CHAPTER 1

Introduction: Maxwell Equations

1.1 FIELD EQUATIONS

Computational electromagnetism is concerned with the numerical study of *Maxwell equations*,

$$(1) \qquad -\partial_t \mathrm{d} + \mathrm{rot}\, \mathrm{h} = \mathrm{j}, \qquad (2) \qquad \partial_t \mathrm{b} + \mathrm{rot}\, \mathrm{e} = 0,$$

$$(3) \qquad \mathrm{d} = \varepsilon_0 \mathrm{e} + \mathrm{p}, \qquad (4) \qquad \mathrm{b} = \mu_0 (\mathrm{h} + \mathrm{m}),$$

completed by *constitutive laws*, in order to account for the presence of matter and for the field-matter interaction. This introductory chapter will explain the symbols, discuss constitutive laws, and indicate how a variety of mathematical models derive from this basic one.

The vector fields e, h, d, b are called *electric field*,[1] *magnetic field*, *magnetic induction*, and *electric induction*, respectively. These four[2] vector fields, taken together, should be construed as the mathematical representation of a physical phenomenon, that we shall call the *electromagnetic field*. The distinction thus made between the physical reality one wants to model, on the one hand, and the mathematical structure thanks to which this modelling is done, on the other hand, is essential. We define a *model* as such a mathematical structure,[3] able to account,

[1] *Italics*, besides their standard use for emphasis, signal notions which are implicitly defined by the context.

[2] Two should be enough, after (3) and (4). Reasons for this redundancy will come.

[3] The structure, in this case, is made of the equations *and* of the framework in which they make mathematical sense: Euclidean three-dimensional space, and time-dependent entities, like scalar or vector fields, living there. There are other possible frameworks: the algebra of differential forms ([Mi], Chapter 4), Clifford algebra [Hs, Ja, Sa], etc. As Fig. 1.1 may suggest, Maxwell's theory, as a *physical* theory, should not be confused with any of its mathematical descriptions (which are historically transient; see [Cr, Sp]).

within some reasonably definite limits, for a class of concrete physical situations. To get a quick start, no attempt is made here either to *justify* the present model, on physical grounds, or to *evaluate* it, in comparison with others. (In time, we'll have to pay for this haste.)

The *current density* j, *polarization* p, and *magnetization*[4] m are the *source*-terms in the equations. Each contributes its own part, as we shall see, to the description of electric charges, at rest or in motion, whose presence is the physical cause of the field. Given j, p, and m, as well as initial values (at time t = 0, for instance) for e and h, Eqs. (1–4) determine e, h, d, b for $t \geq 0$. (This is no trivial statement, but we shall accept it without proof.) Maxwell's model (1–4) thus accounts for situations where j, p, and m are known in advance and independent of the field. This is not always so, obviously, and (1–4) is only the head of a *series* of models, derived from it by adding features and making specific simplifications, some of which will be described at the end of this chapter.

$$4\pi C = \mathrm{V}\cdot\nabla\mathcal{H}, \quad C = \mathrm{C}\mathcal{E} + \dot{\mathcal{D}}, \quad \mathcal{B} = \mathrm{V}\cdot\nabla\mathcal{U}, \quad \mathcal{E} = \mathrm{V}\cdot\dot{\mathcal{R}}\mathcal{B} - \dot{\mathcal{U}} - \nabla\Psi,$$
$$\mathcal{B} = \mu\mathcal{H}, \qquad \mathcal{D} = (4\pi)^{-1}\,\mathrm{K}\mathcal{E},$$

$$\mathrm{dF} = 0, \quad \mathrm{G} = *\mathrm{F}, \quad \mathrm{dG} = \mathrm{J}$$

FIGURE 1.1. Maxwell equations: as they appear in [Ma], Art. 619 (top box), and in modern differential geometric notation (bottom box). Maxwell's formalism, still influenced by quaternionism (∇ is the operator $\mathrm{i}\,\mathrm{d}/\mathrm{dx} + \mathrm{j}\,\mathrm{d}/\mathrm{dy} + \mathrm{k}\,\mathrm{d}/\mathrm{dz}$, and the V means "vector part" of a quaternionic product), is not so remote from today's standard vector notation, once the symbols have been identified (Ψ and $\mathcal{U}$ are scalar and vector potential, $\dot{\mathcal{R}}$ is material velocity). In this book's notation, and apart from factors 4π, the upper line would read rot h = $C = \sigma e + \partial_t d$, b = rot a, $e_{mat} = v \times b - \partial_t a - \nabla\psi$, where e_{mat} is the electric field in the comoving frame of reference.

You may be intrigued, if not put off, by the notation. The choice of symbols goes against recommendations of the committees in charge of such matters, which promote the use of **E**, **H**, **D**, **B**, capital and boldface. Using e, h, d, b instead is the result of a compromise between the desire to keep the (spoken) *names* of the symbols as close as possible to accepted practice and the notational habits of mathematics, capitals for functional spaces and lower case for their elements, according to a hierarchy which reflects

[4]Magnetization could more symmetrically be defined as m such that $b = \mu_0 h + m$. The present convention conforms to the dominant usage.

the *functional point of view* adopted in this book. (Explanations on this fundamental point will recur.) Boldface, still employed in the Preface for 3D vectors, according to the standard convention due to Heaviside [Sp], will from now on be reserved for another use (see p. 71). I should also perhaps call attention to the use of the ∂ symbol: If b is a time-dependent vector field, $\partial_t b$ is the field obtained by differentiating b with respect to time. Having thus $\partial_t b$ instead of $\partial b/\partial t$ is more than a mere ink-saving device: It's a way to establish the status of ∂_t as an *operator*, on the same footing as grad, div, and rot (this will denote the curl operator) which all, similarly, yield a field (scalar- or vector-valued, as the case may be) when acting on a field—the functional viewpoint, again. Other idiosyncrasies include the use of constructs such as $\exp(i\omega t)$ for $e^{i\omega t}$ and, as seen here, of i for the square root of -1, instead of j.[5]

This being said, let's return to our description. Equation (1) is *Ampère's theorem*. Equation (2) is *Faraday's law*. The term $\partial_t d$, whose introduction by Maxwell[6] was the crowning achievement of electromagnetic theory, is called *displacement current*. One defines *electric charge* (expressed in *coulombs* per cubic meter) by

$$q = \operatorname{div} d, \tag{5}$$

a scalar field. According to (1), one has thus

$$\partial_t q + \operatorname{div} j = 0, \tag{6}$$

with j expressed in *ampères* per square meter. Notice that if j is given, from the origin of times to the present, one gets the charge by integration with respect to time: assuming j and q were both null before time 0, then $q(t, x) = -\int_0^t (\operatorname{div} j)(s, x)\, ds$.

If the local differential relation (6) is integrated by applying the Ostrogradskii (Gauss) theorem to a regular[7] spatial domain D bounded by some surface S (Fig. 1.2), one finds that

[5]The shift from i to j was motivated by the desire, at a time when the power of complex numbers in alternating currents theory began to be realized, to denote intensities with the i symbol. Since almost everybody calls "jay" the current density vector, notwithstanding, it makes little sense to perpetuate the dual use of j as the square root of -1. (This remark is respectfully brought to the attention of the above-mentioned Committees.)

[6]Around 1860, and 1873 saw the first edition of his treatise. The classic version we read nowadays [Ma] is the third edition.

[7]"Domain" has a technical meaning: an open set in one piece. Cf. A.2.3. "Regular" means that D is enclosed by one or several surfaces, themselves smooth at all points, with the possible exception of a finite number of corners and edges (Fig. 1.2).

(7) $\frac{d}{dt}\int_D q + \int_S n \cdot j = 0,$

where n denotes the field of normal vectors, of length 1 and outwardly directed with respect to D, on surface S.[8] The first term in this equality is the increase, per unit of time, of the charge contained in D, whereas the second term is the outgoing flux of charge. They balance, after (7), so (6) is the local expression of *charge conservation*.

