# CHAPTER 2

# Magnetostatics: "Scalar Potential" Approach

## 2.1 INTRODUCTION: A MODEL PROBLEM

Let us now tackle problem (31) from Chapter 1: magnetostatics. We need a model problem for this discussion; we need it to be as simple as possible, and still come from the real world.

The following, known as the "Bath cube" problem [DB], will do. It is concerned with a device, built around 1979 at Bath University, which was essentially a hollow box between the poles of a large electromagnet (Fig. 2.1). In this almost closed experimental volume, various conducting or magnetizable objects could be placed, and probes could be installed to measure the field. The purpose was to confront what computational codes would predict with what these probes recorded. The problem was one in a series of such benchmark problems, regularly discussed in an ad-hoc forum (the TEAM international workshop [T&]). Comparative results for this one (known as "Problem 5") can be found in [B5].

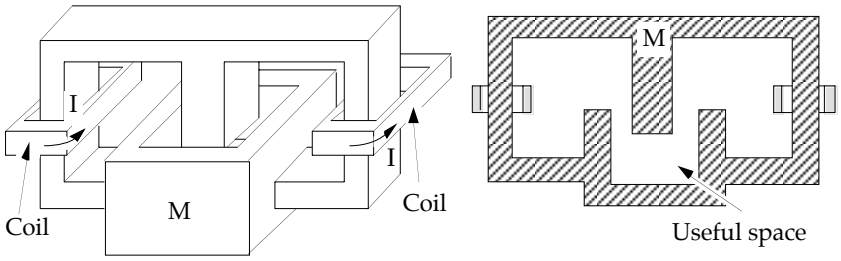



**FIGURE 2.1.** The "Bath cube" benchmark. Both coils bear the same intensity I. The magnetic circuit M is made of laminated iron, with high permeability ($\mu > 1000\ \mu_0$). Various objects can be placed in the central experimental space.

Problem 5 was actually an eddy-current problem, with alternating current in both coils, and we shall address it in Chapter 8. What we discuss here is the corresponding *static* problem, with DC currents: given the coil-current, find the field inside the box.

It will be some time before we are in a position to actually *solve* this problem, despite its obvious simplicity. For before solving it, we must *set* it properly. We have a physical situation on the one hand, with a description (dimensions, values of physical parameters) and a query (more likely, an endless series of queries) about this situation, coming from some interested party (the Engineer, the Experimenter, the Customer, . . .). To be definite about that here, we shall suppose the main query is, "What is the reluctance of the above device?" The task of *our* party (the would-be Mathematical Modeller, Computer Scientist, and Expert in the Manipulation of Electromagnetic Software Systems) is to formulate a *relevant* mathematical problem, liable to approximate solution (usually with a computer), and this solution should be in such final form that the query is answered, possibly with some error or uncertainty, but within a controlled and predictable margin. (Error *bounds* would be ideal.)

*Mathematical modelling* is the process by which such a correspondence between a physical situation and a mathematical problem is established.[1] In this chapter, a model for the above situation will be built, based on the so-called "scalar potential variational formulation". We shall spiral from crude *attempts* to set a model to refined ones, via *criticism* of such attempts. Some points about modelling will be made along the way, but most of the effort will be spent on sharpening the mathematical tools.

First attempt, based on a literal reading of Eqs. (1.31). We are given a scalar field $\mu$, equal to $\mu_0$ in the air region, and a time-independent vector field $j$ (actually, the $j^g$ of (1.31), but we may dispense with the superscript $g$ here). From this data, *find vector fields* $b$ *and* $h$ *such that*

(1) $\quad \text{rot}\, h = j,$

(2) $\quad b = \mu h,$

(3) $\quad \text{div}\, b = 0,$

in all space.

The first remark, predictable as it was, may still come as a shock: *This formulation doesn't really make sense; the problem is* not *properly posed this way.*

[1] It requires from *both* parties a lot of give and take.

## 2.2 HONING OUR TOOLS

At least two things disqualify (1–3) as a proper formulation. One is the non-uniqueness of $b$ and $h$, a mild problem which we'll address later. The other is the implicit and unwarranted assumption of *regularity*, or smoothness, of these fields. For instance, $\mathrm{div}\, b = 0$ makes perfect sense if the three components $b^1, b^2, b^3$, in Cartesian coordinates, are differentiable. Then $(\mathrm{div}\, b)(x) = \partial_1 b^1(x) + \partial_2 b^2(x) + \partial_3 b^3(x)$, a well-defined function of position $x$, and the statement "$\mathrm{div}\, b = 0$" means that this function is identically $0$. No ambiguity about that. But we can't assume such differentiability.[2] As one knows, and we'll soon reconfirm this knowledge, the components of $b$ are *not* differentiable, not even continuous, at some material interfaces. Still, conservation of the induction flux implies a very definite "transmission condition" on $S$.

### 2.2.1 Regularity and discontinuity of fields

Since smoothness, or lack thereof, is the issue, let's be precise, while introducing some shorthands. $D$ being a space domain,[3] the set of all functions continuous at all points of $D$ is denoted $C^0(D)$. A function is *continuously differentiable* in $D$ if all its partial derivatives are in $C^0(D)$, and one denotes by $C^1(D)$ the set of such functions (an infinite-dimensional linear space). Similarly, $C^k(D)$ or $C^\infty(D)$ denote the spaces composed of functions which have continuous partial derivatives of all orders up to $k$ or of all orders without restriction, inside $D$. In common parlance, one says that a function "is $C^k$", or "is $C^\infty$" in some region, implying that there is a domain $D$ such that $C^k(D)$, or $C^\infty(D)$, includes the restriction of this function to $D$ as a set element. "Smooth" means by default $C^\infty$, but is often used noncommittally to mean "as regular as required", that is, $C^k$ for $k$ high enough. (I'll say "k-smooth" in the rare cases when definiteness

[2]This is not mere nit-picking, not one of these gratuitous "rigor" or "purity" issues. We have here a tool, differential operators, that fails to perform in some cases. So it's not the right tool, and a better one, custom-made if necessary, should be proposed, one which will work also in borderline cases. Far from coming from a position of arrogance, this admission that a mismatch exists between some mathematical concepts and the physical reality they are supposed to model, and the commitment to correct it, are a manifestation of modesty: When the physicist says "this tool works well almost all the time, and the exceptions are not really a concern, so let's not bother", the mathematician, rather than hectoring, "But you have no *right* to do what you do with it," should hone the tool in order to make it able *also* to handle the exceptions.

[3]Recall the dual use of "domain", here meaning "open connected set" (cf. Appendix A, Subsection A.2.3).

on this point is important.) These notions extend to vector fields by applying them coordinatewise.

In principle, the gradient of a function is only defined at *interior* points of its domain[4] of definition, since the gradient is a record of variation rates in all directions. Depending on the local shape of the boundary, it may still be possible to define a gradient at a boundary point, by taking directional derivatives. How to do that is clear in the case of a smooth boundary (on each line through a boundary point, there is a half-line going *inwards).* But it's more problematic at a corner, at the tip of a cusp, etc. This is why the concept of smoothness *over* a region (not only *inside* it), including the boundary, is delicate. To avoid ambiguities about it, I'll say that a function f is *smooth over* a region R (which may itself be very irregular, devoid of a smooth boundary) if there is a domain D containing R *in* which some extension of f (cf. A.1.2) is smooth. (See Fig. 2.2.)

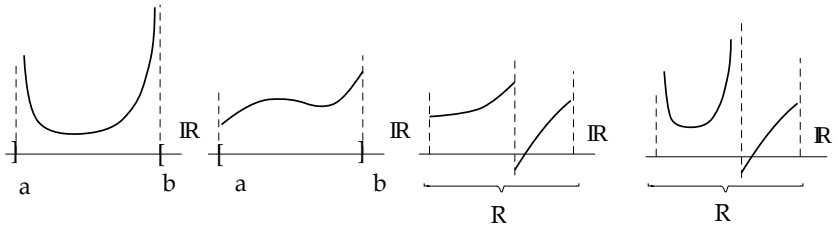



**FIGURE 2.2.** Notions of smoothness, for a function of a real variable. Left to right, functions which are: smooth *in* ]a, b[, smooth *over* [a, b], *piecewise* smooth in region R, *not* piecewise smooth.

*Piecewise smooth*, then, has a precise meaning: It refers to a function, the domain of which can be partitioned into a mosaic of regions, in finite number, *over* each of which the function is smooth. This does not exclude discontinuities across inner boundaries, but allows only frank discontinuities (of the "first kind"), or as we shall say below, "jumps".

**Exercise 2.1.** Check that a piecewise smooth function f has a definite integral $\int_D |f|$ on a *bounded* domain. Is this latter assumption necessary?

Now let's return to the case at hand and see where exceptions to smoothness can occur. In free space ($\mu = \mu_0$, and $j = 0$), rot h = 0 and div h = 0, and the same is true of b. We have this well-known formula which says that, for a $C^2$-vector field u,

(4) $\qquad$ rot rot u = grad div u – $\Delta$u,

[4]The other meaning of the word (Subsection A.1.2).

where $\Delta u$ is the field, the components of which are $\{\Delta u^1, \Delta u^2, \Delta u^3\}$. So both $h$ and $b$ are *harmonic*, $\Delta h = 0$ and $\Delta b = 0$, in free space. A rather deep result, *Weyl's lemma*, can then be invoked: *harmonic functions are* $C^\infty$. So both $b$ and $h$ are smooth.[5] The same argument holds unchanged in a region with a uniform $\mu$, instead of $\mu_0$.

