ of Chapter 7 (Subsection 7.4.5, Eq. (48)).

8.2.4 Final discrete formulation

Let $\mathrm{K} = \{\mathbf{H}_e : e \in \mathcal{E}^0(C);\ \mathbf{\Phi}_n : n \in \mathcal{N}(S)\}$ be the vector of all degrees of freedom (complex valued): one DoF $\mathbf{H}_e$ for each edge *inside* C (that is, not contained in S) and one DoF $\mathbf{\Phi}_n$ for each surface node. The expression of H in C is thus

$$\mathrm{H} = \sum_{e \in \mathcal{E}^0(C)} \mathbf{H}_e\, w_e + \sum_{n \in \mathcal{N}(S)} \mathbf{\Phi}_n\, \mathrm{grad}\, w_n.$$

(This is an element of $W^1{}_m(C)$, thanks to the inclusion $\mathrm{grad}\, W^0 \subset W^1$.) Then (17) becomes

(18) $$i\omega\, (\mathbf{M}(\mu) + \mu_0\, \mathbf{P})\, \mathrm{K} + \mathbf{N}(\sigma)\, \mathrm{K} = i\omega\, \mu_0\, (\mathbf{P\Phi}^g - \mathrm{L}^g),$$

with obvious notations, except for L^g, defined by $\mathrm{L}^g{}_m = \int_S \mathrm{n} \cdot \mathrm{H}^g\, w_m$ for all

$m \in \mathcal{N}(S)$, and other components null. (Beware, though $\mathbf{M}$ is very close to the mass-matrix $\mathbf{M}_1(\mu)$ of the mesh of C, it's not quite the same, just as $\mathbf{N}(\sigma)$ is not quite $\mathbf{R}^t\mathbf{M}_2(\sigma^{-1})\mathbf{R}$.) As in Chapter 7, $\mathbf{P}$ bears only on the "Φ part" of vector κ, a priori, but is bordered by zeroes in order to give sense to (18). (Same remark about $\mathbf{P}\Phi^g$.) Matrices $\mathbf{M}$, $\mathbf{P}$, and $\mathbf{N}$ are symmetric, but because of the factor i, the matrix of the linear system (18) is not Hermitian. In spite of this, the conjugate gradient method, the convergence of which is only guaranteed in the Hermitian case, in theory, works fairly well in practice, with proper conditioning.

Computing the right-hand side of (18) is straightforward: H^g is known by the Biot and Savart formula, and the vector Φ^g of nodal values is derived from the circulations of H^g along the edges of S.

8.2.5 Transient regimes

From (18) to a scheme for the temporal evolution problem is but a short trip, compared to what has been done, so let's show the way, even though this is beyond the objectives of this chapter. Now the given current j^g is real-valued and time-dependent, and the field h is given at time 0, with $\text{div}(\mu h(0)) = 0$. The DoFs are real-valued again. The evolution scheme, discrete in space but not yet in time, is

$$(19) \qquad \partial_t[(\mathbf{M}(\mu) + \mu_0 \mathbf{P})\,\mathbf{k}] + \mathbf{N}(\sigma)\,\mathbf{k} = \mu_0\,\partial_t(\mathbf{P}\boldsymbol{\varphi}^g - \mathbf{l}^g)$$

($\mathbf{k}(0)$ given by the initial conditions, and $\mathbf{l}$ similar to L, but real-valued). Over a temporal interval $[0, T]$, with a time step δt, one may treat this by a Crank–Nicolson scheme [CN] : $\mathbf{k}^0 = \mathbf{k}(0)$, then, for each integer m from 0 to $T/\delta t - 1$,

$$(20) \qquad (\mathbf{M}(\mu) + \mu_0 \mathbf{P})\,(\mathbf{k}^{m+1} - \mathbf{k}^m) + \delta t\;\mathbf{N}(\sigma)\,(\mathbf{k}^{m+1} + \mathbf{k}^m)/2$$
$$= \mu_0\,\mathbf{P}\,[\boldsymbol{\varphi}^g((m+1)\,\delta t) - \boldsymbol{\varphi}^g(m\delta t)] + \mathbf{l}^g((m+1)\,\delta t) - \mathbf{l}^g(m\,\delta t).$$

If $\mathbf{k}^m$ is known, this is a linear system with respect to the unknown $\mathbf{k}^{m+1}$. (Actually, better take $\mathbf{k}^{m+1/2} \equiv (\mathbf{k}^{m+1} + \mathbf{k}^m)/2$ as unknown. Then $\mathbf{k}^{m+1} - \mathbf{k}^m = 2(\mathbf{k}^{m+1/2} - \mathbf{k}^m)$. Unconditionally stable as it may be, the Crank–Nicolson scheme may suffer from numerical oscillations ("weak instability"), to which the sequence of the $\mathbf{k}^{m+1/2}$s is less sensitive than the $\mathbf{k}^m$s.)

This scheme can easily be adapted to the case of a nonlinear b–h law. See [Bo] for details.

Remark 8.4. When j^g is sinusoidal, using this scheme is a viable alternative to solving (18) directly. One should use an informed guess of the solution as initial condition, and monitor the time average over a period, $(2\pi)^{-1}\omega\int_{t-2\pi/\omega}^{t} \exp(i\omega\, s)\, \mathbf{k}(s)\, ds$, duly approximated by a sum of the kind $N^{-1}\sum_{j=1,N} \exp(i\omega(m+1/2-j)\delta t)\ \mathbf{k}^{m+1/2-j}$, where $N\delta t = 2\pi/\omega$, the period. This will converge, relatively rapidly (no more than three or four periods, in practice) towards the solution κ of (18). Such a "time domain" approach to the harmonic problem can thus be conceived as another iterative scheme to solve (18). Fast Fourier Transform techniques make the calculation of time averages quite efficient. ◊

8.3 BOUNDED DOMAINS: TREES, H–Φ

Now we change tack. Let D be a simply connected domain containing the region of interest (here the conductor C and its immediate neighborhood, Fig. 8.6), the inductor I, magnetic parts where $\mu \neq \mu_0$, if such exist, and suppose either that D is big enough so that one can assume a zero field beyond, or that $n \times h = 0$ on ∂D for physical reasons (as in the case of a cavity with ferromagnetic walls, used as a shield to confine the field inside). We thus forget about the far field and concentrate on difficulties linked with the *degeneracy* of the eddy-current equations in regions where $\sigma = 0$.