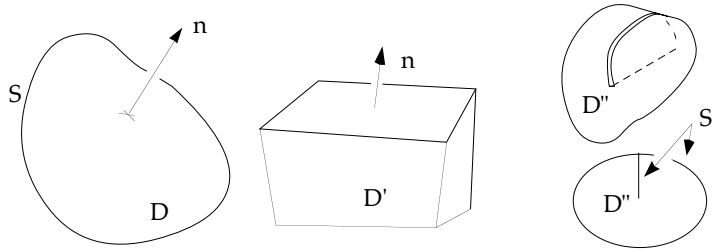
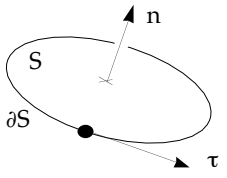


FIGURE 1.2. Notion of *regular domain* (D, on the left), and notations (cf. Note 7). In spite of singularities, D' can still pass as regular (edges and corners form a negligible set), but domain D" on the right (shown in cut view) doesn't qualify, because D" is "on both sides" of a part of its boundary. This geometrical idealization is still useful in the case of small air gaps, cracks, etc., but some care must then be exercised in regard to formulas like (7).

Quite similarly, Eqs. (1) and (2) can be integrated by using the Stokes theorem, hence global (integral) expressions which express flux and current conservation. For instance, the integral form of Faraday's law is

$$\frac{d}{dt}\int_S n \cdot b + \int_{\partial S} \tau \cdot e = 0,$$

where S is a surface, ∂S its boundary, and τ a field of unitary tangent vectors on ∂S (inset),



[8]See Appendix A, Subsection A.4.2, for the notions of *flux* $\int n \cdot j$ and *circulation* $\int \tau \cdot e$, and justification of this notation. When necessary, I denote by dx the volume element, or the area element, according to whether the integral is over a volume or a surface, but each time this does not foster confusion, I'll omit this symbol: $\int f$ rather than $\int f(x)\,dx$. If you do that, don't stop half-way: never $\int f(x)$ alone, without the dx. The symbol x in dx is meant to match with the x in $f(x)$, as demonstrated by the fact that you may substitute at both places some other letter, say y, without changing the meaning. Avoid also constructs such as $\int_S f\,dS$ (although one could make a case for them): it is understood that the integration is with respect to the measure of areas that exists on S, and thus dS is superfluous. The construct $\int_S f$ makes perfect sense by itself (cf. A.4.2): It's the effect on f and S, taken as a pair, of the integration operator.

oriented with respect to n as prescribed by Ampère's rule. Historically, such integral formulations came first and are arguably more germane to physics. Indeed, we shall have to spend some time on correcting some drawbacks of the local differential formulation (or rather, of a too literal interpretation of this formulation).

Treatises on electromagnetism often add two equations to (1–4), namely (5) and div b = 0. But the latter stems from Faraday's law (2), if one assumes a null b (or even just a null div b) before initial time, and (5) is here a definition. So there would be little justification in according to these relations the same status as (1) and (2).

A (rightful) concern for formal symmetry might suggest writing (2) as ∂_t b + rot e = – k, where k would be a given field, the *magnetic current*, and defining *magnetic charge*, expressed in *webers* per cubic meter, as q_m = div b (electric charge q would then be denoted by q_e), hence the equation $\partial_t q_m$ + div k = 0, which would express magnetic charge conservation. But since k and q_m are null in all known physical situations,[9] this generalization seems pointless.

Now, let us address Eqs. (3) and (4). As the next Section will make clear, the (mathematical) fields e and b suffice to describe the effect of the (physical) electromagnetic field on the rest of the world, in particular on charged particles, whose motion is described by j, p, m, and which in turn constitute the source of the field. The electromagnetic field is thus kinematically[10] characterized by the pair {e, b}, and fields d and h are auxiliaries in its dynamic description. Moreover, there is some leeway in the very definition of d and h, because the bookkeeping on charge motion can be shared between j, p, and m in different ways.

Exercise 1.1.[11] Rewrite (1–4) by eliminating d and h. Discuss the interchangeability of j, p, and m.

Equations (3) and (4) thus seem to define redundant entities, and indeed, many classical presentations of electromagnetism make do with two vector

[9]Magnetic monopoles, the density of which would be the above q_m, "should exist" [GT], according to theoreticians, but have not been observed yet. (Reports of such observations have been made [Ca, P&], but were not confirmed.) It is comforting to know that the discovery of such particles would not jeopardize Maxwell's theory.

[10]*Kinematics* is about description: which mathematical entities depict the system's state at any instant. *Dynamics* is about evolution laws: how the state will change under external influences.

[11]Texts of exercises are either at the end of each chapter or, when short enough, given on the spot, in which case **Exercise** is in boldface. Look for the "Hints" and "Solutions" sections at the end of each chapter.

fields[12] instead of four. The main advantage of their presence, which explains why this formalism is popular in the computational electromagnetics community, is the possibility this offers to express material properties in a simple way, via "constitutive laws" which relate j, p, and m to the electromagnetic field they generate.

The vacuum, in particular, and more generally, matter that does not react to the field, is characterized by $p = 0$ and $m = 0$, and thus by the coefficients ε_0 and μ_0. In the MKSA system, $\mu_0 = 4\pi\, 10^{-7}$ H/m and $\varepsilon_0 = 1/(\mu_0 c^2)$ F/m, where c is the speed of light (H for *henry* and F for *farad*). These values reflect two things: A fundamental one, which is the very existence of this constant c, and a more contingent one, which is the body of conventions by which historically established units for electric and magnetic fields and forces have been harmonized, once the unity of electromagnetic phenomena was established.

Let us now review these constitutive laws, which we will see are a condensed account ot the laws of charge–matter interaction in specific cases.

1.2 CONSTITUTIVE LAWS

In all concrete problems, one deals with composite *systems,* analyzable into subsystems, or *compartments:* electromagnetical, mechanical, thermal, chemical, etc. Where to put the boundaries between such subsystems is a modelling decision, open to some arbitrariness: elastic forces, for instance, can sensibly be described as electromagnetic forces, at a small enough scale. Each compartment is subject to its own equations (partial differential equations, most often), whose right-hand sides are obtained by solving equations relative to *other* compartments. For instance, Eqs. (1–4) govern the electromagnetic compartment, and we'll soon see how j, p, and m are provided by others. If one had to deal with all compartments at once, and thus with *coupled* systems of partial differential equations of considerable complexity, numerical simulation would be very difficult. *Constitutive laws*, in general, are the device that helps bypass this necessity: They are an approximate but simple summary of a very complex interaction between the compartment of main interest and secondary ones, detailed modelling of which can then be avoided.

[12]As a rule, e and b, but there are dissidents. In Chu's formulation [FC, PH], for instance, e and h are the basic entities.

1.2.1 Dynamics of free charges: the Vlasov–Maxwell model

A concrete example will illustrate this point. Let's discuss the problem of a population of charged particles moving in an electromagnetic field which they significantly contribute to produce. Coupled problems of this kind occur in astrophysics, in plasma physics, in the study of electronic tubes, and so forth. To analyze such a physical system, we may consider it as made of two compartments (Fig. 1.3): the electromagnetic one (EM), and the "charge motion" compartment (CM), which both require a kinematical description, and influence each other's dynamics, in a circular way.

Let's enter this circle at, for instance, CM. A common way to describe its kinematics is to treat charge carriers as a fluid, characterized by its charge density "in configuration space", a function $\tilde{q}(t, x, v)$ of time, position, and (vector-valued) velocity. The actual charge density and current density are then obtained by summing up with respect to v:

$$(8) \qquad q(t, x) = \int \tilde{q}(t, x, v)\, dv, \qquad j(t, x) = \int \tilde{q}(t, x, v)\, v\, dv,$$

where dv is the volume element in the three-dimensional space of velocities. CM thus influences EM by providing a source $\{q, j\}$ for it. (Later we'll see that q is redundant, (6) being satisfied.)

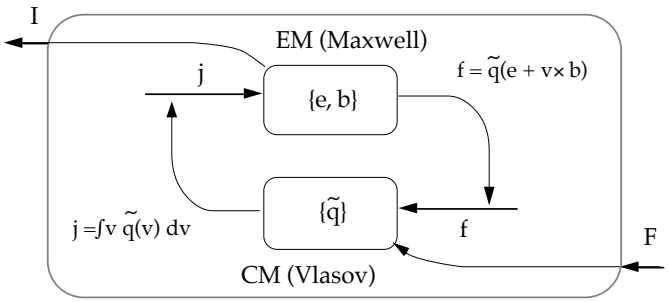


FIGURE 1.3. A typical example of coupling between compartments: the "Vlasov–Maxwell" model. "Vlasov" rules the behavior of a fluid of non-interacting free charges (a "collisionless plasma", for instance). "Maxwell" governs the electromagnetic field. Each compartment influences the other's behavior. External influences (such as, in the present case, the forces F, of non-electromagnetic origin) will in general intervene, and such "input" can then be seen as cause for the coupled system's evolution. Symmetrically, a global parameter (here some macroscopic intensity I) can be designated as "output", and the whole system (here, with its I—F dynamics) can become one compartment in some higher-level description.

