

Bazele Electrotehnicii

5. Similitudini si echivalente

Daniel Ioan

Universitatea Politehnica din Bucuresti –
PUB - CIEAC/LMN

daniel@lmn.pub.ro

5.1. Operatorii de impedanta si admitanta

Doua circuite sau elemente sunt **echivalente** daca asigura aceeasi relatie intre curentii si potentialele terminalelor. Presupunem relatia de forma:

$$\mathbf{y} = \mathcal{H}(\mathbf{x}) = \mathcal{H}\mathbf{x}; \quad \mathcal{H} : \mathcal{F} \rightarrow \mathcal{G}$$

unui operator linear sau nelinier, algebric, diferential sau integral, definit pe spatiul semnalelor de intrare si cu valori in spatiul semnalelor de iesire:

$$\mathbf{x} = \mathbf{f}(t), \text{ cu } \mathbf{f} : (t_{\min}, t_{\max}) \rightarrow \mathbb{R}^{n-1}$$

$$\mathbf{y} = \mathbf{g}(t), \text{ cu } \mathbf{g} : (t_{\min}, t_{\max}) \rightarrow \mathbb{R}^{n-1}$$

$$\mathbf{x} = [\mathbf{i}_a^T, \mathbf{v}_a^T]^T \in \mathbb{R}^{n-1}, \mathbf{i}_a \in \mathbb{R}^m, \mathbf{v}_a \in \mathbb{R}^{n-m-1}$$

$$\mathbf{y} = [\mathbf{v}_d^T, \mathbf{i}_d^T]^T \in \mathbb{R}^{n-1}, \mathbf{v}_d \in \mathbb{R}^m, \mathbf{i}_d \in \mathbb{R}^{n-m-1}$$

Cazuri particulare:

- **Operator tabelar de admitanta:** $\dot{\mathbf{i}} = \mathcal{Y}(\mathbf{v}) = \mathcal{Y}\mathbf{v};$ - pentru $m=0$
- **Operator tabelar de impedanta** $\mathbf{v} = \mathcal{Z}(\dot{\mathbf{i}}) = \mathcal{Z}\dot{\mathbf{i}};$ - pentru $m=n-1$
- **Operator de admitanta** $\dot{\mathbf{v}} = \mathcal{y}(\mathbf{v}) = \mathcal{y}\mathbf{v};$ - pentru $n=2$
- **Operator de impedanta** $\mathbf{v} = \mathcal{z}(\dot{\mathbf{v}}) = \mathcal{z}\dot{\mathbf{v}};$ - pentru $n=2$

Elementele controlabile in \mathbf{I} si \mathbf{v} au operatorii inversabili: $\mathcal{Y} = \mathcal{Z}^{-1}; \mathcal{y} = \mathcal{z}^{-1};$

Echivalenta elementelor de circuit electric

Conditia necesara si suficienta

ca doua **elemente sa fie**

echivalente este ca ele sa aiba

operatori de transfer: impedanta,

admitanta sau hibrizi egali:

$$\mathcal{H}_1 = \mathcal{H}_2; \quad \mathcal{Y}_1 = \mathcal{Y}_2 \text{ sau } \mathcal{Z}_1 = \mathcal{Z}_1$$

Operatorii de transfer ai elementelor ideale:

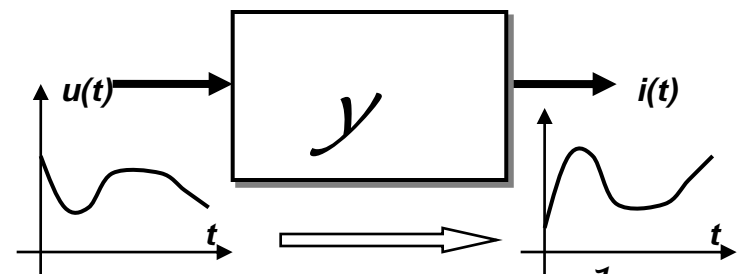
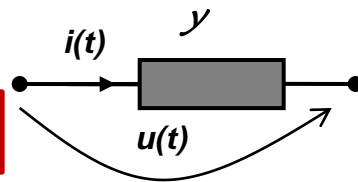
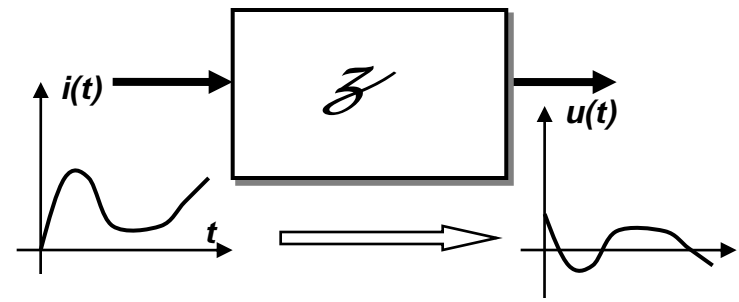
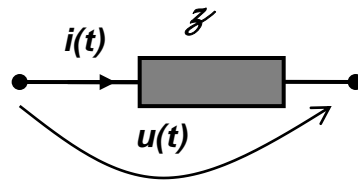
• **Rezistorul ideal liniar:**

$$u = Ri \Rightarrow \mathcal{Z} = R \quad \bullet \quad i = Gu \Rightarrow \mathcal{Y} = \mathcal{Z}^{-1} = G \quad \bullet$$

• **Bobina ideala liniara:**

$$u = L \frac{di}{dt} \Rightarrow \mathcal{Z} = L \frac{d}{dt} \quad \bullet \quad i(t) = i_0 + \frac{1}{L} \int_0^t u(t') dt' \Rightarrow \mathcal{Y} = \mathcal{Z}^{-1} = i_0 + \frac{1}{L} \int_0^t \bullet dt'$$

Element afin (liniar doar daca $i_0=0$), controlat in tensiune si impropriu in curent (cu semnale-functii generalizate, altfel in sens clasic curentul trebuie sa fie continuu si derivabil). Conditia de echivalenta: $L_1 = L_2; i_{01} = i_{02}$



Echivalenta elementelor ideale

- Condensatorul linear ideal:**

$$i = C \frac{du}{dt} \Rightarrow y = C \frac{d\bullet}{dt}; \quad u(t) = u_0 + \frac{1}{C} \int_0^t i(t') dt' \Rightarrow \mathcal{Z} = y^{-1} = u_0 + \frac{1}{C} \int_0^t \bullet dt'$$

Element afin (linear doar daca $i_0=0$), controlat in curent si impropriu in tensiune (cu semnale-functii generalizate, altfel in sens clasic tensiunea trebuie sa fie continua si derivabila). Conditia de echivalenta: $C_1 = C_2; u_{01} = u_{02}$

- Sursa independenta de tensiune:** $u(t) = e(t) \Rightarrow \mathcal{Z} = e(t)$

- Sursa independenta de curent:** $i(t) = j(t) \Rightarrow y = j(t)$

- Rezistorul neliniar:** $u = f(i) \Rightarrow \mathcal{Z} = f(\bullet); \quad i = g(u) \Rightarrow y = g(\bullet)$

- SUCI** $u_1 = 0; u_2 = \rho i_1 \Leftrightarrow \mathbf{u} = \mathbf{Ri} \Rightarrow \mathcal{Z}\bullet = \mathbf{R} = [0, 0 | \rho, 0]$

- SICU** $i_1 = 0; i_2 = \gamma u_1 \Leftrightarrow \mathbf{i} = \mathbf{Gu} \Rightarrow \mathcal{Y}\bullet = \mathbf{G} = [0, 0 | \gamma, 0]$

- SUCU** $i_1 = 0; u_2 = \alpha u_1 \Leftrightarrow \mathbf{y} = [i_1, u_2] = \mathbf{Hx} \Rightarrow \mathcal{H}\bullet = \mathbf{H} = [0, 0 | \alpha, 0]$

- SICI** $u_1 = 0; i_2 = \beta i_1 \Leftrightarrow \mathbf{y} = [u_1, i_2] = \mathbf{H}'\mathbf{x} \Rightarrow \mathcal{H}\bullet = \mathbf{H}' = [0, 0 | \beta, 0]$

- Bobine liniare cuplate mutual**

$$\mathbf{u} = \mathbf{L} \frac{d\mathbf{i}}{dt} \Rightarrow \mathcal{Z} = \mathbf{L} \frac{d\bullet}{dt} \quad \mathbf{i}(t) = \mathbf{i}_0 + \mathbf{L}^{-1} \int_0^t \mathbf{u}(t') dt' \Rightarrow \mathcal{Y} = \mathcal{Z}^{-1} = \mathbf{i}_0 + \mathbf{L}^{-1} \int_0^t \bullet dt'$$

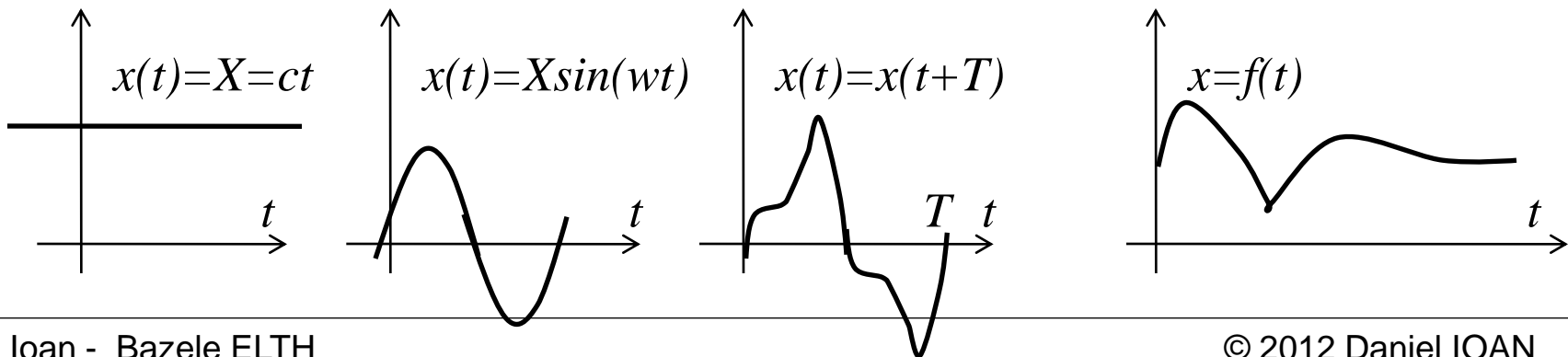
$\mathbf{L}_1 = \mathbf{L}_2; \mathbf{i}_{01} = \mathbf{i}_{02}$

Semnale si regimuri ale circuitelor electrice

- **Regimul stationar (de curent continuu – c.c.):** marimile i, u, v, e, j , sunt invarante in timp
- **Regimul sinusoidal (de curent alternativ c.a. – armonic):** circuite liniare, in care toate semnalele i, u, v, e, j au o variatie in timp, care poate fi adusa la forma standard: $x(t) = X_{\max} \sin(\omega t + \varphi)$ cu ω comun.
- **Regimul periodic (permanent sau c.a. deformat):** toate semnalele din circuit au o variatie periodica in timp: $x(t) = x(t + T)$ cu perioada T comuna
- **Regimul tranzitoriu:** semnalele i, u, v, e, j , din circuit au variatii arbitrare in timp:

$$x = f(t), \quad f : (0, \infty) \rightarrow \mathbb{R}$$

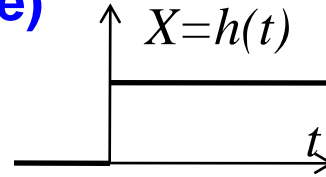
In particular, in regimul tranzitoriu, circuitul evolueaza de la o stare initiala la o stare finala corespunzatoare unui regim permanent (periodic, sinusoidal sau stationar).



Semnale tipice in circuitele electrice

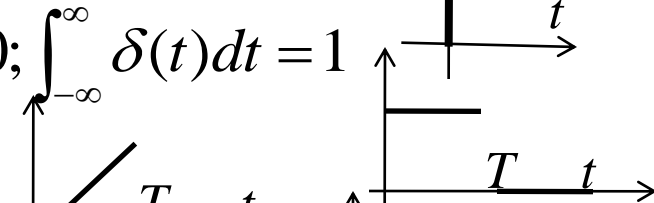
- **Semnalul treapta unitate (al lui Heavisde)**

$$x = h(t) =_{def} \begin{cases} 0 & \text{pentru } t < 0 \\ 1 & \text{pentru } t \geq 0 \end{cases}$$



- **Impuls Dirac** $x = \delta(t) = h'(t) \Rightarrow \delta(t) = 0$ pt $t = 0$; $\int_{-\infty}^{\infty} \delta(t) dt = 1$

- **Impuls dreptunghiular** $x = X[h(t) - h(t - T)]$



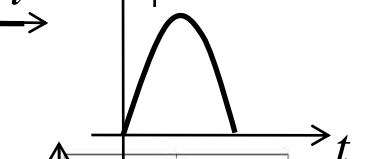
- **Impuls rampa**

$$x = Xt[h(t) - h(t - T)]$$



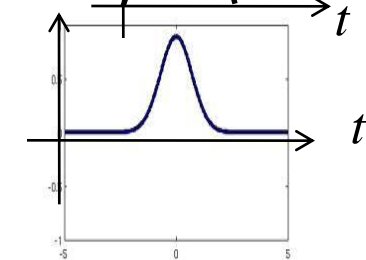
- **Impuls sinusoidal**

$$x = X[h(t) - h(t - T)] \sin(2\pi t / T)$$



- **Impuls Gauss**

$$x = X e^{-(t-T)^2 / (2c^2)}$$

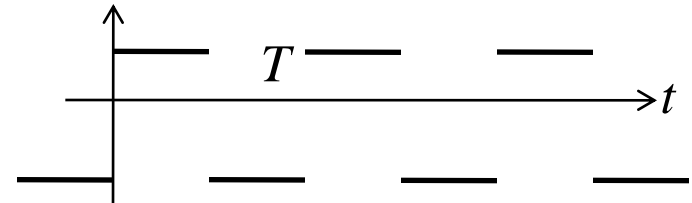


- **Periodic dreptunghiular:**

$$x = x(t - nT) = X[h(t) - h(t - T/2)] - X[h(t - T/2) - h(t - T)]$$

- **Periodic cu un numar finit de armonici**

$$x = \sum_{k=0}^n a_k \sin k\omega t + b_k \cos k\omega t$$



http://en.wikipedia.org/wiki/Impulse_function

5.2. Semnale sinusoidale. Reprezentarea in complex

Semnalul sinusidal – forma standard, caracteristici:

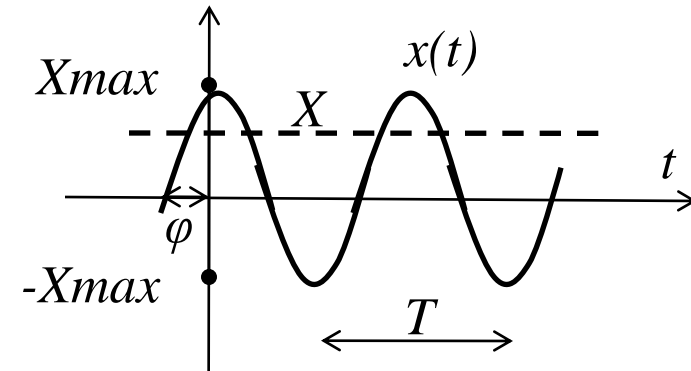
$$x(t) = X \sqrt{2} \sin(\omega t + \varphi)$$

- X – valoare efectiva,
- ω – pulsatie [rad/s] $\omega = 2\pi f = 2\pi / T$
- φ – faza initiala [rad]

$$X_{\max} = X \sqrt{2} \quad \text{Valoare maxima}$$

; T [s] – perioada, f - frecventa [Hz]

$$X = \sqrt{\frac{1}{T} \int_0^T x^2(t) dt} = \sqrt{\frac{X_{\max}^2}{T} \int_0^T \sin^2(\omega t + \alpha) dt} = X_{\max} / \sqrt{2}$$



Valoarea efectiva este valoarea medie patratica.

Curentul I continuu are acelasi efect termic ca un c.a. cu valoare efectiva I

Ampermetrele si voltmetrele de c.a. masoara valoarea efectiva.

Frecventa industrială: 50Hz (60 Hz in SUA). Alocarea frecventelor:

<http://www.ntia.doc.gov/osmhome/allochrt.pdf>

Reprezentarea in complex

Multimea semnalelor sinusoidale de frecventa data

$$\mathcal{S} = \left\{ f : \mathbb{R} \rightarrow \mathbb{R} \mid f(t) = X \sqrt{2} \sin(\omega t + \alpha) \right\}$$

este un spatiu linear bidimensional cu functiile de baza sin si cos:

$$x(t) = X \sqrt{2} \sin(\omega t + \alpha) = X \sqrt{2} (\sin \omega t \cos \alpha + \cos \omega t \sin \alpha) =$$

$$X \sqrt{2} \cos \alpha \sin \omega t + X \sqrt{2} \sin \alpha \sin(\omega t + \pi / 2) = X_s \sin \omega t + X_c \cos \omega t$$

Produsul scalar defineste norma, egala cu valoarea efectiva:

$$(x_1, x_2) = \frac{1}{T} \int_0^T x_1(t) x_2(t) dt = \frac{2X_1 X_2}{T} \int_0^T \sin(\omega t + \alpha_1) \sin(\omega t + \alpha_2) dt =$$

$$\frac{X_1 X_2}{T} \int_0^T \cos(\alpha_1 - \alpha_2) dt - \frac{X_1 X_2}{T} \int_0^T \cos(2\omega t + \alpha_1 + \alpha_2) dt = X_1 X_2 \cos(\alpha_1 - \alpha_2)$$

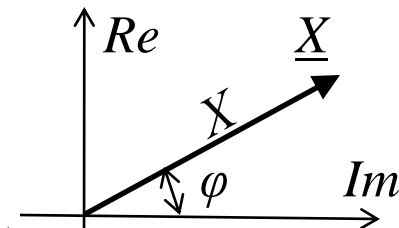
$$\|x\| = \sqrt{(x, x)} = \sqrt{\frac{2X^2}{T} \int_0^T \sin^2(\omega t + \alpha) dt} = \sqrt{\frac{X^2}{T} \int_0^T [1 - \cos 2(\omega t + \alpha)] dt} = X$$

Baza ortonormala: $s(t) = \sqrt{2} \sin \omega t$; $c(t) = \sqrt{2} \cos \omega t$

Reprezentarea complexa:

$$x(t) = X \sqrt{2} \sin(\omega t + \varphi) \xrightarrow{\mathcal{L}} \underline{X} = X e^{j\varphi}$$

$$\Rightarrow \mathcal{L}(s) = 1; \mathcal{L}(c) = j$$



Reprezentarea in complex

Relatia lui Euler: $e^{j\alpha} = \cos \alpha + j \sin \alpha$

Forma exponentiala a nr. complex: $\underline{X} = X e^{j\alpha}; \quad X = |\underline{X}|, \quad \alpha = \arg[\underline{X}]$

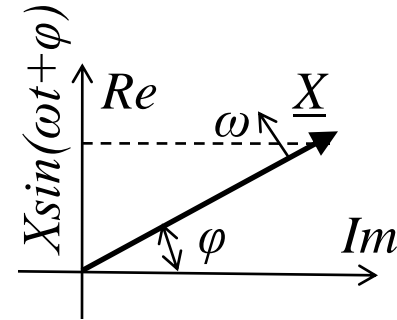
Forma algebrica a nr. complex: $\underline{X} = a + jb; \quad a = \text{Re}[\underline{X}], \quad b = \text{Im}[\underline{X}],$

Trecerea intre forme:

$$a = X \cos \alpha, \quad b = X \sin \alpha; \quad \Leftrightarrow \quad X = \sqrt{a^2 + b^2}, \quad \alpha = \text{atan}[b/a] + (\pi, \text{daca } b < 0)$$

Expresiile reprezentarii complexe directa si inversa:

$$\mathcal{L}[x(t)] = \frac{\sqrt{2}}{T} \int_0^T x(t) e^{j\omega t} dt, \quad \mathcal{L}^{-1}[\underline{X}] = \sqrt{2} \text{Im}[\underline{X} e^{j\omega t}]$$



Proprietatile reprezentarii complexe: - liniaritatea:

$$\mathcal{L}\left[\sum_{k=1}^n \lambda_k x_k(t)\right] = \sum_{k=1}^n \lambda_k \mathcal{L}[x_k(t)] \Leftrightarrow \mathcal{L}^{-1}\left[\sum_{k=1}^n \lambda_k \underline{X}_k\right] = \sum_{k=1}^n \lambda_k \mathcal{L}^{-1}[\underline{X}_k]$$

- Transforma operatiile de derivare/integrare in operatii algebrice:

$$\mathcal{L}\left[\frac{d}{dt} x(t)\right] = j\omega \mathcal{L}[x(t)] \Leftrightarrow \mathcal{L}\left[\int x(t) dt\right] = \frac{1}{j\omega} \mathcal{L}[x(t)]$$

Demonstratie:

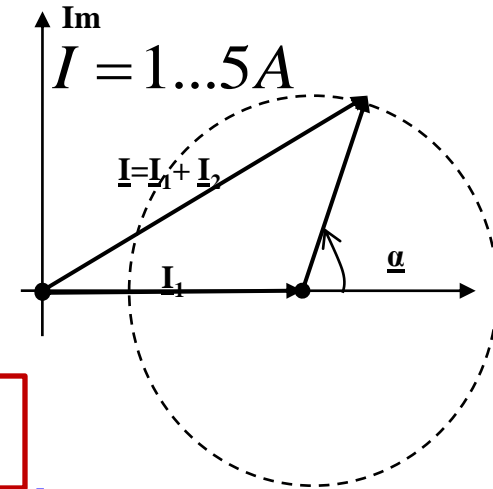
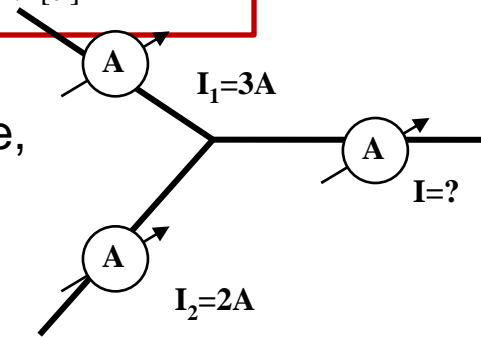
$$\mathcal{L}\left[\frac{d}{dt} X\sqrt{2} \sin(\omega t + \alpha)\right] = \omega \mathcal{L}\left[X\sqrt{2} \cos(\omega t + \alpha)\right] = \omega \mathcal{L}\left[X\sqrt{2} \sin(\omega t + \alpha + \pi/2)\right] = X\omega e^{j(\alpha+\pi/2)} = X e^{j\alpha} \omega e^{j\pi/2} = j\omega X$$

5.3. Impedanta complexa. Forma complexa a ecuatiilor reg. armonic

Forma complexa a ecuatiilor lui Kirchhoff:

$$\sum_{k \in (n)} i_k = 0, \quad \sum_{k \in [b]} u_k = 0 \Rightarrow \sum_{k \in (n)} \underline{I}_k = 0, \quad \sum_{k \in [b]} \underline{U}_k = 0; \quad \text{cu } \underline{I}_k = \mathcal{C}[i_k(t)], \quad \underline{U}_k = \mathcal{C}[u_k(t)]$$

Relatiile lui Kirchhoff sunt satisfacute de marimile instantanee si de cele complexe, dar nu si de valorile efective (indicatiile aparateleor), deoarece suma modulelor nu este egala cu modulul sumei!