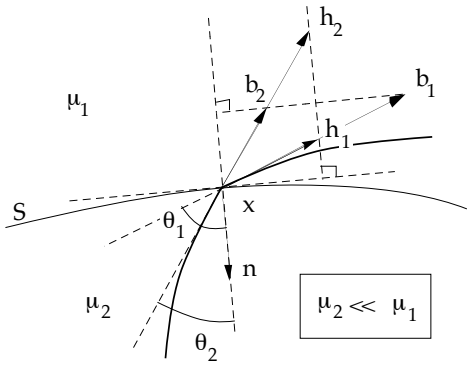



**FIGURE 2.3.** Flux line deviation at a material interface. The pair $\{h_i, b_i\}$ is the field on side $i$, where $i = 1$ or 2.

In case two regions with different permeabilities $\mu_1$ and $\mu_2$ are separated by a smooth surface $S$ (Fig. 2.3), $b$ and $h$ will therefore be smooth on both sides, and thus have well-defined flux lines.[6] But the latter will not go straight through $S$. They deviate there, according to the following "law of tangents":

$$\mu_2 \tan\theta_1 = \mu_1 \tan\theta_2, \tag{5}$$

where $\theta_1$ and $\theta_2$ are the angles the flux half-lines make with the normal $n$ at the traversal point $x$. So if $\mu_1 \neq \mu_2$, neither $b$ nor $h$ can be continuous at $x$. Formula (5) is an immediate consequence of the two equalities, illustrated in Fig. 2.3,

$$n \cdot b_1 = n \cdot b_2 \text{ on } S, \qquad (6)$$

$$n \times h_1 = n \times h_2 \text{ on } S, \qquad (7)$$

called *transmission conditions*, which assert that the normal part of $b$

[5] A similar, stronger result by Hörmander [Hö] implies that $h$ and $b$ are smooth if $\mu$ itself is $C^\infty$. Cf. [Pe]. All this has to do with one (number 19) of the famous Hilbert problems [Br].

[6] A *flux line* of field $b$ through point $x_0$ is a trajectory $t \to x(t)$ such that $x(0) = x_0$ and $(\partial_t x)(t) = b(x(t))$. If $b$ is smooth and $b(x_0) \neq 0$, there is such a trajectory in some interval $]-\beta, \alpha[$ including 0, by general theorems on ordinary differential equations. See, e.g., [Ar], [CL], [Fr], [LS].

and the tangential part of $h$ are continuous across S, and which we now proceed to prove.

The proof of (6) comes from an integral interpretation of Faraday's law. By the latter, the induction flux through any *closed* surface vanishes. Let's apply this to the surface of the "flat pillbox" of Fig. 2.4, built from the patch $\Omega$ (lying in S) by extrusion. This surface is made of two surfaces $\Omega_1$ and $\Omega_2$ roughly parallel to S, joined by a thin lateral band. Applying Ostrogradskii[7] and letting the box thickness $d$ go to 0, one finds that $\int_\Omega (n \cdot b_1 - n \cdot b_2) = 0$, because the contribution of the lateral band vanishes at the limit, whereas $n \cdot b$ on $\Omega_1$ and $\Omega_2$ respectively tend to the values $n \cdot b_1$ and $n \cdot b_2$ of $n \cdot b$ on both sides of $\Omega$. Hence (6), since $\Omega$ is arbitrary.

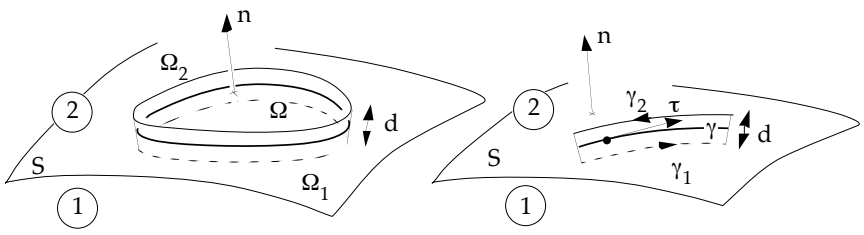



**FIGURE 2.4.** Setup and notations for the proof of (6) and (7).

As for (7), we rely on the integral interpretation of Ampère's theorem: the circulation of $h$ along a closed curve is equal to the flux of $j + \partial_t d$ through a surface bound by this curve. Here we apply this to the "thin ribbon" of Fig. 2.4, built by extrusion from the curve $\gamma$ lying in S. Since $j + \partial_t d = 0$ in the present situation, the circulation of $h$ along the boundary of the ribbon is zero. Again, letting the ribbon's width $d$ go to 0, we obtain $\int_\gamma (\tau \cdot h_1 - \tau \cdot h_2) = 0$, which implies, since $\gamma$ is arbitrary, the equality of the projections (called "tangential parts") of $h_1$ and $h_2$ onto the plane tangent to S. This equality is conveniently expressed by (7).

Fields therefore fail to be regular at all material interfaces where $\mu$ presents a discontinuity, and div or rot cease to make sense there. Some regularity subsists, however, which is given by the interface conditions (6) and (7). For easier manipulation of these, we shall write them $[n \cdot b]_S = 0$ and $[n \times h]_S = 0$, and say that the *jumps* of the normal part of $b$ and of the tangential part of $h$ vanish at all interfaces. Before discussing the possibilities this offers to correctly reformulate (1–3), let's explain the notation and digress a little about jumps.

[7] Flux, circulation, and relevant theorems are discussed in detail in A.4.2.

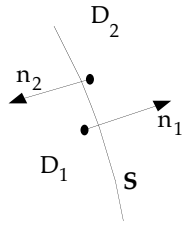
### 2.2.2 Jumps

This section is a partly independent development about the bracket notation [ ] for jumps, which anticipates further uses of it.

Consider a field (scalar- or vector-valued) which is smooth on both sides of a surface $S$, but may have a discontinuity across $S$, and suppose $S$ is provided with a crossing direction. The *jump* across $S$ of this quantity is by definition equal to its value just before reaching the surface, minus its value just after. (The jump is thus counted downwards; rather a "drop", in fact.) Giving a crossing direction through a surface is equivalent to providing it with a continuous field of normals. One says then that the surface has been *externally oriented*.[8]

Not all surfaces can thus be oriented. For one-sided surfaces (as happens with a Möbius band), defining a continuous normal field is not possible, and the crossing direction can only be defined locally, not consistently over the whole surface. For surfaces which enclose a volume, and are therefore two-sided, the convention most often adopted consists in having the normal field point outwards. This way, if a function $\varphi$ is defined inside a domain $D$, and equal to zero outside, its jump $[\varphi]_S$ across the surface $S$ of $D$ is equal to the trace $\varphi_S$ of $\varphi$, that is, its restriction to $S$ if $\varphi$ is smooth enough, or its limit value from inside otherwise. The conventions about the jump and the normal thus go together well.

For interfaces between two media, there may be no reason to favor one external orientation over the other. Nonetheless, some quantities can be defined as jumps in a way which does not depend on the chosen crossing direction.




Consider for instance the flux of some vector field $j$ through an interface $S$ between two regions $D_1$ and $D_2$ (inset). Let $n_1$ and $n_2$ the normal fields defined according to both possible conventions: $n_1$ points from $D_1$ towards $D_2$, and $n_2$ points the other way. Suppose we choose $n_1$ as the crossing direction, and thus set $n = n_1$. Then the jump of the normal component $n \,.\, j$ across $S$ is by definition equal to its value on the $D_1$ side, that is, $n_1 \,.\, j$, minus the value of $n \,.\, j$ on the $D_2$ side, which is $-n_2 \cdot j$. The jump is thus the *sum* $n_1 \cdot j + n_2 \cdot j$. This is symmetrical with respect to $D_1$ and $D_2$, so we are entitled to speak of "the jump of $n \cdot j$ across $S$" without specifying a crossing direction. Whichever this direction, the decrease of $n \cdot j$ when crossing will be the same, because the sign of $n$ intervenes twice, in the choice of direction,

[8] Which suggests there is also a different concept of *internal* orientation (cf. Chapter 5).

and in the choice of which side one "jumps from". Hence the definition of the jump of $j$ as $[n \cdot j] = n_1 \cdot j_1 + n_2 \cdot j_2$, where $j_1$ and $j_2$ are the values on each side of $S$.

Such jumps often have interesting physical interpretations. For instance, if $j$ is a current density, the jump is equal to the intensity that is "instilled in $S$", and is withdrawn by some mechanism. When no such mechanism exists, as for instance at the interface between two conductors, the jump must vanish. But it may happen otherwise. For instance, if $S$ corresponds to a highly conducting inclusion inside a normal conductor, the current $[n \cdot j]$ withdrawn at some place will be conveyed along $S$ and reinjected at other places, where $[n \cdot j]$ will be negative. Note that such considerations would apply to a Möbius band without any problem.

In the case of the electric induction $d$, the jump $[n \,.\, d]_S$ is the density of electric charge present on surface $S$; hence the interface condition $[n \,.\, d]_S = 0$, unless there is a physical reason to have electric charge concentrated there. Same thing with $b$, and magnetic charge. Our proof above that $[n \cdot b] = 0$ across all interfaces made implicit use of the absence of such charge. But there are problems in which the jump of a quantity denoted $n \cdot b$ can be nonzero. This happens, for instance, when fictitious surface magnetic charges are used as auxiliary quantities in integral methods, and then $[n \cdot b] = q$, the fictitious charge density.