The influence of EM on CM is via "Lorentz force". Recall that the force exerted by the field on a point charge Q passing at point x at time t with the (vector-valued) speed v is Q times the vector $e(t, x) + v \times b(t, x)$. (The part independent of celerity, that is $Q\, e(t, x)$, is "Coulomb force".) Here we deal with a continuum of charge carriers, so let us introduce, and denote by $\tilde{f}$, the *density* of force in configuration space: $\tilde{f}(t, x, v)\, dx\, dv$ is thus the force exerted on the packet of charges which are in volume dx around x, and whose speeds are contained in the volume dv of velocity space around v, all that at time t. So we have, in condensed notation,[13]

$$\tilde{f} = \tilde{q}\,(e + v \times b). \tag{9}$$

These forces do work: We note for further use that the power density π thus communicated from EM to CM is what is obtained by integrating with respect to v:

$$\begin{aligned} \pi(t, x) &= \int \tilde{q}\,(t, x, v)\,(e(t, x) + v \times b(t, x)) \cdot v\, dv \\ &= \int \tilde{q}\,(t, x, v)\, e(t, x) \cdot v\; dv = j(t, x) \cdot e(t, x), \end{aligned} \tag{10}$$

after (8).

The circle around Fig. 1.3 will be closed once we know about the dynamics of CM. Let's suppose (which is a *huge*, but often acceptable over-simplification) that particles do not exchange momentum by collision or other non-electromagnetic interaction. One may as well suppose, in that case, that there is a single species of charge carriers (only electrons, for instance), since otherwise their effects will just add, in all respects, and can be computed separately. Let us call Q_c (specific charge of carrier c) the charge-to-mass ratio[14] of these particles. Charge conservation (or equivalently, mass conservation) then implies the *Vlasov equation*,

$$\partial_t \tilde{q} + v \cdot \nabla_x \tilde{q} + Q_c\,(e + v \times b) \cdot \nabla_v \tilde{q} = 0, \tag{11}$$

where ∇_x and ∇_v denote partial gradients with respect to position and speed. Exercises 1.2 to 1.5, at the end of this chapter, suggest a road to this result. Exercise 1.6 will then invite you to prove that (6) holds for q and j as given by (8), when (11) is satisfied.

[13] The notation makes sense if v in (9) is understood as a vector *field*, the value of which is v at point $\{x, v\}$ of configuration space. Then $v \times b$ is a field of the same kind.

[14] For electrons, therefore, $Q_e = -1.602 \times 10^{-19}/9.109 \times 10^{-31}$ C/kg.

So this is a typical example of a coupled system: Given j, and assuming $\rho = 0$ and $m = 0$, since all charges are accounted for by j, system (1–4) determines e and b, hence the forces by (9), to which one adds other known causative forces, symbolized by F in Fig. 1.3, hence the movement of charge carriers (more generally, of charged matter), hence j again, which must be the same we started from. From a mathematical viewpoint, this is a "fixed point condition", which translates into an equation in terms of j, which will in general have a unique solution. One may then get the field by solving (1–4) with the j just found as source.[15] This can serve as a model for other multi-compartment situations: In general, the coupled problem may be proven, by a similar reasoning, to be well-posed,[16] though overwhelmingly difficult to solve.

With this, we may now elaborate on the notion of constitutive laws as summaries of interactions, or more bluntly, proxies which can take the place of secondary compartments in a modelling. For instance, in Fig. 1.3, it would be nice to have an explicit dependence of j on e and b, allowing us to bypass consideration of the "charge motion" compartment. A *constitutive law* is such a direct, more or less complex, dependence (of course, with limited and conditional validity).

1.2.2 Dynamics of conduction charges: Ohm's law

And indeed there are cases in which the "dynamics" part of the problem is especially simple to solve, at least approximately and accurately enough *as far as the main compartment is concerned*. Two such cases are especially important: conductors and generators.

Conductors (metals, etc.) are those bodies where exists a population of electric charges which are not bound to atoms, but still tightly interact with matter. Stirred by the field, the carriers accelerate for a while, but soon are stopped by collision, and the energy and momentum they acquired via Lorentz force are then transferred to the supporting matter, hence heating and also, possibly, movement of the conductor. Carriers move

[15] This is how the coupled problem is showed to be "well-posed" (see next note), not the way it is solved numerically. The favored technique for that is the "particle-in-cell" method used in "particle pusher" codes [BL, HE], which simulates the electron cloud with a finite family of particles and alternates between determining the motion of charge in a constant known field for one time-step, and updating the field values.

[16] *Well-posed* has a technical meaning: It refers to a problem of which one can prove it has a solution and a unique one, with, moreover, continuity of this solution with respect to the data. (The notion is due to Hadamard [Hd].)

anyway by thermal agitation, and at speeds of much higher magnitude than the additional velocity gained from electromagnetic action. But whereas thermal speeds cancel on the average at macroscopic scales, such is not the case of the motions due to Lorentz force. Their nonzero average is the so-called *drift velocity* (see, e.g., [Fe], II, Sections 32–36, or [We]). This slow[17] but collective motion, which can easily be detected and measured [Kl], results in a macroscopic current density.

This picture of the phenomenon is relatively complex, and one can simplify it as follows: Imagine the carrier population as a fluid, moving at the speed at which Lorentz force is balanced by all "friction-like" forces which tend to slow it down. Friction forces are in general, and are here found to be, *proportional* to the drift velocity. Since the current density due to a particular kind of carriers (ions, electrons, "holes" . . .) is proportional to their speed, one may conclude to a proportionality between e and the current density:

$$\mathbf{j} = \sigma \mathbf{e}, \tag{12}$$

where σ, the *conductivity*,[18] depends on the material. This is *Ohm's law*.[19] One has $\sigma \geq 0$, and $\sigma = 0$ in *insulators* (dry air, vacuum, etc.).

The law itself is subject to experimental verification and holds with excellent accuracy in many cases, but the explanation behind it was, let's be candid, a *myth*: an explanation of rational appearance, relevant and reasonably consistent, but which openly glosses over fine points of physics, and whose main merit is to get rapidly to the point. (Indeed, the consistency of the foregoing explanation, in spite of its relevance to Ohm's law, can be challenged: cf. Exer. 1.11.)

As for *generators*, they are by definition these regions of space where the current density (then denoted by $\mathbf{j}^g$, g for "given") can be considered as imposed, independently of the local electromagnetic field, and where, therefore, Ohm's law (12) doesn't apply. It is then convenient to set $\sigma = 0$

[17] In Cu, about 0.6 mm/s for 10 A/mm^2. The direction of the drift with respect to the field tells about the sign of the carriers, which are most often electrons, but can also be "holes" [Kl].

[18] Conductivity is measured in siemens per meter. (The siemens, or mho, Ω^{-1}, is the unit of conductance, and the dimension of σ is $(\Omega\,\mathrm{m})^{-1}$.) Fe : 5 to 10×10^6, Al : 36×10^6, Cu : 58×10^6. Living tissues: ~ 0.1.

[19] This, in the case of *nonmoving* conductors. The v in (9) is sum of the speed of the free charge with respect to the conductor and of the latter's own speed, V. In case $V \neq 0$, one will have $\mathbf{j} = \sigma (\mathbf{e} + V \times \mathbf{b})$ instead of (12). Problems involving moving conductors will not be addressed in this book (with the advantage of always working within a unique reference frame).

in such regions[20] and to write a generalized Ohm's law, valid for generators, conductors, and insulators alike (and thus, most often, uniformly valid in all space):

(13) $\quad j = \sigma e + j^g.$

It all goes then as if the charge dynamics problem had been solved in advance, the result being given by (13). One often calls *passive* conductors those ruled by (12), generators being then dubbed *active*.

One may then append (13) to (1–4), with $p = 0$ and $m = 0$. The system of equations thus obtained (or "Maxwell's model with linear conductors") embodies the theory of *nonmoving* (cf. Note 19) active and passive conductors which are neither polarizable nor magnetizable (cf. next section). It deals with a two-compartment system, EM and CM again, but the theory we have accepted for the latter is so simple, being all contained in (13), that one may easily overlook the coupled nature of the whole system. (One should not.)

