Advances in Mathematical and Computational Methods Applied in Electrical Engineering

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***Abstract*—When dealing with the simulation of coupled problems in the context of domain decomposition methods, one often has to deal with curvilinear interfaces. Several issues and results for handling these interfaces with nonmatching grids in a numerically stable manner are presented. Moreover, an application of nonconforming discretization techniques to an elasto-acoustic problem is considered.**

Keywords—domain decomposition, curvilinear interface, mortar method, elasto-acoustics

I. Introduction

The approximative solution of complex heterogeneous problems is characterized by the necessity of being able to combine different model equations, discretizations, spatial and temporal scales, triangulations, and/or spatial dimensions. Nonconforming discretization techniques cope with this necessity by providing numerically robust discrete problem formulations based on a geometrical decomposition of the computational domain corresponding to the different interacting fields. In this paper, we focus on the handling of curvilinear interfaces and consider an application to an elasto-acoustic problem. In particular, we will summarize our results for the scalar case from [1] in Section II, whereas in Section III, we will deal with vector fields as discussed in [2], [3]. Building up on [4], we apply nonconforming techniques to simulate the sound radiation of an piezo-electric structure in Section IV.

II. The Scalar Case

We consider the classical model problem $-\Delta u = f$ in Ω with $u = 0$ on $\partial\Omega$. The domain Ω is subdivided into two non-overlapping subdomains Ω^{m} and Ω^{s}, sharing the possibly curved interface $\Gamma = \partial\Omega^{\mathrm{m}} \cap \partial\Omega^{\mathrm{s}}$ with the unit normal vector $\boldsymbol{n}$. For simplicity, we assume that Γ is a closed curve. Transforming to the weak setting and defining the Lagrange multiplier $\lambda \in M = H^{-1/2}(\Gamma)$ by $\lambda = -\operatorname{grad} u \cdot \boldsymbol{n}$, it is then possible to check that the pair $(u, \lambda) \in X \times M$, with $X = \{v = (v_{\mathrm{m}}, v_{\mathrm{s}}) \in H^1(\Omega^{\mathrm{m}}) \times H^1(\Omega^{\mathrm{s}})$ respecting the Dirichlet conditions$\}$, satisfies the following saddle point problem:

$$\begin{array}{rcll} a(u,v) + b(v,\lambda) &=& f(v) & v \in X, \\ b(u,\mu) &=& 0 & \mu \in M. \end{array} \qquad (1)$$

Above, the (bi-)linear forms $a(\cdot,\cdot)$ and $f(\cdot)$ are defined in the obvious way, while the coupling bilinear form $b(\cdot,\cdot)$ is given by

$$b(v,\mu) = \langle [v],\, \mu\rangle_\Gamma, \quad v \in X,\ \mu \in M,$$

with $\langle \cdot,\cdot\rangle_\Gamma$ the duality product on $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$.

For each subdomain Ω^j, $j = \mathrm{m}, \mathrm{s}$, we have a triangulation $\mathcal{T}_j$ of a domain Ω^j_h such that $\partial\Omega^j_h$ is a piecewise linear interpolation of $\partial\Omega^j$. In this way, we obtain two piecewise linear approximations of the curvilinear interface Γ that we denote by Γ^{s}_h and Γ^{m}_h. The space of piecewise linear functions on $\mathcal{T}_j$ respecting the Dirichlet boundary conditions is denoted by $X_{j,h}$ and we set $X_h = X_{\mathrm{m},h} \times X_{\mathrm{s},h}$. The discrete Lagrange multiplier space M_h is associated with the discrete $(d-1)$-dimensional interface Γ^{s}_h. While the approximation of the (bi-)linear forms $a(\cdot,\cdot)$ and $f(\cdot)$ by $a_h(\cdot,\cdot)$ and $f_h(\cdot)$ is straightforward, the discretization of the coupling bilinear form $b(\cdot,\cdot)$ is more involved. By employing a suitable projection $P_{\mathrm{s}} : L^2(\Gamma^{\mathrm{m}}_h) \to L^2(\Gamma^{\mathrm{s}}_h)$, we are able to define the discrete jump across Γ^{s}_h as

$$[v]_h = v_{\mathrm{s}} - P_{\mathrm{s}} v_{\mathrm{m}}, \qquad v \in X_h,$$

and replace $b(\cdot,\cdot)$ by

$$b_h(v,\mu) = ([v]_h, \mu)_{\Gamma^{\mathrm{s}}_h}, \quad v \in X_h,\ \mu \in M_h.$$

