

# An exploration of the Integrated Field Equations Method for Maxwell's Equations

Z. Sheng, R. Remis, A.T. de Hoop and P. Dewilde  
 Delft University of Technology  
 POB 5031, 2600GA Delft, the Netherlands  
 Z.Sheng@ewi.tudelft.nl

**Abstract**—This paper presents a survey and a number of experiments with the (relatively new) Integrated Field Equations method for Maxwell's equations. It starts out with a summary of the method, followed by a number of test cases in which its special properties, in particular its ability to handle high contrast using coarse grids is demonstrated. The method gives rise to a rather special set of equations and new strategies to solve these are presented as well, in particular the use of hierarchical semi-separable representations either in a direct solver or through a Krylov iteration strategy. New in the paper are the time-domain integration version of the method and the solution strategies.

**Keywords**—Integrated Field Equations, Krylov method, hierarchically semi-separable solver, time domain integration, high contrasts

## I. OUTLINE OF THE METHOD

A computational technique (the Integrated Field Equations or IFE method) is presented that models time-domain (pulsed) electromagnetic (EM) fields in strongly heterogeneous media. In media of this kind, the constitutive parameters can jump by large amounts upon crossing the boundary surface of any of the elements of the geometrical discretization. On a global scale, the EM field components are, therefore, not differentiable and Maxwell's equations in differential form cannot be used: one has to resort to some sort of integral form of the EM field relations as a basis for the method of computation. An appropriate integral form is provided by the classical interrelations between the circulation of the electric/magnetic field strength along a closed curve and the (time rate of change of) the magnetic/electric flux passing through some surface with the circulation loop as boundary. For these to hold, only integrability is needed, which condition is imposed in accordance with the physical condition of boundedness of the field quantities. To satisfy the constitutive relations (that are representative of physical volume effects), an analytical continuation of the boundary representations of the field components on the boundary of an element into its interior is needed. A consistent algorithm that meets all of these requirements is constructed, using a simplicial geometrical discretization, combined with piecewise linear representation of the electric and magnetic field components along the edges of the elements, a piecewise linear representation of the electric and the magnetic flux density components along the normals to

the faces, piecewise linear extrapolations into the interior of the elements and taking constant values of the constitutive coefficients (or relaxation functions) in these interiors. In time, also piecewise linear representations are used. This procedure can be proved to converge to the local EM field equations and constitutive relations on a scale where the discontinuous material behavior no longer applies.

As in any EM problem, radiation takes place into an unbounded embedding of the configuration of interest. This embedding we take to be free space. To handle this aspect, the bounded domain of computation is embedded in a Cartesian Perfectly Matched Embedding that is constructed via a (causal) space-time coordinate-stretching procedure, truncating this embedding, and invoking a periodic boundary condition on the boundary of the remaining 3-rectangle. The causality requirement on the coordinate-stretching functions ensures the uniqueness of the solution of the resulting field problem. In the domain of interest, the field values are only disturbed by the amount of spurious field propagation due to the periodic repetition of the configuration, which disturbance can be made as small as desired by providing the matching layers with the proper amount of attenuation. Note that in the (artificial) embedding configuration large jumps in parameters may occur, in addition to the ones in the physical domain of interest.

Since the whole procedure only uses continuously varying quantities in space-time in the field representations, irrespective of how wildly the constitutive properties of the configuration vary with position in space, the method is believed to be fully compatible with the physics of the type of configuration at hand, which no numerical method based on the differentiability of the field quantities can claim.

## II. EQUATION SOLVERS

The IFE method in the frequency domain gives rise to a system of equations which, in the 2D case, contains roughly twice as many equations as in the more direct differential versions (such as FDTD), because of the extra constitutive equations. It is, however, very sparse (as in the other methods based on volume finite elements), and has a double band structure with somewhat irregularly ragged bands. Much of the difficulties with this method can be solved by using adequate (and new) numerical methods. For this, the structure of the matrices has to be exploited systematically, especially in the 3D

case where the number of equations can be very large. The goal is to find the Moore-Penrose solution of the set of equations, i.e. a solution that averages the extra equations out. We present both a direct method and an iterative method to solve these. In both cases the structure of the matrices is captured through the use of 'Hierarchical Semi-separable' solvers, a method pioneered in [1]. The method is based on ideas from [2], [3] and [4].