1.2.3 Dynamics of bound charges: dielectric polarization

Now, another case of two-compartment system for which the same approach leads to a specific constitutive law. It deals with polarizable materials, in which charges are too strongly bound to separate from their original sites, but loose enough to be pulled a little off their equilibrium position by Coulomb forces, when the material is subject to a macroscopic electric field. This *polarization* phenomenon is important for some materials, dubbed *dielectric*. The simple reasoning (or myth . . .) that follows shows how to account for it, by a simple relation between e and the p of (3).

Despite its electrical neutrality at a macroscopic scale, matter contains positive and negative charges (+ and − for brevity) which we may imagine as being attached by pairs at certain material sites. Suppose the density of + charges is equal to q_+, a function of position x. In the absence of any macroscopic electric field, the density of − charges must equal $-q_+$, by electric neutrality. Now, a field e being applied, let's represent by a vector field u the separation of charged pairs that results, as follows: A + charge [resp. a − charge] that was at point x is now at $x + u(x)/2$ [resp. at $x - u(x)/2$]. To easily compute the new charge density q_p due to this

[20]This amounts to neglecting the internal resistance of the generator. In some modellings, having a nonzero σ there can be useful. Note this wouldn't change the form of (13).

change in localization, let us treat it as a mathematical *distribution*,[21] that is, as the mapping $\psi \to \int q_p \psi$, where ψ denotes a so-called "test function". Expanding ψ to first order and integrating by parts, we have

$$\int q_p \psi = \int q_+(x)\,[\psi(x + u(x)/2) - \psi(x - u(x)/2)]\,dx$$
$$\sim \int q_+ \, u \cdot \mathrm{grad}\,\psi \equiv \int - \mathrm{div}(q_+ u)\,\psi,$$

hence $q_p = -\,\mathrm{div}\,p$, where $p = q_+ u$. This field p, soon to be identified with the one in (3), is the polarization of the dielectric.

Exercise 1.7. Try to do the same computation "the other way around", by starting from $\int q_p \psi = \int q_+(x - u(x)/2)\,\psi(x)$, etc. Why does it go wrong this way?

The macroscopic manifestation of this local charge splitting is thus the appearance of a distribution of charges in what was initially an electrically neutral medium. Moreover, if the polarization changes with time, the motion of charges $+$ and $-$ in opposite directions amounts to a current density $j_p = \partial_t(qu) = \partial_t\, p$. (Note that $\partial_t\, q_p + \mathrm{div}\, j_p = 0$, as it should be.)

We might treat this current density on the same footing as j, and replace the polarized matter by vacuum plus polarization current. Then $d = \varepsilon_0 e$, and Eqs. (1) and (3) would combine to give

$$(14) \qquad -\partial_t(\varepsilon_0\, e) + \mathrm{rot}\, h = j + \partial_t\, p.$$

Instead, we use our option (cf. Exer. 1.1) to charge $\partial_t\, p$ on the account of Eq. (3), by setting $d = \varepsilon_0\, e + p$, hence $-\partial_t d + \mathrm{rot}\, h = -\partial_t(\varepsilon_0\, e + p) + \mathrm{rot}\, h = j + \partial_t\, p - \partial_t\, p \equiv j$, leaving e unchanged. This separates macroscopic currents j,

[21] In the theory of distributions [Sc], functions are not defined by their values at points of their domain of definition, but via their effect on other functions, called *test functions*. So, typically, a function f over some domain D is known if one is given all integrals $\int_D f\psi$, for all smooth ψ supported in D. It is thus allowable to identify f with the linear mapping $\psi \to \int_D f\psi$. (The arrowed notation for maps is discussed in A.1.9.) This has the advantage of making functions appear as special cases of such linear *TEST_FUNCTION* $\to$ *REAL_NUMBER* mappings, hence a useful generalization of the notion of function: One calls such maps *distributions*, provided they satisfy some reasonable continuity requirements. For instance, the map $\psi \to \psi(a)$, where a is some point inside D, is a distribution ("Dirac's mass" at point a, denoted δ_a). The generalization is genuine, since there is no function f_a such that $\psi(a) = \int_D f_a\psi$ for all ψ. It is useful, because some theories, such as Fourier transformation, work much better in this framework. The Fourier tranform of the constant 1, for instance, is not defined as a function, but makes perfect sense as a distribution: It's $(2\pi)^{d/2}$ times a Dirac mass at the origin, i.e. $(2\pi)^{d/2}\,\delta_0$, in spatial dimension d.

which continue to appear on the right-hand side of the expression of Faraday's law, and microscopic (polarization) currents $j_P = \partial_t p$, now hidden from view in the constitutive law. Notice that $\mathrm{div}\, d = q$, where q is the macroscopic charge, and $\mathrm{div}(\varepsilon_0 e) = q + q_p$.

All this shuffling, however, leaves the polarization current to be determined. The "(coupled) problem of bound charges" would consist in simultaneously computing p and the electromagnetic field, while taking into account specific laws about the way charges are anchored to material sites. Just as above about conduction, one makes do with a simple—and empirically well confirmed—solution to this problem, which consists in pretending (by invoking a "myth", again) that p and e are proportional: $p = \chi e$, as would be the case if charges were elastically bound, with a restoring force proportional to e, and without any inertia.[22] Now, let us set $\varepsilon = \varepsilon_0 + \chi$. Then, Eqs. (1) and (3) become

$$(1') \qquad -\partial_t d + \mathrm{rot}\, h = j, \qquad\qquad (3') \qquad d = \varepsilon e.$$

The coefficient ε in (3') (called *permittivity*, or *dielectric constant* of the medium[23]) thus appears as the simple summary of a complex, but microscopic-scale interaction, which one doesn't wish to know about at the macroscopic scale of interest.

Another, simpler solution of the coupled problem obtains when one may consider the field p, then called *permanent polarization*, as independent of e. The corresponding behavior law, $d = \varepsilon_0 e + p$ with fixed p, is well obeyed by a class of media called *electrets*. Of course one may superpose the two behaviors (one part of the polarization being permanent, the other one proportional to e), whence the law $d = \varepsilon e + p$ instead of (3), with a fixed p.

1.2.4 Magnetization

It is tempting to follow up with a similar presentation of magnetization, where a proportionality between m and h would be made plausible by a simple myth about the interaction of magnetic moments (due to the electrons' spins, mainly) with the magnetic field. This would be a little artificial, however, because too remote from the real physics of magnetism

[22] The latter hypothesis will be reconsidered in the case of high frequencies. Note that χ can be a tensor, to account for anisotropy.

[23] Terminology wavers here. Many authors call "permittivity" the *ratio* between ε and ε_0, and speak of "dielectric constant" when it comes to ε, or even to its real part in the case when ε is complex (see below). Note that ε may be a tensor.

(cf., e.g., [OZ]), and the point is already made anyway: Constitutive laws substitute for a detailed analysis of the interaction, when such analysis is either impossible or unproductive. So let us just review typical constitutive laws about magnetization.

Apart from *amagnetic* materials ($m = 0$), a simple case is that of *paramagnetic* or *diamagnetic* materials, characterized by the linear law $m = \chi h$ (whence $b = \mu h$, with $\mu = (1 + \chi)\mu_0$), where the *magnetic susceptibility* χ is of positive or negative sign, respectively. It can be a tensor, in the case of anisotropic materials. For most bodies, χ is too small to matter in numerical simulations, the accuracy of which rarely exceeds 1% ($\chi \sim 10^{-4}$ for Al or Cu).

Ferromagnetic metals (Fe, Co, Ni) and their alloys are the exception, with susceptibilities up to 10^5, but also with a nonlinear (and hysteretic[24]) behavior beyond some threshold. In practice, one often accepts the linear law $b = \mu h$ as valid as far as the modulus of b does not exceed 1 tesla.[25]

For *permanent magnets* [La, Li] a convenient law is $m = \chi h + h_m$, where h_m is a vector field independent of h and of time, supported by the magnet (that is, zero-valued outside it), with χ roughly independent of h, too, and on the order of 1 to 4, in general [La]. This law's validity, however, is limited to the normal working conditions of magnets, that is, for h and b of opposite signs, and not too large. The characteristic $b = \mu h + \mu_0 h_m$ is then called the "first order reversal curve".