A bit different is the case of vector quantities, such as the magnetic field. The jump $[h]_S$ is simply the field, defined on $S$, obtained by taking the jumps of the three coordinates. The subject of interest, however, is more often the jump of the *tangential* part of $h$.

If $h$ is smooth, we call *tangential part* and denote by $h_S$ the field of vectors tangent to $S$ obtained by projecting $h(x)$, for all $x$ in $S$, on the tangent plane $T_x$ at $x$ (Fig. 2.5, left). If $h$ is smooth on both faces of $S$ but discontinuous there, there are two bilateral projections $h_{S1}$ and $h_{S2}$, and the jump of $h_S$, according to the general definition, is $[h_S]_S = h_{S1} - h_{S2}$ in the case of Fig. 2.5. The sign of this of course depends on the crossing direction. But the remark

$$(8) \qquad [h_S]_S = -n \times [n \times h]_S = -n \times (n_1 \times h_1 + n_2 \times h_2)$$

points to the orientation-independent surface vector field $[n \times h]_S$. This is equal to the jump $[h_S]_S$ of the tangential part, up to a 90° rotation, counterclockwise, around the normal. The cancellation of the tangential jump is thus conveniently expressed by $[n \times h]_S = 0$.

This field $[n \times h]$ is interesting for another reason. As suggested by Fig. 2.5, right, $[n \times h]$ is always equal to minus the current density $j_S$ (a surface vector field, thus modelling a "current sheet") supported by the interface. For instance, if the crossing direction is from region 1 to region 2, and thus $n = n_1 = -n_2$, then $[n \times h] = n_1 \times h_1 + n_2 \times h_2$, which is $-j_S$ by Ampère's theorem. We find the same result with the other choice. Again, if there is no way to carry along the excess current (such as, for instance, a thin sheet of high conductivity borne by S), then $j_S = -[n \times h] = 0$, which is the standard transmission condition about h we derived earlier.

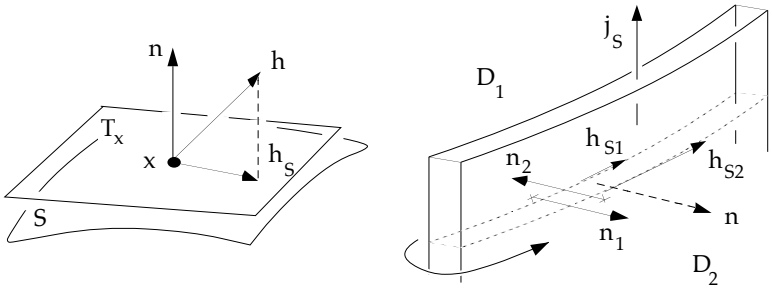



**FIGURE 2.5.** Left: Definition of $h_S$. Right: Relation between $j_S$ and the jump of $h_S$. (Take the circulation of h along the circuit indicated.)

In a quite similar way, $[n \times e]$ is equal, irrespective to the choice of normal, to the time-derivative of the induction flux vector, $\partial_t b_S$, along the surface. This is most often 0—hence the transmission condition $[n \times e] = 0$—but not always so. By way of analogy with the previous example, the case of a thin highly permeable sheet will come to mind. But there are other circumstances, when modelling a thin air gap, for instance, or a crack within a conductor in eddy-current testing simulations, when it may be necessary to take account of the induction flux in a direction tangential to such a surface.

### 2.2.3 Alternatives to the standard formalism

Back to our critical evaluation of the ill-specified equation $\mathrm{div}\, b = 0$: What can be done about it? A simple course would be to explicitly acknowledge the exceptions, and say, "We want $\mathrm{div}\, b = 0$ wherever b is effectively differentiable, and $[n \cdot b] = 0$ across all material interfaces and surfaces where a singularity might occur." Indeed, many textbooks list transmission conditions as equations to be satisfied, and add them to Maxwell equations, on almost the same footing.

It would be quite awkward, however, always to be forced to dot i's and cross t's that way. Besides, and more importantly, the practice of finite elements does not suggest that material interfaces should contribute *additional* equations. So there must be a way to stretch the meaning of statements such as $\mathrm{div}\, b = 0$ in order to *imply* the transmission conditions. In fact there are two main ways. A radical one: differential forms; and a moderate one: weak formulations, laid on top of the bedrock of classical vector calculus.

The radical way will not be followed here, but must be mentioned, because being aware of its existence helps a lot in understanding the surprising analogies and formal symmetries that abound in the classical approach. When looking for substitutes for the *differential, local* equations $\mathrm{div}\, b = 0$ and $\mathrm{rot}\, h = 0$, we invoked *integral, global* relations: flux conservation, Ampère's theorem. All electromagnetic laws (apart from constitutive laws) say things like "This circulation along that line is equal to this flux across that surface, this volume integral equals that charge," and so forth, with line, surface, and volume in a definite and simple relationship, such as "is the boundary of". The laws thus appear as relations between real quantities assigned to geometrical elements (points, lines, surfaces, volumes), and the scalar or vector fields are there as a way to compute these quantities.

Once we begin to see things in this light, some patterns appear. Fields like $e$ and $h$ are definitely associated with *lines:* One takes their *circulations*, which are electromotive forces (emf) and magnetomotive forces (mmf). The same can be said about the vector potential $a$. And it can't be a coincidence either if when a curl is taken, one of these fields is the operand. Fields like $b$ and $d$, or $j$, in contrast, are *surface* oriented, their *fluxes* matter, and it's div, rarely rot, which is seen acting on them. Even the scalar fields of the theory (charge density $q$, magnetic potential $\varphi$, electric potential $\psi$) have an associated dimensionality: Point values of $\varphi$ and $\psi$ matter, but only volume integrals of $q$ are relevant, and terms like $\mathrm{grad}\, q$ are never encountered, contrary to $\mathrm{grad}\, \varphi$ or $\mathrm{grad}\, \psi$.

This forces us to shift attention from the fields to the linear mappings of type $GEOMETRIC\_ELEMENT \to REAL\_NUMBER$ they help realize. For instance, what matters about $h$, physically, is not its pointwise values, but its circulations along lines (mmf). Thus, the status of $h$ as a $LINE \to REAL$ linear map is more important than its status as a vector field. The status of $b$ as a $SURFACE \to REAL$ linear map is what matters (and in this respect, $b$ and $h$ are different kinds of vector fields). The (mathematical) fields thus begin to appear as mere props, auxiliaries in

the description of the (physical) field as a connector between geometrical entities.

Which somewhat devalues differential operators, too: grad, rot and div, in this light, appear as auxiliaries in the expression of conservation relations, as expressed by the Ostrogradskii and Stokes theorems. Their failure to make sense locally is thus not to be taken too seriously.

Proper form is given to the foregoing ideas in *differential geometry*. There, one forgets about the scalar or vector fields and one focuses on the mappings they represent (and thus, to some extent, hide). Fields of linear mappings of $GEOMETRIC\_ELEMENT \to REAL$ type are called *differential forms*, of *degree* 0 to 3 according to the dimension of the geometric objects they act upon, and under regularity assumptions which are milder than for the scalar or vector proxies, one defines a unique operator $\mathsf{d}$, the *exterior differential*, which is realized as grad, rot, or div, depending on the dimension. *All* laws of electromagnetism can be cast in this language (including constitutive laws, which are mappings from p-forms (p = 0 to 3) to (3 – p)-forms).

The moderate approach we now follow does not go so far, and keeps the fields as basic objects, but stretches the meaning of the differential operators, so that they continue to make sense for some discontinuous fields. The main idea is borrowed from the theory of distributions: Instead of seeing fields as collections of pointwise values, we consider how they act on other fields, by integration. But the full power of the theory of distributions is not required, and we may eschew most of its difficulties.

## 2.3 WEAK FORMULATIONS

First, some notation. Symbols $C^k$ and $C^\infty$ for smoothness have already been introduced, compact support[9] is usually denoted by a subscripted 0, and blackboard capitals are used in this book to stress the vector vs scalar opposition when referring to spaces of fields. Putting all these conventions together, we shall thus have the following list of infinite-dimensional linear spaces:

- $C^k(E_3)$ : The vector space of all k-smooth functions in $E_3$,
- $\mathbb{C}^k(E_3)$ : All k-smooth vector fields in $E_3$,

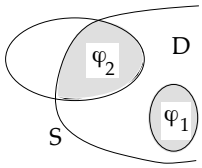
[9]The *support* of a function, real- or vector-valued, is the closure of the set of points where it doesn't vanish. Cf. A.2.3.

- $C_0^k(D)$, $\mathbb{C}_0^k(D)$ : Same, with compact support contained in $D$,

being understood that domain $D$ can be all $E_3$, and finally,

- $C^k(\overline{D})$, $\mathbb{C}^k(\overline{D})$,

for the vector spaces of restrictions to $\overline{D}$ (the closure of D), of k-smooth functions or vector fields. In all of these, $k$ can be replaced by $\infty$. (In the inset, the supports of a $\varphi_1 \in C_0^\infty(D)$ and a $\varphi_2 \in C^\infty(\overline{D})$, which is thus the restriction of some function defined beyond D, whose support is sketched.)