The presentation will give a survey of the new type of efficient algorithms we propose. With an efficient hierarchically semi-separable representation of the system matrix, both matrix-vector multiplication and direct solutions can be done efficiently, or more precisely, their complexity goes linearly with the size of the matrices. However, the direct solution method does not scale well with the rank of the off-diagonal sub-matrices. This limits the usage of the direct method in 2D and 3D configuration. We can then apply a model reduction method on the original HSS representation to find low rank approximations for the off-diagonal matrices. The resulting overall approximations can be used as *preconditioner*. With such preconditioning methods, efficient matrix-vector method as well as other efficient HSS algorithms, any Krylov iterative method can be combined with the HSS representation.

In the case of time-integration and with the use of the trapezium rule, a concatenation of locally coupled matrices with the structure described above arises. Because the integration method is implicit, we have devoted special care to combine the Moore-Penrose (least square) method with the progressing integration. More precisely, we propose a local schema with a limited time horizon, so that the time progression can be solved efficiently as well.

### III. NUMERICAL RESULTS

We present in addition a number of numerical results for the 2D case, in particular results that illustrate (1) the contrast properties of the method and (2) the use of PML boundary conditions. The prototype case is the EM field for a problem in which the field quantities are independent of the z-coordinate, with a current source (a wire in the z-direction), producing an electric field that is non-zero only in the z-direction and a magnetic field that lays entirely in the xy-plane. The xy-plane is partitioned in domains with starkly varying contrasts. In addition, we show experimental properties of the various methods used to solve the resulting system of equations efficiently.

### IV. FURTHER WORK

The extension of the proposed method to the 3D case poses a number of new issues. We briefly indicate how they can be tackled. In the 3D case the redundancy amounts roughly to a factor 4, i.e. the number of equations is roughly four times the number of unknowns. The structure, while still very sparse, is now of the type 'multiple block band of multiple block bands' where 'multiple' stands for a factor 3, due to the lexicographic ordering of the finite elements. In addition, there is the progression of the integration in the time domain. While such

a structure may seem very hard to handle in a Moore-Penrose context, we do have new methods to apply, and even methods that claim to be linear in the number of unknowns [5]. They are based on a combination of HSS techniques [1], [6]–[8] and model reduction methods derived from time-varying system theory [4]. Combined with an iterative Krylov-space solver and a time-horizon progression, it should be possible to squeeze as much efficiency as possible out of the structure of the matrices.

### V. CONCLUSIONS

The IFE method holds considerable promise to model electromagnetic effects in integrated circuits, where high contrasts between different types of materials is the rule and very few regular structures are present.

### REFERENCES

- [1] S.Chandrasekaran P.Dewilde M.Gu W.Lyons and T.Pals, "A fast solver for hss representation via sparse matrices," August 2005.
- [2] L. Greengard and V. Rokhlin, "A fast algorithm for particle simulations," *J. Comp. Phys.*, vol. 73, pp. 325–348, 1987.
- [3] V. Rokhlin, "Applications of volume integrals to the solution of pde's," *J. Comp. Phys.*, vol. 86, pp. 414–439, 1990.
- [4] P. Dewilde and A.-J. van der Veen, *Time-varying Systems and Computations*, Kluwer, 1998.
- [5] Patrick Dewilde and Shiv Chandrasekaran, "A hierarchical semi-separable moore-penrose equation solver," Nov 2005.
- [6] S.Chandrasekaran M.Gu and T.Pals, "A fast ulv decomposition solver for hierachically semiseparable representations," 2004.
- [7] W.Lyons S.Chandrasekaran and M.Gu, "Fast lu decomposition for operators with hierachically semiseparable structure," April 2005.
- [8] W.Hackbusch leipzig, "A sparse matrix arithmetic based on h-matrices. part 1: Introduction to h-matrices," *Computing*, December 1998.