1.2.5 Summing up: Linear materials

Hysteresis, and nonlinearity in general, are beyond our scope, and we shall restrict to the "Maxwell model of memoryless linear materials with Ohm's law":

[24] *Hysteresis* occurs when the value of b at time t depends not only on $h(t)$, but on past values. *Linearity* does not preclude hysteresis, for it just means that if two field histories are physically possible, their superposition is possible too. This does not forbid behavior laws "with memory", but only allows "convolution laws" of the form $b(t) = \int^t \mathcal{M}(t - s)\, h(s)\, ds$. As we shall see in Section 1.4, this amounts to $B = \mu H$, in Fourier space, with a complex and frequency-dependent μ.

[25] The unit for b is the tesla (T), or weber (Wb) per square meter. (One tesla is 10 000 gauss, the cgs unit still in use, alas.) The field h is measured in ampères per meter (A/m). An ordinary magnet creates an induction on the order of .1 to 1 T. The Earth field is about 0.4×10^{-4} tesla.

(15) $-\partial_t d + \mathrm{rot}\, h = j \equiv j^g + \sigma e,$ (16) $\partial_t b + \mathrm{rot}\, e = 0,$

(17) $d = \varepsilon e,$ (18) $b = \mu h,$

plus occasionally some constant term on the right of (17) or (18), in order to model electrets or permanent magnets. In most modellings, these equations correctly describe what we shall from now on call "the electromagnetic compartment" (and still denote by EM, although it has been slightly enlarged). But let's not forget the complexity of field-matter interactions that are thus hidden beyond a neat façade, and the relative arbitrariness with which compartment boundaries have been moved in order to incorporate microscopic interactions in (15–18), leaving only macroscopic interactions with other compartments to describe. We now turn to this.

1.3 MACROSCOPIC INTERACTIONS

Most engineering applications have to do with power conversion. In this respect, what we have established in (10) has general validity:

Proposition 1.1. *The power density yielded by the electromagnetic compartment of a system to other compartments is given by* $\pi(t, x) = j(t, x) \cdot e(t, x)$, that is, as an equality between scalar fields,

(19) $\pi = j \cdot e,$

at all times. (Be aware that j is the total current, $j = \sigma e + j^g$.)

In the case of a passive and immobile conductor, $j \cdot e = \sigma |e|^2$, so this is Joule loss, and therefore, thermal power. In the case of generators, $-\pi(t, x)$ is the density of power needed to push charges up the electric field (and thus given to the EM compartment). In the case of moving conductors, $j \cdot e$ is in part Joule heating, and for the other part mechanical work. In all cases, the total yielded power is thus[26] $\Pi = \int_{E_3} \pi(x)\, dx$, that is, $\Pi = \int_{E_3} j \cdot e$. This is the bottom-line figure in the inter-compartment trade balance.

[26]See Appendix A, Subsections A.2.4 and A.2.5, for E_3. This symbol stands for "oriented three-dimensional Euclidean affine space": ordinary space, equipped with a notion of orientation (i.e., a way to distinguish direct and skew reference frames, cf. A.2.5), and with the dot-product here denoted by " $\cdot$ ", which gives sense to the notions of distance, area, volume, etc.

1.3.1 Energy balance

Compartmentalization, however, is not limited to physically distinct subsystems, and may concern distinct regions of space too. In this respect, energetical exchanges through spatial boundaries are important. Let thus a closed surface S separate a domain D from the rest of space. Take the scalar product of both sides of (1) and (2) by $-e$ and h, respectively, add, and integrate over D:

$$\int_D (h \cdot \partial_t b + e \cdot \partial_t d) + \int_D (h \cdot \mathrm{rot}\, e - e \cdot \mathrm{rot}\, h) = -\int_D j \cdot e.$$

The result is then transformed by the following integration by parts formula, to which we shall return in the next chapter:

$$\int_D h \cdot \mathrm{rot}\, e = \int_D e \cdot \mathrm{rot}\, h - \int_D (n \times h) \cdot e, \tag{20}$$

and by setting

$$W_D(t) = \tfrac{1}{2} \int_D (\mu\, |h(t)|^2 + \varepsilon\, |e(t)|^2),$$

hence

$$\partial_t W_D + \int_S n \cdot (e \times h) = -\int_D j \cdot e. \tag{21}$$

A special case of this[27] is when D is all space:

$$\partial_t W = -\Pi, \tag{22}$$

where $W(t) = W_{E_3}(t)$, a quantity that may thus legitimately be called *electromagnetic energy:* indeed, (22) points to it as being the energy stored in the electromagnetic compartment[28] of the system.

So if we turn to (21), its interpretation in similar terms is immediate: The "subcompartment EM-D" cedes the power $\int_D j \cdot e$ to D-based subsidiaries of all non-EM compartments, and exports $\int_S n \cdot (e \times h)$ to other regions of the EM compartment, which themselves, of course, may trade with non-EM entities in their own domain. The vector field $e \times h$, which records these trans-boundary exchanges, is *Poynting's vector*.[29]

[27] One has $\int_{E_3} h \cdot \mathrm{rot}\, e = \int_{E_3} e \cdot \mathrm{rot}\, h$ (no "surface term at infinity") provided e and h both belong to the functional space $\mathbb{L}^2_{\mathrm{rot}}(E_3)$, to be studied in more detail in Chapter 5, where this assertion will be proved. Its physical content is just that h and e *decrease fast enough* at infinity; hence the absence of the boundary term when S recedes to infinity, and this we can accept without qualms for the moment.

[28] In the extended sense in which we now understand "electromagnetic" compartment. If $\varepsilon \neq \varepsilon_0$, for instance, part of this energy is in dipole vibration.

Note that, by applying Ostrogradskii's formula to (21),

(23) $\partial_t w + j \cdot e + \operatorname{div}(e \times h) = 0,$

where w is the scalar field $x \to \frac{1}{2}(\mu\,|h(x)|^2 + \varepsilon\,|e(x)|^2)$. Just as (6) expressed "local" charge conservation, (23) is the local expression of energy conservation, the integrated or "global" form of which is (21). It is tempting to call w the (electromagnetic) *energy density*, and we shall do that. See, however, Remark 1.1 below.

As an illustration, let us mention thermal exchanges (induction heating, direct heating, microwave heating, welding . . .). The "thermal compartment" (TM) of a system is governed by the heat equation, in all its guises, the best known of which, valid when most thermal exchanges are by conduction and diffusion, is

(24) $\partial_t(c\,\theta) - \operatorname{div}(\kappa \operatorname{grad} \theta) = \pi,$

where θ (a scalar field) stands for the temperature, c for the volumic heat, κ for the thermal conductivity, and π for the injected power density. When this power is Joule loss, one has $\pi(x) = \sigma(x)\,|e(x)|^2$. Since σ, as well as coefficients ε and μ for that matter, may depend on temperature, studying electrothermal interactions amounts to studying the coupled system (15–18)(24).

Most often, there is a natural division into subcompartments. In induction heating, for instance, if the workpiece (the passive conductor) occupies domain D, TM will be restricted to D, with of course adequate boundary conditions for (24) on its boundary S. A partition of EM into D and $E_3 - D$ is then the obvious thing to do, especially at low frequencies, where the equations in the non-conducting region take a simple form, as we shall see in Chapter 8. The flux of $e \times h$ through S is then the heating power, and thus of particular significance.

Remark 1.1. Just as W_{E_3} is the energy of EM, we may *define* W_D as "the energy of EM-D". But to say that this energy *is inside* D, which amounts to saying that $w(x)\,dx$ is the energy "effectively present" in volume dx, goes much further, since it asserts that energy is *localized*, as a substance can be, and this is controversial. Some authors, comparing this with localizing the beauty of a painting at specific parts of it, protest they "do

[29] An instance of this appalling habit many physicists have to call "vector" what is actually a vector *field*. Such sloppiness about the *type* (cf. A.1.2) of the entities one deals with, harmless as it may be in the present case, should not be condoned. Vector fields are objects of type $A_3 \to V_3$, in the notation of A.2.2 and A.2.4, vectors being elements of V_3.

not believe that 'Where?' is a fair or sensible question to ask concerning energy. Energy is a function of configuration, just as (. . .) beauty (. . .)". (Cf. [MW], pp. 266–267.) The problem is inherent in field theory, and not special to electromagnetism [KB]. ◊

Remark 1.2. The Poynting vector field also is a bone of contention. There are totally *static* situations in which the energy flux $e \times h$ is not zero (**Exercise 1.8:** find one). It all goes then as if energy was perpetually flowing in circles. The idea may seem unattractive, and alternatives have been proposed, based on the fact that the flux of a curl through a closed surface is always zero, so one may add to $e \times h$ the curl of any vector field u one fancies, without changing *any* power flux, whatever the domain of interest. (This is clear on the local expression (23), since $\mathrm{div}(\mathrm{rot}\, u) = 0$.) Slepian [Sl] thus could list no fewer than eight plausible expressions for the energy-flow vector, including Poynting's. The debate rebounds regularly [Ly, Lo, He]. There is an old argument ([Bi], discussed in [Ro]) to the effect that if $\mathrm{rot}\, u$ is to be a function of e and b only, then $\mathrm{rot}\, u$ is a constant, so Poynting's vector is the natural choice (the "gauge invariant" one) among these alternatives. But this leaves some unconvinced [BS]. ◊

1.3.2 Momentum balance

Even more controversial[30] is the question of momentum: One century ago, Abraham and Minkowski disagreed about the correct expression of the linear momentum of the electromagnetic field [Cs]. The question is still debated, and what follows will not resolve it. But having discussed energy, we cannot elude momentum, since they are two observer-dependent aspects of one and the same objective entity (the four-dimensional energy–momentum, or "momenergy" [TW]). Moreover, the expression of forces exerted by EM on conductors and polarized or magnetized matter derives from momentum conservation, and forces are an often-desired output in computations, even those restricted to immobile bodies, to which we limit consideration here.