### 2.3.1 The "divergence" side

Now, let's establish a technical result, which generalizes integration by parts. Let $D$ be a regular domain (not necessarily bounded), $S$ its boundary, $b$ a smooth vector field, and $\varphi$ a smooth function, both with compact support in $E_3$ (but their supports may extend beyond $D$). Form $u = \varphi\, b$. Ostrogradskii's theorem asserts that $\int_D \operatorname{div} u = \int_S n \cdot u$, with $n$ pointing outwards, as usual. On the other hand, we have this vector analysis formula,

$$\operatorname{div}(\varphi\, b) = \varphi \operatorname{div} b + b \cdot \operatorname{grad} \varphi.$$

Both things together give

$$\int_D \varphi \operatorname{div} b = -\int_D b \cdot \operatorname{grad} \varphi + \int_S n \cdot b\ \varphi, \tag{9}$$

a fundamental formula.

By (9), we see that a 1-smooth divergence-free field $b$ in $D$ is characterized by

$$\int_D b \cdot \operatorname{grad} \varphi = 0 \quad \forall\, \varphi \in C_0^1(D), \tag{10}$$

since with $\varphi = 0$ on the boundary, there is no boundary term in (9). But (10) makes sense for fields $b$ which are only *piecewise* smooth.[10] We now take a bold step:

**Definition 2.1.** *A piecewise smooth field* $b$ *which satisfies* (10) *will be said to be* divergence-free, *or solenoidal*, in the weak sense.

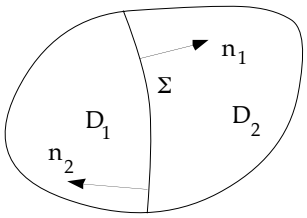
The $\varphi$'s in (10) are called *test functions*.

[10] All that is required is the integrability of $b \cdot \operatorname{grad} \varphi$ in (10), so 0-smoothness, that is, continuity, of each "piece" of $b$ is enough.

A solenoidal smooth field is of course weakly divergence-free. But from our earlier discussion, we know that the physical $b$, in magnetostatics, is only piecewise smooth, and satisfies transmission conditions. Hence the interest of the following result:

**Proposition 2.1.** *For a piecewise-smooth* $b$, (10) *is equivalent to* $\operatorname{div} b = 0$ *inside regularity regions and* $[n \cdot b] = 0$ *at their interfaces.*

*Proof.* Recall that "piecewise" means that $D$ can be partitioned into a *finite* number of subdomains $D_i$ in which $b$ is smooth, so a proof with *two* subdomains will be enough. It's a long proof, which will require two steps.




We begin with the shorter one, in which $b$ is supposed to be solenoidal in $D_1$ and $D_2$ separately (notation in inset), with $[n \cdot b] = 0$ on the interface $\Sigma$. Our aim is to prove (10). Let $\varphi$ be a test function. Formula (9) holds in $D_1$ and $D_2$ separately, and gives

$$\begin{aligned}\int_D b \cdot \operatorname{grad} \varphi &= \int_{D_1} b \cdot \operatorname{grad} \varphi + \int_{D_2} b \cdot \operatorname{grad} \varphi \\ &= -\int_{D_1} \varphi \operatorname{div} b + \int_\Sigma n_1 \cdot b\ \varphi - \int_{D_2} \varphi \operatorname{div} b + \int_\Sigma n_2 \cdot b\ \varphi \\ &= \int_\Sigma [n \cdot b]\, \varphi = 0\end{aligned}$$

since $[n \cdot b] = 0$ has been assumed, hence (10). The absence of surface terms on $S$ is due to the inclusion $\operatorname{supp}(\varphi) \subset D$.

Conversely, suppose (10) holds. Since (10) says "*for all* test functions $\varphi$", let's pick one which is supported in $D_1$, and apply (9). There is no surface term, since $\operatorname{supp}(\varphi_1)$ does not intersect $\Sigma$, so $0 = \int_{D_1} \operatorname{div} b\, \varphi$. (Note that $\operatorname{div} b$ is a smooth function there.) This holds for *all* $\varphi \in C_0^1(D_1)$.[11] But the only way this can happen (see A.2.3 if this argument doesn't sound obvious) is by having $\operatorname{div} b = 0$ in $D_1$. Same reasoning in $D_2$, leading to $\operatorname{div} b = 0$ also in $D_2$. Now, start from (10) again, with a test function which does not necessarily vanish on $\Sigma$, and use the newly acquired knowledge that $\operatorname{div} b = 0$ in $D_1$ and $D_2$:

$$\begin{aligned}0 = \int_D b \cdot \operatorname{grad} \varphi &= \int_{D_1} b \cdot \operatorname{grad} \varphi + \int_{D_2} b \cdot \operatorname{grad} \varphi = \\ &= \int_\Sigma n_1 \cdot b\, \varphi + \int_\Sigma n_2 \cdot b\, \varphi = \int_\Sigma [n \cdot b]\, \varphi\end{aligned}$$

for all $\varphi \in C_0^1(D)$. But such test functions can assume any value on $\Sigma$, so

[11] Which is why the presence of the quantifier $\forall$ in Eq. (10) is mandatory. Without it, the meaning of the statement would change totally.

again, the only way $\int_\Sigma [n \cdot b]\, \varphi$ can vanish for all of them is by $[n \cdot b]_\Sigma$ being 0. ◊

*Interface conditions are thus implicit* in the "weak solenoidality" condition (10). We shall therefore acquire the "weak formulation reflex": Each time a statement of the form "div b = 0" appears in the formulation of a problem (this is what one calls the "strong formulation"), replace it by the weak formulation (10). This does no harm, since there is equivalence in case b has a divergence in the ordinary ("strong") sense. It does some good if b is only piecewise smooth, since there is no need to make explicit, or even mention, the transmission conditions $[n \cdot b] = 0$, which are implied by (10), as Prop. 2.1 has shown.

We now see why the $\varphi$'s are called "test functions": By carefully selecting them, we were able to "test" the equality div b = 0 inside regularity regions, to "test" the transmission condition over $\Sigma$, etc. The function div b and the constant 0 were thus deemed equal not because their *values* would coincide, but on the ground that their *effects* on test functions were the same. (This principle, duly abstracted, was the foundation of the theory of distributions.)

**Remark 2.1**. The reader aware of the "virtual work principle" in mechanics will have recognized the analogy: There too, fields of forces are tested for equality by dot-multiplying them by fields of virtual displacements and integrating, and two force fields are equal if their virtual works always coincide. ◊

An obvious generalization of (10) is

$$(11) \qquad -\int_D b \cdot \operatorname{grad} \varphi = \int_D f\, \varphi \quad \forall\, \varphi \in C_0^1(D),$$

where f is a given function (piecewise smooth). This means "div b = f in the weak sense". (**Exercise 2.2**: Check that.)

**Remark 2.2.** One may wonder to which extent weak solenoidality depends on the regularity of test functions, and this is a good question, since of course the following statement, for instance,

$$(10') \qquad \int_D b \cdot \operatorname{grad} \varphi = 0 \quad \forall\, \varphi \in C_0^\infty(D),$$

is logically *weaker* than (10): There are fewer test functions, hence fewer constraints imposed on b, and hence, conceivably, more weakly solenoidal fields in the sense of (10') than in the sense of (10). The notion would be of dubious value if things went that way. But fortunately (10) *and* (10') *are equivalent:* This results from the density property proved in Section A.2.3: Given a $C_0^1$ test function $\varphi$, there exists a sequence $\{\varphi_n\}$ of $C_0^\infty(D)$ functions

such that $\int_D |\text{grad}(\varphi_n - \varphi)|^2 \to 0$. Then (**Exercise 2.3**: Check it) $\int_D b \cdot \text{grad}\, \varphi = \lim_{n\to\infty} \int_D b \cdot \text{grad}\, \varphi_n$ if (10') holds, which implies (10). ◊

**Remark 2.3.** As a corollary of Remark 2.2, any function $\varphi$ such that grad $\varphi$ is the limit, in the above sense, of a sequence of gradients of test functions, also qualifies as a test function. We shall remember this in due time. ◊

**Remark 2.4.** If the superscript $k$ in $C_0^k(D)$ is thus not crucial (all that is required of $\varphi$, in terms of regularity, is to have a square-integrable gradient), what about the subscript 0, denoting compact support ? *That* is essential. If test functions could assume nonzero values on the boundary, this would put *more* constraints on $b$ than mere solenoidality. We shall make use of this too, when treating boundary conditions. ◊

### 2.3.2 The "curl" side

All of this cries out for symmetrization: What we did with the divergence operator should have counterparts with the curl operator. This time we know the way and will go faster.

Let $a$ and $h$ both belong to $\mathbb{C}_0^1(E_3)$, and let $D$ be as above. Form $u = h \times a$. We have this other vector analysis formula,

$$\text{div}(h \times a) = a \cdot \text{rot}\, h - h \cdot \text{rot}\, a,$$

and by Ostrogradskii again, we get

$$\int_D h \cdot \text{rot}\, a = \int_D a \cdot \text{rot}\, h - \int_S n \times h \cdot a, \tag{12}$$

the second fundamental integration by parts formula, on a par with (9).

**Remark 2.5.** Note the formal analogies, and also the differences, between (9) and (12): grad became rot, – div became rot too, × replaced the dot, signs changed in puzzling patterns . . . Obviously these two formulas are, in some half-veiled way, realizations of a *unique* one, which would call for different notation and concepts: those of differential geometry. ◊

By (12), a smooth curl-free field $h$ in $D$ is characterized by

$$\int_D h \cdot \text{rot}\, a = 0 \quad \forall\, a \in \mathbb{C}_0^1(D). \tag{13}$$

Again (13) makes sense for non-smooth fields $h$, if they are square-integrable, and hence (now the obvious thing to do):

**Definition 2.2.** *A piecewise smooth field* $\mathrm{h}$ *which satisfies* (13) *will be said to be* curl-free, *or irrotational,* in the weak sense.