Forma complexa a relatiei lui Ohm: $u = zi \Rightarrow \underline{U} = \underline{Z} \cdot \underline{I}$

Impedanta complexa – forma exponentiala si cea algebrica:

$$\underline{Z} = \frac{\underline{U}}{\underline{I}} = \frac{U e^{j\alpha_u}}{I e^{j\alpha_i}} = \frac{U}{I} e^{j(\alpha_u - \alpha_i)} = \underline{Z} e^{j\varphi} = \underline{Z} = R + jX = Z \cos \varphi + jZ \sin \varphi$$

Z - modulul impedntei, R - rezistenta de c.a. ; X - reactanta; $\varphi = \alpha_u - \alpha_i$ defazajul

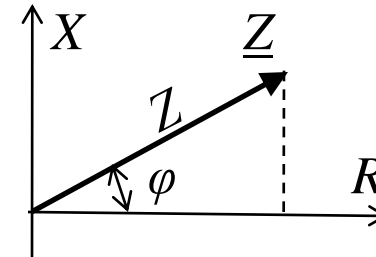
Admitanta complexa

$$\underline{Y} = 1/\underline{Z} = \frac{I}{U} e^{j(\alpha_i - \alpha_u)} = \underline{Y} e^{-j\varphi} = \underline{Y} = G + jB = Y \cos \varphi - jY \sin \varphi$$

Impedanta elementelor ideale liniare

Element	Rezistor	Bobina	Condensator
Impedanta complexa \underline{Z}		$j\omega L$	$1/(j\omega C)$
Admitanta complexa \underline{Y}	$G = 1/R$	R	$j\omega C$
Defazajul φ	0	$\pi/2$	$-\pi/2$
Impedanta Z	R	ωL	$1/(\omega C)$
Admitanta Y	$G = 1/R$	$1/(\omega L)$	ωC
Rezistenta de c.a. R	R	0	0
Reactanta X	0	ωL	$-1/(\omega C)$
Conductanta de c.a. G	$G = 1/R$	0	0
Susceptanta B	0	$-1/(\omega L)$	ωC
Planul \underline{Z}			

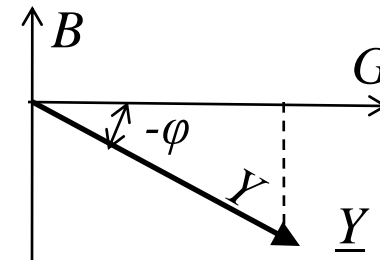
Planul (triunghiul) impedantei



$$Z = \sqrt{R^2 + X^2}; \operatorname{tg} \varphi = X / R;$$

$$R = Z \cos \varphi; X = Z \sin \varphi;$$

Planul (triunghiul) admitantei



$$Y = \sqrt{G^2 + B^2}; \operatorname{tg} \varphi = -B / G;$$

5.4. Teorema similitudinii c.c. – c.a.

Se considera un circuit linear de c.c. si unul de c.a. cu laturi surse reale de tensiune (SRT) sau de curent (SRC). Prin reprezentare in complex:

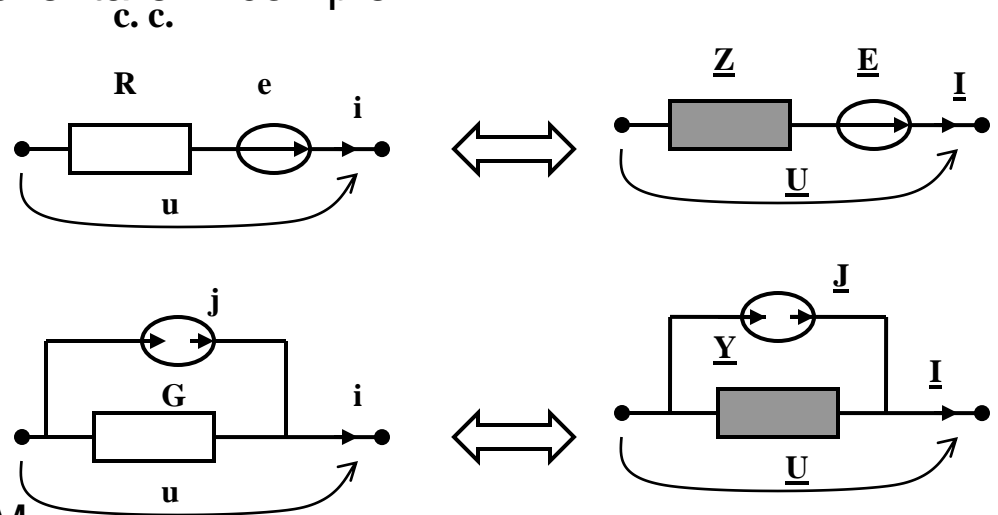
cele doua circuite au ecuatiile

similare: la circuitul de c.a. marimile sunt complexe, iar locul rezistentelor/ conductantelor din c.c. este luat de impedantele/admitantele complexe.

$$\mathcal{C}[u = e(t); i = j(t)] \Rightarrow \underline{U} = \underline{E}; \underline{I} = \underline{J}$$

$$\mathcal{C}[\mathbf{u} = \mathbf{L}d\mathbf{i} / dt] \Rightarrow \underline{\mathbf{U}} = \underline{\mathbf{Z}}\underline{\mathbf{I}}; \underline{\mathbf{Z}} = j\omega\mathbf{L}$$

Bobine cuplate = pereche SUCI cu $\underline{z}=j\omega\mathbf{M}$



Circuitul de c.c.

$$TK1: \sum_{k \in (n)} i_k = 0; \quad TK2: \sum_{k \in [b]} u_k = 0;$$

$$SRT: u_k = R_k i_k - e_k; \quad SRC: i_k = G_k u_k + j_k$$

$$SUCI: e_k = r_{km} i_m, \quad SICU: j_k = g_{km} u_m,$$

$$SUCU: e_k = \alpha_{km} u_m, \quad SICI: j_k = \beta_{km} i_m.$$

Circuitul de c.a.

$$TK1: \sum_{k \in (n)} \underline{I}_k = 0; \quad TK2: \sum_{k \in [b]} \underline{U}_k = 0;$$

$$SRT: \underline{U}_k = \underline{Z}_k \underline{I}_k - \underline{E}_k; \quad SRC: \underline{I}_k = \underline{Y}_k \underline{U}_k + \underline{J}_k$$

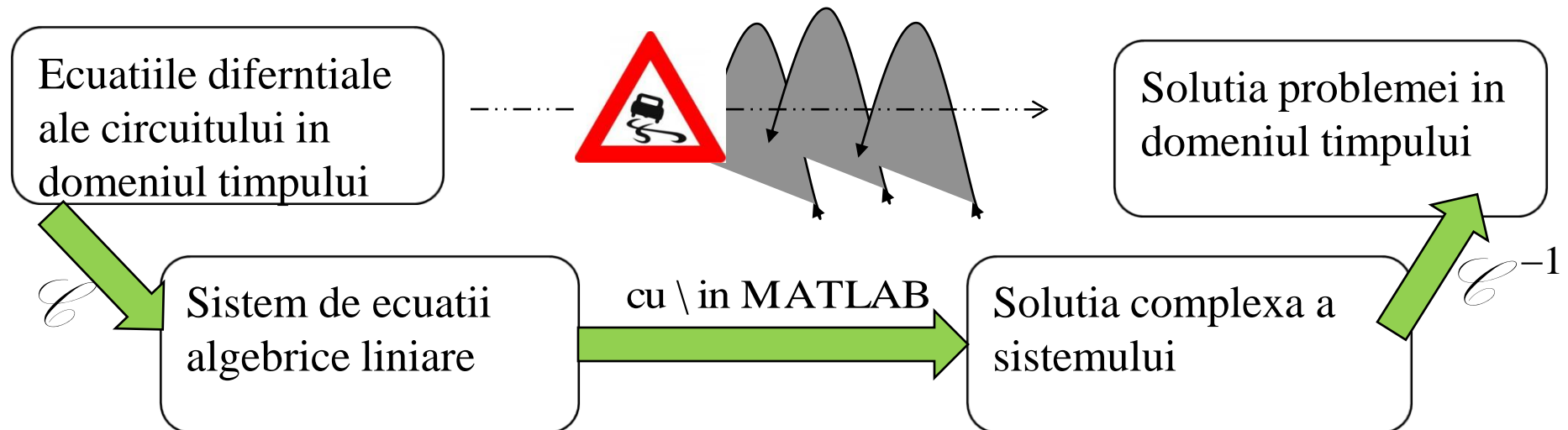
$$SUCI: \underline{E}_k = \underline{z}_{km} \underline{I}_m, \quad SICU: \underline{J}_k = \underline{y}_{km} \underline{U}_m,$$

$$SUCU: \underline{E}_k = \underline{a}_{km} \underline{U}_m, \quad SICI: \underline{I}_k = \underline{b}_{km} \underline{I}_m.$$

Analiza in domeniul frecventei, prin reprezentare complexa

- Se reprezinta in complex t.e.m si respectiv c.e.m. ale surselor independente de tensiune respectiv de curent din circuit.
- Se calculeaza impedantele complexe ale tuturor elementelor liniare din circuit $\underline{Z} = R, j\omega L, j\omega M, 1/(j\omega C)$. Bobinele cuplate se pot inlocui cu SUCI.
- Se analizeaza acest circuit cu una din metodele specifice circuitelor electrice liniare de c.c. si se determina valorile complexe ale curentilor si tensiunilor din laturile circuitului.
- Se determina solutia problemei in domeniul timpului, aplicand transformarea complexa inversa.

Un drum mai lung, dar mai usor - prin domeniul frecventei:



5.5. Puteri in c.a.

- **Puterea instantanee** – pentru un element dipolar este

$$p(t) = u(t)i(t) = 2UI \sin(\omega t + \alpha_u) \sin(\omega t + \alpha_i) = UI \cos(\alpha_u - \alpha_i) - UI \cos(2\omega t + \alpha_u + \alpha_i)$$

- **Puterea activa** este valoarea medie a puterii instantanee:

$$P = \frac{1}{T} \int_0^T p(t) dt = UI \cos(\alpha_u - \alpha_i)$$

- **Puterea complexa:** $\underline{S} = \underline{UI}^* = Ue^{j\alpha_u} Ie^{-j\alpha_i} = UIe^{j(\alpha_u - \alpha_i)} = UIe^{j\varphi} = Se^{j\varphi}$

cu partile reala si imaginara: **puterea activa P [W]** si **puterea reactiva Q[Var]:**

$$\underline{S} = P + jQ \Rightarrow P = UI \cos \varphi; \quad Q = UI \sin \varphi$$

si modulul **puterea aparenta [VA]** $S = UI$

Raportul $k = P / S = \cos \varphi$ se numeste **factor de putere.**

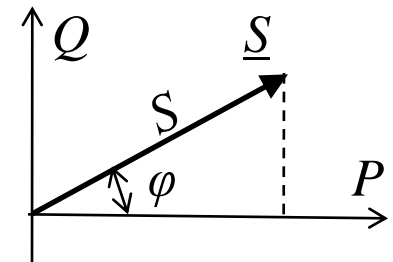
$$S = \sqrt{P^2 + Q^2}; \quad \text{tg } \varphi = Q / P; \quad P = S \cos \varphi; \quad Q = S \sin \varphi;$$

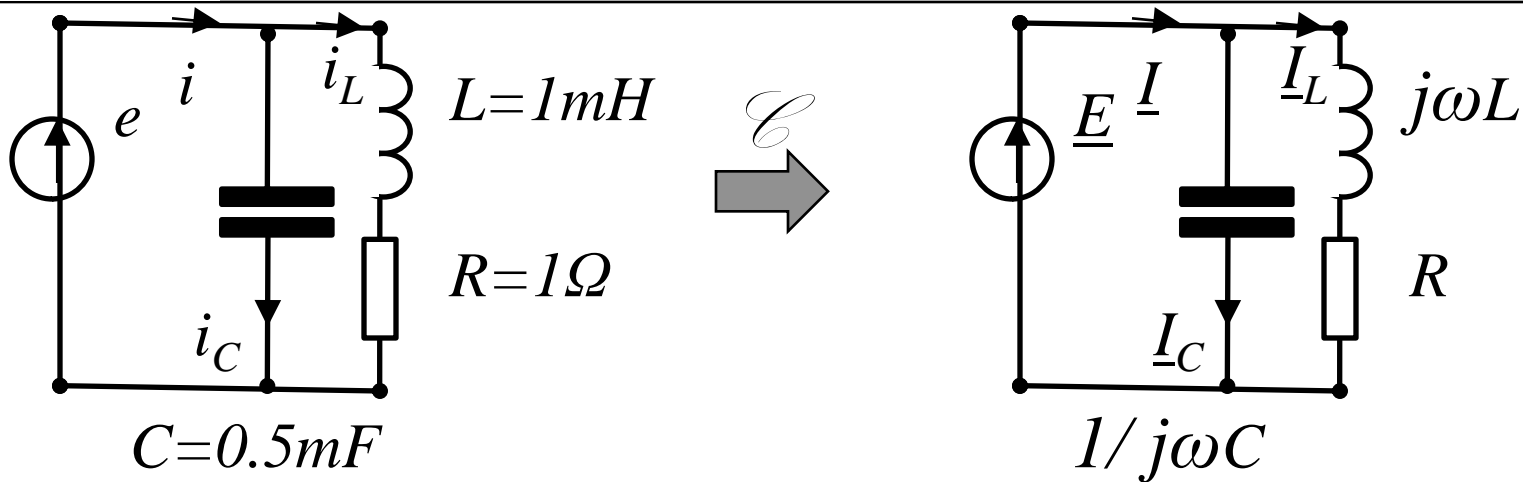
Elementele multipolare: $\underline{S} = \underline{\mathbf{V}} \underline{\mathbf{I}}^* = \sum_{k=1}^{n-1} \underline{V}_k \underline{I}_k^* = P + jQ$

Conditia de pasivitate:

$$\underline{S} = \underline{\mathbf{V}}^T \underline{\mathbf{I}}^* = \underline{\mathbf{I}}^{*T} \underline{\mathbf{V}} = \sum_{k=1}^{n-1} \underline{V}_k \underline{I}_k^* = \underline{\mathbf{I}}^T \underline{\mathbf{Z}}^T \underline{\mathbf{I}}^* = \underline{\mathbf{I}}^{*T} \underline{\mathbf{Z}} \underline{\mathbf{I}} \Rightarrow P = \text{Re}[\underline{\mathbf{I}}^T \underline{\mathbf{Z}}^T \underline{\mathbf{I}}^*] > 0$$

Planul (triunghiul)
puterilor





$$e(t) = 100\sqrt{2} \sin(\omega t + \pi/4) \Rightarrow \underline{E} = 100e^{j\pi/4} = 50\sqrt{2}(1+j); \omega = 2\pi f = 100\pi$$

$$\underline{I} = \underline{I}_C + \underline{I}_L = \underline{E}j\omega C + \underline{E}/(R + j\omega C) = 50\sqrt{2}(1+j)j100\pi 5 \cdot 10^{-4} +$$

$$50\sqrt{2}(1+j)/(1 + j100\pi \cdot 10^{-3}) = \sqrt{2}\pi 2.5(j-1) + \sqrt{2}(1+j)(50 - 5\pi j)/(1 + \pi^2 \cdot 10^{-2}) =$$