First let us introduce a notation (local to this section): if v is a vector field and φ a scalar field, $\nabla_v\varphi$ may conveniently denote the scalar field $v \cdot \mathrm{grad}\, \varphi$, so that $\nabla_v\varphi(x)$ is "the derivative of φ in the direction of v", at point x. Now if u is another vector field, one can form $\nabla_v u^i$ for its three Cartesian coordinates u^i, hence the three scalar components of a vector, which will be denoted $\nabla_v u$. Next move, please, is yours:

[30] See R.H. Romer: "Question #26: Electromagnetic field momentum", **Am. J. Phys., 63,** 9 (1995), pp. 777–779, and the answers provided in **Am. J. Phys., 64**, 1 (1996), pp. 15–16.

Exercise 1.9. Show that $\int_D \nabla_v u = -\int_D u \ \mathrm{div}\, v + \int_S n \cdot v \ u$ (with D, S, and n as usual, cf. Fig. 1.2).

Exercise 1.10. Show that, in case $v = \alpha u$, where α is a scalar field,

$$\int_D v \times \mathrm{rot}\, u = -\int_D \nabla_v u + \tfrac{1}{2} \int_S u \cdot v \ n - \tfrac{1}{2} \int_D |u|^2 \nabla\alpha. \tag{25}$$

We can then do the following calculation. Starting from (15) and (16), take the cross product (from the left) of both sides by b and d, respectively, add, and integrate. This gives

$$\partial_t \int_D d \times b + \int_D b \times \mathrm{rot}\, h + \int_D d \times \mathrm{rot}\, e = -\int_D j \times b.$$

Now apply (25) and Exer. 1.9 to d and e, with $\alpha = \varepsilon$:

$$\int_D d \times \mathrm{rot}\, e = -\int_S n \cdot d \ e + \int_D e \ \mathrm{div}\, d + \tfrac{1}{2} \int_S d \cdot e \ n - \tfrac{1}{2} \int_D |e|^2 \nabla\varepsilon$$

(recall that div d = q), then to b and h, quite similarly, and gather the results to get

$$\partial_t \int_D d \times b + \int_S (\tfrac{1}{2} \ b \cdot h \ n - n \cdot b \ h) + \int_S (\tfrac{1}{2} \ d \cdot e \ n - n \cdot d \ e) = \tag{26}$$
$$-\int_D [j \times b + q\, e - \tfrac{1}{2} \ |h|^2 \nabla\mu - \tfrac{1}{2} \ |e|^2 \nabla\varepsilon].$$

This (to be compared with (21), which had the same structure) is the momentum balance: $\int_D d \times b$ is the momentum "of" (same caveats as above) subcompartment EM-D, its flux is governed by the so-called "Maxwell stress tensor", here[31] $M = \tfrac{1}{2} \ b \cdot h - b \otimes h + \tfrac{1}{2} \ d \cdot e - d \otimes e$, and the right-hand side of (26) is, up to sign, the resultant of body forces.[32] Note the unexpected gradients, which should be interpreted in the sense of distributions when ε or μ are discontinuous.

The local version of (26), quite similar to (23), is

$$\partial_t [d \times b] + j \times b + q\, e - \tfrac{1}{2} \ |h|^2 \nabla\mu - \tfrac{1}{2} \ |e|^2 \nabla\varepsilon + \mathrm{div}\, M = 0. \tag{27}$$

But the global version (26) is more popular: A standard way to obtain the total force on some object (in a time-independent situation) is to compute the flux of M through some boundary S enclosing it.

[31] This tensor product symbol $\otimes$ will not be used again. Owing to our sign conventions, M is actually *minus* the Maxwell tensor of tradition.

[32] No proof has been offered here as to the validity of these interpretations. But if one accepts the expression of body-force density (which is standard, cf. e.g., [Rb]), the rest follows. See [Bo] for a *direct* derivation of the body force expression.

When studying the dynamics of moving conductors, one should take into account the momentum of the moving bodies *and* the momentum of the field in the expression of momentum conservation.[33] In an interaction between two solids, for instance, momentum lost by one of them may temporarily be stored in the field, before being restituted to the other body. Thus, action and reaction may seem not to balance, in apparent violation of Newton's third law [Ke]. See for instance [Co], [Ho], and the abundant literature on Feynman's "disk paradox", a situation in which a disk, initially at rest in a static field, can acquire angular momentum without any mechanical action, just because of a change in the electromagnetic environment [Lm].

Remark 1.3. So there are *static* configurations in which $\int \mathsf{d} \times \mathsf{b} \neq 0$: Surprising as this may appear, a static electromagnetic field *can* possess linear momentum. (Cf. R.H. Romer, **Am. J. Phys., 62,** 6 (1994), p. 489. See also [PP].) $\Diamond$

Remark 1.4. The cross product is an orientation-dependent operation: its very definition requires a rule for orienting ambient space. Yet we see it appear in expressions such as $\mathsf{e} \times \mathsf{h}$ or $\mathsf{d} \times \mathsf{b}$, which account for energy or momentum flux, physical quantities which obviously do *not* depend on orientation conventions. How come? It must be that some of the vector fields e, d, b, h themselves depend on orientation. No surprise in that: The *mathematical* entities by which the physical field is represented may depend on the structures of Euclidean space, whereas the objective phenomena do not. The question is further discussed in Section A.3 of Appendix A. $\Diamond$

1.4 DERIVED MODELS

Concrete problems in electromagnetism rarely require the solution of Maxwell equations in full generality, because of various simplifications due to the smallness of some terms. The displacement currents term $\varepsilon\, \partial_t \mathsf{e}$, for instance, is often negligible; hence an important submodel, *eddy-currents* theory, which we shall later study in its own right:

$$\partial_t \mathsf{b} + \mathrm{rot}\, \mathsf{e} = 0, \quad \mathrm{rot}\, \mathsf{h} = \mathsf{j}, \quad \mathsf{j} = \sigma \mathsf{e} + \mathsf{j}^g, \tag{28}$$

[33] Many papers in which this commonsense rule is neglected get published, notwithstanding, in refereed Journals. It has been asserted, for example, that the operation of a railgun cannot be explained in terms of classical electrodynamics. See a refutation of this crankish claim in [AJ].

with in particular, in passive conductors (where one may eliminate e from (28) after division by σ), $\partial_t(\mu h) + \mathrm{rot}\,(\sigma^{-1}\,\mathrm{rot}\,h) = 0$.

Another frequent simplification is the passage to complex numbers representations. If the source current j^g is sinusoidal in time,[34] that is, of the form $j^g(t, x) = \mathrm{Re}[J^g(x)\,\exp(i\omega t)]$, where J^g is a *complex-valued* vector field, and *if* all constitutive laws are linear, one may[35] look for the electromagnetic field in similar form, $h(x) = \mathrm{Re}[H(x)\,\exp(i\omega t)]$, etc., the unknowns now being the complex fields H, E, etc., independent of time. Maxwell's model with Ohm's law (15–18) then assumes the following form:

$$(29) \qquad -i\omega D + \mathrm{rot}\,H = J^g + \sigma E, \quad i\omega B + \mathrm{rot}\,E = 0, \quad D = \varepsilon E, \quad B = \mu H.$$

It is convenient there to *redefine* ε by assigning to this symbol the complex value $\varepsilon + \sigma/(i\omega)$, which allows the incorporation of the term σE into $i\omega D$, whence the model

$$(29') \qquad -i\omega D + \mathrm{rot}\,H = J^g, \quad i\omega B + \mathrm{rot}\,E = 0, \quad D = \varepsilon E, \quad B = \mu H,$$

which is, with appropriate boundary conditions, the *microwave oven* problem. In (29'), ε is now complex, and one often writes it as $\varepsilon = \varepsilon' - i\varepsilon''$, where the real coefficients ε' and ε'', of same physical dimension as ε_0, are nonnegative. (They often depend on temperature, and are measured and tabulated for a large array of products, foodstuffs in particular. Cf. eg., [FS, St, Jo]. Figure 1.4 gives an idea of this dependence.)