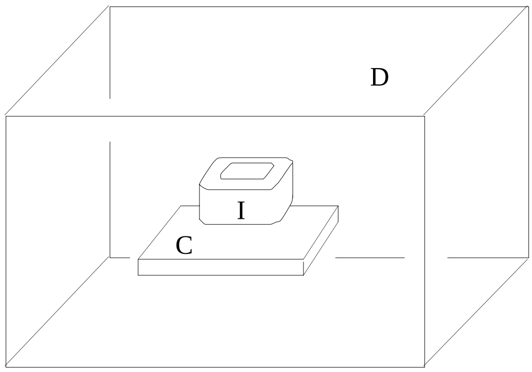


FIGURE 8.6. Computational domain D, containing the region of interest, and large enough for the boundary condition $n \times h = 0$ on ∂D to be acceptable. (In practice, the size of the elements would be graded in such a case, the farther from C the bigger, the same as with Fig. 7.2.)

8.3.1 A constrained linear system

Let m be a simplicial mesh of D. Functional spaces, a bit different now, are $\mathbb{H} = \{H \in \mathbb{L}^2_{rot}(D) : n \times H = 0 \text{ on } \partial D\}$ and

(21) $\quad \mathbb{H}^g = \{H \in \mathbb{H} : \text{rot } H = J^g \text{ in } D - C\},$

(22) $\quad \mathbb{H}^0 = \{H \in \mathbb{H} : \text{rot } H = 0 \text{ in } D - C\}.$

Now we know the paradigm well, and we can state the problem to solve without further ado: *find* $H \in \mathbb{H}^g$ *such that*

(23) $\quad \int_D i\,\omega\,\mu\, H \cdot H' + \int_C \sigma^{-1} \text{rot } H \,.\, \text{rot } H' = 0 \quad \forall\, H' \in \mathbb{H}^0.$

As we intend to enforce null boundary conditions on the boundary of D, let us remove from $\mathcal{N}$, $\mathcal{E}$, $\mathcal{F}$ the boundary simplices, as we did earlier in 7.3.1, and for convenience, still call $\mathcal{N}$, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{T}$ the simplicial sets of this "peeled out" mesh. Apart from this modification, the notation concerning the spaces W^p and the incidence matrices is the same as before. In particular, W^1_m is the span—with *complex* coefficients, this time—of Whitney edge elements w_e, for all e in $\mathcal{E}$. As this amounts to have null circulations along the boundary edges, a field in W^1_m can be prolongated by 0 to all space, the result being tangentially continuous and therefore an element of $\mathbb{L}^2_{rot}(E_3)$. So we can identify W^1_m with a subspace of $\mathbb{L}^2_{rot}(E_3)$. Let us set $\mathbb{H}_m = W^1_m$, denote by $\mathrm{I}\mathbf{H}$ the isomorphic finite-dimensional space $\mathbb{C}^{\mathcal{E}}$, composed of all vectors $\mathbf{H} = \{\mathbf{H}_e : e \in \mathcal{E}\}$, and call $E = \#(\mathcal{E})$ the number of inner edges. For $\mathbf{U}$ and $\mathbf{U}'$ both in $\mathrm{I}\mathbf{H}$, we set

$$(\mathbf{U}, \mathbf{U}') = \sum_{e \in \mathcal{E}} \mathbf{U}_e \cdot \mathbf{U}'_e$$
$$\equiv \sum_{e \in \mathcal{E}} (\text{Re}[\mathbf{U}_e] + i\,\text{Im}[\mathbf{U}_e]) \cdot (\text{Re}[\mathbf{U}'_e] + i\,\text{Im}[\mathbf{U}'_e]).$$

(Again, beware: This is not the Hermitian scalar product.) This way, an integral of the form $\int_D \alpha\, U \cdot U'$, where α is a function on D (such as μ, for instance), possibly complex-valued, is equal to $(\mathbf{M}_1(\alpha)\, \mathbf{U}, \mathbf{U}')$ when $U = \sum_{e \in \mathcal{E}} \mathbf{U}_e w_e$ and $U' = \sum_{e \in \mathcal{E}} \mathbf{U}'_e w_e$.

We know from experience the eventual form of the discretized problem: It will be *find* $H \in \mathbb{H}^g_m$ *such that*

(24) $\quad \int_D i\,\omega\,\mu\, H \cdot H' + \int_C \sigma^{-1} \text{rot } H \,.\, \text{rot } H' = 0 \quad \forall\, H' \in \mathbb{H}^0_m,$

where $\mathbb{H}^g_m$ and $\mathbb{H}^0_m$ are parallel subspaces of $\mathbb{H}_m$.

But these subspaces must be constructed with some care. One cannot simply set $\mathbb{H}^g_m = W^1_m \cap \mathbb{H}^g \equiv \{H \in W^1_m : \text{rot } H = J^g \text{ in } D - C\}$, because J^g has no reason to be mesh-wise constant (which rot H is, if $H \in W^1_m$), although this happens frequently. Failing that, $W^1_m \cap \mathbb{H}^g$ may very well reduce to {0}. So we need to find a subspace of W^1_m that closely approximate $W^1_m \cap \mathbb{H}^g$. For this, let $\mathcal{F}_C \subset \mathcal{F}$ be the subset of "conductive" faces, i.e., those *inside* C, faces of its boundary being excluded. Let us set

$$\mathbb{H}^g_m = \{H \in W^1_m : \int_f n \cdot \text{rot } H = \int_f n \cdot J^g \ \ \forall f \notin \mathcal{F}_C\}.$$