We can prove, in quite the same way as Prop. 2.1, what follows:

**Proposition 2.2.** *For a piecewise-smooth* $\mathrm{h}$, (13) *is equivalent to* $\mathrm{rot}\,\mathrm{h} = 0$ *inside regularity regions and* $[\mathrm{n} \times \mathrm{h}] = 0$ *at their interfaces.*

*Proof.* This should be an exercise. If $\mathrm{rot}\,\mathrm{h} = 0$ in the regularity regions $\mathrm{D}_1$ and $\mathrm{D}_2$, and $[\mathrm{n} \times \mathrm{h}] = 0$ at the interface $\Sigma$, then

$$\begin{aligned} \int_{\mathrm{D}} \mathrm{h} \cdot \mathrm{rot}\,\mathrm{a} &= \int_{\mathrm{D}_1} \mathrm{h} \cdot \mathrm{rot}\,\mathrm{a} + \int_{\mathrm{D}_2} \mathrm{h} \cdot \mathrm{rot}\,\mathrm{a} \\ &= \int_{\mathrm{D}_1} \mathrm{a} \cdot \mathrm{rot}\,\mathrm{h} - \int_{\Sigma} \mathrm{n}_1 \times \mathrm{h} \cdot \mathrm{a} + \int_{\mathrm{D}_2} \mathrm{a} \cdot \mathrm{rot}\,\mathrm{h} - \int_{\Sigma} \mathrm{n}_2 \times \mathrm{h} \cdot \mathrm{a} \\ &= -\int_{\Sigma} [\mathrm{n} \times \mathrm{h}] \cdot \mathrm{a} = 0 \end{aligned}$$

for all test fields in $\mathbb{C}_0^1(\mathrm{D})$, which is (13). Conversely, assuming (13), we first obtain $\mathrm{rot}\,\mathrm{h} = 0$ in $\mathrm{D}_1$ and $\mathrm{D}_2$ separately by the same maneuvers as above, then, backtracking,

$$\begin{aligned} 0 = \int_{\mathrm{D}} \mathrm{h} \cdot \mathrm{rot}\,\mathrm{a} &= \int_{\mathrm{D}_1} \mathrm{h} \cdot \mathrm{rot}\,\mathrm{a} + \int_{\mathrm{D}_2} \mathrm{h} \cdot \mathrm{rot}\,\mathrm{a} = \\ &= -\int_{\Sigma} \mathrm{n}_1 \times \mathrm{h} \cdot \mathrm{a} - \int_{\Sigma} \mathrm{n}_2 \times \mathrm{h} \cdot \mathrm{a} = -\int_{\Sigma} [\mathrm{n} \times \mathrm{h}] \cdot \mathrm{a} \end{aligned}$$

for all $\mathrm{a} \in \mathbb{C}_0^1(\mathrm{D})$. Surface values of $\mathrm{a}$ are not constrained on $\Sigma$, so the only way this equality can hold is by having $[\mathrm{n} \times \mathrm{h}]_{\Sigma} = 0$. ◊

One may generalize there too:

$$\int_{\mathrm{D}} \mathrm{h} \cdot \mathrm{rot}\,\mathrm{a} = \int_{\mathrm{D}} \mathrm{j} \cdot \mathrm{a} \quad \forall\, \mathrm{a} \in \mathbb{C}_0^1(\mathrm{D}), \tag{14}$$

with $\mathrm{j}$ given, piecewise smooth. This means (**Exercise 2.4:** Make sure you understand this) "$\mathrm{rot}\,\mathrm{h} = \mathrm{j}$ in the weak sense". Here also, $\mathbb{C}_0^1(\mathrm{D})$ can be replaced by $\mathbb{C}_0^\infty(\mathrm{D})$.

This was only a first brush with weak formulations, and the full potential of the idea has not been exploited yet. Instead of (10) or (10'), we could have characterized divergence-free vector fields by the weak formulation

$$\int_{\mathrm{D}} \mathrm{b} \cdot \mathrm{grad}\,\varphi = \int_{\mathrm{S}} \mathrm{n} \cdot \mathrm{b}\ \varphi \quad \forall\, \varphi \in \mathrm{C}^\infty(\bar{\mathrm{D}}), \tag{15}$$

for instance, which suggests (cf. Remark 2.4) that not only right-hand sides, as in (11), but also some boundary conditions may be accommodated. The symmetrical formula on the curl side is

$$\int_{\mathrm{D}} \mathrm{h} \cdot \mathrm{rot}\,\mathrm{a} = \int_{\mathrm{S}} \mathrm{n} \times \mathrm{h} \cdot \mathrm{a} \quad \forall\, \mathrm{a} \in \mathbb{C}^\infty(\bar{\mathrm{D}}). \tag{16}$$

With experience, this flexibility turns out to be the most compelling reason to use weak formulations.

### 2.3.3 The uniqueness issue

So we got rid of the ambiguities hidden in the "strong" formulations rot h = j and div b = 0. A different kind of problem arises about the *uniqueness* of the solution, assuming there is one. Take $j = 0$ in (1), and $\mu = \mu_0$ in all space. The physical solution is then $h = 0$ and $b = 0$. But this is *not* implied by Eqs. (1–3): Take $h = \text{grad}\ \varphi$, where $\varphi$ is a harmonic function in all space (for instance, to exhibit only one among an infinity of them, $\varphi(x, y, z) = xy$, in x–y–z Cartesian coordinates). Then rot h = 0, and $\text{div}(\mu_0 h) = \mu_0 \Delta\varphi = 0$. So we have here an example of a nonzero static field that satisfies the equations, although there is no source to create it.

All fields of this kind, however, have in common the property of carrying infinite energy, which is the criterion by which we shall exclude them: We want[12] fields with *finite* energy. From Chapter 1, the expression of the energy of the magnetic field is

$$W_{mag} = {}^1/_2 \int_{E_3} \mu\, |h|^2 = {}^1/_2 \int_{E_3} \mu^{-1}\, |b|^2. \tag{17}$$

Since $\mu \geq \mu_0$ all over, the first integral is bounded from below by $\mu_0 \int_{E_3} |h|^2/2$; hence the eligible h's are *square-integrable:* $\| h \| < \infty$, where $\| \ \|$ denotes the quadratic norm, thus defined:

$$\| h \| = [\int_{E_3} |h(x)|^2\, dx]^{1/2}.$$

If there is also an upper bound $\mu_1$ to $\mu$, which we assume, the same reasoning with the other integral shows that b should be square-integrable as well.

To be consistent with this requirement of finite energy, we shall also assume that j, besides being piecewise smooth, has compact support: this excludes cases such as, for instance, that of a uniform current density in all space, which would generate a field of infinite energy.

All that is required of $\mu$, then (last item in our critical review of (1–3)), is not to spoil these arrangements. We want the integrals in (17) to make sense for all eligible fields h and b, that is, square-integrable fields, and this is the case if $\mu$ is piecewise smooth (a reasonable requirement, as regards a material property) and if there exist two positive

[12]Note this is a *modelling choice*, justified in the present situation, not a dogma.

constants $\mu_0$ and $\mu_1$ such that[13]

(18) $\mu_0 \le \mu(x) \le \mu_1$ a.e. in $E_3$.

**Remark 2.6.** The two values in (17) are equal when $b = \mu h$. But notice we have there two different expressions of the energy, one in terms of $h$, the other in terms of $b$. It's customary to call *energy* of a vector field $b$ (any vector field $b$, not necessarily the physical induction) the integral $\frac{1}{2}\int \mu^{-1} |b|^2$, and *coenergy* of $h$ the integral $\frac{1}{2}\int \mu |h|^2$. Note that energy(b) + coenergy(h) $\ge \int b \cdot h$, with equality only when $b = \mu h$. ◊

## 2.4 MODELLING: THE SCALAR POTENTIAL FORMULATION

At last, we found a problem that, first, is relevant to the situation, and second, makes mathematical sense:

*Given* $\mu$ *and* j, *piecewise smooth, with* $\mu$ *as in* (18) *and* j *with compact support, find piecewise smooth fields* b *and* h *such that*

$$
\left|
\begin{array}{l}
\int_{E_3} b \cdot \operatorname{grad} \varphi = 0 \quad \forall\, \varphi \in C_0^\infty(E_3), \\
b = \mu\, h, \\
\int_{E_3} h \cdot \operatorname{rot} a = \int_{E_3} j \cdot a \quad \forall\, a \in \mathbb{C}_0^\infty(E_3).
\end{array}
\right. \tag{19}
$$

Whether there is a solution and how to get it is another story, but at least we have, for the first time so far, a *model*.

### 2.4.1 Restriction to a bounded domain

For the moment, let us return to physics, and criticize this model on the grounds of an element of the situation which has been neglected up to now: the large value of $\mu$ in the magnetic core $M$ of the apparatus. A look at Fig. 2.6 shows that flux lines will arrive almost orthogonally to the "magnetic wall" $\partial M$ (the boundary of $M$). On the other hand, if $\mu$ is large, $h$ must be small in $M$, since the magnetic energy is finite.[14] We are thus entitled to neglect $M$ in the eventual calculation, and to set

[13]The abbreviation "a.e." stands for "almost everywhere", meaning "at all points except those of some negligible set". (The latter notion is discussed in Appendix A, Subsection A.4.2.) The a.e. clause is a necessary precaution since $\mu$ has no definite value at discontinuity points.