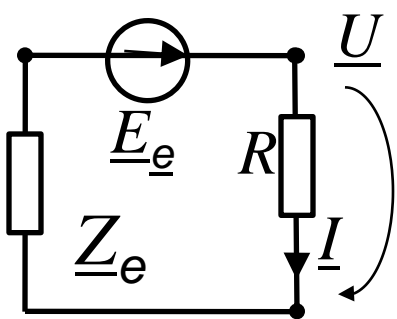
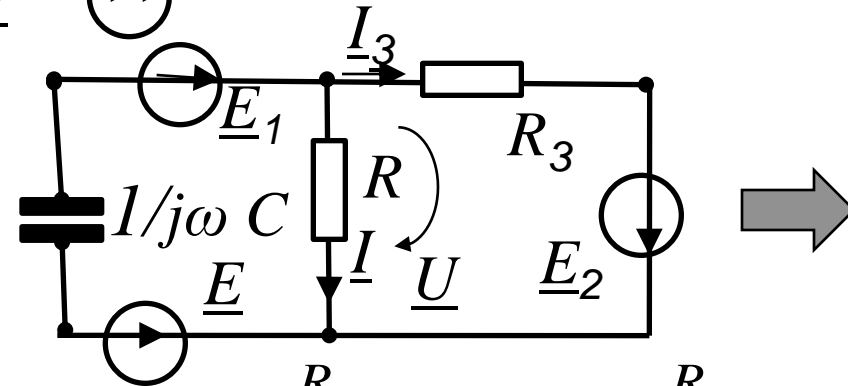
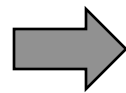
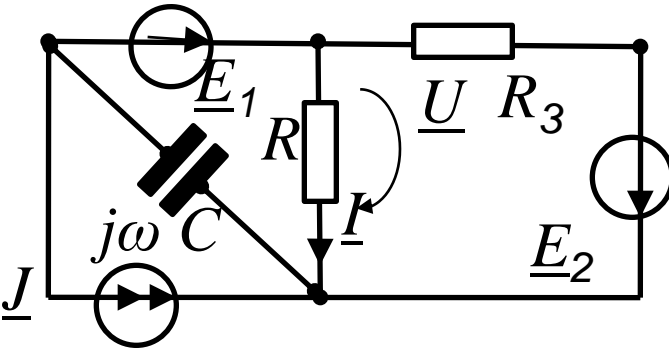
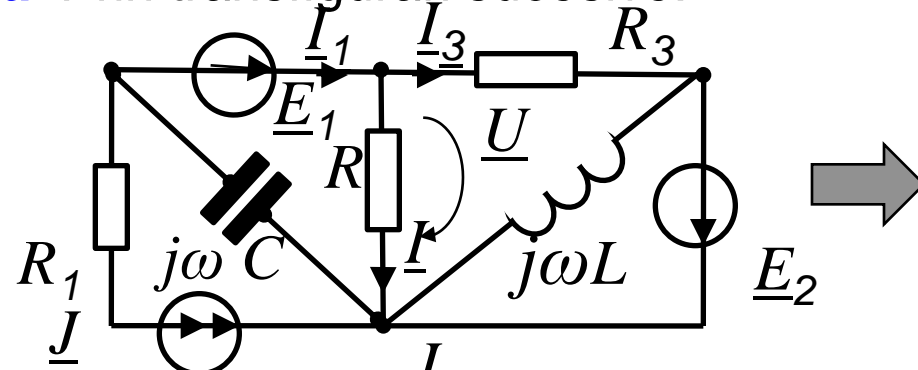
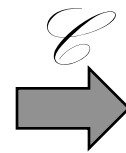
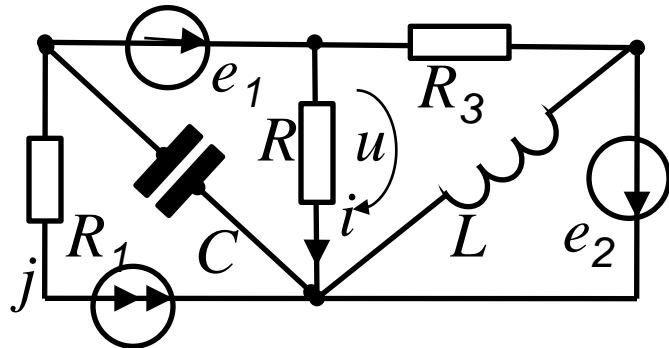
$$11.107(j-1) + 1.287(1+j)(50 - 5\pi j) = 73.459 + j55.2408 \Rightarrow i(t) = I\sqrt{2} \sin(\omega t + \varphi);$$

$$I = |73.459 + j55.2408| = 95.1366A; \varphi = \arctg(55.2408/73.459) = 0.644rad$$

$$\underline{S}_g = P_g + jQ_g = \underline{E}\underline{I}^* = 50\sqrt{2}(1+j)(73.459 - j55.2408) = 9100 + j1288 \Rightarrow$$

$$P = 9.1kW; Q = 1.288kVAr$$

Exemplu de analiza unui circuit de c.a. Prin transfigurari succesive:



$$\underline{E} = \underline{J} / (j\omega C); \underline{Z}_e = C \parallel R_3 = \frac{R_3}{j\omega C(R_3 + 1/j\omega C)} = \frac{R_3}{j\omega CR_3 + 1};$$

$$\underline{E}_e = \frac{(\underline{E}_1 - \underline{E})R_3 - \underline{E}_2 / j\omega C}{R_3 + 1/j\omega C} = \frac{(\underline{E}_1 - \underline{E})j\omega CR_3 - \underline{E}_2}{j\omega CR_3 + 1} \Rightarrow$$

$$\underline{I} = \underline{E}_e / (\underline{Z}_e + R) \Rightarrow \underline{U} = R\underline{I} \Rightarrow \underline{I}_3 = (\underline{U} + \underline{E}_2) / R_3; \underline{I}_1 = \underline{I} + \underline{I}_3$$

Aplicatii. Rezonanta in circuitul RLC serie

$$\underline{U} = \underline{U}_R + \underline{U}_L + \underline{U}_C = R\underline{I} + j\omega L\underline{I} + \underline{I}/(j\omega C) = [R + j[\omega L - 1/(\omega C)]]\underline{I}$$

- **Impedanta complexa:**

$$\underline{Z} = \underline{U} / \underline{I} = R + j[\omega L - 1/(\omega C)]$$

- **Reactanta se anuleaza la pulsatia de rezonanta,**

cand $U_L = U_C = U\omega_0 L/R$: $X = \text{Im}(\underline{Z}) = \omega L - 1/(\omega C)$

$$\omega L = 1/(\omega C) \Rightarrow \omega^2 = \frac{1}{LC} \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}}$$

- **Curentul efectiv:** $I = \frac{U}{Z} = \frac{U}{\sqrt{R^2 + X^2}} = \frac{U}{\sqrt{R^2 + [\omega L - 1/(\omega C)]^2}}$

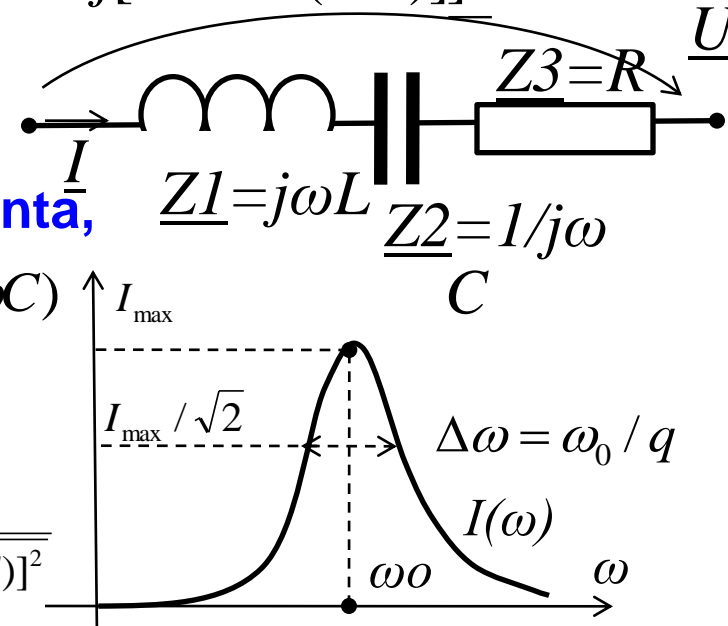
- **Factorul de calitate** (cu cat este mai mare, cu atat curba este mai ascutita si tensiunile pe elementele reactive sunt mai mari $U_L = U_C = qU$)

$$q = \frac{\omega_0 L}{R} = \frac{L}{R\sqrt{LC}} = \frac{1}{R} \sqrt{\frac{L}{C}}$$

$$\omega_1 L - 1/(\omega_1 C) = R \Rightarrow \omega_1^2 L - R\omega_1 - 1/C = 0$$

$$-\omega_2 L + 1/(\omega_2 C) = R \Rightarrow \omega_2^2 L + R\omega_2 - 1/C = 0$$

$$\Delta\omega = \omega_2 - \omega_1 = R/L = \omega_0 / q$$



La rezonanta elementele
reactive isi anuleaza efectele.

Rezonanta in circuitul dual RLC paralel

$$\underline{I} = \underline{I}_R + \underline{I}_L + \underline{I}_C = \underline{U} / R + j\omega C \underline{U} + \underline{U} / (j\omega L) = [1/R + j[\omega C - 1/(\omega L)]] \underline{U}$$

- Admitanta complexa:**

$$\underline{Y} = \underline{I} / \underline{U} = G + j[\omega C - 1/(\omega L)]$$

- Susceptanta se anuleaza la pulsatia de rezonanta,**

cand $I_L = I_C = I \omega_0 RC$: $B = \text{Im}(\underline{Y}) = \omega C - 1/(\omega L)$

$$\omega C = 1/(\omega L) \Rightarrow \omega^2 = \frac{1}{LC} \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}}$$

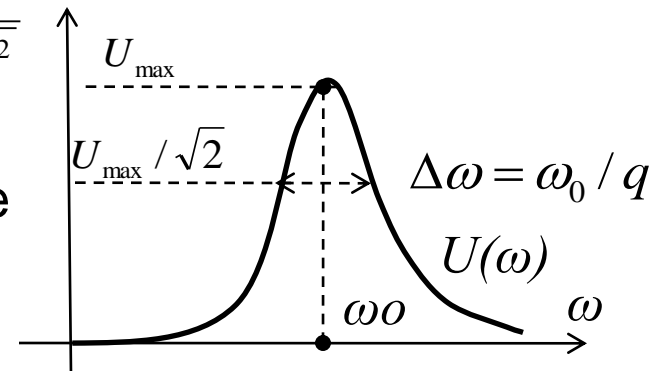
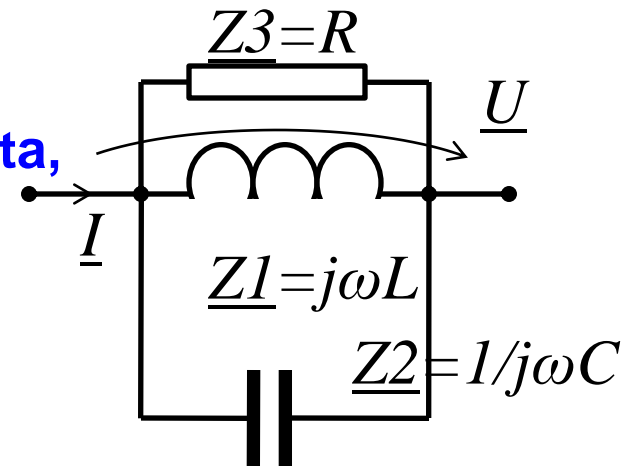
- Tens. efectiva:**

$$U = \frac{I}{Y} = \frac{U}{\sqrt{G^2 + B^2}} = \frac{I}{\sqrt{1/R^2 + [\omega C - 1/(\omega L)]^2}}$$

- Factorul de calitate** (cu cat este mai mare,

cu atat curba este mai ascutita si curentii elementele reactive sunt mai mari $I_L = I_C = qI$)

$$q = \omega_0 RC = \frac{RC}{\sqrt{LC}} = R \sqrt{\frac{C}{L}}$$



La rezonanta: elementele reactive isi anuleaza efectele, iar circuitul se comporta ca un rezistor: LC paralel = izolator perfect, LC serie = conductor perfect

complet rezistive (omnirezonante)

Daca

$$R = \sqrt{L/C} \Rightarrow C = L/R^2$$

circuitul din figura are impedanta complexa pur rezistiva si deci independenta de frecventa:

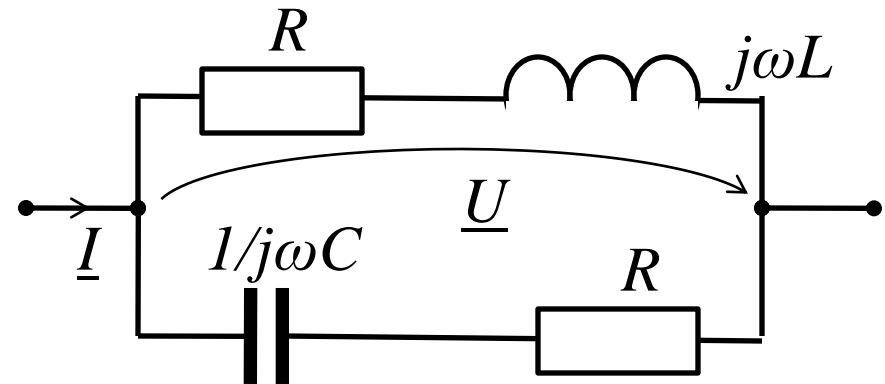
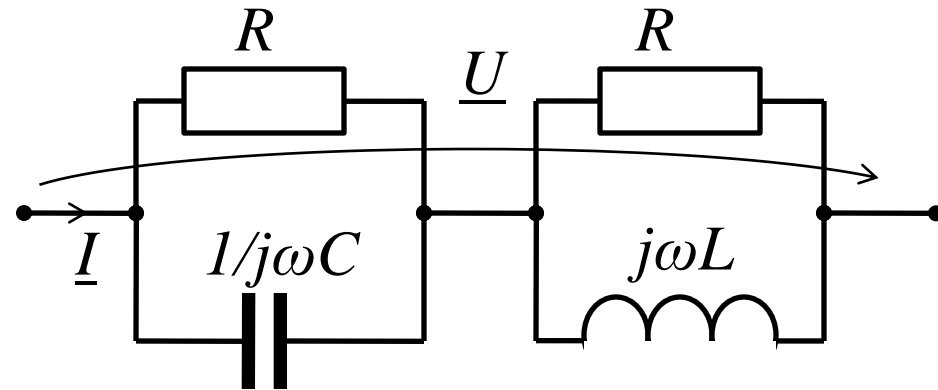
$$\underline{Z} = \frac{j\omega LR}{R + j\omega L} + \frac{R}{j\omega C(R + 1/(j\omega C))} =$$

$$\frac{j\omega LR}{R + j\omega L} + \frac{R^3}{j\omega L(R + R^2/(j\omega L))} =$$

$$\frac{j\omega LR}{R + j\omega L} + \frac{R^2}{(j\omega L + R)} = R$$

Circuitul dual alcatuit din doua ramuri in paralel, una RL serie si alta RC serie are deasemenea o comportare globala pur rezistiva, daca

$$R = \sqrt{L/C} \Rightarrow C = L/R^2$$



Curentul din rezistenta

$$\underline{I} = \frac{E}{\underline{Z}_1 + \frac{\underline{Z}\underline{Z}_2}{\underline{Z} + \underline{Z}_2}} \frac{\underline{Z}_2}{\underline{Z}_2 + \underline{Z}} = \frac{E\underline{Z}_2}{\underline{Z}_1(\underline{Z} + \underline{Z}_2) + \underline{Z}\underline{Z}_2} = \frac{E\underline{Z}_2}{\underline{Z}(\underline{Z}_1 + \underline{Z}_2) + \underline{Z}_2\underline{Z}_1}$$

are valoarea

$$\underline{I} = \frac{E}{\underline{Z}_1}$$

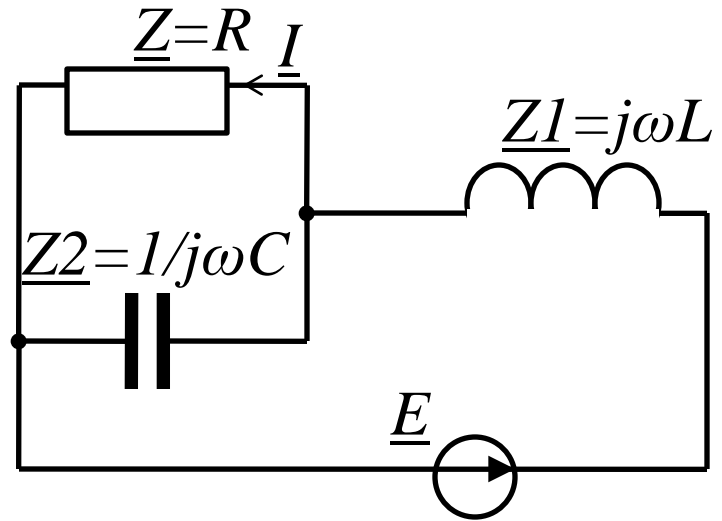
daca $\underline{Z}_1 + \underline{Z}_2 = 0$

Conditia are loc la rezonanta LC.: $j\omega L - j/(\omega C) = 0 \Rightarrow \omega = 1/\sqrt{LC}$

Fata de rezistor, circuitul se comporta la frecventa de rezonanta ca o sursa ideal de curent, generand un curent independent de R.

Explicatia este ca pasivizand circuitul, la bornele rezistorului de sarcina se afla un izolator perfect (LC in paralel la rezonanta).

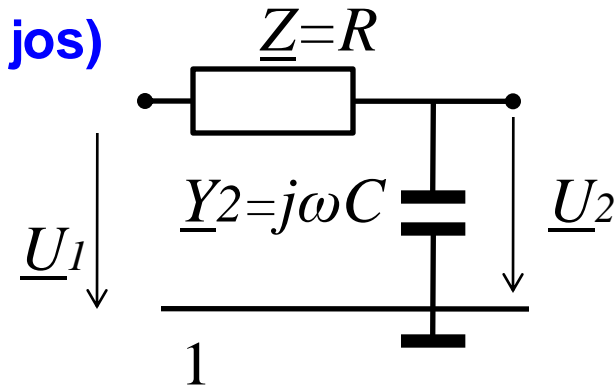
- **Compensarea factorului de putere.** Considerati un generator, o linie si un consumator inductiv (RL). Calculati pierderile pe linie si analizati cum se modifica, daca in paralel cu consumatorul montam un condensator rezonant.



Circuit RC - de ordinul unu (filtru TJ- trece jos)

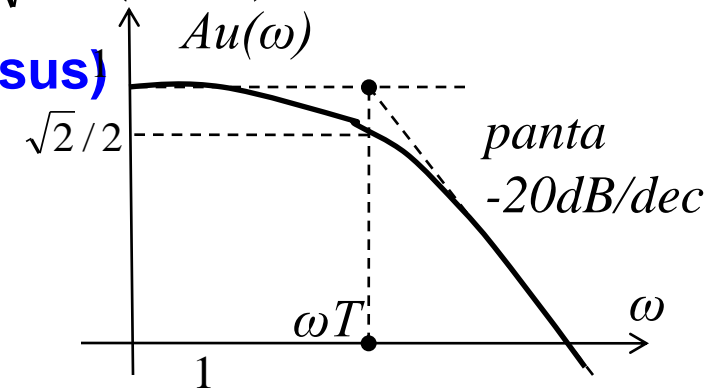
are factorul de transfer in tensiune,
la mers in gol la iesire:

$$\underline{A}_u = \frac{\underline{U}_2}{\underline{U}_1} = \frac{1/(j\omega C)}{R + 1/(j\omega C)} = \frac{1}{1 + j\omega RC} \Rightarrow A_u = \frac{1}{\sqrt{1 + (\omega RC)^2}}$$



Circuit dual RC de ordinul unu (filtru TS- trece sus)

are caracteristica crescatoare cu aceeaasi
pulsatia de taiere $\omega_T = 1/RC$



Circuit RLC de ordinul doi: $\underline{Z}=R+j\omega L$

$$\underline{A}_u = \frac{\underline{U}_2}{\underline{U}_1} = \frac{1/(j\omega C)}{R + j\omega L + 1/(j\omega C)} = \frac{1}{1 + j\omega RC - \omega^2 LC} \Rightarrow A_u = \frac{1}{\sqrt{(1 - \omega^2 LC)^2 + (\omega RC)^2}}$$

e tot un filtru TJ, dar cu panta -40dB/dec. Caracteristica poate avea un maxim

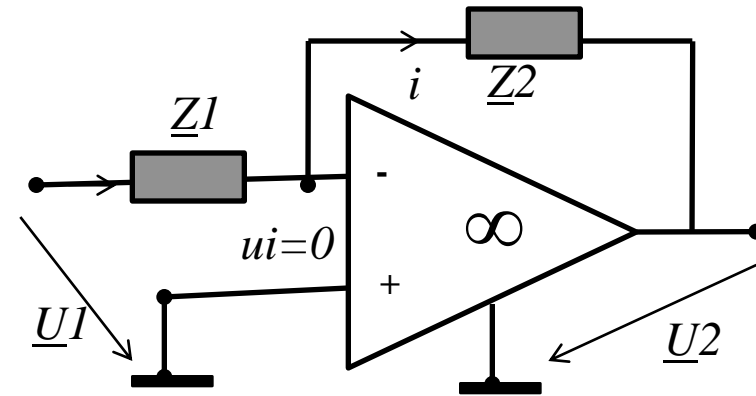
$$A_{u_{\max}} = q / \sqrt{1 - 1/(4q^2)} > 1, \text{ pentru } \omega_c = \omega_0 \sqrt{1 - 1/(2q^2)}; \omega_0 = 1/\sqrt{LC}; q = \omega_0 L / R$$

In baza teoremei de similitudine c.c.-c.a.

$$\underline{A}_u(\omega) = \frac{U_2}{U_1} = -\frac{Z_2}{Z_1} \text{ pentru circuitul inversor}$$

$$\underline{A}_u(\omega) = \frac{U_2}{U_1} = 1 + \frac{Z_2}{Z_1} \text{ pentru circuitul ne - inversor}$$

$$\underline{Z}_{21}(\omega) = \frac{U_2}{I_1} = \underline{Z} \text{ pentru circuitul convertor i - u (SUCI)}$$



Filtrele active au avantajul unidirectionalitatii, intrarea nu este influentata de iesire iar sarcina nu influenteaza amplificarea. Pentru orice realizare RLC a impedantelor Z , functia de transfer intrare-iesire este rationala in $j\omega \Rightarrow P(j\omega)/Q(j\omega)$ - raport de polinoame

Filtrul trece banda (de ordinul 4):

$$\underline{A}_u(\omega) = \frac{U_2}{U_1} = -\frac{Z_2}{Z_1} = -\frac{1}{\underline{Z}_1 \underline{Y}_2} = -\frac{1}{(R_1 + j\omega L_1 + 1/(j\omega C_1))(G_2 + j\omega C_2 + 1/(j\omega L_2))}$$

are la intrare un circuit RLC serie si in reactie un circuit RLC paralel.

Functia de transfer are patru poli. Pozitia lor determina frecventele de taiere.