In [1], we provide a rigorous mathematical analysis of the resulting discrete saddle point formulation. Proceeding in two steps, we first analyze an auxiliary problem given in terms of blending elements, where the curved interface is resolved in an exact manner. Then, the original discrete problem is handled as a perturbation of the blending approach. The main result, which is developed for a decomposition into many subdomains, is given as follows.

Proposition 1 *For (u, λ) sufficiently regular and (u_h, λ_h) given by the discrete form of (1), we have*

$$\|u - u_h\|_{X_h} + \|\lambda - \mathcal{S}\lambda_h\|_M \le C(u)h, \qquad (2)$$

where $\mathcal{S}$ is a stable mapping onto the curved interface Γ.

III. The Vector Field Case

We intend to solve (1) with spaces and (bi-)linear forms given by the weak form of the linear elasticity problem of finding a displacement vector field $\boldsymbol{u}$ such that $-\operatorname{div}\sigma(\boldsymbol{u}) = f$ in Ω, supplemented by boundary conditions, by the Saint-Venant Kirchhoff law $\sigma = \lambda_{\mathrm{L}}(\operatorname{tr}\varepsilon)\mathrm{Id} + 2\mu_{\mathrm{L}}\,\varepsilon$, with the Lamé constants $\lambda_{\mathrm{L}}, \mu_{\mathrm{L}}$, and by the linearized strain tensor $\varepsilon(\boldsymbol{u}) = \frac{1}{2}(\operatorname{grad}\boldsymbol{u} + [\operatorname{grad}\boldsymbol{u}]^{\mathrm{T}})$. Here, the Lagrange multiplier $\boldsymbol{\lambda}$ corresponds to the surface tractions on Γ, namely, $\boldsymbol{\lambda} = -\sigma(\boldsymbol{u})\boldsymbol{n}$. The spaces X and M consist of vector fields with component functions being in the corresponding spaces for the scalar case.

The a priori result (2) transfers to the linear elasticity setting by standard arguments. Nevertheless, the use of dual Lagrange multipliers may exhibit an undesired behavior in form of unphysical oscillations, if the Lagrange multiplier space is chosen with respect to the coarse grid. Figure 1 illustrates this misbehavior by means of a simple example. The domain is a spherical shell experiencing a

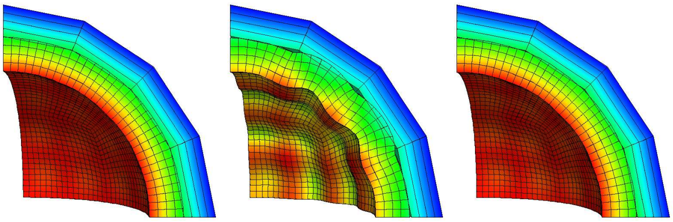

Fig. 1. Distorted domains: standard Lagrange multipliers (left), unmodified dual (middle), modified dual (right).

deformation which is constant in normal direction to the inner and outer boundary. While the use of standard Lagrange multipliers results in a visually correct result, the use of dual multipliers yields a rather poor approximation of the exact solution. However, often it is strongly desirable to employ the advantages of the dual approach.