This time, $\mathbb{H}^g_m$ is in $\mathbb{H}_m \equiv W^1_m$, and the isomorphic space $\mathbf{\mathbb{H}}^g$ is characterized by

$$\mathbf{\mathbb{H}}^g = \{\mathbf{H} \in \mathbf{\mathbb{H}} : (\mathbf{R}\,\mathbf{H})_f = \mathbf{J}^g_f \ \ \forall f \notin \mathcal{F}_C\}, \tag{25}$$

where $\mathbf{J}_f^g$ is the intensity $\int_f n \cdot J^g$ through face f and $\mathbf{R}$ the edge-to-faces incidence matrix. As in Chapter 6, we shall abridge all this as follows: $\mathbf{\mathbb{H}}^g = \{\mathbf{H} \in \mathbf{\mathbb{H}} : \mathbf{L}\mathbf{H} = \mathbf{L}^g\}$, where $\mathbf{L}$ is a submatrix of $\mathbf{R}$, the dimensions of which are $(F - F_C) \times E$ (F inner faces in D, minus the F_C conductive faces) and $\mathbf{L}^g$ a known vector. Denoting by $\mathbf{\mathbb{H}}^0$ the kernel of $\mathbf{L}$ in $\mathbf{\mathbb{H}}$, we may now reformulate (24) like this: *find* $\mathbf{H} \in \mathbf{\mathbb{H}}^g$ *such that*

$$i\,\omega\,(\mathbf{M}_1(\mu)\,\mathbf{H}, \mathbf{H}') + (\mathbf{M}_2(\sigma^{-1})\,\mathbf{R}\,\mathbf{H}, \mathbf{R}\,\mathbf{H}') = 0 \ \ \forall \mathbf{H}' \in \mathbf{\mathbb{H}}^0. \tag{26}$$

This is, as in Chapter 6, a constrained linear system, which can be rewritten as follows:

$$\begin{vmatrix} i\,\omega\,\mathbf{M}_1(\mu) + \mathbf{R}^t\,\mathbf{M}_2(\sigma^{-1})\mathbf{R} & \mathbf{L}^t \\ \mathbf{L} & 0 \end{vmatrix} \begin{vmatrix} \mathbf{H} \\ \mathbf{V} \end{vmatrix} = \begin{vmatrix} 0 \\ \mathbf{L}^g \end{vmatrix}, \tag{27}$$

where the dimension of the vector-valued Lagrange multiplier $\mathbf{V}$ is $F - F_C$.

Exercise 8.2. Find a physical interpretation for the $\mathbf{V}_f$s.

One can very well tackle the system (27) as it stands (cf. Appendix B). But for the same reasons as in Chapter 6, one may prefer to use a representation of $\mathbf{\mathbb{H}}^g$ and $\mathbf{\mathbb{H}}^0$ in terms of *independent* degrees of freedom. There are two main ways to do that.

8.3.2 The tree method

We suppose C contractible for a while. Let $\mathcal{N}_C$, $\mathcal{E}_C$, $\mathcal{F}_C$ denote the sets of nodes, edges, and faces inside C. Let us form a spanning tree $\mathcal{E}^T$ (cf. 5.3.2) for the mesh of the closure of $D - C$. (Some edges of the interface ∂C will belong to $\mathcal{E}^T$, and form a spanning tree for this surface.) Let's recall that for each edge of $\mathcal{E} - \mathcal{E}_C - \mathcal{E}^T$, or co-edge, one can build a closed chain over $D - C$ by adding to it some edges of $\mathcal{E}^T$, with uniquely determined coefficients. The idea is to select as independent DoFs the mmf's along the E_C edges inside C and the E^T edges of the tree. Then, the mmf's along the co-edges can be retrieved by using (25), as explained in the next paragraph. This vector of independent DoFs will be denoted U. The corresponding vector space, isomorphic to $\mathbb{C}^{E_C + E^T}$, is denoted $\mathbf{U}$.

For each edge e of $\mathcal{E}_C \cup \mathcal{E}^T$, set $\text{H}_e = \text{U}_e$. Any other edge $e \in \mathcal{E}$ is a co-edge, and is thus the closing edge of a circuit all other edges of which come from $\mathcal{E}^T$, by construction. Call $C(e) \subset \mathcal{E}^T$ the set composed of these other edges, and $\mathbf{c}_\varepsilon$ the chain-coefficient assigned to edge $\varepsilon \in C(e)$ by the procedure of 5.3.2. The circuit they form bounds a two-sided[6] and hence orientable polyhedral surface Σ_e, formed of faces of $\mathcal{F} - \mathcal{F}_C$. Each of the two possible fields of normals on Σ_e orients its boundary $\partial\Sigma_e$, as we saw in Chapter 5. Let n be the one for which e and $\partial\Sigma_e$ are oriented the same way. Now, assign the value

(28) $$\text{H}_e = \int_{\Sigma_e} n \cdot \text{J}^g + \sum_{\varepsilon \in C(e)} \mathbf{c}_\varepsilon \, \text{U}_\varepsilon,$$

as DoF to co-edge e. This completes the mmf vector H, hence a field $\text{H} = \sum_{e \in \mathcal{E}} \text{H}_e \, w_e$, associated with U. We'll denote this correspondence by $\text{H} = f(\text{U}, \text{J}^g)$. (Beware: U is a **vector**, H is a field.) Let now

(29) $$\mathbb{K}_m{}^g = \{\text{H} : \text{H} = \sum_{e \in \mathcal{E}} \text{H}_e \, w_e\} \equiv \{f(\text{U}, \text{J}^g) : \text{U} \in \mathbf{U}\}$$

be the span of these fields in $\mathbb{H}_m$, and $\mathbb{K}_m{}^0$ be the parallel subspace, which is obtained by exactly the same construction, but with $\text{J}^g = 0$. Thus constructed, $\mathbb{K}_m{}^g$ and $\mathbb{K}_m{}^0$ coincide with $\mathbb{H}_m{}^g$ and $\mathbb{H}_m{}^0$.