[14]Be wary of this line of reasoning, which is correct in the present case, but can lead to unexpected trouble in some topologically complex situations [Bo].

$$n \times h = 0 \quad \text{on } \partial M$$

as a boundary condition for a problem that will now be posed in the complementary domain $E_3 - M$.

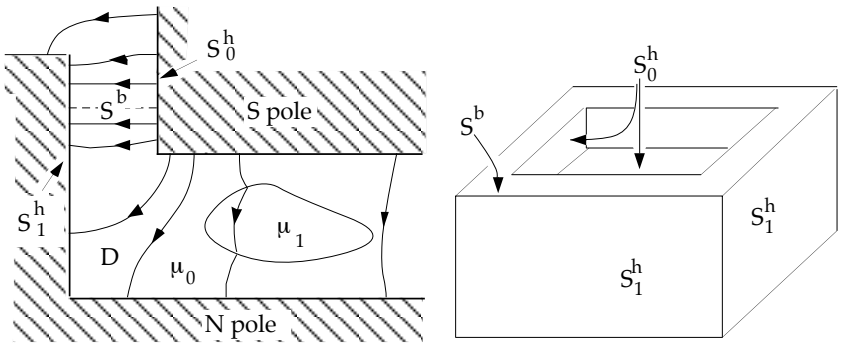



**FIGURE 2.6.** Left: Expected pattern of field lines inside the box, showing the existence of a horizontal plane on which $n \cdot b = 0$, an annular part of which, called $S^b$, will close the box. Right: Persective view of the "computational domain" $D$ thus delimited, and of its surface. One has $S = S^h \cup S^b$, and the "magnetic wall" $S^h$ is in two parts, $S^h_0$ and $S^h_1$.

But one can go farther here, and restrict the domain of interest to the "central box" of Fig. 2.1, the experimental volume. Fig. 2.6 explains why: The air region $E_3 - M$ is almost cut in two by the magnetic circuit, and between the North and South poles of the electromagnet, there is an air gap in which the flux lines go straight from one magnetic wall to the opposite one, horizontally. So we can introduce there an artificial boundary ($S^b$ in Fig. 2.6), horizontal, which one can assume is spanned by flux lines (this is only approximately true, but a legitimate approximation), and therefore

$$n \cdot b = 0 \quad \text{on } S^b.$$

Consequently, let us restrict our computational domain to the part of the inner box below the plane of $S^b$, and call $D$ this region. Its boundary $S$ is thus made of $S^b$ and of the part of $\partial M$ which bounds the inside of the box, which we shall denote by $S^h$. Hence our boundary conditions, which, combined with the strong form of the magnetostatics equation, lead to

(20) $\operatorname{rot} h = 0$ in D, (21) $n \times h = 0$ on $S^h$,

(22) $b = \mu h$ in D,

(23) $\operatorname{div} b = 0$ in D, (24) $n \cdot b = 0$ on $S^b$.

We note that $S^h$ is in two parts, $S^h_0$ and $S^h_1$, corresponding to the two poles of the electromagnet.

This calls for a few remarks. First, let's not forget that materials of various permeabilities can be put inside D, so we must expect discontinuous fields, and weak formulations are still in order.

The second remark is about the symmetry of the box, and of its content. In the "Bath cube" experiment, four identical aluminum cubes (hence the nickname) were put inside the box, symmetrically disposed, so that it was possible to solve for only a quarter of the region, since the whole field is then symmetrical with respect to the vertical symmetry planes, hence $n \cdot b = 0$ there. The equations are thus the same, provided D and $S^b$ are properly redefined: D as a quarter of the cavity, and $S^b$ as a quarter of the former $S^b$ plus the part of the symmetry planes inside the box. We shall do that in Chapter 6, but we may ignore the issue for the time being.

Third remark, the above display (20–24) does not say anything about the source of the field. That was j, the current density in the coil, which is now out of the picture. This lost information must be reintroduced into the formulation in some way.

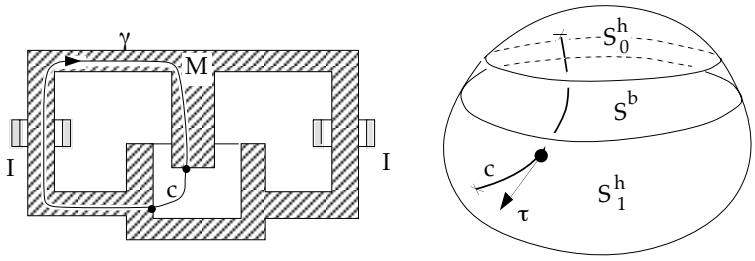



**FIGURE 2.7.** Left: Applying Ampère's theorem to the path $\gamma$ shows that the mmf along c is approximately equal to the DC intensity I. Right: topological aspects of the situation.

Figure 2.7 suggests how it can be done. Consider a circuit $\gamma$ which, except for the part c inside D that links opposite poles, is entirely contained in M. By Ampère's theorem, the circulation of h along $\gamma$ is equal to I, the DC intensity in the coil.[15] But $\mu$ being very large in M,

the field $h$ is so small there that the circulation along $\gamma$ is approximately equal to the circulation along the sub-path $c$. Since we already assumed $h = 0$ in $M$ in this modelling, we consistently set

$$\int_c \tau \cdot h = I, \tag{25}$$

where $\tau$ is the field of unit tangent vectors along $c$ (Fig. 2.7). (**Exercise 2.5:** Show that any path $c$ from $S^h_0$ to $S^h_1$ will give the same circulation.) Now, common sense says that (20–24) and (25) do uniquely determine the field, and the mathematical model we are building had better have this property (which we'll eventually see is the case).

There is another possibility: We could specify the magnetic flux $F$ through the box instead, like this:

$$\int_{S_1^h} n \cdot b = F. \tag{26}$$

(**Exercise 2.6:** Show that other surfaces than $S^h_1$ can be used in (26) with the same result. How would you characterize them?) Of course, $F$ is not known here, but this is not important for a *linear* problem: Just solve with some value for $F$, get $I$, and scale. In fact, since we want to compute the *reluctance* of the system, which is by definition the ratio $R = I/F$, the flux is the objective of the computation if $I$ is known, and the other way around. We may thus solve (20–24) (25) and then compute $F$, using some approximation of formula (26), or solve (20–24) (26), with an arbitrary nonzero value for $F$, and then compute $I$ by (25). This alternative reflects the symmetry between $b$ and $h$ in the problem's formulation.

We shall return to this symmetry (Chapter 6). We now break it by playing the obvious move in the present situation, which is to introduce a *magnetic potential*.

### 2.4.2 Introduction of a magnetic potential

Indeed, since the field $h$ we want must be curl-free, it is natural to look for it as the gradient of some function $\varphi$. The boundary condition $n \times h = 0$ on $S^h$ is then satisfied by taking $\varphi$ equal to a constant there. (This is general: Magnetic walls are equipotentials for $\varphi$ in static contexts.) Since $S^h$ is in two pieces, there are two such constants, one of which can be $0$. The other one must then be equal to $I$, after (25).

[15]Notice how the equality of intensities in the energizing coils is necessary in this reasoning: Otherwise, we could not assume $\mu$ infinite in $M$ without contradiction. This is a well-known difficulty of the theory of the transformer, which we shall ignore here.

All these considerations lead us to the definition of a class of *admissible* potentials: piecewise smooth functions $\varphi$, which satisfy all the a priori requirements we have about $\varphi$ (finite energy, being equal to 0 or I on $S^h$), and we shall select in this class *the* potential which solves the problem. This is, still grossly sketched, the *functional point of view:* Define a functional space of eligible candidates, characterize the right one by setting tests it will have to pass, and hence an *equation*, which one will have to solve, exactly or approximately.

To define admissible potentials, let's proceed by successive reductions. First, a broad enough class:

$$\Phi = \{\text{all } \varphi\text{'s piecewise smooth (over the closure of D)}\}.$$

(If D was not, as here, bounded, we should add "such that $\int_D |\operatorname{grad}\varphi|^2$ is finite", in order to take care of the finite energy requirement. This is implicit in the present modelling, but should be kept in mind.) Next,

$$\Phi^I = \{\text{all } \varphi \in \Phi : \varphi = 0 \text{ on } S^h_0 \text{ and } \varphi = I \text{ on } S^h_1\} \tag{27}$$

where I is just a real parameter for the moment. In particular, we shall have $\Phi^0 = \{\varphi \in \Phi : \varphi = 0 \text{ on } S^h\}$. If $\varphi$ is in $\Phi^I$ for *some* value of I, it means that $n \times \operatorname{grad}\varphi = 0$ on $S^h$, and thus Eqs. (20) and (21) are satisfied by $h = \operatorname{grad}\varphi$, if $\varphi$ is any of these potentials. Last, we select the given value of I, and now, if $\varphi$ is in *this* $\Phi^I$, (25) is satisfied.

Eligible potentials thus fulfill conditions (20), (21), and (25). To deal with the other conditions, we request b ($\equiv \mu \operatorname{grad}\varphi$) to satisfy (23) by using the weak solenoidality condition. But since *the set of test functions is left to our choice*, we may do better and also check (24), all in one stroke:

**Proposition 2.3.** *If* $\varphi \in \Phi^I$ *is such that*

$$\int_D \mu \operatorname{grad}\varphi \cdot \operatorname{grad}\varphi' = 0 \quad \text{for all test functions } \varphi' \text{ in } \Phi^0, \tag{28}$$

*then the field* $b = \mu \operatorname{grad}\varphi$ *verifies* (23) *and* (24).

