Finite-Difference Simulations of Electromagnetic Solitary Waves

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***Abstract*—A suitable correction of the Maxwell model brings to an enlargement of the space of solutions, allowing for the existence of solitons in vacuum. We review the basic achievements of the theory and propose some approximation techniques based on explicit finite-difference methods.**

Keywords— electromagnetic wave-fronts, solitons, finite-differences.

I. Preliminary considerations

It is known that Maxwell equations in empty space do not admit finite-energy solitary waves among their solutions. One of the reasons to explain this fact is attributed to the linearity of these equations, not allowing a due focussing of the signal on a constrained path. Such an impediment has stimulated the research of alternative nonlinear models allowing for the existence of soliton-like solutions.

A further step in the comprehension of electromagnetic solitary waves has been given in [4], where, in the frame of a new self-consistent theory, explicit solutions are carried out. The main argument is that the classical equations of electromagnetism are not capable to follow the evolution of finite-energy wave-fronts in the proper way (i.e., the one described for instance by the Huygens principle). The reason is that the two conditions $\text{div}\mathbf{E} = 0$ and $\text{div}\mathbf{B} = 0$ cannot hold at the same time on each point of the same wave-front, unless the wave-front itself assumes a very unnatural topology. Therefore, the above conditions have been dropped, without this implying the existence of electrical charges or magnetic monopoles.

Here we devote our attention to numerical simulations. It is soon evident how the removal of relations $\text{div}\mathbf{E} = 0$ and $\text{div}\mathbf{B} = 0$ is important for the construction of numerical algorithms in general, since most of the difficulties in the approximation of Maxwell equations come exactly from imposing these kind of constraints. This includes the efforts made to build up approximation spaces satisfying some divergence-free conditions (see [1], [3], [5]) or divergence corrections techniques (see for instance [6]). Modifications of the Maxwell model have been proposed at numerical level, in order to get stable schemes ([8], [7]), to handle boundary conditions (see [2]), or for the treatment of wave propagation in linear non-dispersive lossy materials. These methods are mainly proposed to overcome numerical troubles and are not intended to modifying the Maxwell model itself, as we are doing here. For simplicity, we use explicit finite-differences. The aim is to validate the theory with a series of simple experiments.

II. The model equations

In [4] the theory of electromagnetism was entirely reviewed by replacing the Maxwell equations by a suitable nonlinear set of equations. Compared to the classical approach, the new formulation provides a far more accurate description of wave phenomena, and, not only improves (with no contradictions) the existing models, but also allows links and generalizations not possible with the standard Maxwell theory.

We assume to be in vacuum. No electric charges or masses will be present. We are only concerned with the evolution of pure electromagnetic waves. We claim that the correct modelling of wave-fronts is given by the following set of equations:

$$\frac{\partial \mathbf{E}}{\partial t} = c^2 \text{curl}\mathbf{B} - (\text{div}\mathbf{E})\mathbf{V} \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\text{curl}\mathbf{E} - (\text{div}\mathbf{B})\mathbf{V} \tag{2}$$

$$\frac{D\mathbf{V}}{Dt} = \mu\big(\mathbf{E} + \mathbf{V}\times\mathbf{B}\big) \tag{3}$$

$$|\mathbf{V}| = c \tag{4}$$

The substantial derivative in (3) is given by: $\frac{D}{Dt}\mathbf{V} = \frac{\partial}{\partial t}\mathbf{V} + (\mathbf{V}\cdot\nabla)\mathbf{V}$. Moreover, c is the speed of light and μ is a constant (not better specified) whose dimension is *charge/mass*. In (4), $|\cdot|$ denotes the standard norm.

Here, $\mathbf{E}$ and $\mathbf{B}$ are the usual electric and magnetic fields, while $\mathbf{V}$ is a new velocity field. The important fact is that, even in vacuum, $\text{div}\mathbf{E}$ and $\text{div}\mathbf{B}$ are allowed to be different from zero.

By taking the divergence of (1), we can write:

$$\frac{\partial \rho}{\partial t} = -\text{div}(\rho\mathbf{V}) \tag{5}$$

where $\rho = \text{div}\mathbf{E}$. This is the continuity equation for a nonsingular density distribution of charge travelling with the wave. The nonlinear term, on the right-hand side of (1), can be now interpreted as the source term due to the Ampère law, for the density ρ, moving at the speed c in the direction of $\mathbf{V}$. Even if there are no real classical electric charges, the density ρ lives inside the wave, evolving with it.