As a bonus, the above construction gives an approximation $\text{H}^g{}_m \in W^1{}_m$ of the source-field H^g: the field corresponding to $\text{U} = 0$, that is $\text{H}^g{}_m = f(\mathbf{0}, \text{J}^g)$. Note that $\mathbb{K}^g{}_m = \text{H}^g{}_m + \mathbb{K}^0{}_m$, and hence $\mathbb{H}^g{}_m = \text{H}^g{}_m + \mathbb{H}^0{}_m$ as well, that is to say,

(30) $$\mathbb{H}^g{}_m = \{\text{H}^g{}_m + \text{H} : \text{H} \in \mathbb{H}^0{}_m\}.$$

[6] This is *not* supposed to be obvious (but please read on, and return to the present note at leisure). The circuit based on a co-edge can be a knot of arbitrary complexity, so it's not so clear that it always bounds an orientable and non-self-intersecting surface. But this is true, being a theorem in knot theory. Such a surface, called a *Seifert surface* (cf., e.g., [Ro]), always exists [Se], however tangled the knot may be. See Fig. 8.9 in Exercise 8.3 at the end.

Since $\mathbb{K}^g_m$ and $\mathbb{K}^0_m$ coincide with $\mathbb{H}^g_m$ and $\mathbb{H}^0_m$, all we have to do now is throw into (24) the expressions $\text{H} = f(\text{U}, \text{J}^g)$ and $\text{H}' = f(\text{U}', 0)$ in order to obtain a linear system in terms of U, the form of which is

$$(31) \qquad (i\omega\, \mathbf{M} + \mathbf{N})\, \text{U} = \text{L}^g,$$

with $\mathbf{M}$ and $\mathbf{N}$ symmetrical, non-negative definite, and $\mathbf{M} + \mathbf{N}$ regular. But this time $\mathbf{M}$ and $\mathbf{N}$ largely differ from $\mathbf{M}_1(\mu)$ and $\mathbf{R}^t\, \mathbf{M}_2(\sigma^{-1})\mathbf{R}$ (only the blocks relative to the edges of $\mathcal{E}_C$ coincide), and overall, their conditioning greatly depends on the tree construction. (To each spanning tree corresponds a particular basis for the space $\mathbb{H}_m{}^0$.) Not all trees are thus equivalent in this respect, and finding methods that generate good spanning trees is a current research subject.

The matrix $i\omega\, \mathbf{M} + \mathbf{N}$ is not Hermitian, and this raises specific algorithmic problems. So here begins the *numerical* work (to say nothing of the *programming* work, which is far from run-of-the-mill), but we shall stop there, because the *modelling* work is done—at least in the case when C is contractible.

So how can the technique be generalized to the non-contractible case? If there are only "holes", i.e., if C is simply connected but with a non-connected boundary, no difficulty: Just build a spanning tree for each connected component of D – C. The problem is with "loops". Suppose for definiteness there is a single loop in C, as in Fig. 8.6. Then, by a deep but intuitively obvious result of topology ("Alexander's duality", cf. [GH]), there is also one loop in D – C. There are now two kinds of co-edges, depending on whether the circuits they close surround the conductive loop or not. (Note that those which do surround the loop do *not* bound a polyhedral surface of the kind discussed above, that is, made of faces in $\mathcal{F} - \mathcal{F}_C$, and this is what characterizes them.) Next, select *one* of these loop co-edges, and add it to the initial tree, thus obtaining a "belted tree". Thanks to this added edge, the circuits of all remaining co-edges do bound, as we noticed in 5.3.2. Obviously (by Ampère), the DoF of the belt fastener is the intensity in the current loop. There is one additional DoF of this kind for each current loop. With this, the key result ($\mathbb{K}_m{}^g$ and $\mathbb{K}_m{}^0$ coincide with $\mathbb{H}_m{}^g$ and $\mathbb{H}_m{}^0$) stays valid, and everything elses goes unchanged.

8.3.3 The H–Φ method

The H–Φ method is "edge elements and nodal elements in association" and stems from the second way to obtain a set of independent degrees of freedom. With the previous method, the DoFs were all magnetomotive forces, those along the selected edges. Now, we'll have two different kinds of DoF: Besides the mmf's along edges inside C, there are others, associated with the nodes in the air and on the conductor's surface, which can be interpreted as nodal values of the magnetic scalar potential, as we shall see. Again, let us first treat the contractible case.

Let $\mathcal{E}_C$, as above, be the subset of edges *inside* C, that is, entirely contained, apart from the extremities, in the interior of C. The set $\mathcal{N} - \mathcal{N}_C$ is composed of the nodes which are neither in int(C), nor in ∂D. Let E_C be the number of edges in $\mathcal{E}_C$ and N_0 the number of nodes in $\mathcal{N} - \mathcal{N}_C$. Last, call **U** (isomorphic to $\mathbb{C}^{E_C + N_0}$) the space of vectors $\mathbf{u} \equiv \{\mathbf{H}, \mathbf{\Phi}\} = \{\mathbf{H}_e : e \in \mathcal{E}_C, \mathbf{\Phi}_n : n \in \mathcal{N} - \mathcal{N}_C\}$, where the degrees of freedom $\mathbf{H}_e$ and $\mathbf{\Phi}_n$ are now unconstrained complex numbers. Let at last $\mathbb{K}^0{}_m$ be the space of vector fields of the form

(32) $$\text{H} = \sum_{e \in \mathcal{E}_C} \mathbf{H}_e \, w_e + \sum_{n \in \mathcal{N} - \mathcal{N}_C} \mathbf{\Phi}_n \, \text{grad } w_n.$$

Proposition 8.5. $\mathbb{K}^0{}_m$ *is isomorphic to* **U**.