(Pay attention to the notational shift: Since from now on we shall have the eligible potentials on the one hand, and the test functions on the other hand, the latter will be denoted with a prime. This convention will be used throughout.)

*Proof.* Set $b = \mu \operatorname{grad}\varphi$. This is a piecewise continuous field. Since $\Phi^0$ contains $C_0^\infty(D)$, we have div b = 0 in the weak sense, as required. But since there are test functions in $\Phi^0$ which do not belong to $C_0^\infty(D)$ (all those that do not vanish on $S^b$), the implications of (28) may not have been all derived. Starting from (28), and integrating by parts with formula

(9), we get

$$0 = \int_D b \cdot \text{grad}\, \varphi' = -\int_D \varphi' \,\text{div}\, b + \int_S n \cdot b\ \varphi' = \int_{S^b} n \cdot b\ \varphi' \quad \forall \varphi' \in \Phi^0,$$

since $\text{div}\, b = 0$ a.e. and $\varphi' = 0$ on $S^h$ *by our choice* of test functions. What is thus left is the following implication of (28):

$$\int_{S^b} n \cdot b\ \varphi' = 0 \quad \forall \varphi' \in \Phi^0,$$

which can be satisfied, since values of $\varphi'$ are unconstrained on $S^b$, only by $n \cdot b$ vanishing on this part of the boundary. ◊

We are thus entitled to set a *problem:*

(29) *find* $\varphi$ *in* $\Phi^I$ *such that* (28) *hold.*

This (mathematical) problem, more accurately described as an *equation,*[16] is *the weak formulation, in scalar potential*, of our (physical) problem. We just proved that if there is a solution, it will satisfy all the requirements of the modelling.

### 2.4.3 Uniqueness

No need to underline what this proof owes to that of Prop. 2.1. (Notice that the ideas of Remark 2.4 and Eq. (15) also have been exploited, to some extent.) But the serendipity by which $\Phi^0$ happened to be the right space of test functions calls for an explanation, which Fig. 2.8 will suggest: In the linear space $\Phi$, the $\Phi^I$s form a family of *parallel* affine subspaces, and are thus all isomorphic with the vector subspace $\Phi^0$. In particular the difference between two eligible potentials $\varphi_1$ and $\varphi_2$, being in $\Phi^0$, qualifies as a test function.

Now, (29) can be construed as a system of linear equations, to be satisfied by $\varphi$, one equation for each test function engaged. Even though we are dealing here with infinite-dimensional spaces, and thus, so to speak, with an infinity of unknowns, the general rule of algebra that there should be "as many equations as unknowns" in a properly formed linear system is still in force: Fig. 2.8 shows that our choice of test functions obeys this

[16] In the more precise language of Appendix A, an equation is the problem consisting in finding all the values of the free variable in some predicate. Here the free variable is $\varphi$, and the predicate is (28); it consists of a *list* of subpredicates, indexed by the bound variable $\varphi'$. Note again the importance of the "for all" clause in (28) in this mechanism. Without it, we wouldn't have an equation, only nonsense.

rule automatically, thanks to the one-to-one correspondence between $\Phi^I$ and its parallel vector subspace.

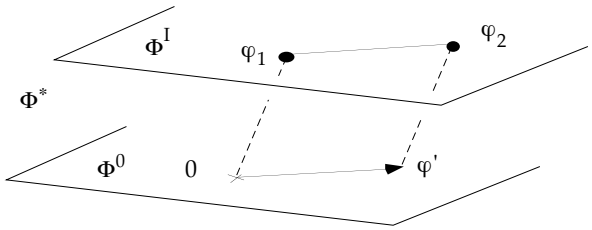



**FIGURE 2.8.** Geometry of the variational method. The "space" of the picture represents $\Phi$, and the parallel "planes" represent $\Phi^I$ and $\Phi^0$. The latter contains the origin. Dots and arrows signal points and vectors, respectively, in these infinite-dimensional spaces. $\Phi^*$ is an ad-hoc notation for the set union $\cup\{\Phi^I : I \in \mathbb{R}\}$, which does not fill out $\Phi$.

This *proves* nothing yet, of course. But the heuristic principle thus suggested is of enormous value: *To find the weak form of a problem, set up the affine space of all a priori eligible solutions, then use the elements of the parallel vector subspace as test functions.*

This principle is quite flexible: "Eligible" depends on which equations and boundary conditions we can, and wish to, enforce a priori, and the others are automatically taken into account by weak formulation of the remaining requirements of the model (cf. Exer. 2.9). Here, we chose to enforce the equations relative to h (which is why this method can be depicted as "h-oriented"), but we might as well have focused on the equations relative to b, hence a b-oriented method (the opening move of it, of course, would be to introduce a *vector* potential, b = rot a). We'll do this in Chapter 6. There is also some leeway with the constant I, which was imposed here, but could have been left in charge of the weak formulation, as we shall see also.

As a first testimony of the power of the principle, let us prove this "uniqueness" result:

**Proposition 2.4.** *Problem* (29) *has at most one solution.*

*Proof.* Suppose there are two solutions $\varphi_1$ and $\varphi_2$. Then

$$\int_D \mu \, \mathrm{grad}(\varphi_1 - \varphi_2) \cdot \mathrm{grad}\, \varphi' = 0 \quad \forall\ \varphi' \text{ in } \Phi^0. \tag{30}$$

But (see Fig. 2.8), $\varphi_1 - \varphi_2$ is one of the test functions, and for *that* one, (30)

yields $\int_D \mu\, |\mathrm{grad}(\varphi_1 - \varphi_2)|^2 = 0$, hence (cf. (18)) $\mathrm{grad}(\varphi_1 - \varphi_2) = 0$, which means $\varphi_1 = \varphi_2$, since they coincide on $S^h$. $\Diamond$

**Remark 2.7.** This prompts the question, irrelevant here, but sensible in other circumstances, "What if $S^h$ is empty?" Then, simply, the potential is not unique, but the field $h = \mathrm{grad}\,\varphi$ is, which is generally what one is interested in. $\Diamond$

If this was linear algebra, Prop. 2.4 would solve the problem! For in finite dimension, *uniqueness forces existence,* as the old saying goes, when the number of equations and unknowns coincide. (**Exercise 2.7:** Why?) But here we deal with elements of an infinite-dimensional space, in which things are not that simple. Whether and when problem (29) has a solution, the *existence* issue, will be the concern of the next chapter. But before leaving the present one, something you may have been surprised to see de-emphasized:

### 2.4.4 Laplace, Poisson, Dirichlet, and Neumann

As a consequence of (23) and (24), the solution of (29) will satisfy

$$-\mathrm{div}(\mu\, \mathrm{grad}\,\varphi) = 0 \quad \text{in } D, \tag{31}$$

(or at least, inside regions of regularity—but we shall stop reminding that all the time, from now on), and

$$\varphi = 0 \ \text{ on } S^h_0, \quad \varphi = I \ \text{ on } S^h_1, \tag{32}$$

$$\mu\, \partial_n \varphi = 0 \ \text{ on } S^b, \tag{33}$$

where $\partial_n\varphi$ is the notation in force here for the normal derivative of $\varphi$ at the boundary, often denoted as $\partial\varphi/\partial n$. Equation (31) is an immediate generalization of the *Laplace equation* $\Delta\varphi = 0$, to which it reduces if $\mu = \mu_0$ in all $D$. The expression *Poisson problem* refers to (31) with a nonzero right-hand side, which we don't have here, but could easily handle (cf. (11) and Exer. 2.2). One calls (32) and (33) the *Dirichlet* and *Neumann boundary conditions*, respectively. Here the latter are homogeneous (right-hand side equal to 0), but non-homogeneous similar conditions can be accommodated by the above method, as we'll see later.

This Dirichlet vs Neumann opposition is classical and quite important, but here we should rather focus on the *fields* $h$ and $b$ than on the potentials, and the $n \times h$ vs $n \cdot b$ contrast is thus more topical. Also more important conceptually is the distinction between *essential* boundary conditions, like (32), which are built into the very definition of the set of admissible

solutions, and *natural* conditions like (33), which are enforced by the weak formulation.

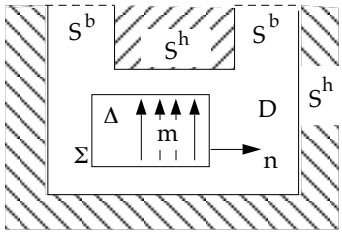
## EXERCISES

Exercise 2.1 is on p. 34. Exers. 2.2 and 2.3 are on p. 44, and Exer. 2.4 p. 46. Exercises 2.5 and 2.6 are on p. 51, and Exer. 2.7 on p. 55.

**Exercise 2.8.** Suppose a part of $D$ contains a permanent magnet, characterized by $b = \mu_0(h + m)$, where $m$ is a given field, the rest of $D$ being air. Show that the weak formulation, *find* $\varphi \in \Phi^I$ *such that*

$$(34) \qquad \int_D \mu_0 \, \text{grad}\, \varphi \cdot \text{grad}\, \varphi' = - \int_D \mu_0 \, m \cdot \text{grad}\, \varphi' \quad \forall \varphi' \in \Phi^0,$$

is a correct interpretation of the problem. In case $m$ is uniform over a region $\Delta \subset D$, and 0 outside (inset), show that (34) can be written




$$(35) \qquad \int_D \mu_0 \, \text{grad}\, \varphi \cdot \text{grad}\, \varphi' = \int_\Sigma n \cdot (\mu_0 \, m) \; \varphi' \quad \forall \varphi' \in \Phi^0,$$

where $\Sigma = \partial\Delta$.