III. Evolution of free electromagnetic waves

We start by defining $\mathbf{J} = (\mathbf{E}\times\mathbf{B})/|\mathbf{E}\times\mathbf{B}|$ to be the (adimensional) normalized Poynting vector. In the special case in which $\mathbf{V} = c\mathbf{J}$, the set of equations becomes:

$$\frac{\partial \mathbf{E}}{\partial t} = c^2 \mathrm{curl}\mathbf{B} - c(\mathrm{div}\mathbf{E})\mathbf{J} \qquad (6)$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\mathrm{curl}\mathbf{E} - c(\mathrm{div}\mathbf{B})\mathbf{J} \qquad (7)$$

$$\mathbf{E} + c\,\mathbf{J}\times\mathbf{B} = 0 \qquad (8)$$

The equation (4) is always satisfied. The equation (8) is instead satisfied when $\mathbf{E}$ and $\mathbf{B}$ are orthogonal and $|\mathbf{E}| = |c\mathbf{B}|$, which is a standard requirement.

Some important aspects of the theory are the following.

- If div$\mathbf{E}$ and div$\mathbf{B}$ are zero, we obviously reobtain the Maxwell equations. Hence, the new set of equations admits more solutions than the classical theory.
- Perfect spherical waves and travelling signal-packets (solitons) of finite energy, are also solutions. It is well-known that this is not true for the ordinary Maxwell model.
- If div$\mathbf{E}$ and div$\mathbf{B}$ are relatively small (as it actually happens in many practical circumstances), then (6) and (7) are as accurate as the corresponding standard Maxwell equations. Therefore, we expect the new model to be consistent with the existing ones, for a broad range of applications.
- It can be proven that the electromagnetic free waves described by the new set of equations are perfectly compatible with the Huygens principle and the eikonal equation.
- For div$\mathbf{B} = 0$, the equation (6) follows from minimizing the standard Lagrangian of classical electromagnetism, after imposing the constraint $\mathbf{A} = \Phi\mathbf{J}$ to the potentials.

IV. Constrained waves

Let us define $\mathbf{G} = \frac{D}{Dt}\mathbf{V}$, so that $\mathbf{G}$ is an acceleration. Basically, if $\mathbf{V}$ is the normalized vector field tangent to the light rays, then the field $\mathbf{G}$ gives a measure to their curvature.

The equation (3), even if there are no classical moving charges, generalizes the Lorentz law. If $\mathbf{G}$ is zero, then one is dealing with a free electromagnetic wave, and the corresponding rays are straight-lines (see previous section). Actually, the equation (8), corresponding to $\mathbf{G} = 0$, says that the development of the wave is free from constraints, that is, there are no external perturbations (forces) acting on it.

When for some external reasons, (8) is not satisfied, then $\frac{D}{Dt}\mathbf{V}$ is different from zero, so that $\mathbf{V}$ changes direction (the rays are curving) and the electromagnetic wave-fronts locally follow the evolution of the new normalized Poynting vector $\mathbf{J} = \mathbf{V}/c$.

Thus, when $\mathbf{G}$ is different from zero, the wave is no longer free, and it will be called *constrained wave*. This may happen when the wave is subjected to external electromagnetic fields, for instance during the interaction with matter at atomic level, such as in reflection-refraction, diffraction, scattering, etc.

Therefore, the equations (1), (2) and (3) provide the coupling between the curvature of the rays and the motion of the wave-fronts. Note that, during the change of trajectory of the rays, the polarization may also vary.

V. Approximation

For simplicity, we just discuss examples in the two-dimensional space (x, z). For instance, for any y, we may consider the following fields:

$$\mathbf{E} = (cf(x)g(ct-z), 0, 0) \qquad \mathbf{B} = (0, f(x)g(ct-z), 0) \quad (9)$$

where f and g are arbitrary functions with compact support. These fields are solutions to (6), (7), (8). On the contrary, they do not solve the Maxwell system unless f is constant (plane wave). The continuity equation now becomes:

$$\frac{\partial \rho}{\partial t} = -c\,\frac{\partial \rho}{\partial z} \qquad (10)$$

corresponding to a shift, at the speed of light, along the direction of the z-axis. We obtained in this way a perfect electromagnetic soliton.

We used this exact solutions to check the validity of the second-order explicit Lax-Wendroff method, implemented for systems of equations. It can be shown that the algorithm only acts on those vector components that were initially different from zero. Therefore, up to errors due to the interpolation of the initial data, the discrete solution also behaves as a perfect soliton.

Successively, experiments can be carried out concerning solitons subjected to external perturbations. If, according to (3), a constant gravitational field $\mathbf{G}_0$ is applied in such a way that $D\mathbf{V}/Dt = \mathbf{G}_0$, with $|\mathbf{V}| = c$, then the trajectory of the soliton is deflected. Such a behavior can be actually observed numerically.

These preliminary tests are encouraging in view of more sophisticated applications.

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