Proof. This amounts to saying that degrees of freedom are independent, that is to say, $\text{H} = 0$ in (32) implies all $\mathbf{H}_e$ and $\mathbf{\Phi}_n$ are zero. We know this is the case of the $\mathbf{H}_e$'s, by restriction to C (cf. Remark 5.2). As for the $\mathbf{\Phi}_n$'s, $0 = \sum_n \mathbf{\Phi}_n \text{grad } w_n \equiv \text{grad}(\sum_n \mathbf{\Phi}_n w_n)$ implies $\sum_n \mathbf{\Phi}_n w_n$ equal to a constant in the only connected component of D – C, a constant which is the value of this potential on ∂D, that is, 0. Again we know (cf. Exer. 3.8) that all $\mathbf{\Phi}_n$'s must vanish in this case. ◊

Proposition 8.6. *If* C *is contractible,* $\mathbb{K}^0{}_m = \mathbb{H}^0{}_m \equiv \{\text{H} \in W^1{}_m : \text{rot H} = 0$ out of C$\}$.

Proof. After (32), one has $\text{rot H} = \sum_{e \in \mathcal{E}_C} \mathbf{H}_e \text{ rot } w_e$, and $\text{supp}(\text{rot } w_e)$ is contained in the closure of C, so $\text{rot H} = 0$ out of C. Conversely, if $\text{H} \in W^1{}_m$ and if $\text{rot H} = 0$ in D – C, which is simply connected, there exists a linear combination Φ of the w_n, for $n \in \mathcal{N} - \mathcal{N}_C$, such that $\text{H} = \text{grad } \Phi$ in D – C, hence (32). ◊

Now let $\text{H}^g{}_m \in W^1{}_m$ be an approximation of the source field. Again, $\mathbb{H}^g{}_m = \text{H}^g{}_m + \mathbb{H}^0{}_m$, and we can "suffix everything with m", hence the desired Galerkin approximation for problem (23), the same, formally, as (24): *find* $\text{H} \in \mathbb{H}^g{}_m$ *such that*

(33) $$\int_D i\,\omega\,\mu\,\text{H} \cdot \text{H}' + \int_C \sigma^{-1} \text{rot H} \,.\, \text{rot H}' = 0 \quad \forall\, \text{H}' \in \mathbb{H}^0{}_m.$$

This is, in an obvious way, a linear system with respect to the unknowns $\mathbf{H}_e$ and $\mathbf{\Phi}_n$, the form of which is similar to (31).

To build $\text{H}^g{}_m$, two techniques are possible. The first one consists of first computing H^g by the Biot and Savart formula, then evaluate the circulations $\mathbf{H}_e{}^g$ of H^g along all edges inside D. (For edges on the boundary, one sets $\mathbf{H}_e{}^g = 0$, which does introduce some error, but compatible with the desired accuracy,[7] if the mesh was well designed.) One then sets $\text{H}^g{}_m = \sum_{e \in \mathcal{E}} \mathbf{H}^g{}_e \, w_e$.

However, this does not warrant $\text{rot } \text{H}^g{}_m = 0$ where $\text{J}^g = 0$, as it should be, for the Biot and Savart integral is computed with some error, and the sum of circulations of H^g along the three edges of a face where no current flows may come out nonzero, and this numerical error can be important when the edges in question happen to be close to inductor parts. This is a serious setback.

Hence the idea of again using the spanning tree method, which automatically enforces these relations. But contrary to the previous section, it's not necessary to deal with all the outside mesh to this effect. One will treat only a submesh, as small as possible, covering the support of J^g. Of the set $\mathcal{E}^g$ of edges of this submesh, one extracts a spanning tree $\mathcal{E}^t$, and one attributes a DoF to each co-edge the same way as in (28). One finally sets $\text{H}^g{}_m = \sum_{e \in \mathcal{E}^g - \mathcal{E}^t} \mathbf{H}_e{}^g \, w_e$.

8.3.4 Cuts

Difficulties with the non-contractible case are about the same as in Subsection 8.3.2. Holes are no problem: Just pick one node n inside each non-conductive cavity within C and set $\boldsymbol{\Phi}_n = 0$ for this node. But the "loop problem" arises again, for if D – C is not simply connected, $\text{IK}^0{}_m$ is strictly included in $\text{IH}^0{}_m$: Missing are the fields H that, although curl-free in D – C, are only *local* gradients, or if one prefers, gradients of *multivalued* potentials Φ.

Hence the concept of "cuts", that is, for each current-loop, a kind of associated Seifert surface (cf. Note 6 and Exer. 8.3), formed of faces of the mesh. (Figure 8.7 will be more efficient than any definition to suggest what a cut is, but still, recall the formal definition of 4.1.2: a surface in $\overline{\text{D}}$, closed mod C, that doesn't bound mod C.) One then doubles the nodal DoF for each node of this surface (Fig. 8.7, right): to n_+ is assigned the DoF $\boldsymbol{\Phi}^*_{n_+} = \boldsymbol{\Phi}_n$, and to n_- the nodal value $\boldsymbol{\Phi}^*_{n_-} = \boldsymbol{\Phi}_n + J$, where J is the loop-intensity. Let us denote by $\mathcal{N}^*$ the system of nodes thus obtained,

[7] If this is not the case, one may always resort to solving the magnetostatics problem, $\text{rot H}^g = \text{J}^g$ and $\text{div H}^g = 0$ in D, with the same boundary conditions.

and set $\Phi = \sum_{n \in \mathcal{N}^*} \boldsymbol{\Phi}^*_n w_{n'}$, where the $w_{n\pm}$ are supported on one side of Σ, as suggested on Fig. 8.7: Then Φ is multivalued in D – C, and the fields $H = \sum_{e \in \mathcal{E}_C} \mathbf{H}_e w_e + \sum_{n \in \mathcal{N}^*} \boldsymbol{\Phi}^*_n \operatorname{grad} w_n$ do fill out IH^0_m. It all goes again as if there was one extra DoF (the unknown intensity J) for each current loop.