<http://users.ece.gatech.edu/~alan/ECE3040/Lectures/Lecture29OP%20Amp%20Frequency%20Response.pdf>

5.6. Semnale tranzitorii. Transformata Laplace

Clasa semnalelor tranzitorii: $\mathcal{T} = \{f : (0, \infty) \rightarrow \mathbb{R}\}$

Funcțiile extinse la întreaga axa reală, cu valori nule pentru $t < 0$ se numesc *funcții originale*.

Transformata Laplace $\mathcal{L} : \mathcal{T} \rightarrow \mathcal{F} = \{F : \mathbb{C} \rightarrow \mathbb{C}\}$, definită de integrala:

$$\mathcal{L}[x(t)] = \int_0^{\infty} x(t)e^{-st} dt$$

face să corespundă fiecărui semnal tranzitoriu o funcție complexă analitică de variabilă complexă $s = \sigma + j\omega$ numită și *frecvență complexă*. Elementele mulțimii \mathcal{F} se numesc *semnale operaționale* sau *funcții imagine*.

Pentru ca funcția complexă să fie corect definită trebuie ca integrala să fie convergentă.

Următoarea teoremă asigură convergența și definiția corectă în sens clasic. Dacă $x(t)$:

- este continuă pe porțiuni în orice interval marginit din $0 < t < \infty$ (adică are cel mult un număr finit de discontinuități de tip salt marginit) și
- este marginită de o funcție exponențială, adică există constantele reale M și a , astfel încât $|x(t)| < Me^{at}$ pentru orice $t > 0$,

atunci integrala este convergentă pentru orice s cu $\text{Re}[s] > a$ și definește corect semnalul operațional $X(s)$, care este o funcție complexă analitică pe pe domeniul de convergență. Funcția se extinde apoi prin continuitate într-o mare parte din restul planului complex (exceptând punctele singulare).

Proprietatile transformatei Laplace

- **Teorema liniaritatii transformatei Laplace:**

$$\mathcal{L}\left[\sum_{k=1}^n \lambda_k x_k(t)\right] = \sum_{k=1}^n \lambda_k \mathcal{L}[x_k(t)];$$

Atat domeniul cat si codomeniul transformatei Laplace sunt spatii liniare.

- **Teorema derivarii si a integrarii functiilor original**

$$\mathcal{L}\left[\frac{d}{dt} x(t)\right] = sX(s) - x(0); \quad \mathcal{L}\left[\int_0^t x(\tau) d\tau\right] = \frac{1}{s} X(s) \iff \mathcal{L}[y(t)] = \frac{1}{s} \mathcal{L}\left[\frac{dy}{dt}\right] \text{ cu } y(0) = 0$$

Demo:
$$\mathcal{L}\left[\frac{d}{dt} x(t)\right] = \int_0^{\infty} \frac{dx}{dt} e^{-st} dt = xe^{-st} \Big|_0^{\infty} - \int_0^{\infty} x(t) \frac{de^{-st}}{dt} dt = -x(0) + s \int_0^{\infty} x(t) e^{-st} dt$$

- **Teorema intarzierii/ deplasarii:**

$$\mathcal{L}[x(t - t_0)] = e^{-st_0} X(s); \quad \mathcal{L}[e^{\alpha t} x(t)] = X(s - \alpha);$$

- **Teorema convolutiei:**

$$\mathcal{L}[f(t) * g(t)] = F(s)G(s); \quad f * g = \int_0^t f(\tau) g(t - \tau) d\tau$$

- **Teorema valorilor finale/ initiale:**

$$sF(s) \xrightarrow{s \rightarrow \infty} f(0); \quad sF(s) \xrightarrow{s \rightarrow 0} f(\infty)$$

- Funcția treaptă a lui Heaviside**

$$\mathcal{L}[h(t)] = \int_0^{\infty} h(t)e^{-st} dt = \int_0^{\infty} e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^{\infty} = \frac{1}{s} \stackrel{\mathcal{L}}{\leftarrow} h(t)$$

- Impulsul Dirac** – funcție generalizată, definită ca derivata funcției treapta:

$$\delta(t) = \frac{dh(t)}{dt} \Rightarrow \mathcal{L}[\delta(t)] = s\mathcal{L}[h(t)] - h(-0) = 1 \stackrel{\mathcal{L}}{\leftarrow} \delta(t)$$

Această funcție are două proprietăți fundamentale, care în sens clasic sunt contradictorii: funcția este nulă peste tot cu excepția originii (are suportul în origine); integrala sa pe întreaga axă este egală cu 1 (funcția este nemărginită în origine). În produs cu altă funcție ea filtrează valoarea acesteia în origine. Pentru a include suportul impulsului Dirac, este necesar ca integrala din definiția transformatei Laplace să se facă de la -0 :

- Funcția exponențială**

$$\mathcal{L}[e^{\alpha t}] = \int_0^{\infty} e^{\alpha t} e^{-st} dt = \int_0^{\infty} e^{(\alpha-s)t} dt = \frac{1}{\alpha-s} e^{(\alpha-s)t} \Big|_0^{\infty} = \frac{1}{s-\alpha} \stackrel{\mathcal{L}}{\leftarrow} e^{\alpha t}$$

- Funcții sinusoidale**

$$\mathcal{L}[e^{j\omega t}] = \mathcal{L}[\cos \omega t] + j\mathcal{L}[\sin \omega t] = \int_0^{\infty} e^{j\omega t} e^{-st} dt = \int_0^{\infty} e^{(j\omega-s)t} dt = \frac{1}{j\omega-s} e^{(j\omega-s)t} \Big|_0^{\infty} =$$

$$\frac{1}{s-j\omega} = \frac{s+j\omega}{s^2+\omega^2} \Rightarrow \cos \omega t \stackrel{\mathcal{L}}{\Rightarrow} \frac{s}{s^2+\omega^2}; \sin \omega t \stackrel{\mathcal{L}}{\Rightarrow} \frac{\omega}{s^2+\omega^2}$$

- **Functii sinusoidale amortizate:**

$$e^{\alpha t} \cos \omega t \xrightarrow{\mathcal{L}} \frac{s - \alpha}{(s - \alpha)^2 + \omega^2}; \sin \omega t \xrightarrow{\mathcal{L}} \frac{\omega}{(s - \alpha)^2 + \omega^2}$$

- **Functie impuls**

$$\mathcal{L}[h(t)] = \int_0^{\infty} f(t)[h(t) - h(t - \tau)]e^{-st} dt = F(s) - \int_{\tau}^{\infty} f(t)e^{-st} dt = (1 - e^{s\tau})F(s)$$

Referinte despre transformata Lapalce:

http://en.wikipedia.org/wiki/Laplace_transform

http://amotion.pub.ro/cercetare/laboratoare/ra/download/RA_Anexa_01A.pdf

http://www.shiva.pub.ro/PDF/TRA/Anexa_1_Tabel_de_Transformate_Laplace.pdf

http://users.utcluj.ro/~gurzau/an%20I%20AR%20sem.ii/laplace_tr_bun.pdf

<http://www.stanford.edu/~boyd/ee102/laplace.pdf>

<http://www.math.utah.edu/~gustafso/laplaceTransform.pdf>

<http://www.syscompdesign.com/AppNotes/laplace-cookbook.pdf>

MATLAB:

http://thor.info.uaic.ro/~fliacob/An2/2009-2010/Resurse/Introducere%20in%20Matlab_index%20cu%20exemple.pdf

Aplicatii. Tabel cu transformate Laplace

Denumire functie	Functia original f(t)	Transf. Lapalce F(s)
Impuls Dirac unitar	$\delta(t)$	1
Constanta (treapta unitara)	$h(t)$	$\frac{1}{s}$
Exponentiala	$e^{\alpha t}$	$\frac{1}{s - \alpha}$
Cosinusoidala	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
Sinusoidala	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
Cosinusoidala amortizata	$e^{\alpha t} \cos \omega t$	$\frac{s - \alpha}{(s - \alpha)^2 + \omega^2}$
Sinusoidala amortizata	$e^{\alpha t} \sin \omega t$	$\frac{\omega}{(s - \alpha)^2 + \omega^2}$
Rampa	t	$1/s^2$
Monom	t^n	$n!/s^{n+1}$

Transformata Laplace inversa

Pentru a defini transformata Laplace inversa trebuie restrans sau extins domeniul si codomeniul transformarii a.i. ea sa devina bijectiva:

$$\mathcal{L} : \mathcal{F} \rightarrow \mathcal{F} = \{F : \mathbb{C} \rightarrow \mathbb{C}\} \Rightarrow \mathcal{L}^{-1} : \mathcal{F} \rightarrow \mathcal{F} ; \mathcal{L} \circ \mathcal{L}^{-1} = I$$

Aplicand recurent teorema derivarii, rezulta ca vom considera codomeniul clasa functiilor complexe analitice, ar trebui ca domeniul sa contina functii infinit derivabile. Pentru a avea mai mult flexibilitate vom extinde domeniul la clasa functiilor generalizate (distributiilor), care sunt derivabile fara restrictie.

Deoarece semnalele tranzitorii de interes practic se reprezinta operational prin functii rationale ne vom referi la acest caz, in care functia imagine este raportul a doua polinome in s. Daca polii sunt simpli restul se descompune in fractii simple de forma:

$$X(s) = \frac{P(s)}{Q(s)} = C(s) + \frac{R(s)}{Q(s)} = C(s) + \sum_{k=1}^n \frac{b_k}{s - s_k}; \text{ cu polii } s_k : Q(s_k) = 0 \text{ si reziduurile : } b_k = \frac{R(s_k)}{Q'(s_k)}$$

$$C(s) = c_0 + c_1 s + c_2 s^2 + \dots + c_n s^n \xrightarrow{\mathcal{L}^{-1}} c_0 \delta(t) + c_1 \frac{d\delta}{dt} + c_2 \frac{d^2\delta}{dt^2} + \dots + c_n \frac{d^n\delta}{dt^n}$$

$$\mathcal{L}^{-1} \left[\frac{R(s)}{Q(s)} \right] = \mathcal{L}^{-1} \left[\sum_{k=1}^n \frac{b_k}{s - s_k} \right] = \sum_{k=1}^n b_k e^{s_k t}$$

Semnalul original este deci o suma de derivate ale impulsurilor Dirac plus o suma de exponentiale

Transformata Laplace inversa (cont)

Daca polii sunt complecsi, atunci ei sunt grupati in perechi complex-conjugate.

Pentru fiecare pereche: $a_k \pm j\omega_k$

$$\mathcal{L}^{-1} \left[\frac{b_k}{s - s_k} + \frac{b_k^*}{s - s_k^*} \right] = b_k \exp(a_k t + j\omega_k t) + b_k^* \exp(a_k t - j\omega_k t) =$$

$$e^{a_k t} [(b_k \exp j\omega_k t) + (b_k \exp j\omega_k t)^*] = 2e^{a_k t} \operatorname{Re}[b_k \exp j\omega_k t] = 2e^{-t/\tau_k} |b_k| \cos(\omega_k t + \varphi_k)$$

semnalul original este oscilant amortizat. Pulsatia sa este egala cu parte imaginara iar constanta de timp este inversa opusului partii imaginare a polului. $\varphi_k = \arg(b_k)$

Daca polii sunt dubli, atunci semnalul original este de tip rampa.

$$X(s) = \mathcal{L}[te^{-at}] = \frac{1}{(s+a)^2} \Leftrightarrow x(t) = \mathcal{L}^{-1} \left[\frac{1}{(s+a)^2} \right] = te^{-at} \Rightarrow \mathcal{L}^{-1} \left[\frac{1}{s^2} \right] = t$$

Transformata Laplace inversa in cazul polilor multipli este data de relatia lui Heaviside

$$x(t) = \mathcal{L}^{-1} \left[\frac{R(s)}{Q(s)} \right] = \sum_{k=1}^n \sum_{j=1}^{m_k} b_{kj} \frac{t^{j-1}}{(j-1)!} e^{s_k t}; b_{kj} = \frac{1}{(m_k - j)!} \left[\frac{d^{(m_k - j)}}{ds^{(m_k - j)}} \frac{(s - s_k)^{m_k} R(s)}{Q(s)} \right]_{s=s_k}$$

Daca $X(s)$ are polii in semiplanul stang atunci $x(t)$ este marginit si tinde catre 0.

5.7. Similitudinea intre ecuatiile regimurilor armonic si tranzitoriu

Forma operationala a ecuatiilor lui Kirchhoff:

$$\sum_{k \in (n)} i_k = 0; \quad \sum_{k \in [b]} u_k = 0 \Rightarrow \sum_{k \in (n)} I_k(s) = 0; \quad \sum_{k \in [b]} U_k(s) = 0; \quad \text{cu } I_k(s) = \mathcal{L}[i_k(t)], \quad U_k(s) = \mathcal{L}[u_k(t)]$$

In cazul elementelor dipolare liniare \rightarrow **forma operationala a relatiei lui Ohm:**

$$u = zi \Rightarrow U(s) = Z(s)I(s); \quad i = yu \Rightarrow I(s) = Y(s)U(s)$$

unde $Z(s)$ este **impedanta operationala** si $Y(s) = 1/Z(s)$ - **admitanta operationala**

In cazul elementelor liniare multipolare controlate hibrid:

$$\begin{bmatrix} \mathbf{U}_d \\ \mathbf{I}_d \end{bmatrix} = \begin{bmatrix} \mathbf{Z}' & \mathbf{A} \\ \mathbf{B} & \mathbf{Y}' \end{bmatrix} \begin{bmatrix} \mathbf{I}_a \\ \mathbf{U}_a \end{bmatrix}$$

$\mathbf{Y}(s)$ – vectorul semnalelor op. de iesire

$\mathbf{X}(s)$ – vectorul semnalelor op. de intrare

$\mathbf{H}(s)$ – matrice hibrida operationala

$$\mathbf{Y}(s) \quad \mathbf{H}(s) \quad \mathbf{X}(s) \Leftrightarrow \mathbf{Y}(s) = \mathbf{H}(s)\mathbf{X}(s)$$

In cazul elementelor reactive, teoreme derivarii obliga ca la trecerea in operational sa se tina cont si de conditiile initiale.

Ecuatiile operationale ale elementelor ideale

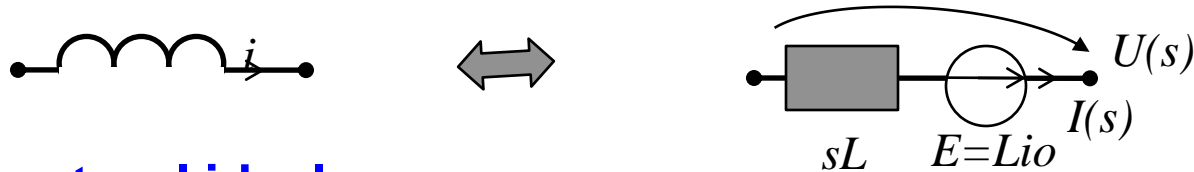
Rezistorul ideal:

$$u = Ri \Rightarrow U(s) = RI(s) \Rightarrow \boxed{Z(s) = R} \Leftrightarrow Y(s) = 1/R = G$$

Bobina ideala:

$$u = Ldi/dt \Rightarrow U(s) = sLI(s) - Li_0 \Rightarrow \boxed{Z(s) = sL; E(s) = Li_0}$$

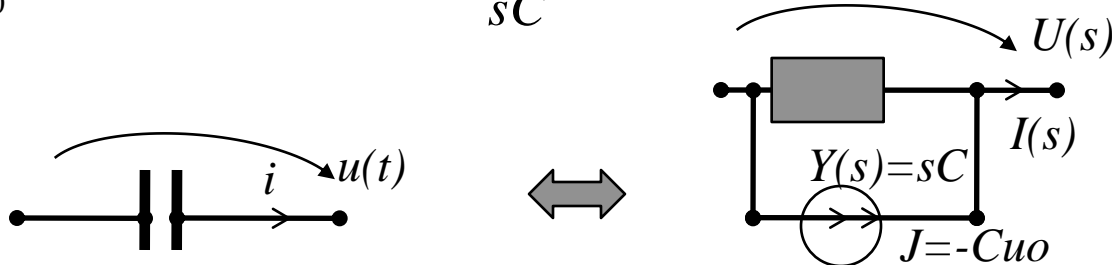
$$i = i_0 + \frac{1}{L} \int_0^t u(t') dt' \Rightarrow I(s) = i_0 + \frac{U(s)}{sL} \Rightarrow Y(s) = 1/(sL); J(s) = i_0/s$$



Condensatorul ideal:

$$i = Cdu/dt \Rightarrow I(s) = sCU(s) - Cu_0 \Rightarrow \boxed{Y(s) = sC; J(s) = -Cu_0}$$

$$u = u_0 + \frac{1}{C} \int_0^t i(t') dt' \Rightarrow U(s) = u_0 + \frac{I(s)}{sC} \Rightarrow Z(s) = 1/(sC); E(s) = u_0/s$$



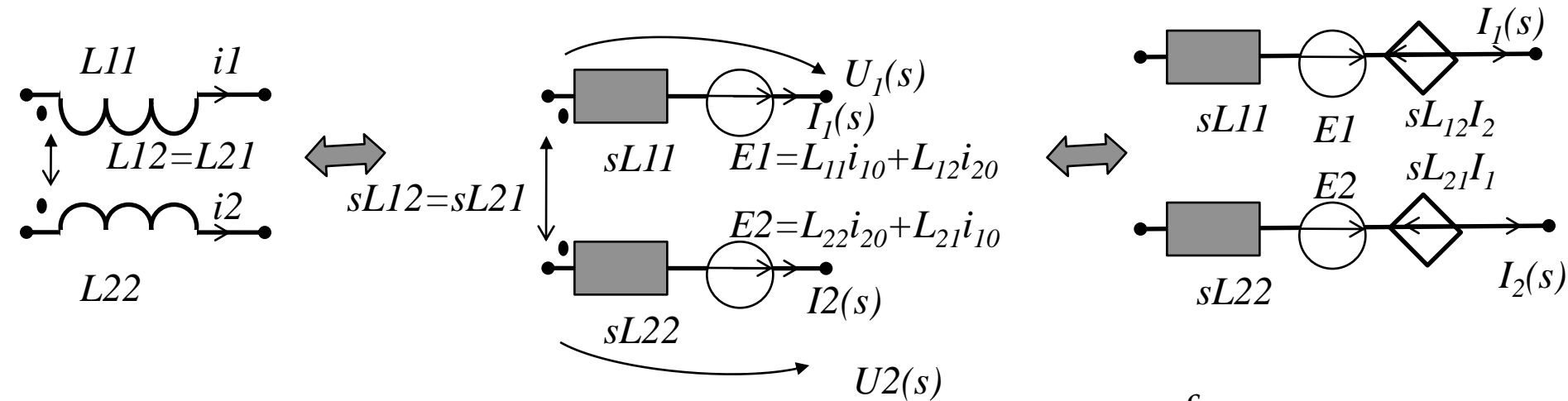
Ecuatiile operationale ale elementelor ideale (cont.)

Bobinele cuplate:

$$\mathbf{u} = \mathbf{L}d\mathbf{i} / dt \Rightarrow \boxed{\mathbf{U}(s) = s\mathbf{L}\mathbf{I}(s) - \mathbf{L}\mathbf{i}_0} \Rightarrow$$

$$\mathbf{Z}(s) = s\mathbf{L}; \mathbf{E}(s) = \mathbf{L}\mathbf{i}_0; \mathbf{Z} = \mathbf{Z}^T; \text{echiv cu perechi de SUCI}$$

Fiecare bobina are inseriata o sursa echiv. c.i. cu tem op egala cu fluxul initial.



Sursele independente:

$$u = e(t) \xRightarrow{\mathcal{L}} \boxed{U(s) = E(s)}$$

$$i = j(t) \xRightarrow{\mathcal{L}} \boxed{I(s) = J(s)}$$

Teorema similitudinii c.a. – tranzitoriu operational.