Nothing forbids accepting complex μ's as well, and not only for the sake of symmetry. This really occurs with ferrites[36] [La, Li], and also in some modellings, a bit simplistic[37] perhaps, of hysteresis.

[34] One often says "harmonic", but be wary of this use, not always free of ambiguity.

[35] This procedure is valid, a priori, each time one is certain about the *uniqueness* of the solution of the problem "in the time domain", for if one finds a solution, by whatever method, it's bound to be the right one. But it's the *linearity* of constitutive laws (cf. Note 24) that makes the procedure effective. Moreover, linearity allows one to extend the method to non-periodic cases, thanks to Laplace transform (then one has p, complex-valued, in lieu of $i\omega$). The passage to complex numbers is *in principle* of no use in nonlinear cases (for instance, when iron or steel is present), and the notion of "equivalent (complex) permeability", often invoked in applications to induction heating, is not theoretically grounded. (Its possible empirical value is another question, to be considered in each particular instance.)

[36] One refers to *linear* behavior there, and this complex permeability is not of the same nature as the one of the previous note.

[37] Because of their essentially *linear* nature. Law $B = (\mu' - i\mu'')H$ amounts to $\mu''\,\partial_t h = \omega(\mu'\,h - b)$ in the time domain.

An even more drastic simplification obtains when one may consider the phenomena as independent of time (steady direct current at the terminals, or current with slow enough variations). Let us review these models, dubbed *stationary*, derived from Maxwell's model by assuming that all fields are independent of time.

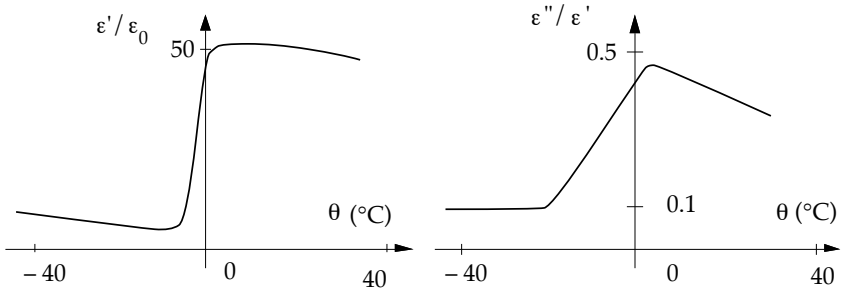


FIGURE 1.4. Typical curves for ε' and ε'' as functions of temperature, for a stuff with high water content. The ratio $\varepsilon''/\varepsilon'$, shown on the right, is often denoted by $\tan\delta$.

In this case, one has in particular $\partial_t b = 0$, and thus $\operatorname{rot} e = 0$. So, after (5) and (17),

(30) $\qquad \operatorname{rot} e = 0, \quad d = \varepsilon e, \quad \operatorname{div} d = q,$

and this is enough to determine e and d in all space, if the electric charge q is known: Setting $e = -\operatorname{grad}\psi$, where ψ is the *electric potential*, one has indeed $-\operatorname{div}(\varepsilon \operatorname{grad}\psi) = q$, a Poisson problem which is, as one knows, well posed. In the case where $\varepsilon = \varepsilon_0$ all over, the solution is given by

$$\psi(x) = \frac{1}{4\pi\varepsilon_0}\int_{E_3} \frac{q(y)}{|x-y|}\, dy,$$

as one will check (cf. Exers. 4.9 and 7.5) by differentiating under the summation sign in order to compute $\Delta\psi$. Model (30) is the core of linear *electrostatics*.

In a similar way, one has $\operatorname{rot} h = j$, after (1), whence, taking into account $\operatorname{div} b = 0$ and (18), the model of linear *magnetostatics*:

(31) $\qquad \operatorname{rot} h = j, \quad b = \mu h, \quad \operatorname{div} b = 0,$

and this determines b and h in all space when j is given. If $\mu = \mu_0$ all

over, the solution is obtained in closed form by introducing the vector field

$$a(x) = \frac{\mu_0}{4\pi}\int_{E_3} \frac{j(y)}{|x-y|}\, dy,$$

called *magnetic vector potential,* and by setting $b = \text{rot } a$. (By differentiating inside the integral, one will find *Biot and Savart's formula,* which directly gives h in integral form:

$$h(x) = \frac{1}{4\pi}\int_{E_3} \frac{j(y)\times(x-y)}{|x-y|^3}\, dy.) \tag{32}$$

When, as in the case of ferromagnetic materials, constitutive laws more involved than $b = \mu h$ occur, problem (31) appears as an intermediate in calculations (one step in an iterative process, for instance), with then in general a position-dependent μ. An important variant is the magnetostatics problem for a given distribution of currents and magnets, the latter being modelled by $b = \mu h + \mu_0 h_m$ with known μ and (vector-valued) h_m.[38] Setting $h_m = 0$ in the air, one gets

$$\text{rot } h = j, \quad b = \mu h + \mu_0 h_m, \quad \text{div } b = 0.$$

An analogous situation may present itself in electrostatics: $d = \varepsilon e + p$, with p given, as we saw earlier.

Still under the hypothesis of stationarity, one has $\partial_t q = 0$, and thus $\text{div } j = 0$, after (6), hence

$$\text{rot } e = 0, \quad j = \sigma e, \quad \text{div } j = 0, \tag{33}$$

in passive conductors. This is the *conduction* or *electrokinetics model*. In contrast to the previous ones, it does not usually concern the whole space, and thus requires boundary conditions, at the air-conductor interfaces, in order to be properly posed.

The formal similarity between these static models is obvious, and we need examine only one in detail to master the others. We'll focus on

[38] A legitimate question, at this stage, would be, "How does one know h_m, for a given permanent magnet?". Giving a rigorous answer would require the knowledge of the conditions under which the material has been magnetized, as well as the details of its hysteretic response, and a feasible simulation method of this process. In practice, most often, a uniform magnetization field parallel to one of the edges of the magnets is a fair representation. However, as more and more complex magnetization patterns are created nowadays, the problem may arise to find h_m from measurements of b by a computation (solving an inverse problem).

magnetostatics in this book, with only a few indications about the other models in Chapters 8 and 9. This disproportion is to some extent mitigated by the paradigmatic character of the magnetostatics model. As pointed out in the Preface, the difficulties encountered in computational electromagnetism in the 1970s, when one tried to extend then well-established finite element or boundary integral 2D methods to three-dimensional situations, appear in retrospect to be due not to the increased dimensionality per se, but to the essential difference between the "curl–curl" operator and the "div–grad" operator to which it reduces in two dimensions, and fortunately, all essential points about the curl–curl operator can be understood in the simple, limited, and well-defined framework of linear magnetostatics.

EXERCISES

The text for Exer. 1.1 is on p. 5.

Exercise 1.2. Let X be an affine space and V the associated vector space, $f: \mathbb{R} \times X \times V \to \mathbb{R}$ a *repartition function,* interpreted as the time-dependent density of some fluid in configuration space $X \times V$. Let $\gamma(t, x)$ be the acceleration imparted at time t to particles passing at x at this instant, by some given external force field. Show that

$$\partial_t f + v \cdot \nabla_x f + \gamma \cdot \nabla_v f = 0 \tag{34}$$

expresses *mass conservation* of this fluid. What if γ, instead of being a data, depended on velocity?

Exercise 1.3. What is the divergence of the field $x \to a \times x$, of type $E_3 \to V_3$, where a is a fixed vector? Its curl? Same questions for $x \to x$. (Cf. Subsection A.1.2 for the notion of type, and the notational convention, already evoked in Note 13, and Note 29.)

Exercise 1.4. In the context of Exer. 1.2, what is the divergence of the field $v \to e + v \times b$?