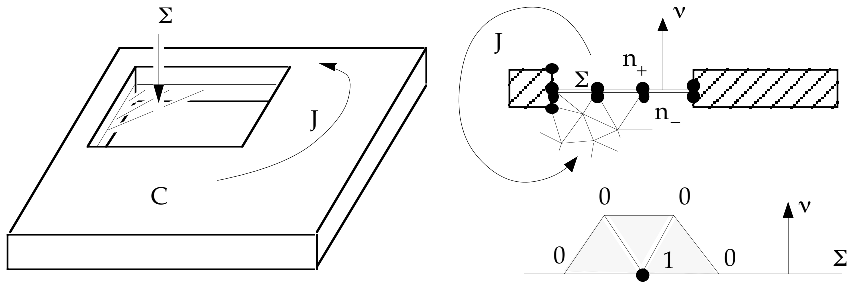


FIGURE 8.7. Cutting surface Σ, and doubling of nodes in Σ. The loop intensity J is also the circulation of the magnetic field along a circuit crossing Σ (along with the normal ν) and is therefore equal to the jump of Φ. Bottom right: support of the nodal function w_{n+}.

The big difficulty with this method is the construction of cuts. Several algorithms have been proposed [Br, HS, LR, VB], all more or less flawed because of a faulty definition of cuts. All these early works assumed, explicitly or not, that cuts must constitute a system of orientable (i.e., two-sided) surfaces, having their boundaries on ∂C ("closed modulo C but non-bounding", in the parlance of Chapter 4)—which is all right up to now—but also, *such that the complement in* D *of their set union with* C, that is, what remains of air after cuts have been removed, *be simply connected*. And this makes the definition defective, as Fig. 8.8 suffices to show: The complement of the set union of C and of the Seifert surface Σ is *not* simply connected, but the three components of Σ do qualify as cuts notwithstanding, for the magnetic potential Φ is effectively single-valued outside $C \cup \Sigma$. See the controversy in the IEE Journal [B&] triggered by the publication of [VB] for a discussion of this point. What cuts should do is make every curl-free field equal to a gradient in the outer region minus cuts. Credit is due to Kotiuga and co-workers [Ko] for the first correct definition of a cut, a constructive algorithm, and an implementation [GK].

It cannot be assessed, as the time this is written, whether this method is preferable to the "belted tree" approach. This is the matter of ongoing research [K&]. Let us, however, acknowledge that the problem of "knotted loops" is really marginal. Even common loops are infrequent in everyday work, because good modelling, taking symmetries into account, often allows one to dodge the difficulties they might otherwise raise. Eddy-current codes that were implemented, years ago, with cut-submodules based on some of the above-mentioned premature methods, on which current research is trying to improve, still work superbly in their respective domains of validity [BT, RL].

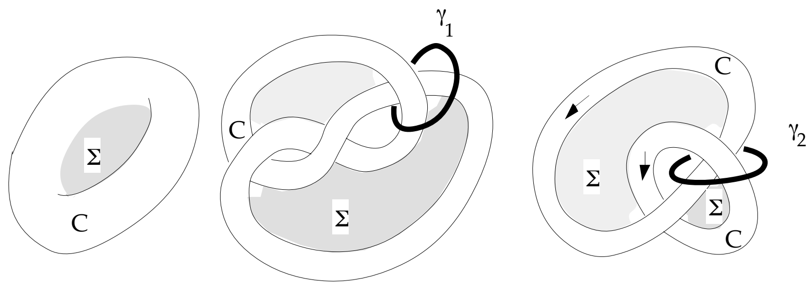


FIGURE 8.8. (Look first at Fig. 8.9 for the way Σ, here in three components, is constructed.) If some parts of the conductor (here made of four distinct connected components) are knotted or linked, it may happen that the complement of C and Σ is *not* simply connected, although the cuts Σ do play their role, which is to forbid multivalued potentials in the outside region. (Apply Ampère to circuits γ_1 and γ_2.)

The adaptation of the previous ideas to evolutionary regimes is straightforward, along the lines of 8.2.5.

8.4 SUMMING UP

What is special with eddy-current problems, and explains the almost unclassifiable variety of methods to deal with them, is the difference in the nature of the equations in the conductor and in the air. From the mathematical point of view, we have "parabolic equations" in conductors, "elliptic equations" in the air. For sure, passing in complex representation makes them elliptic all over, but different operators apply in the two main categories of regions: the "curl–curl" operator in the conductors, the "div–grad" operator in the air. We saw in Chapter 6 how intricate the relations between these two basic differential operators could be, marked by deep symmetries as well as striking differences. In the eddy-current problem, they coexist and must be compatible in some way at the common boundary. No wonder there is such a variety of methods for eddy currents! A few general ideas emerge, however:

1. Inside conductors, edge elements are the natural choice.

2. Outside, the magnetic potential is the most natural representation.

3. The structural relation $\text{grad } W^0 \subset W^1$ makes the two previous approaches compatible, be it in air volumes (H–Φ method) or on air–conductor interfaces (the "Trifou" hybrid method).

4. Two types of numerical treatment of the magnetic potential are available: integral equations (as we used to precompute the Dirichlet-to-Neumann map) and finite elements.

5. Multivaluedness of the magnetic potential, in the case of current loops, compounds difficulties, but tree and cotree methods offer solutions.

All these ideas can be combined. For instance, one may associate the H–Φ method in a bounded domain, thus dealing with current loops, and the integral equation method to account for the far field, provided the computational domain is simply connected. (There are cases when the latter restriction is still too much, however. It can be lifted thanks to the notion of *vectorial* Dirichlet-to-Neumann operator (end of Chapter 7). Cf. [Ve] for the early theory, and [RR] for an implementation, and an application to the problem of Fig. 8.2.) The variety of associations thus made possible is far from being exhausted, as witnessed by an abundant production of research papers (cf. especially **IEEE Trans. on Magnetism**, **IEE Proc., Series A**, **COMPEL**).

EXERCISES

Exercises 8.1 and 8.2 are on p. 227 and p. 233.