Se considera un circuit linear de c.a. si unul in regim tranzitoriu cu laturi surse reale de tensiune (SRT) sau de curent (SRC). Prin reprezentare in operational:

cele doua circuite au ecuatiile similare:

locul impedantelor/admitantelor complexe

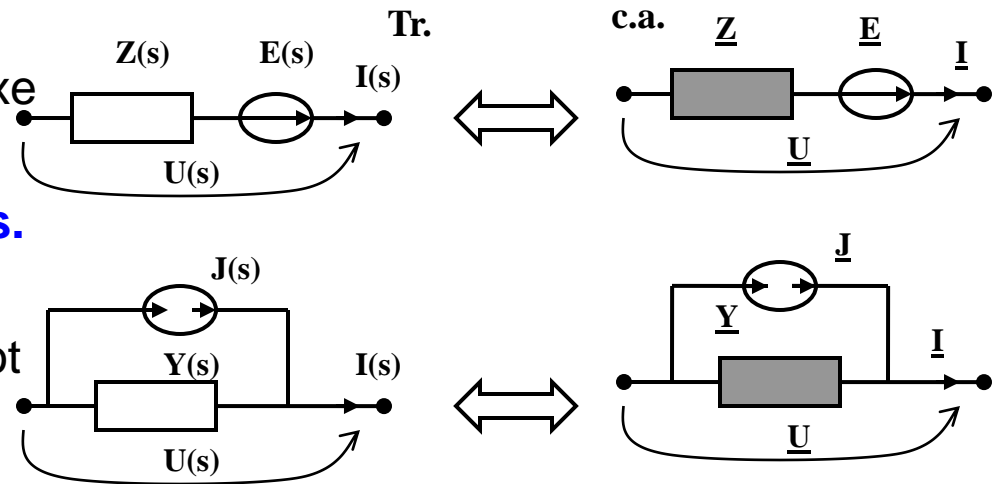
este luat de impedantele/admitantele

operationale. **Locul lui $j\omega$ este luat de s .**

Conditiiile initiale sunt reprezentate prin

surse echivalente iar bobinele cuplate pot

fi Inlocuite de surse comandate.



Circuitul in regim tranzitoriu repr. operational

Circuitul de c.a.

$$TK1: \sum_{k \in (n)} I_k(s) = 0; \quad TK2: \sum_{k \in [b]} U_k(s) = 0;$$

$$SRT: U_k(s) = Z_k(s)I_k(s) - E_k(s); \quad SRC: I_k(s) = Y_k(s)U_k(s) + J_k(s)$$

$$SUCI: E_k(s) = Z_{km}(s)I_m(s), \quad SICU: J_k(s) = Y_{km}(s)U_m(s),$$

$$SUCU: E_k(s) = A_{km}(s)U_m(s), \quad SICI: J_k(s) = B_{km}(s)I_m(s).$$

$$TK1: \sum_{k \in (n)} \underline{I}_k = 0; \quad TK2: \sum_{k \in [b]} \underline{U}_k = 0;$$

$$SRT: \underline{U}_k = \underline{Z}_k \underline{I}_k - \underline{E}_k; \quad SRC: \underline{I}_k = \underline{Y}_k \underline{U}_k + \underline{J}_k$$

$$SUCI: \underline{E}_k = \underline{z}_{km} \underline{I}_m, \quad SICU: \underline{J}_k = \underline{y}_{km} \underline{U}_m,$$

$$SUCU: \underline{E}_k = \underline{a}_{km} \underline{U}_m, \quad SICI: \underline{I}_k = \underline{b}_{km} \underline{I}_m.$$

Etapele analizei operationale prin transformata Lapalace

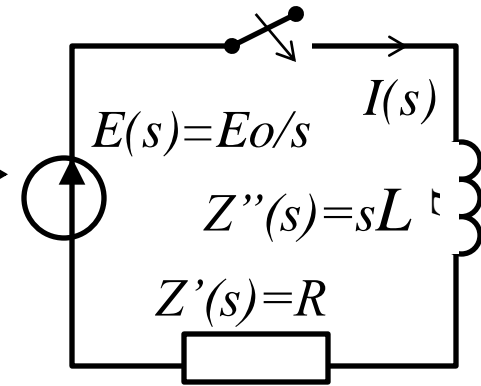
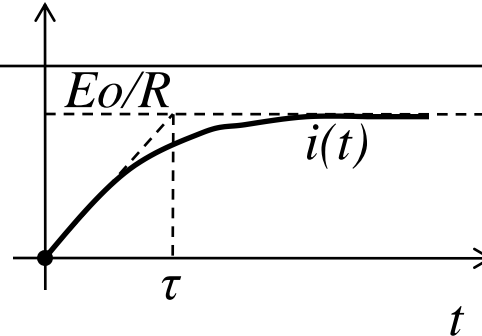
- Se determina transformatele Lapalce ale tuturor t.e.m si respectiv c.e.m. ale surselor independente de tensiune respectiv de curent din circuit.
- Se determina valorile conditiilor initiale: curentii prin bobine si tensiunile la bornele condensatoarelor, analizand circuitul in regimul stationar anterior regimului tranzitoriu.
- Se calculeaza impedantele operationale ale tuturor elementelor liniare din circuit (R, L, M, C), care sunt functii rationale de s : $R, sL, sM, 1/(sC)$.
- Se alcatuieste circuitul operational ce contine sursele cu parametrii lor operationali, elementele liniare cu impedantele/admitantele lor si sursele ce reprezinta conditiile initiale (φ_{em} =fluxul unitial in bobine, c_{em} =sarcina initiala).
- Se analizeaza acest circuit cu una din metodele specifice circuitelor electrice liniare de c.c. sau c.a. si se determina expresiile operationale ale curentilor si tensiunilor din laturile circuitului. Rezultatele vor fi functii rationale de s .
- Se determina solutia problemei, calculand transformatele Lapalce inverse ale curentilor si tensiunilor operationale cu relatia lui Heaviside sau din tabel. Functia MATLAB "residue" calculeaza polii si reziduurile unei functii rationale.
- Se verifica solutia, determinand valorile initiale/finale ale variabilelor de stare.

Circuitul RL in regim tranzitoriu

$$I(s) = \frac{E_0}{s(R + sL)} = \frac{E_0/R}{s(1 + s\tau)}$$

$$\frac{E_0}{R} \left[\frac{1}{s} - \frac{\tau}{1 + s\tau} \right] \Rightarrow i(t) = \frac{E_0}{R} \left[1 - e^{-t/\tau} \right]$$

Regimul tranzitoriu dureaza 3-4 constante de timp $\tau = L/R$.



Circuitul serie RC in regim tranzitoriu

$$[Z(s) + 1/Y(s)]I(s) = E(s) \Rightarrow I(s) = \frac{E_0}{s(R + 1/sC)} = \frac{CE_0}{1 + sRC}$$

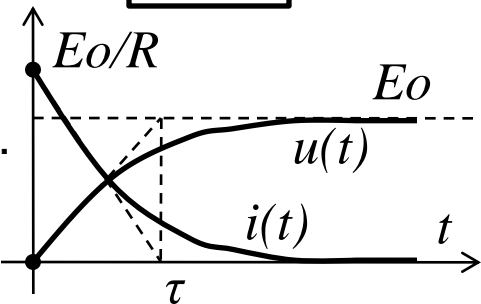
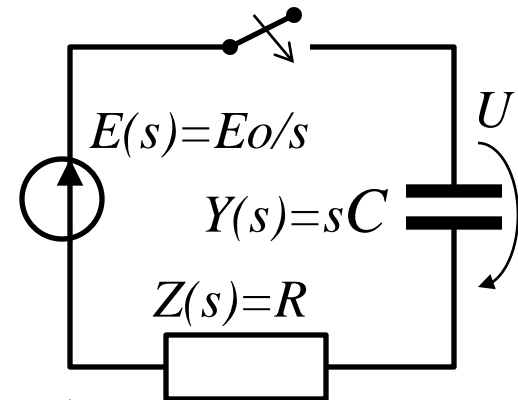
$$i(t) = \mathcal{L}^{-1}[I(s)] = \mathcal{L}^{-1}\left[\frac{E_0/R}{s + 1/\tau}\right] = \frac{E_0}{R} e^{-t/\tau}; \tau = RC;$$

$$U(s) = \frac{I(s)}{sC} = \frac{E_0}{s(\tau s + 1)} = E_0 \left[\frac{1}{s} - \frac{\tau}{(\tau s + 1)} \right] \Rightarrow u(t) = E_0(1 - e^{-t/\tau})$$

Incarcarea condensatorului dureaza 3-4 constante de timp $\tau = RC$.

In conditii initiale nenule (condensatorul incarcat cu u_0):

$$[Z(s) + 1/Y(s)]I(s) = E(s) - u_0/s \Rightarrow I(s) = (E_0 - u_0)/s(R + 1/sC)$$



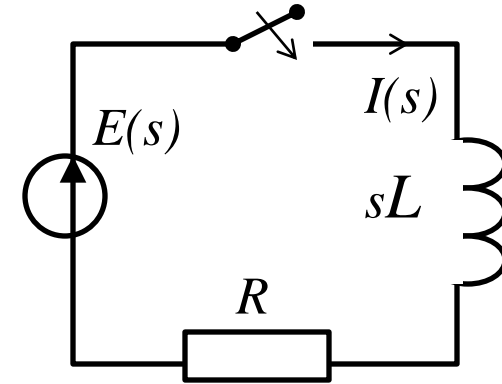
- Conectarea circuitului serie RL la o sursa de c.a.:**

$$e(t) = E_0 \sin \omega t \Rightarrow I(s) = \frac{E_0 \omega}{(s^2 + \omega^2)(R + sL)} = \frac{As + B}{s^2 + \omega^2} + \frac{C}{R + sL};$$

$$\Rightarrow A = -E_0 \omega L / (R^2 + L^2 \omega^2); B = -AR / L; C = -AL \Rightarrow$$

$$i(t) = \frac{E_0 \omega L}{R^2 + L^2 \omega^2} \left(\frac{1}{\omega \tau} \sin \omega t - \cos \omega t + e^{-t/\tau} \right); \tau = L / R$$

Solutia are un termen permanent – sinusoidal si unul tranzitoriu, care se stinge -exponential cu constanta de timp $\tau=L/R$.

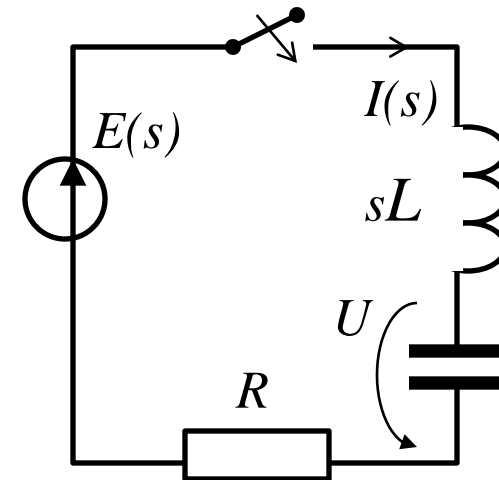


- Circuite de ordinul doi RLC**

$$e(t) = E_0 \Rightarrow I(s) = \frac{E_0}{s(R + sL + 1/(sC))} = \frac{E_0 C}{1 + sRC + s^2 LC};$$

$$i(t) = e^{\alpha t} [A \sin \omega t + B \cos \omega] \rightarrow I(s) = \frac{A \omega + B(s - \alpha)}{(s - \alpha)^2 + \omega^2} \Rightarrow$$

$$i(t) = \frac{E_0}{\omega L} e^{\alpha t} \sin \omega t; \text{ cu } \alpha = -R/(2L); \omega^2 = \sqrt{1/(LC) - \alpha^2}$$



Filtre pasive si active in regim tranzitoriu

Conform teoremei similitudinii ac-tr caracteristicile de frecventa ale filtrelor active si pasive se devin functiile de transfer in regim tranzitoriu – operational, daca se substituie $j\omega$ cu s :

- filtrul TJ de ordinul unu realizat cu un circuit RC: $\underline{A}_u = \frac{1}{1 + j\omega RC} \Rightarrow A_u(s) = \frac{1}{1 + sRC}$

- filtrul TJ de ordinul doi, realizat cu un circuit RLC:

$$\underline{A}_u = \frac{1}{1 + j\omega RC - \omega^2 LC} \Rightarrow A_u(s) = \frac{1}{1 + sRC + s^2 LC}$$

- filtrul activ trece banda de ordinul patru

$$\underline{A}_u(\omega) = -1 / (R_1 + j\omega L_1 + 1 / (j\omega C_1)) (G_2 + j\omega C_2 + 1 / (j\omega L_2))$$

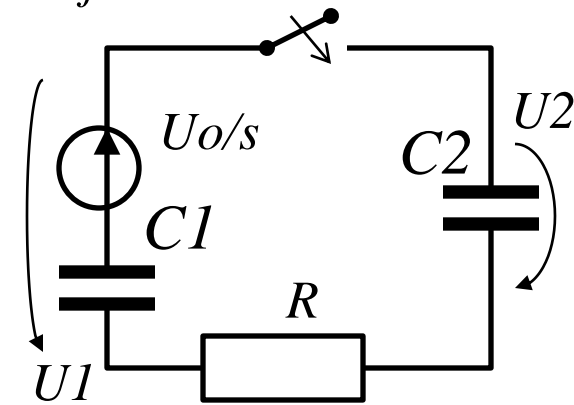
$$\Rightarrow A_u(s) = -1 / (R_1 + sL_1 + 1 / (sC_1)) (G_2 + sC_2 + 1 / (sL_2))$$

Incarcarea unui condensator de la altul:

$$I(s) = \frac{U_{10}}{s[R + 1/(sC_1) + 1/(sC_2)]} = \frac{U_{10}}{sR + 1/C_1 + 1/C_2} \xrightarrow{R=0} \frac{U_{10}}{1/C_1 + 1/C_2} \xrightarrow{L^{-1}} i(t) = \frac{U_{10}\delta(t)}{1/C_1 + 1/C_2}$$

$$U_2(s) = I(s) / (sC_2) = \frac{U_{10}}{sC_2(sR + 1/C_1 + 1/C_2)} \xrightarrow{R=0} \frac{U_{10}}{s(C_2/C_1 + 1)} \xrightarrow{L^{-1}} u_2(t) = \frac{U_{10}h(t)}{C_2/C_1 + 1}$$

In cazul $R=0$ are loc instantaneu, printr-un puls nemarginit de curent!



Recapitularea principiilelor formule ale analizei in frecventa

In curent alternativ:

Reprezentarea complexa :

$$x(t) = X \sqrt{2} \sin(\omega t + \varphi) \xrightarrow{\mathcal{C}} \underline{X} = X e^{j\varphi}$$

$$\mathcal{C}[\lambda_1 x_1(t) + \lambda_2 x_2(t)] = \lambda_1 \underline{X}_1 + \lambda_2 \underline{X}_2$$

$$\mathcal{C}[dx(t)/dt] = j\omega \underline{X}$$

$$\text{Impedanta complexa : } \underline{Z} = R + jX = Z e^{j\varphi}$$

$$\text{Rezistorul: } \underline{Z}_R = R$$

$$\text{Bobina : } \underline{Z}_L = j\omega L$$



$$\text{Condensatorul: } \underline{Z}_C = \frac{1}{j\omega C}$$

$$\text{Cuplajul: } \underline{Z}_M = j\omega M$$

Puterea complexa, activa, reactiva, aparenta :

$$S = UI^* = P + jQ = S e^{j\varphi}$$

In tranzitoriu – operational

Transformata Laplace

$$F(s) = \mathcal{L}[f(t)] = \int_{-0}^{\infty} f(t) e^{-st} dt$$

$$\lambda_1 f_1(t) + \lambda_2 f_2(t) \xrightarrow{\mathcal{L}} \lambda_1 F_1(s) + \lambda_2 F_2(s)$$

$$df(t)/dt \xrightarrow{\mathcal{L}} sF(s) - f(0)$$

$$\text{Impedanta op.: } Z(s) = U(s)/I(s)$$

$$\text{Rezistorul: } Z_R = R$$

$$\text{Bobina : } Z_L = sL$$

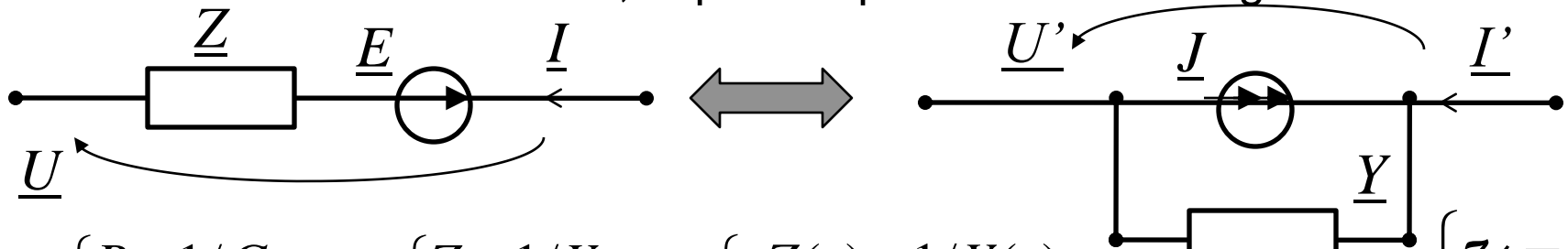
$$\text{Condensatorul: } Z_C = \frac{1}{sC}$$

$$\text{Cuplajul: } Z_M = sM$$

5.8. Teoreme de echivalenta pentru surse

• Teorema generatoarelor liniare (Thevenin-Norton) echivalente

Doua surse afine, una reala de tensiune (Thevenin) si alta reala de curent (Norton) sunt echivalente, daca au aceeasi tensiune de mers in gol si acelasi curent de scurtcircuit, implicit impedante interne egale.



$$\begin{array}{l}
 \text{- c.c.:} \left\{ \begin{array}{l} R = 1/G \\ E = RJ \end{array} \right. \quad \text{c.a.:} \left\{ \begin{array}{l} Z = 1/Y \\ E = ZJ \end{array} \right. \quad \text{tr.:} \left\{ \begin{array}{l} Z(s) = 1/Y(s) \\ E(s) = Z(s)J(s) \end{array} \right.
 \end{array}$$

In general : $\begin{cases} \mathcal{Z} = \mathcal{Y}^{-1} \\ e = \mathcal{Z}j \end{cases}$

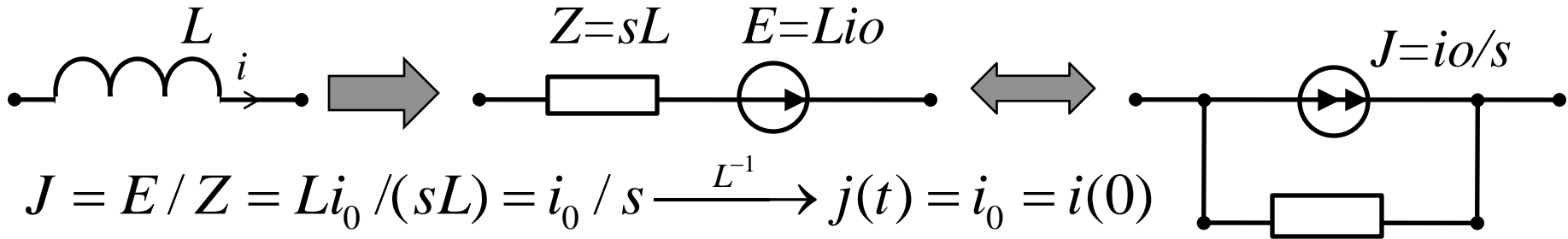
Datorita teoremei de similitudine este suficienta o singura demonstratie.

$$u = \mathcal{Z}i + e; \quad i' = \mathcal{Y}u' - j' \Rightarrow \mathcal{Z}i' = \mathcal{Z}\mathcal{Y}u' - \mathcal{Z}j' \Rightarrow u' = \mathcal{Z}i' + \mathcal{Z}j' = u' = \mathcal{Z}i' + e$$

In conditiile teoremei, cele doua surse reale au relatia u-i de tip afin identica, deci ele sunt echivalente. In conditiile in care impedanta interna este liniara si inversabila, orice sursa reala de curent se poate transfigura in una de tensiune si invers.

Teorema surselor echivalente conditiilor initiale

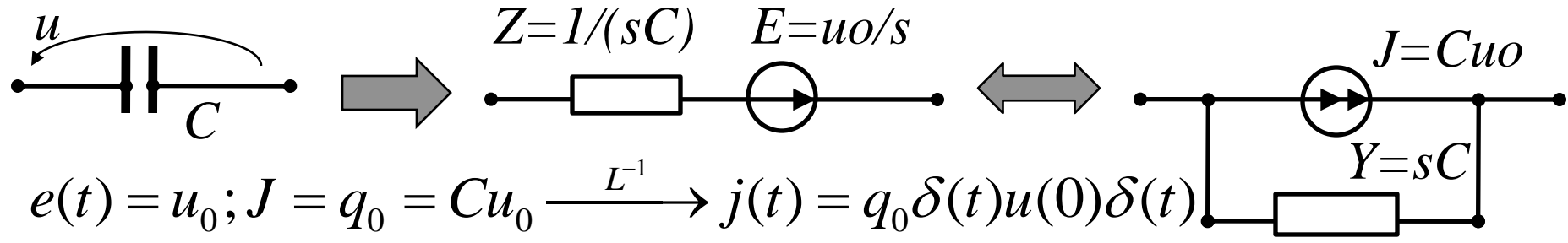
Elementele acumuloare de energie – bobine si condensatoare au circuite echivalente alcatuite din elementele L, C liniare (cu conditii initiale nule) si surse echivalente ce reprezinta conditiile lor initiale. In baza teoremei anterioare, aceste circuite pot fi surse reale te tensiune sau de curent, echivalente intre ele (cu e si j de tip treapta Heaviside sau impuls Dirac):



$$J = E / Z = Li_0 / (sL) = i_0 / s \xrightarrow{L^{-1}} j(t) = i_0 = i(0)$$

$$E = Li_0 \xrightarrow{L^{-1}} e(t) = \varphi_0 \delta(t) = Li(0) \delta(t)$$

$$Y = 1/(sL)$$



$$e(t) = u_0; J = q_0 = Cu_0 \xrightarrow{L^{-1}} j(t) = q_0 \delta(t) u(0) \delta(t)$$

$$J = Cu_0$$

$$Y = sC$$

Atentie la orientarile sensurilor de referinta si semne!