Exercise 1.5. Establish Vlasov's equation (11).

Exercise 1.6. Prove, using (11), that charge and current as given by (8) do satisfy the charge conservation relation (6).

See p. 12 for Exer. 1.7, pp. 18 and 19 for Exers. 1.8 to 1.10.

Exercise 1.11. Show that, in a region of a conductor where σ is not constant (due to variations in temperature, or in the composition of an alloy, etc.), $q = \operatorname{div} d$ may not be zero, and that this can happen in stationary situations (continuous current). Thus, there can exist a permanent charge imbalance at some places in the conductor. But Lorentz force acts on this charge packet. *Why doesn't it move?*

HINTS

1.1. Don't worry about differentiability issues: Assume all fields are smooth.

1.2. Imitate the classical computation about the convective derivative in fluid dynamics (which is very close to our treatment of charge conservation, p. 4).

1.3. For $x \to a \times x$, divergence: 0, curl: 2a. For $x \to x$, curl-free, the divergence is the constant scalar field $x \to 3$.

1.4. Mind the trap. Contrary to e and b, this field does *not* live in 3D Euclidean space! The *type* of the map will tell you unambiguously what "divergence" means.

1.5. Apply Exer. 1.2, acceleration being $\varrho_e(e + v \times b)$. By Exer. 1.4, there is no extra term.

1.6. Ostrogradskii on $\{t, x\} \times V$. Ensure suitable boundary conditions by assuming, for instance, an upper bound for the velocity of charges.

1.7. A careless attempt, like[39] $q_+(x - u(x)/2)\ \psi(x)\ {}^*\!= -\frac{1}{2}\ \nabla q_+ \cdot u$, would seem to lead to $-\int q_+ \operatorname{div}(\psi u)$, and hence to a different result than above if $\operatorname{div} u \neq 0$. This is the key: Why does this divergence matter?

1.8. A bar magnet between the plates of a condenser.

1.9. This is an extension of the integration by parts formula (2.9)[40] of the next chapter, $\int_D v \cdot \operatorname{grad} u^i = -\int_D u^i \operatorname{div} v + \int_S n \,.\, v\, u^i$, $i = 1, 2, 3$.

[39] The star in ${}^*\!=$ serves as a warning that the assertion should not be believed blindly.

[40] As a rule, we'll refer to "Eq. (n)" inside a chapter, and to "Eq. (X.n)" for the equation labelled (n) in Chapter X.

1.10. The simplest way is probably to work in Cartesian coordinates, starting from

$$(\int_D v \times \mathrm{rot}\, u)^i = \sum_j \int_D v^j (\partial_i u^j - \partial_j u^i), \quad i = 1, 2, 3.$$

Then the last term is $-\int_D \nabla_v u$, and $\int_D v^j \partial_i u^j = \int_D \alpha\, u^j \partial_i u^j = \frac{1}{2} \int_D \alpha\, \partial_i |u^j|^2 = \frac{1}{2} \int_D \partial_i(\alpha |u^j|^2) - \frac{1}{2} \int_D |u^j|^2 \partial_i \alpha$.

1.11. It doesn't move, but *they* do: Charge carriers may very well pass through the region of charge imbalance, being accelerated by the electric field and slowed down by the invoked "friction" along the way, and leave the apparent net charge constant. But how does the charge dynamics account for this behavior? Imagine two kinds of carriers, positive and negative but identical in all other respects, and argue against the logical consistency of the myth we used to justify Ohm's law. (This is more than a mere exercise, rather a theme for reflection. See the **Int. Compumag Society Newsletter, 3,** 3 (1996), p. 14.)

SOLUTIONS

1.1. Eliminate h and d: Then $\partial_t b + \mathrm{rot}\, e = 0$, unchanged, and

$$-\varepsilon_0 \partial_t e + \mathrm{rot}(\mu_0^{-1} b) = j + \partial_t p + \mathrm{rot}\, m,$$

so j can "absorb" p and m at leisure. Alternatively, p can assume the totality of charge fluxes (integrate $j + \mathrm{rot}\, m$ in t). But one can't put all of them in $\mathrm{rot}\, m$, since $j + \partial_t p$ may not be divergence-free. One calls $\mathrm{rot}\, m$ the density of *Amperian currents*.

1.2. Consider a domain D in configuration space (Fig. 1.5). The decrease of the mass it contains, which is $-\int_D \partial_t f$, equals outgoing mass. The latter is the flux through the boundary S of the vector field $\{v, \gamma\} f$, which is the speed, not of a particle in physical space, but of the representative point $\{x, v\}$ in configuration space. By Ostrogradskii, $\partial_t f + \mathrm{div}(\{v, \gamma\} f) = 0$. Since γ does not depend on v, $\mathrm{div}(\{v, \gamma\}) = 0$. So $\mathrm{div}(\{v, \gamma\} f) = \{v, \gamma\} \cdot \nabla f \equiv v \cdot \nabla_x f + \gamma \cdot \nabla_v f$. (Be wary of the wavering meaning of the dot, which stands for the dot-product in $V \times V$ left to the $\equiv$ sign, but for the one in V right to it.) If γ depends on v, an additional term $f\, \mathrm{div}_v \gamma$ will appear on the left-hand side of (34). (Here, $\mathrm{div}_v \gamma$ is the divergence of γ considered as a field on V, the x–coordinates being mere parameters.)

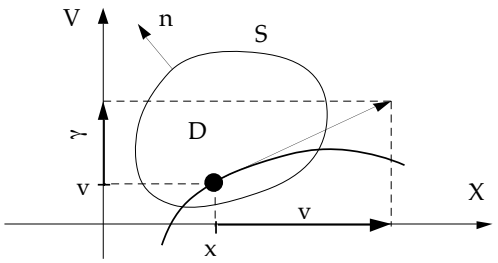


FIGURE 1.5. Notations for Exer. 1.2. The open curve is the trajectory of {x, v} in configuration space.

1.4. Let X be E_3, V the associated vector space (denoted V_3 in A.2.2). With $v \to e + v \times b$, we are dealing with a V-valued function, the domain[41] of which is all or part of the vector space V, considered with its affine structure, and position x and time t (which are what e and b may depend on) are parameters. (This is an illustration of the notion of section, cf. A.1.1: section by {x, t} of the function $\{t, x, v\} \to e(t, x) + v \times b(t, x)$.) Now, e does not depend on v, and since this is also the case for b, one has $\operatorname{div}(v \to v \times b) = 0$, after the result of Exercise 1.3.

1.6. Last term in (11) is $\operatorname{div}(v \to Q_c (e + v \times b)\, \tilde{q})$, the integral of which over V (with t and x as parameters) is zero if $\tilde{q}$ vanishes fast enough. And by (8), $\partial_t q + \operatorname{div} j = \int_V [\partial_t \tilde{q} + \operatorname{div}(v\, \tilde{q})] = \int_V (\partial_t \tilde{q} + v \cdot \nabla_x \tilde{q})$, thus 0 after (11).

1.7. The *density* q_p does not transform like a *function* in the change of reference frame defined by $x \to x + u(x)/2$, because the volume element also changes, unless $\operatorname{div} u = 0$, which characterizes volume-preserving deformations. A correct computation must therefore explicitly take into account the Jacobian of the mapping $x \to x + u(x)/2$. Hence a more involved computation in the case when $\operatorname{div} u \neq 0$, for of course the same final result.

1.11. Let $\rho = 1/\sigma$ be the conductivity. Assume steady currents. Then $\operatorname{div} j = 0$ by (1), $e = \rho j$ if Ohm's law is valid, and $q = \operatorname{div}(\varepsilon_0 e) = \varepsilon_0\, e \cdot \nabla\rho$, nonzero if ρ varies with position. This result clashes with the predictions of the simple-minded model in which there would be two symmetrical, but oppositely charged, kinds of carriers. Charges of opposite signs moving

[41] Be aware that "domain" has a dual meaning, open connected set as in Note 7, or domain of definition of a map, as here. Cf. Appendix A for precise definitions.

in opposite directions yield a net nonzero current, but a zero macroscopic charge. Under the basic assumption of the myth (speed proportional to electric field), the symmetry between the two kinds of charge is total, and hence $q = 0$. This is enough to show there is a problem. See the **Int. Compumag Society Newsletter, 4,** 1 (1997), pp. 13–18, for a discussion, including my own answer (the *inertia* of charge carriers plays a role in suppressing what would be otherwise a logical conundrum) and two other approaches [Cp], [Ni].

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