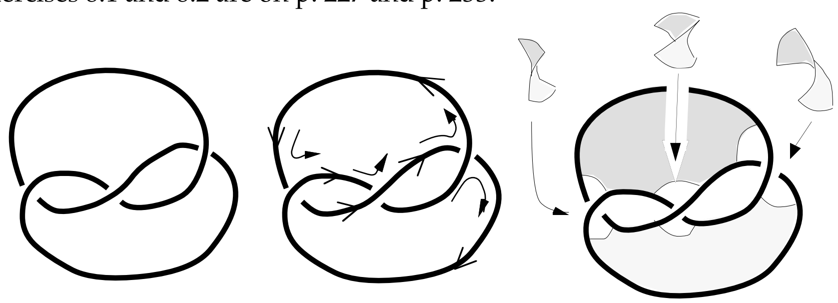


FIGURE 8.9. A knot, its Seifert surface (which has two distinct faces, as one can see, and is thus orientable), and the construction method.

Exercise 8.3. Figure 8.9 explains how to build a Seifert surface (as defined in Note 6) for a relatively simple knot (the procedure makes sense for *links*, too, as on Fig. 8.8, right): Work on an orthogonal plane projection of the knot, and select an arbitrary orientation along it; start from a point and follow the selected direction, never crossing at an apparent intersection, but instead, jumping forward to another part of the knot, and going on in the right direction, as suggested by the middle of Fig. 8.9; do this as many times as possible, thus obtaining as many pieces of the Seifert surface; these pieces are then seamed together by "flaps", as explained on the right of the figure, in order to obtain a single, two-sided surface. Practice on some knots of your own design. Show (by a series of drawings) that the surface thus obtained in the case of Fig. 8.9 is homeomorphic to a punctured torus.

Exercise 8.4. Show that if a circuit γ bounds a surface Σ entirely contained in a current-free region, then $\int_\gamma \tau \cdot h = 0$. The converse happens to be true. For instance, the circulation of h along γ_1 on Fig. 8.8 is null whatever the current in C. Show that γ_1 is indeed the boundary of a surface which does not encounter C.

Exercise 8.5. Can you devise a continuous current distribution on a torus, the way Fig. 8.10 suggests, such that the magnetic field outside be zero, though the induction flux through a surface bounded by γ is not?

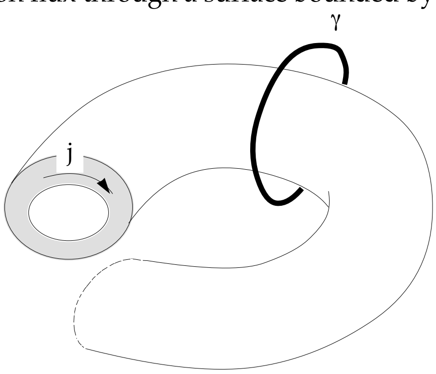


FIGURE 8.10. Such a current distribution can correspond to a null magnetic field in the outside region. Part of the hollow torus is cut off for better view. The conductive section is shaded. Circuit γ will be relevant to the next Exercise.

Exercise 8.6. In the situation of Fig. 8.10, suppose the *intensity* of the current distribution changes in time (but not the shape of its spatial distribution). Assuming γ is a conductive thread, will it support an induced current? Do you see a paradox in this? Can you solve it *within* eddy current theory (that is, without invoking retardation effects and the like)?

Exercise 8.7. The original Bath cube problem is described by Fig. 8.11. Differences with respect to the static version are: The mmf I is alternative, at angular frequency ω, and an aluminum cube is placed inside each quarter of the cavity, away from the bottom and from the walls. Do the modelling (continuous and discrete formulation).

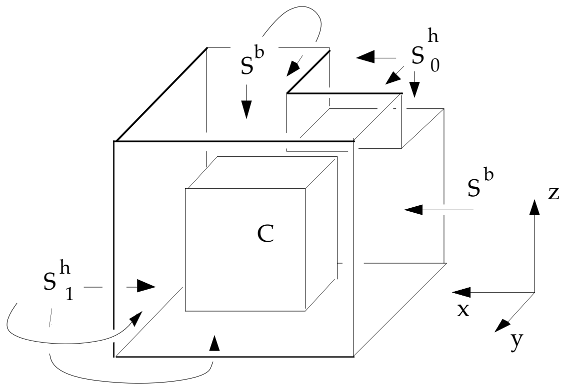
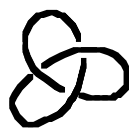


FIGURE 8.11. The Bath-cube problem (one quarter of the cavity, with conductive cube C included).

HINTS

8.3. You may have difficulties with some drawings, for example if you sketch the trefoil knot of Fig. 8.9 like this: (it's the same knot). In that case, imagine the knot as projected on the surface of a sphere of large radius, instead of on the plane.



8.4. Stokes.

8.5. Call D the domain occupied by the torus of Fig. 8.10, including the inner bore, and C the conductive part. Build a smooth solenoidal field a, curl-free out of C, and make sure the circulation of a along γ is non-zero. This means a = grad ψ outside D, but with a multivalued ψ, so use a cut. Similar thing in D – C. Extend a to C so that a is curl-conformal, then rectify a to make it sinusoidal *in all space.* Then take b = rot a, h = μ_0^{-1} b, and j = rot h.

8.6. Let j be the stationary current distribution of Exer. 8.4, h the corresponding field, and assume a current density J(t) j, hence an induction field μ_0 J(t) h. Its flux through the loop γ is not null, and changes with time, so there is an emf along γ, by Faraday's law, which may drive a current in the conductive loop. Now, to quote from [PK], “The commonly asked question is: ‘We know that charges in the loop move in response to an emf produced by a changing magnetic field; but since there is no magnetic field outside the solenoid, how do the charges in the loop know they must move?’.” What of it?

8.7. Use the H–Φ method: edge elements in C, nodal elements for Φ in D – C, scalar DoFs for Φ on ∂C. No constraints on Φ on surface S^b. On S^h_0 and S^h_1, set Φ = 0 and Φ = I respectively, where I is the imposed mmf between the poles.