Teoremele lui Vaschy

- Teorema lui Vaschy, a surselor de tensiune cu actiune nula:**

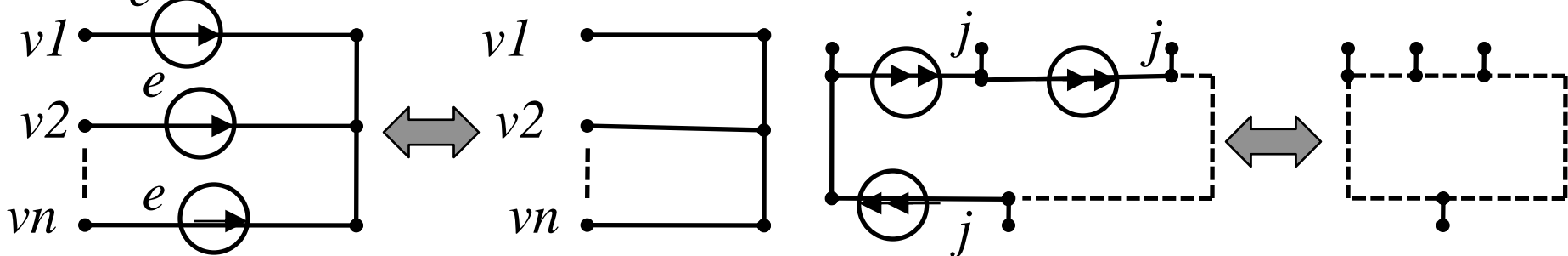
Un circuit multipolar cu graf de tip stea, avand laturile surse ideale de tensiune identice si orientate identic este echivalent cu un circuit stea cu toate laturile conductoare perfecte. In consecinta, daca in laturile care concura la orice nod al unui circuit se inseriaza surse ideale de tensiune identice si orientate identic fata de nod, atunci curentii din circuit nu se modifica.

Demo: $u_{kn} = e_k - e_n = e - e = 0$, pt. $k = 1, 2, \dots, n-1$

- Teorema lui Vaschy, a surselor de curent cu actiune nula:**

Un circuit multipolar cu graf de tip bucla, avand laturile surse ideale de curent identice si orientate identic este echivalent cu o bucla cu toate laturile izolatoare perfecte. In consecinta, daca in paralel cu laturile oricarei bucle dintr-un circuit se conecteaza surse ideale de curent identice si orientate identic, atunci tensiunile din laturile circuitului nu se modifica.

Demo: $i_k = j_k - j_{k+1} = j - j = 0$, pt. $k = 1, 2, \dots, n-1$



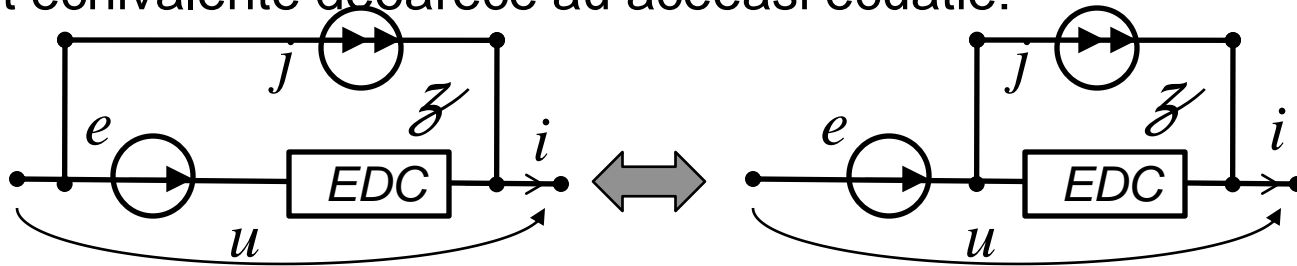
3.9. Teoreme de echivalenta pentru surse (cont)

• Teorema de echivalenta a surselor E-J

Circuitele din figura sunt echivalente deoarece au aceeași ecuație:

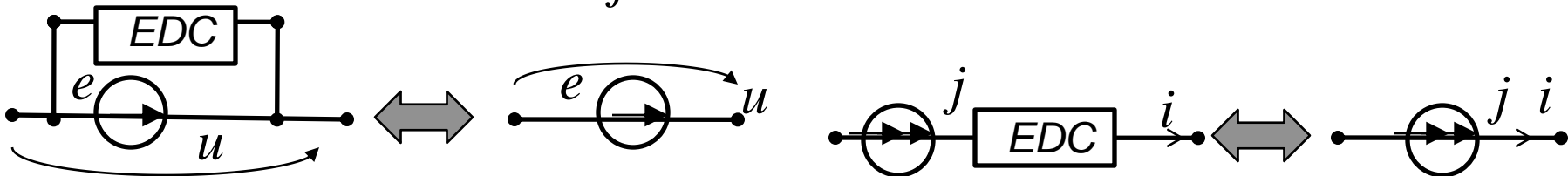
$$u = -e + Z(i - j)$$

indiferent de elementul dipolar EDC.



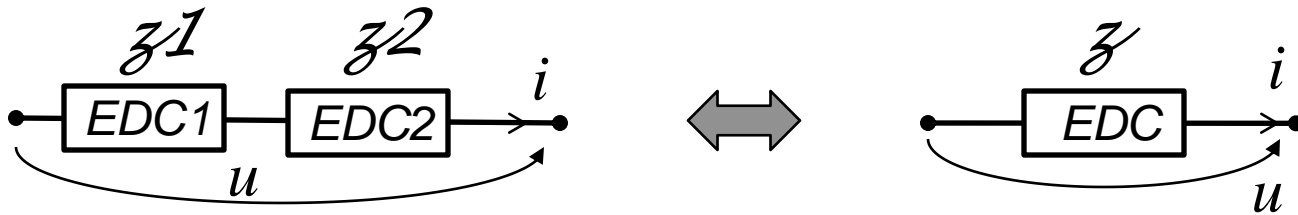
- **Teorema de echivalenta a surselor ideale de tensiune:** O sursă ideală de tensiune e conectată în paralel cu un element compatibil cu ea este echivalentă cu sursă ideală de tensiune: $u=e$. Elementele controlate în tensiune (inclusiv sursele ideale de curent) sunt compatibile cu e , în schimb alte surse de tensiune nu sunt compatibile.

- **Teorema de echivalenta a surselor ideale de curent:** O sursă ideală de curent j conectată în serie cu un element compatibil cu ea este echivalentă cu sursă ideală de curent: $i=j$



Teoreme de echivalenta pentru conexiunea serie

Doua EDC sunt conectate serie daca sunt parcurse de acelasi curent (au un terminal comun, care este nod neramificat).



Elemente dipolare liniare sau nu, controlate in curent, conectate in serie sunt echivalente cu un element cu impedanta egala cu suma impedantelor:

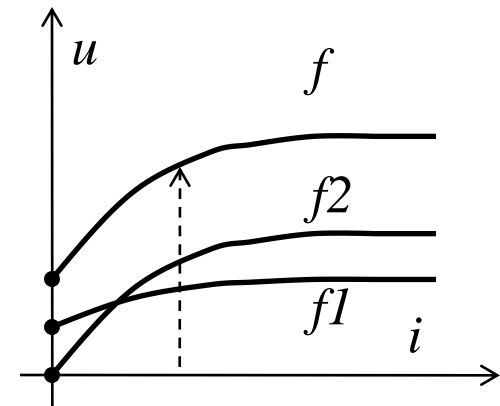
$$u = u_1 + u_2 = \mathcal{Z}_1 i + \mathcal{Z}_2 i = \mathcal{Z} i \quad \forall i \Rightarrow \boxed{\mathcal{Z} = \mathcal{Z}_1 + \mathcal{Z}_2}$$

In general. Pentru n elemente:

$$u = u_1 + u_2 + \dots = \sum_{k=1}^n \mathcal{Z}_k i = \mathcal{Z} i \Rightarrow \boxed{\mathcal{Z} = \sum_{k=1}^n \mathcal{Z}_k}$$

In particular:

$$\text{c.c.: } R = R_1 + R_2; \quad E = E_1 + E_2;$$

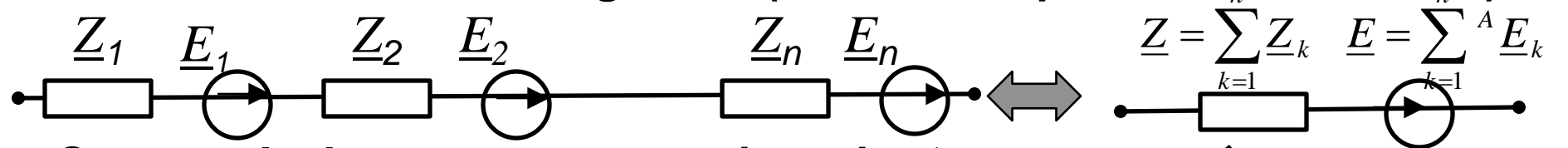


Rezistoare neliniare : $f(i) = f_1(i) + f_2(i)$; se aduna vertical in planul $u-i$ (V - A)

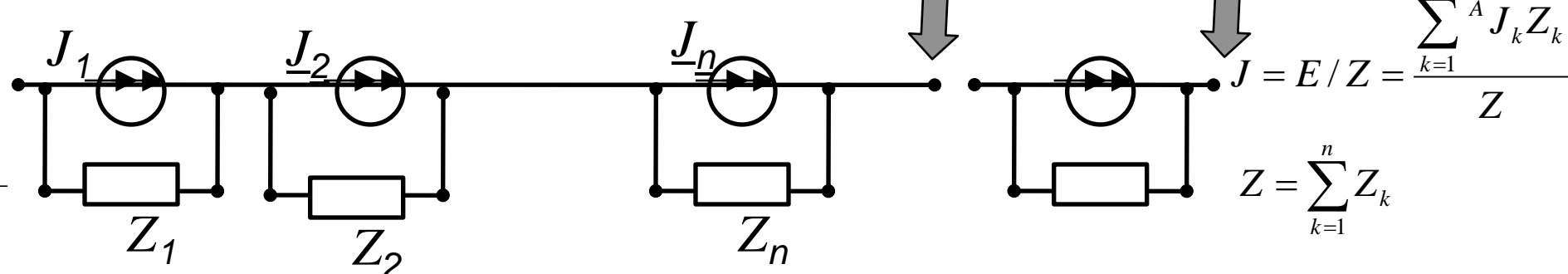
$$\text{c.a.: } \underline{Z} = \underline{Z}_1 + \underline{Z}_2; \quad \underline{E} = \underline{E}_1 + \underline{E}_2; \quad \text{Tr.: } Z(s) = Z_1(s) + Z_2(s)$$

- Surse reale de tensiune conectate in serie

Atentie ! Suma t.e.m. este algebrica (unele surse pot fi orientate invers)



- Surse reale de curent conectate in serie



- Rezistente, bobine, condensatoare si surse ideale de tensiune inseriate

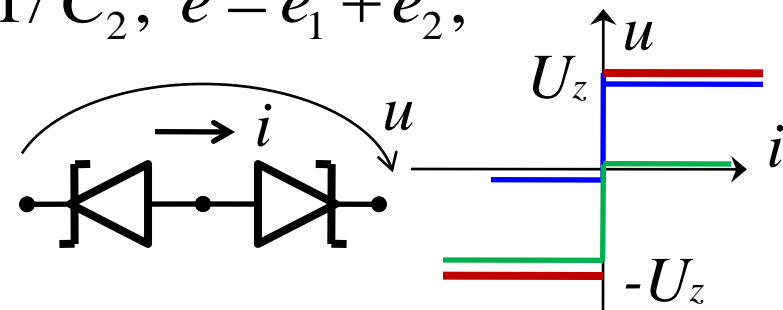
$$R = R_1 + R_2; \quad L = L_1 + L_2; \quad 1/C = 1/C_1 + 1/C_2; \quad e = e_1 + e_2;$$

- Limitatorul de tensiune cu diode Zener

Se aduna cele doua functii

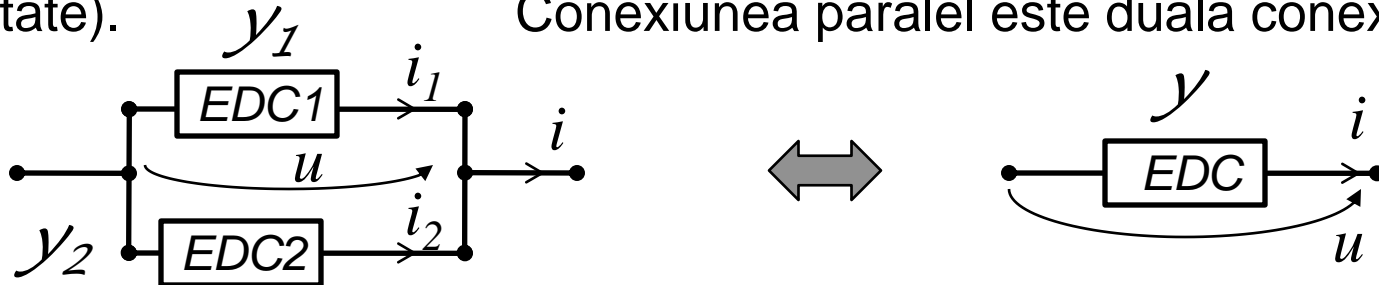
caracteristice idealizate (rezultatul este

Reprezentat cu rosu):



3.10. Teoreme de echivalenta pentru conexiunea paralel

Doua EDC sunt conectate paralel daca au tensiune identica (au un terminalele conectate). Conexiunea paralel este duala conexiunii serie.



Elemente dipolare liniare sau neliniare controlate in curent, conectate in paralel sunt echivalente cu un element cu admitanta egala cu suma admitantelor:

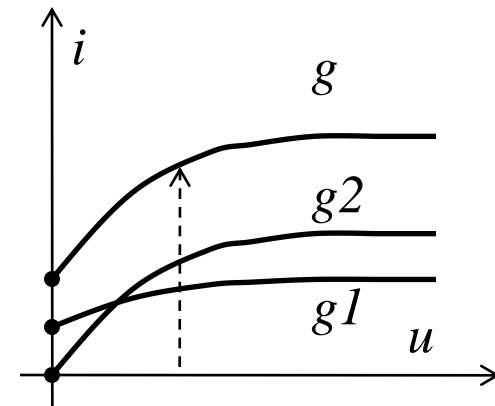
$$i = i_1 + i_2 = y_1 u + y_2 u = y u \quad \forall u \Rightarrow y = y_1 + y_2$$

In general, pentru n elemente:

$$i = i_1 + i_2 + \dots = \sum_{k=1}^n y_k u = y u \Rightarrow y = \sum_{k=1}^n y_k$$

In particular:

$$\text{c.c.: } G = G_1 + G_2; \quad J = J_1 + E_{J2};$$

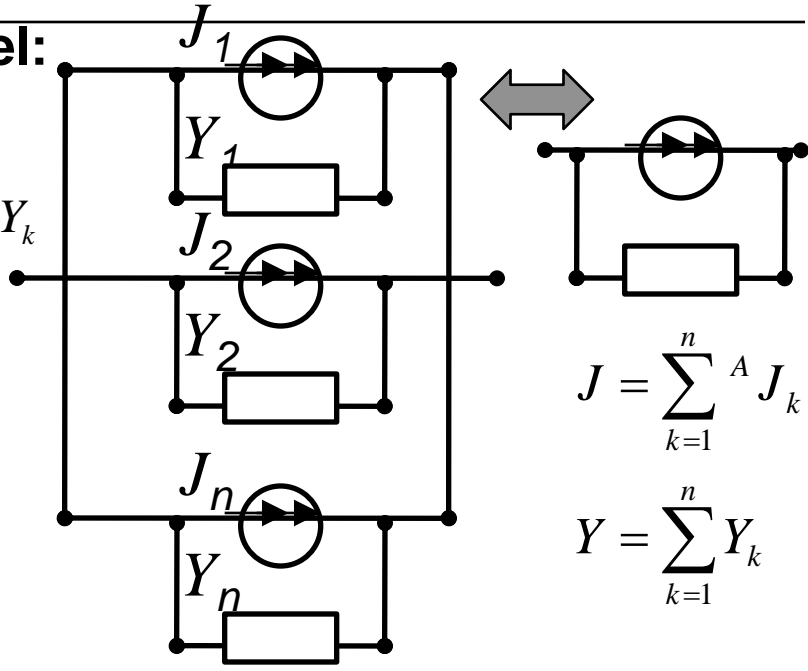
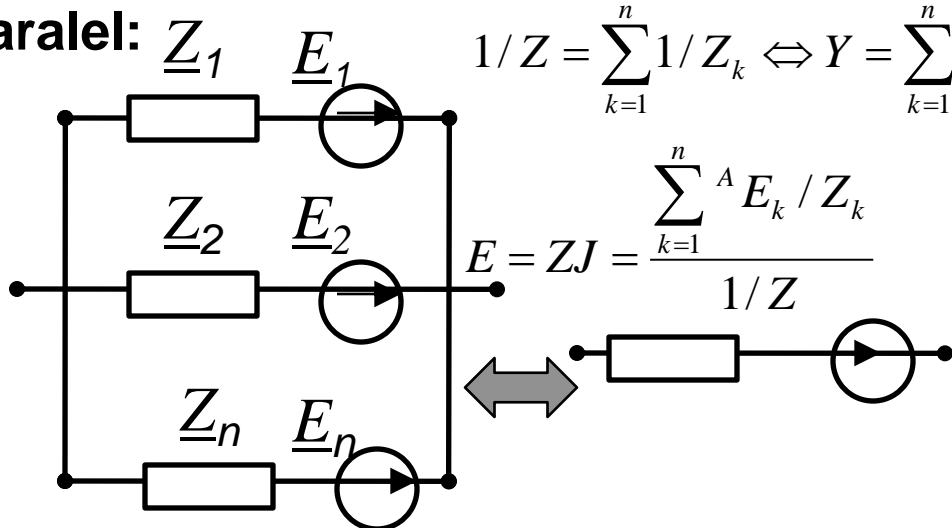


Rezistoare neliniare : $g(u) = g_1(u) + g_2(u)$; se aduna vertical in pl. $i - u$ (A - V)

sau orizontal in pl. $u - i$. c.a.: $\underline{Y} = \underline{Y}_1 + \underline{Y}_2; \underline{J} = \underline{J}_1 + \underline{J}_2; \text{Tr.: } Y(s) = Y_1(s) + Y_2(s)$

- Surse reale de curent conectate in paralel:
- Surse reale de tensiune conectate

in paralel:

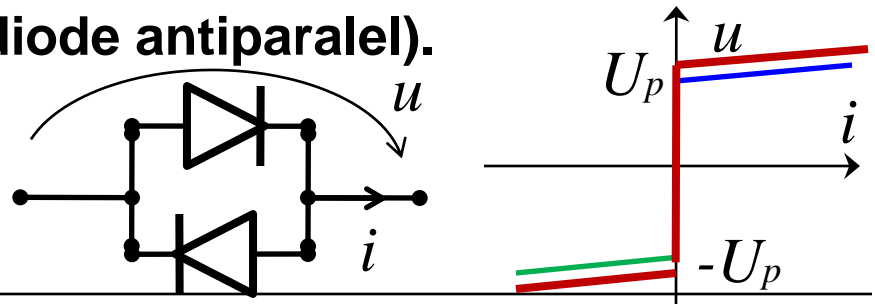


- Rezistente, bobine, condensatoare si surse ideale de curent in paralel

$G = G_1 + G_2 \Leftrightarrow R = R_1 R_2 / (R_1 + R_2); 1/L = 1/L_1 + 1/L_2; C = C_1 + C_2; j = j_1 + j_2$

- Limitatorul de tensiune (protectie cu diode antiparalel).