In [2], [3], we present two alternative modifications. Both have in common that only the coupling of the Lagrange multipliers to the master side is changed, namely, $(P_{\rm s}\boldsymbol{v}_{\rm m}, \boldsymbol{\mu})_{\Gamma_h^{\rm s}}$. Therefore, all the advantages of the dual approach are preserved. The coupling bilinear form $b_h(\cdot,\cdot)$ is replaced by a modification $b_h^{\rm mod}(\cdot,\cdot)$. The modifications incorporate a splitting of vector fields on the interface into normal and tangential components admitting a correct transfer of these quantities across the two non-matching grids. In the first alternative, we replace the L^2-scalar product $(\cdot,\cdot)_{\Gamma_h^{\rm s}}$ by a discrete one. This can also be interpreted by replacing $\boldsymbol{\mu}$ on the master side by $\boldsymbol{\mu} + \Delta\boldsymbol{\mu}$, where $\Delta\boldsymbol{\mu}$ is the sum of Dirac distributions. Due to this non-smooth modification, we are not able to prove the optimality of the resulting modified approach. However, numerical evidence shows that the undesired oscillations vanish, as illustrated in the right picture of Fig. 1. For the second alternative, the modification $\Delta\boldsymbol{\mu}$ is chosen to be momentum-free such that the modified approach still preserves linear momentum. Here, we are able to prove optimal a priori estimates which can be confirmed by numerical tests. In favor of the first modification is the fact that it is easier to implement. Both approaches reduce to the original one in the case of a planar interface.

IV. Application

We present an application of nonconforming domain decomposition techniques to an elasto-acoustic problem setting. For the structure $\Omega_{\rm S}$, we choose a piezo-electric material, where mechanical quantities interact with an electric field, modeled by

$$\begin{aligned} \rho_{\rm S}\ddot{\boldsymbol{u}}_{\rm S} - \operatorname{div}\sigma &= \boldsymbol{f}, \\ \operatorname{div}\boldsymbol{D} &= q, \\ \sigma(\boldsymbol{u}_{\rm S},\phi) &= \mathcal{C}\varepsilon(\boldsymbol{u}_{\rm S}) + \mathcal{B}^{\rm T}\operatorname{grad}\phi, \\ \boldsymbol{D}(\boldsymbol{u}_{\rm S},\phi) &= \mathcal{B}\varepsilon(\boldsymbol{u}_{\rm S}) - \mathcal{E}\operatorname{grad}\phi. \end{aligned}$$

where $\boldsymbol{u}_{\rm S}$, σ, and ε denote the mechanical displacement, stress, and strain, respectively, and ϕ, $\boldsymbol{D}$, $\boldsymbol{E}$ indicate the electric potential, the flux density, and the electric field, respectively. The coupling between the electrical and the mechanical part is characterized by the elastic stiffness tensor $\mathcal{C}$, the piezo-electric tensor $\mathcal{B}$, and the dielectric permittivity tensor $\mathcal{E}$.

Inside the acoustic fluid $\Omega_{\rm A}$, we consider the wave equation for the acoustic velocity potential ψ,

$$c^{-2}\ddot{\psi} - \Delta\psi = 0.$$

The coupling between the acoustic potential and the mechanical displacement field is given by the continuity of the normal velocities and of the surface tractions across the interface Γ, namely,

$$\boldsymbol{n}\cdot\dot{\boldsymbol{u}} = -\frac{\partial\psi}{\partial\boldsymbol{n}}, \qquad \sigma\boldsymbol{n} = -\rho_{\rm A}\boldsymbol{n}\dot{\psi}.$$

In the left picture of Fig. 2, a part of the finite element

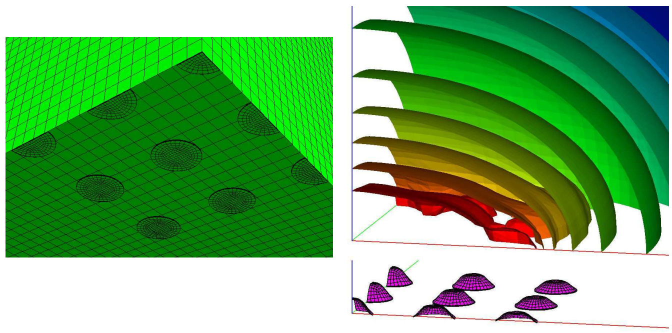

Fig. 2. Left: cylindrical plates attached to the fluid domain, right: isosurfaces of the acoustic potential, deformed plates.

grids used for the computation is shown. The nonconforming approach admits to use the grid desired for each subdomain regardless of the other subdomains. The right picture shows the result of a preliminary calculation where no piezo-electric effect has been taken into account.

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