SOLUTIONS

8.3. Figure 8.12.

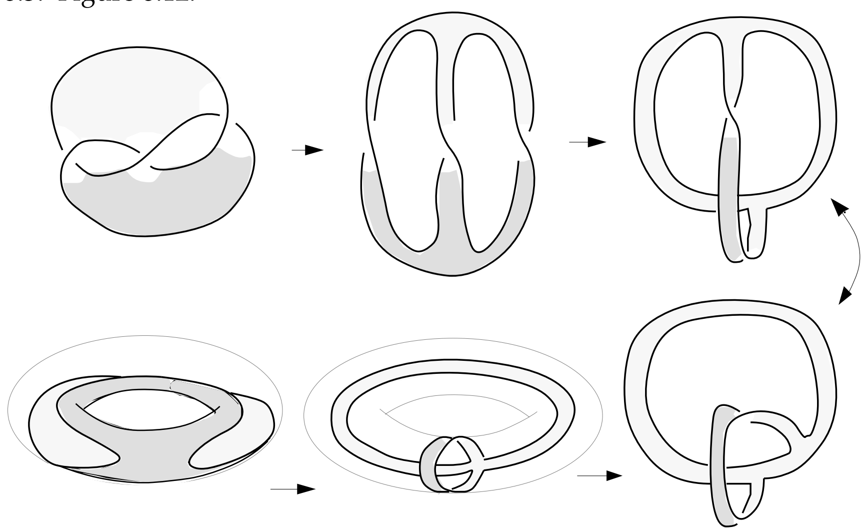


FIGURE 8.12. Homeomorphism between a punctured torus and the Seifert surface of a trefoil knot.

8.4. Figure 8.13 displays an orientable surface (a punctured torus, again) with γ_1 as its boundary. Since rot h = 0 outside C, one has $\int_{\gamma_1} \tau \cdot h = \int_\Sigma n \cdot \text{rot}\, h = 0$, whatever the intensity in C.

8.5. Let Σ_{ext} be a "cut" in E_3 – D, and $O_{ext} = E_3 - D - \Sigma_{ext}$. Let ψ_{ext} be the minimizer of $\int_{O_{ext}} |\text{grad}\, \psi|^2$ in $\{\psi \in BL(O) : [\psi]_{\Sigma_{ext}} = F_{ext}\}$, with $F_{ext} \neq 0$.

Similarly, select $\psi_{ext} \in \operatorname{arginf}\{\int_B |\operatorname{grad} \psi|^2 : \psi' \in BL(O_{int}) : [\psi']_{\Sigma_{int}} = F_{int}\}$ (these minimizers differ by an additive constant), where Σ_{int} is a cut inside $D - C$. Set $a = \operatorname{grad} \psi_i$ in O_i, with $i = ext$ or int. Extend a to a smooth field $\tilde{a}$ in $D - C$, with null jumps $[n \times \tilde{a}]_{\partial C}$ on the air–conductor interface. Now $\tilde{a} \in \mathbb{IL}^2_{rot}(\underset{\sim}{E_3})$ and its circulation along γ is nonzero, but $\tilde{a}$ is not solenoidal. Set $a = \tilde{a} + \operatorname{grad} \tilde{\psi}$, where $\tilde{\psi}$ is the unique element of $BL(E_3)$ such that $\int_{E_3} (\tilde{a} + \operatorname{grad} \tilde{\psi}) \cdot \operatorname{grad} \psi' = 0 \;\; \forall\, \psi' \in BL(E_3)$. Now $\operatorname{div} a = 0$ *in all space*, while its curl hasn't changed. Take $h = \mu_0^{-1} \operatorname{rot} a$ and $j = \operatorname{rot} h$. The field created by j is h, and has the required properties. By playing on the values of F_{ext} and F_{int}, one may control the coil intensity. The *pratical* realization is another issue, but is possible in principle: One may always imagine bundles of thin separate conductors coiled around the inner torus, along the small circles, and fed by small batteries.

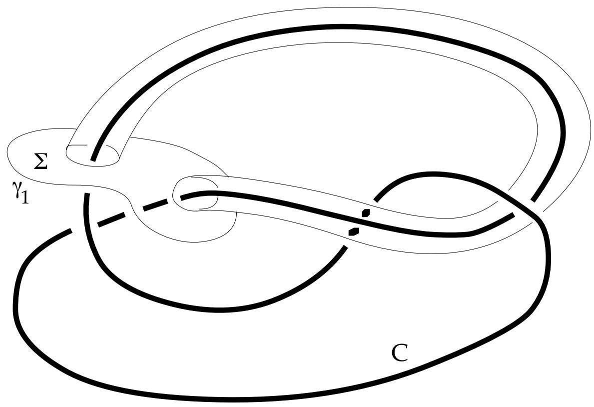


FIGURE 8.13. Circuit γ_1 does bound an orientable surface contained in the current-free region. The conductor C is the black knot.

8.6. Did you feel confused by this argument? Rightly so, because the expression, "emf produced by a changing magnetic field" *is* subtly confusing. The causal phenomenon by which changes of magnetic field generate electric fields is ruled by the equations $\operatorname{rot} e = -\partial_t b$, $\operatorname{div}(\varepsilon_0 e) = 0$ outside C, with tangential e known on ∂C, so it has an inherently *non-local* character. Changes of magnetic field in some region (here, domain D) thus produce emf's away from this region, including at places where the magnetic field is zero and stays zero. (This is no more paradoxical than the fact that changes in electric charge can modify the electric field in regions where there is no charge.) So there is an induced current in the loop, and its value can be predicted by eddy-current theory alone. No need to argue about the "physical unreality" of the situation" (*all* eddy current modellings are "unreal" to a comparable degree!) and to add irrelevant considerations on the way the solenoid is energized, on finite propagation speeds, and so forth [PK, Te].

This problem is relevant to discussions of the Aharonov–Bohm effect (cf. Remark A.2). Most papers on the subject assume a straight, infinite solenoid, instead of the above toroidal one, which makes some analytical computations easier, but also needlessly raises side issues.

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