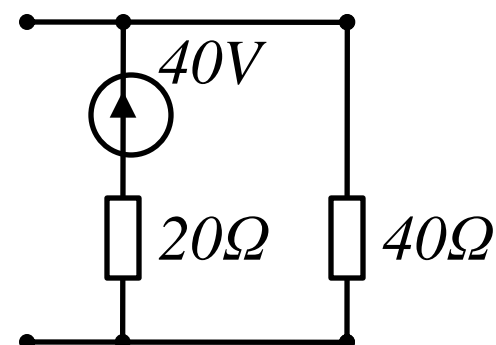
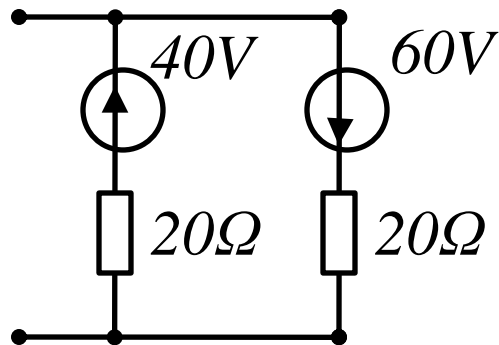
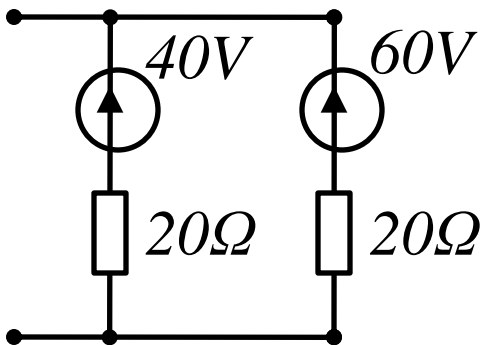
Tensiunea este limitata la nivelul tensiunii de prag a diodelor (cca 0.6V).



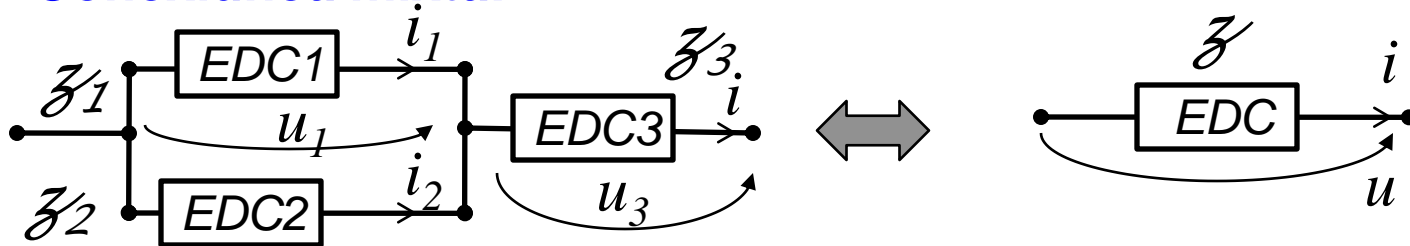
- Alcatuiti un tabel patrat (7x7), in care scrieti **relatiile de echivalenta pentru toate combinatiile serie si paralel** dintre elementele: rezistor, conductor perfect, izolator pefect, sursa ideala de tensiune (SIT), sursa ideala de curent (SIC), sursa real de tensiune (SRT) si sursa reala de curent (SRC).
- O echivalare utila in practica este cea a doua **surse reale de tensiune conectate in paralel**. Aratati ca sursa echivalenta are parametrii:

$$E = \frac{E_1 Z_2 + E_2 Z_1}{Z_1 + Z_2}; Z = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

- Determinati sursele echivalente ale circuitelor:

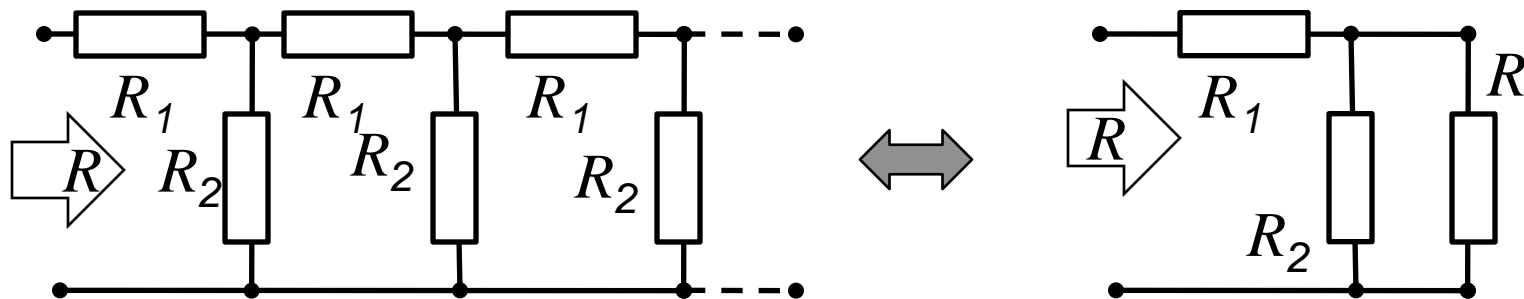


- **Conexiunea mixta:**



$$Z_{12} = (Z_1^{-1} + Z_2^{-1})^{-1} \Rightarrow Z = Z_{12} + Z_3 = (Z_1^{-1} + Z_2^{-1})^{-1} + Z_3 \Rightarrow c.c.: R = R_3 + \frac{R_1 R_2}{R_1 + R_2}$$

- **Rezistenta (impedanta) R de intrare a unui circuit in scara infinita:**



$$R = R_1 + RR_2 / (R + R_2) \Rightarrow (R + R_2)R = R_1(R + R_2) + RR_2 \Rightarrow$$

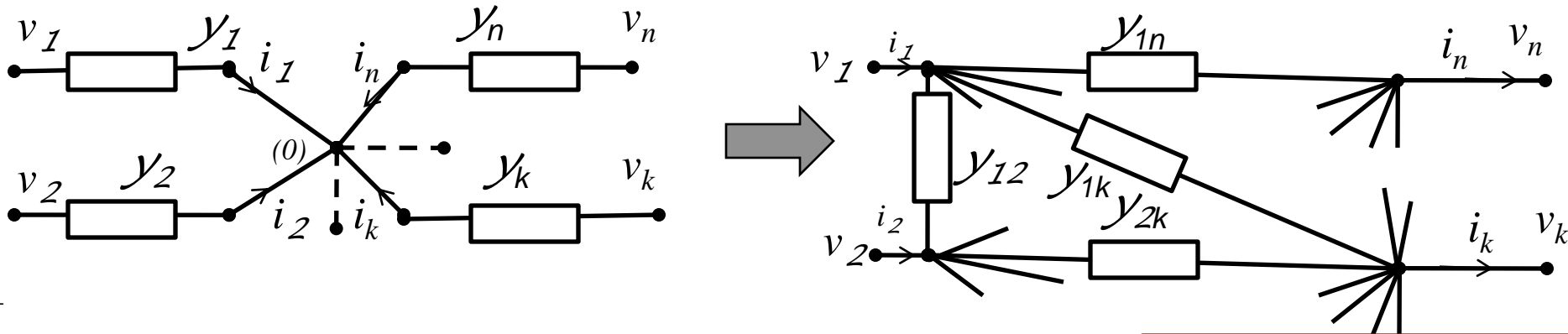
$$R^2 - R_1R - R_1R_2 = 0 \Rightarrow R = \left(R_1 \pm \sqrt{R_1^2 + 4R_1R_2} \right) / 2 \Rightarrow R = R_1 / 2 + \sqrt{R_1^2 / 4 + R_1R_2}$$

- Definiti un **limbaje eficient de descriere a circuitelor de c.a.** cu conexiune mixta oricat de complicata. Implementati un interpretor pentru acest limbaj.

3.11. Teoreme de echivalenta pentru conexiunile stea, triunghi, poligon complet

• Teorema transfigurarii stea-poligon complet

Orice circuit multipolar linear cu topologie stea este echivalent cu un circuit cu topologie poligon complet, in care admitantele laturile au expresiile de mai jos.



$$(0) : \sum_{k=1}^n i_k = 0 \Rightarrow \sum_{k=1}^n y_k (v_k - v_0) = 0 \Rightarrow \sum_{k=1}^n y_k v_k = \left(\sum_{k=1}^n y_k \right) v_0 \Rightarrow v_0 = \left(\sum_{j=1}^n y_j \right)^{-1} \sum_{k=1}^n y_k v_k$$

• Teorema lui Millman: da expresia potentialului nodului central al stelei.

In circuitul stea:
$$i_k = y_k v_k - y_k \left(\sum_{l=1}^n y_l \right)^{-1} \sum_{j=1}^n y_j v_j$$

In poligonul complet:

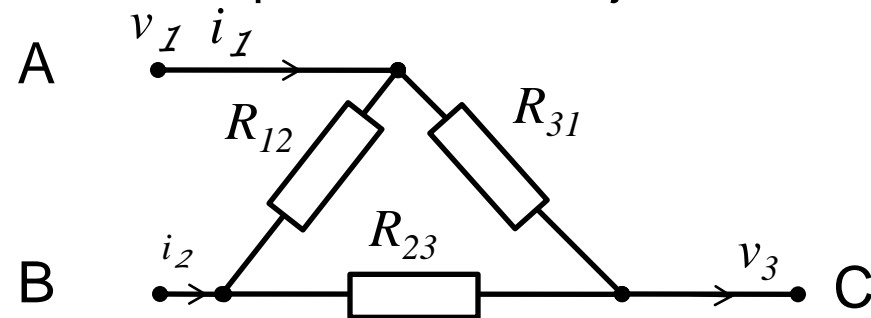
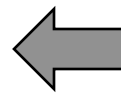
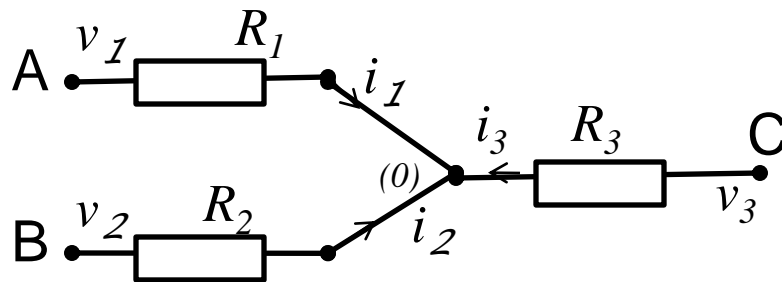
$$i_k = \sum_{\substack{j=1 \\ j \neq k}}^n i_{kj} = \sum_{\substack{j=1 \\ j \neq k}}^n y_{kj} (v_k - v_j) = \left(\sum_{\substack{j=1 \\ j \neq k}}^n y_{kj} \right) v_k - \sum_{\substack{j=1 \\ j \neq k}}^n y_{kj} v_j \Rightarrow y_{kj} = y_k \left(\sum_{l=1}^n y_l \right)^{-1} y_j \quad \text{cond. de echiv.}$$

Teoreme de echivalenta triunghi-stea

Transfigurarea din teorema anterioara nu se poate inversa in cazul general, deoarece numarul de laturi din poligonul complet $n(n-1)/2 > n$ – numarul laturilor stelei. In cazul $n=3$ aceste numere sunt egale si este valabila:

• Teorema transfigurarii triunghi-stea

Orice circuit multipolar linear cu topologie triunghi este echivalent cu un circuit cu topologie stea, in care admitantele laturile au expresiile de mai jos. echiv.



$$R_{AB} = R_1 + R_2 = R_{12} \parallel (R_{23} + R_{31})$$

$$R_{BC} = R_2 + R_3 = R_{23} \parallel (R_{12} + R_{31}) \Rightarrow R_1 + R_2 + R_3 = (R_{AB} + R_{BC} + R_{CA}) / 2 \Rightarrow$$

$$R_{CA} = R_3 + R_1 = R_{31} \parallel (R_{12} + R_{23})$$

$$R_1 = (R_{AB} + R_{BC} + R_{CA}) / 2 - R_{BC} = (R_{AB} - R_{BC} + R_{CA}) / 2 = \text{R2,R3 prin permutari circulare}$$

$$\frac{R_{12}(R_{23} + R_{31}) - R_{23}(R_{12} + R_{31}) + R_{31}(R_{12} + R_{23})}{2(R_{12} + R_{23} + R_{31})} = \frac{R_{12}R_{31}}{R_{12} + R_{23} + R_{31}} = R_1, R_2, R_3$$

- Potentialul punctului neutru al unui circuit stea cu si fara fir neutru**

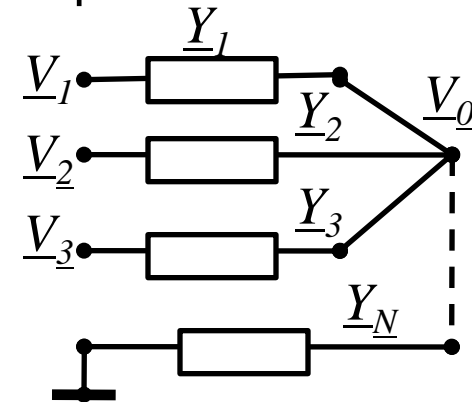
Are valoarea medie ponderata a potentialelor terminalelor cu ponderi admitantele. In c.a. reprezentat in complex:

$$\underline{V}_0 = (\underline{Y}_1 \underline{V}_1 + \underline{Y}_2 \underline{V}_2 + \underline{Y}_3 \underline{V}_3) / (\underline{Y}_1 + \underline{Y}_2 + \underline{Y}_3); \text{ fara f.n.}$$

$$\underline{V}_0 = (\underline{Y}_1 \underline{V}_1 + \underline{Y}_2 \underline{V}_2 + \underline{Y}_3 \underline{V}_3) / (\underline{Y}_1 + \underline{Y}_2 + \underline{Y}_3 + \underline{Y}_N); \text{ cu f.n.}$$

Daca admitantele sunt egale, media devine aritmetica:

$$\underline{V}_0 = (\underline{V}_1 + \underline{V}_2 + \underline{V}_3) / 3$$



- Transfigurarea rezistentelor stea-triunghi**

$$y_{kj} = y_k \left(\sum_{l=1}^n y_l \right)^{-1} y_j \Rightarrow G_{kj} = \frac{G_k G_j}{\sum_{l=1}^n G_l} \Rightarrow R_{12} = \frac{G_1 + G_2 + G_3}{G_1 G_2} = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_3}$$

In cazul rezistentelor egale:

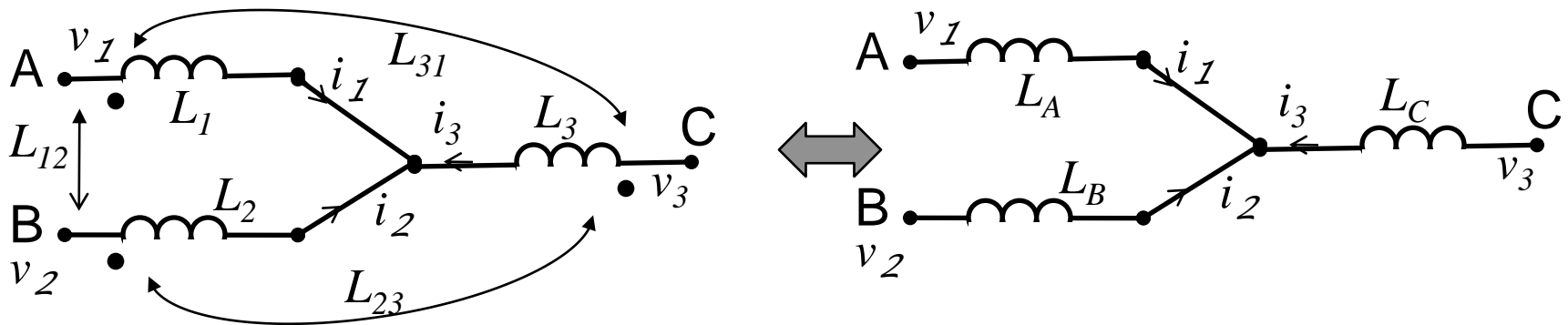
$$\boxed{R_{\Delta} = 3R_Y} \Leftrightarrow R_Y = R_{\Delta} / 3$$

Relatiile similare in celelalte regimuri (c.a.; tr) : $\underline{Z}_{\Delta} = 3\underline{Z}_Y$; $Z_{\Delta}(s) = 3Z_Y(s)$

3.12. Teoreme de echivalenta pentru bobinele cuplate

- Teorema eliminarii cuplajului:**

Trei bobine liniare cuplate mutual conectate in stea sunt echivalente cu trei bobine necuplate, in conditiile de mai jos:



Demonstratia se poate face in domeniul timpului sau in cel al frecventei.

$$\underline{U}_{AB} / (j\omega) = L_1 \underline{I}_1 - L_2 \underline{I}_2 + L_{31} \underline{I}_3 + L_{12} \underline{I}_2 - L_{23} \underline{I}_3 - L_{12} \underline{I}_1 =$$

$$L_1 \underline{I}_1 - L_{12} \underline{I}_1 - L_2 \underline{I}_2 + L_{12} \underline{I}_2 - (L_{31} - L_{23})(\underline{I}_1 + \underline{I}_2) =$$

$$L_1 \underline{I}_1 - L_2 \underline{I}_2 - (L_{31} - L_{23})(\underline{I}_1 + \underline{I}_2) = (L_1 - L_{12} + L_{23} - L_{31}) \underline{I}_1 +$$

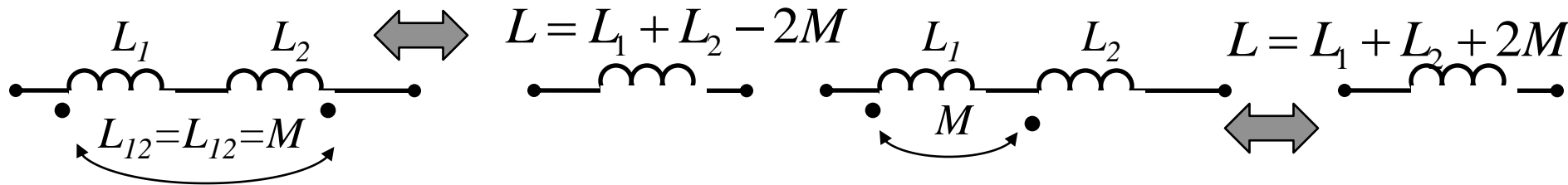
$$(-L_2 + L_{12} + L_{23} - L_{31}) \underline{I}_2 - (L_{31} - L_{23})(\underline{I}_1 + \underline{I}_2) = L_A \underline{I}_1 - L_B \underline{I}_2 \Rightarrow$$

$$L_A = L_1 + L_{23} - L_{12} - L_{31}; \quad L_B = L_2 + L_{31} - L_{12} - L_{23}; \quad L_C = L_3 + L_{12} - L_{23} - L_{31}$$

Teoremele de echivalenta ale bobinelor cuplate conectate serie/paralel

• Teorema de echivalenta pentru bobinele cuplate inseriate

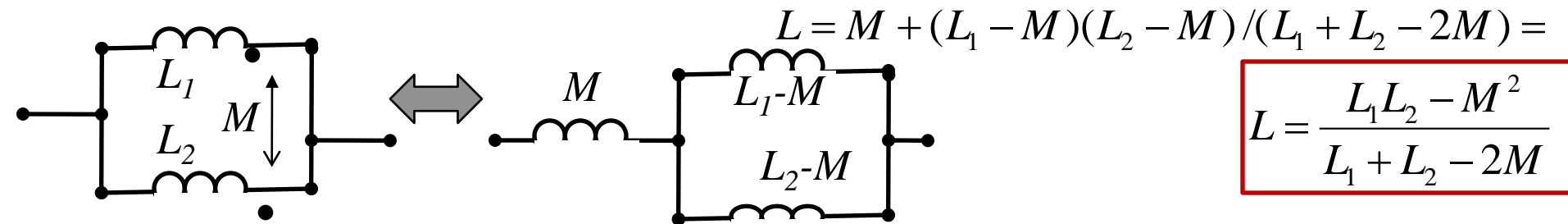
Dupa eliminarea cuplajului: $L = L_A + L_B = L_1 - L_{12} + L_1 - L_{21} = L_1 + L_2 - 2M$
 $L = L_1 + L_2 \pm 2M$ cu - la cuplajului diferential si cu + la cuplajul aditiv.