**Exercise 2.9.** Consider the space $\Phi^* = \cup_I \Phi^I$ of Fig. 2.8. Each $\varphi \in \Phi^*$ belongs to one of the $\Phi^I$s, so let's define $\mathcal{J}$ as the map that assigns to $\varphi$ the corresponding value of I. Let $F$ be given. Show that if $\varphi \in \Phi^*$ satisfies

$$(36) \qquad \int_D \mu \, \text{grad}\, \varphi \cdot \text{grad}\, \varphi' = F \, \mathcal{J}(\varphi') \quad \forall \varphi' \in \Phi^*,$$

then $b = \text{grad}\, \varphi$ verifies (23), (24) and (26).

**Exercise 2.10.** Here the artificial boundary $S^b$ has been placed in a position where it was known in advance that $n \cdot b = 0$. It may happen that the flux is thus known on some conveniently placed surfaces, but not null. What to do then?

## HINTS

2.1. The definition doesn't say *bounded* regions. Recall that continuous functions are bounded on closed *bounded* (hence, compact) regions of a finite-dimensional space.

2.2. Just redo the proof of Prop. 2.1, reintroducing $f$ at the right places. Observe the way a minus sign appears.

2.3. Cauchy–Schwarz. Observe (this is for experts) that a certain condition on the supports of the $\varphi_n$s should be satisfied.

2.4. Same as Exer. 2.2.

2.5. Build a circuit on which to apply Stokes.

2.6. Build a volume to which Ostrogradskii–Gauss may apply, part of its surface being $S^h_1$.

2.7. The question is, if $\mathbf{A}$ is an $n \times n$ matrix, and $\mathbf{b}$ an n-vector, why does uniqueness of $\mathbf{x}$ such that $\mathbf{Ax}=\mathbf{b}$ imply the existence of a solution, whatever the right-hand side?

2.8. All that matters is $\operatorname{div} b = 0$, where $b = \mu_0 (m + \operatorname{grad} \varphi)$, and the proof of Prop. 2.3 handles that. For (35) vs (34), apply (9) to $\Delta$.

2.9. $\Phi^*$ being *larger* than $\Phi^0$, the proof of Prop. 2.3 can be recycled in its entirety, hence (23) and (24). So concentrate on (26), using (9).

2.10. Imitate (35), which can be understood as describing a flux injection $n \cdot (\mu_0 m)$ on $\Sigma$.

## SOLUTIONS

2.1. The constant 1 is not integrable over all $E_3$, so the restriction to bounded domains is certainly necessary. Now suppose $f$ is smooth over each region $R_i$ of a finite family. The way we understand "over", $f$ is smooth, and in particular continuous, in a bounded domain $D_i$ containing the closure of $R_i \cap D$, which is thus compact, so $f$ is bounded there, hence integrable. Pieces being in finite number, $f$ is integrable on $D$.

2.3. Since $\operatorname{supp}(\varphi)$ is compact, one can build the $\varphi_n$s so that there exists a compact $K$ containing all the $\operatorname{supp}(\varphi_n)$. Then (Exer. 2.1) $b$ is bounded on

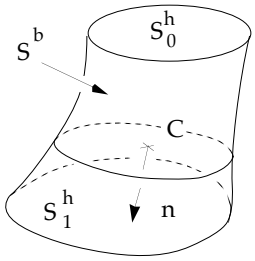
K, and hence, applying the Cauchy–Schwarz inequality,

$$|\int_D b \cdot \mathrm{grad}(\varphi - \varphi_n)|^2 \le \int_D |b|^2 \int_D |\mathrm{grad}(\varphi - \varphi_n)|^2$$

tends to zero.

2.5. Take $c_1$ and $c_2$ from $S^h_0$ to $S^h_1$, with the same orientation, and join the extremities by two paths lying in $S^h_0$ and $S^h_1$ respectively, in order to make a closed circuit. As rot h = 0, and by the Stokes theorem, the circulations along $c_1$ and $c_2$ are equal (those along the boundary links are 0 by (21)). One says that $c_1$ and $c_2$ are *homologous*. (The relation between them is an equivalence, called *relative homology modulo* $S^h$. We'll have more to say about this in Chapter 5. Cf. [GH].)

2.6. See the inset. Surface C is what is commonly called a "cut": Its boundary is entirely in $S^b$, and it separates D into two parts, each containing one piece of $S^h$. Moreover, C has an external orientation (provided by a normal field n), compatible with that of $S^h_1$. Now, as $n \cdot b = 0$ on $S^b$, the fluxes through C and $S^h_1$ are equal, by Ostrogradskii, since div b = 0 in D. All possible cuts of this kind will do, including $S^h_1$ and $S^h_0$, *but* the latter must be oriented the other way with respect to S. Again we have here an equivalence relation (relative homology, but now modulo $S^b$), and "cuts" are elements of a same class of surfaces, of which one says they are *homologous* (mod $S^b$). We'll return to this in Chapter 4, and again, more formally, in 5.2.5.




2.7. Because then **A** is regular.

2.8. $\int_D m \cdot \mathrm{grad}\,\varphi' = \int_\Delta m \cdot \mathrm{grad}\,\varphi' = -\int_\Delta \mathrm{div}\, m\ \varphi' + \int_\Sigma n \,.\, m\varphi'$. If div m ≠ 0, there is no special avantage to this formulation over (34), but otherwise (35) may be easier to implement in the subsequent finite element modelling. Be careful about the correct orientation of the normal on Σ when doing that. (You may worry about what happens when Δ touches S. This is a good question, but no more a simple exercise.)

2.9. By (9), and using the information brought by the proof of Prop. 2.3 (div b = 0, $n \cdot b = 0$ on $S^b$), plus $\varphi' = 0$ on $S^h_0$, (36) reduces to

$$(37) \qquad F\,\mathcal{J}(\varphi') = \int_{S^h} n \cdot b\ \varphi' \equiv \int_{S^h_1} n \cdot b\ \varphi' \quad \forall\, \varphi' \in \Phi^*,$$

and since the value of $\varphi'$ on $S^h_1$ is precisely $\mathcal{J}(\varphi')$, by definition of $\mathcal{J}$, we have $F\,\mathcal{J}(\varphi') = (\int_{S^h} n \cdot b)\,\mathcal{J}(\varphi')$ for all $\varphi'$, hence $\int_{S^h} n \cdot b = F$.

Now, $\varphi$ being known, $\mathcal{J}(\varphi)$ has a definite value I, and the desired reluctance is $R = I/F$. This trick, by which the essential boundary constraint $\varphi = I$ on $S^h_1$ (condition (25)) has been exchanged for the natural boundary condition (37), is known as the *dualization* of the constraint (25).

2.10. Let $g$ be the known value of $n \cdot b$ on $S^b$. Prolongate $g$ to all $S$ by setting $g = 0$ (or any value, it doesn't matter) on $S^h$. The relevant weak formulation is *find* $\varphi \in \Phi^I$ *such that*

$$\int_D \mu \, \text{grad}\, \varphi \cdot \text{grad}\, \varphi' = \int_S g \varphi' \quad \forall \varphi' \in \Phi^0.$$

Indeed, (9) yields $\int_S n \cdot b \; \varphi' = \int_S g \; \varphi' \quad \forall \varphi' \in \Phi^0$, therefore $n \cdot b = g$ on $S^b$, where the values of $\varphi'$ are free.

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The differential geometric point of view alluded to in Subsection 2.2.3 will probably soon gain popularity. Most introductions to differential geometry have a chapter on Electromagnetism, e.g.:

[Bu] W.L. Burke: **Applied Differential Geometry**, Cambridge University Press (Cambridge, U.K.), 1985.

[Sc] B. Schutz: **Geometrical methods of mathematical physics**, Cambridge University Press (Cambridge, U.K.), 1980.

[We] S. Weintraub: **Differential Forms, A Complement to Vector Calculus,** Academic Press (San Diego), 1997.

However, few book-size treatments are available so far:

[Ko] P.R. Kotiuga: **Hodge Decompositions and Computational Electromagnetics** (Thesis), Department of Electrical Engineering, McGill University (Montréal), 1984.

[BH] D. Baldomir, P. Hammond: **Geometry of Electromagnetic Systems,** Oxford U.P. (Oxford), 1996.

The idea by itself is not new, and several authors have devoted work to its promotion. See:

[Bn] F.H. Branin, Jr.: "The algebraic-topological basis for network analogies and the vector calculus", in **Symposium on Generalized Networks** (12–14 April 1966)**,** Polytechnic Institute of Brooklyn , 1966, pp. 453–491.

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[To] E. Tonti: "On the Geometrical Structure of Electromagnetism", in **Gravitation, Electromagnetism and Geometrical Structures** (G. Ferrarese, ed.)**,** Pitagora (Bologna), 1996, pp. 281–308.

I do think this differential geometric approach, far from being merely an esthetically attractive alternative, is mandatory when it comes to the question (not addressed here) of force computation in deformable materials. Cf.:

A. Bossavit: "Differential forms and the computation of fields and forces in Electromagnetism", **Europ. J. Mech., B/Fluids, 10**, 5 (1991), pp. 474–488.

A. Bossavit: "Edge-element Computation of The Force Field in Deformable Bodies", **IEEE Trans., MAG-28,** 2 (1992), pp. 1263–1266.

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