In cazul cu n bobine se aduna toate elementele matricei \mathbf{L} :

$$\underline{U} / (j\omega) = \sum_{k=1}^n L_{1k} \underline{I}_k + \sum_{k=1}^n L_{2k} \underline{I}_k + \dots = \underline{I} \sum_{j=1}^n \left(\sum_{k=1}^n L_{jk} \right) \Rightarrow L = \sum_{j=1}^n \sum_{k=1}^n L_{jk}$$

• Teorema de echivalenta pentru bobinele cuplate conectate in paralel



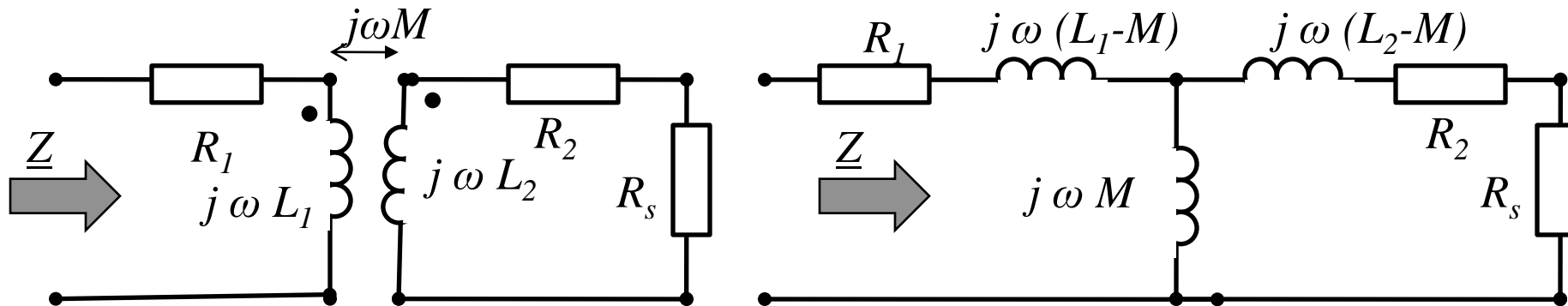
$$L = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$$

Pentru n bobine se sumeaza elementele inversei matricei \mathbf{L} .

- Impedanta de intrare a unui transformator cu sarcina rezistiva**

Se elimina cuplajul si apoi

$$\underline{Z} = R_1 + j\omega(L_1 - M) + j\omega M \parallel [j\omega(L_2 - M) + R_1 + R_s]$$



- Masurarea inductantei de cuplaj**

Se inseriaza cele doua bobine cu cuplaj aditiv si apoi cu cuplaj diferential. Se masoara inductantele bobinelor inseriate (masurand de exemplu modulul impedantei si puterea consumata, din care se extrage factorul de putere si apoi reactanta elementului iar prin impartire la ω se obtine inductanta). Cele doua inductante masurate sunt:

$$L_a = L_1 + L_2 + 2M; L_d = L_1 + L_2 - 2M$$

Diferenta lor impartita la patru este chiar inductanta mutuala.

3.13. Analiza circuitelor electrice prin transfigurari succesive

- Scopul analizei: determinarea curentilor si tensiunilor din laturile circuitului. Datele problemei: schema elctrica (topologia circuitului), valorile parametrilor elementelor pasive si active (sursele).
- Transfigurarea succesiva a circuitului este cea mai eficienta metoda de analiza manuala (fara calculator). Ea poate fi aplicata circuitelor cu topologie simpla (serie, paralel, mixta sau cel mult stea-triunghi). Folosind teoremele de echivalenta se reduce succesiv topologia circuitului la o singura bucla. Se calculeaza curentul din bucla si tensiunile la bornele elementelor si apoi se revine succesiv la topologia initiala, calculand in fiecare etapa din ecuatiile constitutive curentii si tensiunile laturilor. In acest proces sunt utile si urmatoarele relatii.
- **Relatia divizorului de tensiune - exprima tensiunile la bornele elementelor R,L sau C inseriate in functie de tensiunea totala:**
- **Relatia divizorului de curent – exprima curentii din elementele R,L, sau C, conectate paralel in functie de curentul total:**

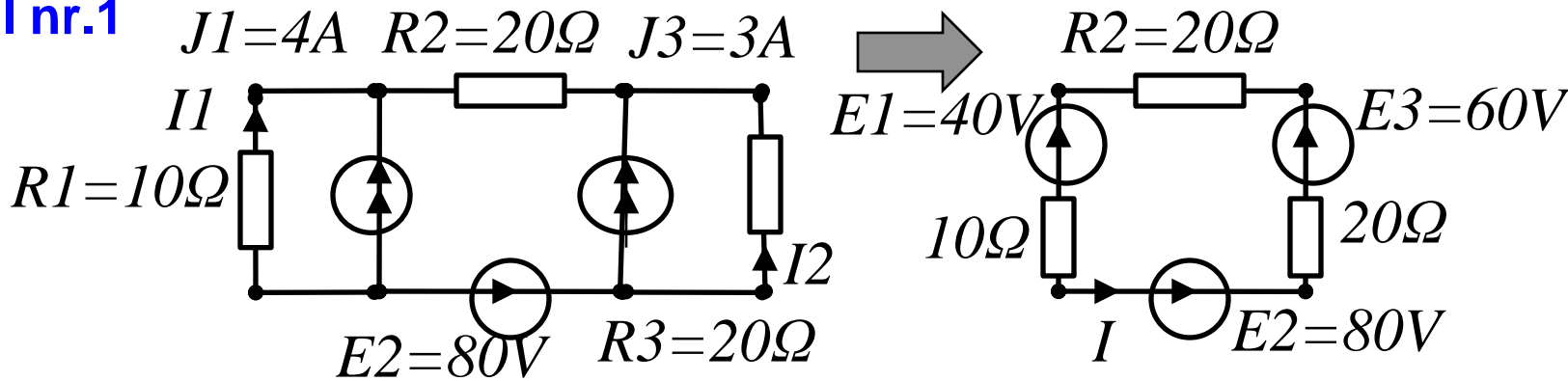
$$u_1 = \mathcal{Z}_1 i = \mathcal{Z}_1 (\mathcal{Z}_1 + \mathcal{Z}_2)^{-1} u$$

$$u_2 = \mathcal{Z}_2 i = \mathcal{Z}_2 (\mathcal{Z}_1 + \mathcal{Z}_2)^{-1} u$$

$$i_1 = y_1 u = y_1 (y_1 + y_2)^{-1} i$$

$$u_2 = y_2 u = y_2 (y_1 + y_2)^{-1} i$$

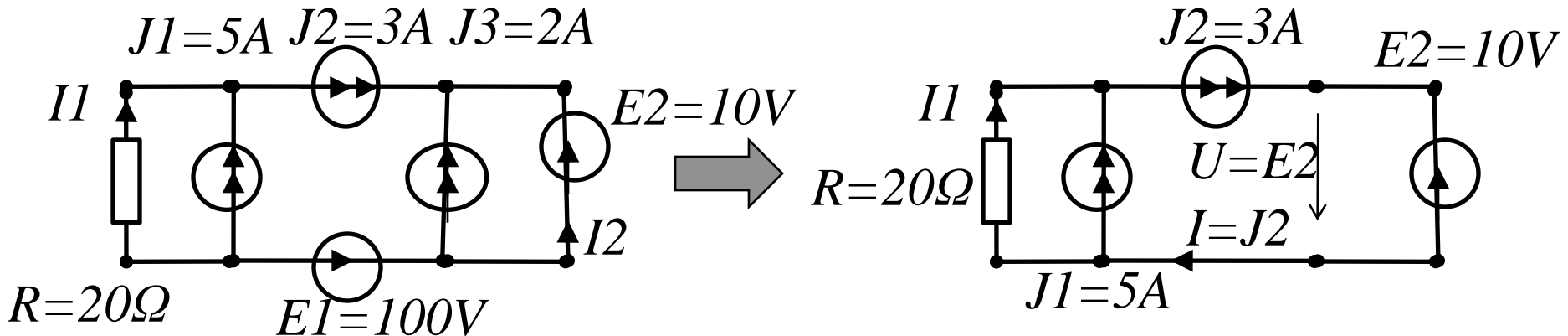
• **Circuitul nr.1**



$$I = (E_2 + E_3 - E_1) / (R_1 + R_2 + R_3) = (80 + 60 - 40) / (10 + 20 + 20) = 2A$$

$$I_1 = -I - J_1 = -2 - 4 = -6A; \quad I_2 = I - J_3 = 2 - 3 = -1A$$

• **Circuitul nr.2**

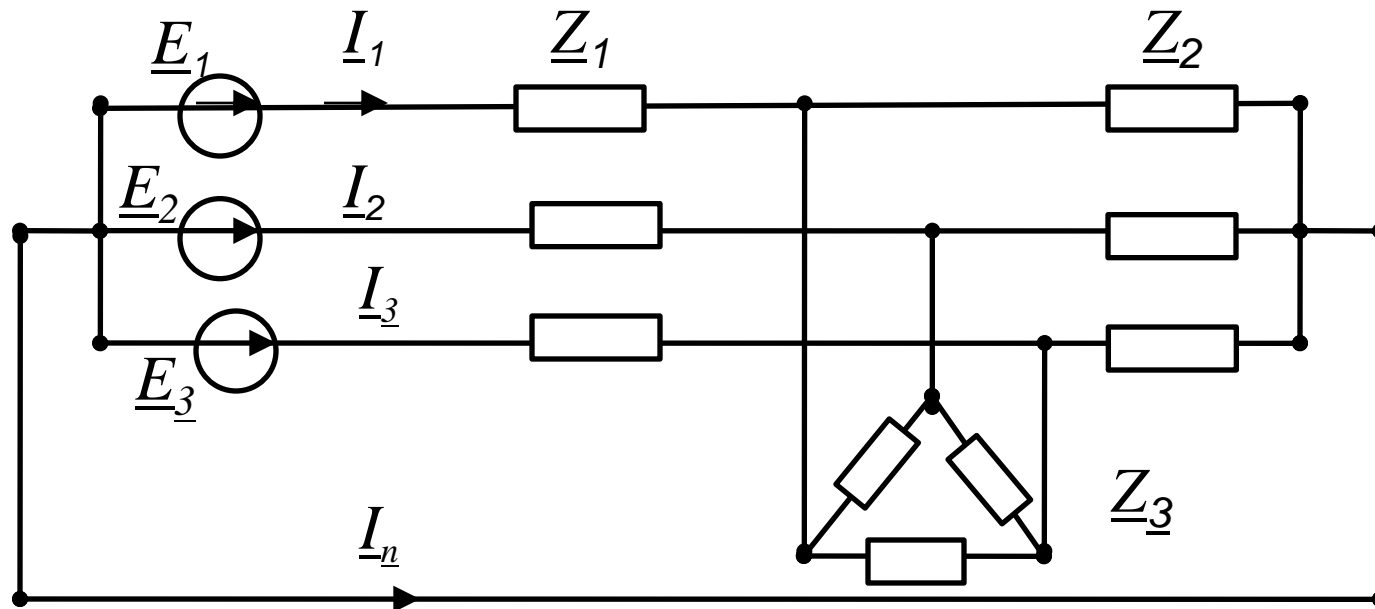


$$I_1 = J_2 - J_1 = 3 - 5 = -2A; \quad I_2 = -J_2 - J_3 = -3 - 2 = -5A$$

3.14. Circuite trifazate (opt.)

- In productia si distributia energiei electrice se foloseste curentul alternativ de frecvenata industriala in circuite trifazate, in care curentii si tensiunile formeaza un triplet de marimi sinusoidale simetrice, care au valori efective egale si sunt defazate cu 120 grade intre ele. Acesta tehnologie are avantaje multiple: reducerea necesarului de material conductor din liniile de transport si usurinta de a produce camp magnetic invariat, lucru ce simplifica structura masinilor electrice – motoare si generatoare. La circuitel echilibrate (cu impedante egele pe cele trei faze) curentul prin firul neutru este nul.

Generator trifazat Impedanta liniei Consumator triunghi Consumator stea

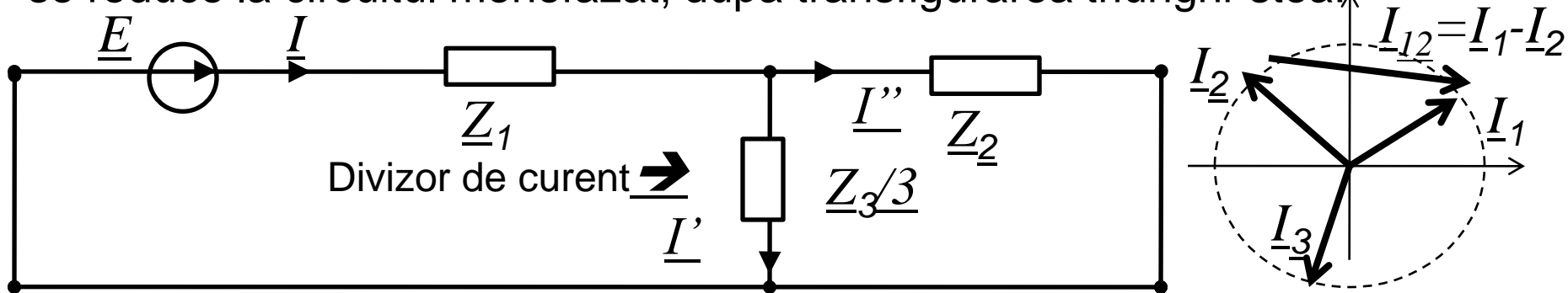


Circuite trifazate (cont.)

Sistemele trifazate simetrice au suma nula:

$$\underline{I}_1 = I e^{j\varphi} = \underline{I}; \underline{I}_2 = I e^{j(\varphi+2\pi/3)} = \underline{I} e^{j2\pi/3}; \underline{I}_3 = I e^{j(\varphi-2\pi/3)} = \underline{I} e^{-j2\pi/3} \Rightarrow$$

$\underline{I}_1 + \underline{I}_2 + \underline{I}_3 = 0$ Deoarece fazele 2, 3 repeta defazat starea fazei 1 analiza se reduce la circuitul monofazat, dupa transfigurarea triunghi-stea:



$$\underline{I} = \underline{E} / (\underline{Z}_1 + \underline{Z}_3 \underline{Z}_2 / (\underline{Z}_3 + 3\underline{Z}_2)); \underline{I}' = \underline{I} \underline{Z}_2 / (\underline{Z}_2 + \underline{Z}_3 / 3); \underline{I}'' = \underline{I} - \underline{I}'$$

Pe linia de transport se masoara *marimile de linie*. Curentii si tensiunile elementelor dipolare ce alcatuiesc consumatorii si generatoarele trifazate se numesc curenti de faza. La conexiunea stea curentii de linie sunt egali cu cei de faza si $\underline{U}_{12} = \underline{V}_1 - \underline{V}_2 \Rightarrow U_l = \sqrt{3}U_f$ iar la conexiunea triunghi tensiunile de linie si de faza sunt egale iar curentii sunt: $I_l = \sqrt{3}I_f$

3.15. Modelarea liniilor lungi, de transmisie

Prin reprezentarea in complex a ecuatiilor liniilor de transmisie in c.a.:

$$-\frac{di}{dx} = C_l \frac{du}{dt} + G_l u \Rightarrow -\frac{d\underline{I}}{dx} = j\omega C_l \underline{U} + G_l \underline{U}; \underline{U}(x) = \mathcal{L}(u(x, t))$$

$$-\frac{du}{dx} = L_l \frac{di}{dt} + R_l i \Rightarrow -\frac{d\underline{U}}{dx} = j\omega L_l \underline{I} + R_l \underline{I}; \underline{I}(x) = \mathcal{L}(i(x, t))$$

$$-\frac{d}{dx} \begin{bmatrix} \underline{I} \\ \underline{U} \end{bmatrix} = \begin{bmatrix} 0 & G_l + j\omega C_l \\ R_l + j\omega L_l & 0 \end{bmatrix} \begin{bmatrix} \underline{I} \\ \underline{U} \end{bmatrix}; \frac{d^2}{dx^2} \begin{bmatrix} \underline{I} \\ \underline{U} \end{bmatrix} = \begin{bmatrix} \gamma^2 & 0 \\ 0 & \gamma^2 \end{bmatrix} \begin{bmatrix} \underline{I} \\ \underline{U} \end{bmatrix};$$

$$\gamma = \sqrt{(R_l + j\omega L_l)(G_l + j\omega C_l)}; \underline{Z}_0 = \sqrt{\frac{R_l + j\omega L_l}{G_l + j\omega C_l}}; \text{ imped. caracteristica}$$

$$\underline{U}(x) = \underline{U}_2 e^{-\gamma x} + \underline{U}_1 e^{\gamma x}; \underline{I}(x) = (\underline{U}_2 e^{-\gamma x} - \underline{U}_1 e^{\gamma x}) / \underline{Z}_0; \Rightarrow \underline{Z}_{in} = \underline{Z}_0 \frac{\underline{Z}_{sarc} + \underline{Z}_0 \text{th} \gamma}{\underline{Z}_0 + \underline{Z}_{sarc} \text{th} \gamma}$$

Se obtin "condensatoare" fara dielectrice si "bobine" fara spirale!

$$\underline{I}_1 = (\underline{U}_2 + \underline{U}_1) / \underline{Z}_0; \underline{I}_2 = (\underline{U}_2 e^{-\gamma l} - \underline{U}_1 e^{\gamma l}) / \underline{Z}_0 \Rightarrow \mathbf{Z}(\omega) = \begin{bmatrix} 1/\underline{Z}_0 & 1/\underline{Z}_0 \\ -e^{\gamma l} / \underline{Z}_0 & e^{-\gamma l} / \underline{Z}_0 \end{bmatrix}$$

Linii lungi: $l > \lambda / 4; \lambda = cT = c / f; c = \omega / |\gamma|.$

Model cu parametri concentrati, prin interpolarea lui Z cu functii rationale in ω .

http://en.wikipedia.org/wiki/Distributed_element_model

3.16. Concluzii

- Similitudinile si echivalentele au un rol importanta atat **teoretic** cat si **aplicativ**.
- Metodele de reprezentare simbolica: in complex pentru c.a. si prin transformata Laplace in regim tranzitoriu transforma ecuatiile diferentiale ale circuitelor in aceste regimuri **in ecuatii algebrice**.
- Aceste ecuatii algebrice sunt mult mai usor de rezolvat si au **forma similara ecuatiilor de curent continuu**.
- Acest deziderat se realizeaza deoarece reprezentarile mentionate sunt **liniare** si transforma operatiile de **derivare (si implicit integrare) in operatii algebrice** de inmultire (si implicit impartire) cu $j\omega$ respectiv s
- Locul rezistenteleor/conductantelor din c.c. este luat de impedantele/admitantele complexe respectiv operationale. Chiar daca in domeniul frecventei (reale sau complexe) se opereaza cu numere complexe, **rezolvarea este mai simpla decat cea din domeniul timpului**.
- Relatiile si teoremele circuitelor electrice liniare se **demonstreaza o singura data**, concluziile fiind transferate si altor regimuri prin similitudine.
- Teoremele de echivalenta stau la baza **metodei transfigurarilor**, o metoda de mare eficienta pentru analiza manuala a circuitelor electrice.

- <http://www.circuiteelectrice.ro/>
- <http://www.em.ucv.ro/eLEE/RO/realisations/CircuitsElectriques/index.htm>
- http://en.wikipedia.org/wiki/Electrical_network
- http://en.wikibooks.org/wiki/Signals_and_Systems
- http://en.wikipedia.org/wiki/Impulse_function
- http://en.wikipedia.org/wiki/Alternating_current
- http://en.wikipedia.org/wiki/AC_power
- <http://web.cecs.pdx.edu/~ece2xx/ECE222/Slides/LaplaceCircuits.pdf>
- http://en.wikipedia.org/wiki/Electrical_impedance#Combining_impedances
- http://en.wikipedia.org/wiki/Equivalent_impedance_transforms
- http://en.wikipedia.org/wiki/Series_and_parallel_circuits
- http://en.wikipedia.org/wiki/Y-%CE%94_transform
- http://en.wikipedia.org/wiki/Current_divider
- http://en.wikipedia.org/wiki/Voltage_divider
- http://en.wikipedia.org/wiki/Millman%27s_theorem

- [http://en.wikipedia.org/wiki/Topology_\(electronics\)](http://en.wikipedia.org/wiki/Topology_(electronics))
- http://www.tifr.res.in/~achanta/Electronics/Lectures/EquivalentTheorems_Proofs.pdf
- http://en.wikipedia.org/wiki/RC_circuit
- http://en.wikipedia.org/wiki/RL_circuit
- http://en.wikipedia.org/wiki/RLC_circuit
- [http://en.wikipedia.org/wiki/Network_analysis_\(electrical_circuits\)](http://en.wikipedia.org/wiki/Network_analysis_(electrical_circuits))
- <http://fourier.eng.hmc.edu/e84/lectures/ch3/node1.html>
- http://en.wikipedia.org/wiki/Active_filter
- <http://www.ti.com/lit/ml/sloa088/sloa088.pdf>
- <http://web.cecs.pdx.edu/~ece2xx/ECE222/Slides/LaplaceCircuits.pdf>
- http://en.wikipedia.org/wiki/Electric_power_distribution
- http://en.wikipedia.org/wiki/Three-phase_electric_power
- <http://en.wikipedia.org/wiki/Three-phase>
- <http://orselj.free.fr/files/et1-triphase.pdf>
- http://www.elect.mrt.ac.lk/EE201_3phase_sym_comp.pdf