

# Hierarchical preconditioning in electromagnetism

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**Abstract**—In this talk a new numerical algorithm solving large sparse linear systems arising in electromagnetic field computation will be presented. This method is based on hierarchical partitioning of the matrix and uses block-wise low-rank approximation in combination with element dropping in order to construct a preconditioner for iterative solution. For the treatment of multiply connected domains different strategies of regularisation will be compared. The efficiency of the presented solver will be shown by means of an electromagnetic application.

**Keywords**—computational electromagnetism, preconditioning, hierarchical methods, clustering, multiply connected domains

## I. INTRODUCTION

In the design of electromagnetic applications numerical field computation of three-dimensional problems plays an important role. Fast and efficient solver concepts are necessary for reliable information about the system in an early state of development. The numerical discretisation is done by a coupling of the boundary element method (BEM) and the finite element method (FEM) both based on edge elements. Fine discretisation of complex problems leads to large systems of equations. The BEM part can be solved with asymptotically optimal complexity by using block-wise adaptive cross approximation (ACA) [1]. In larger problems, the main cost will be caused by the FEM part. In this paper, we address the solution of large sparse linear FE-systems with symmetric, positive definite or semi-definite system matrix.

In case of trivial topologies, a positive definite FE-matrix is constructed by means of a discrete regularisation. For the regularised system, hierarchical ( $\mathcal{H}$ -matrix [2], [3]) based concepts of preconditioner construction using low-rank approximation are considered.

An approximate Cholesky decomposition by  $\mathcal{H}$ -matrices was presented by [4] and will be applied to our system. We want to improve the performance by replacing the geometric bisection required for the block partitioning by a new method called interface clustering [5].

In our investigation of hierarchical concepts, a non-recursive algorithm called HSILLT was developed [5]. It combines a block Cholesky decomposition with low-rank approximation and element dropping. It will be described in this paper.

Especially for BEM-FEM-coupling, non-trivial topologies arise. Owing to topological homology, the system ma-

trix is positive semi-definite with a kernel of dimension of the number of holes in the domain. In the discrete case, representatives of this part of the kernel can be computed by topological considerations of the FE-mesh [6]–[8] or by algebraic considerations. We will solve the system by computing a coarse preconditioner of the singular system by one of the above mentioned algorithms. The iterative solver takes care of the regularisation.

By means of electromagnetic field problems arising from an industrial application, we will show the efficiency of the new method. An evaluation is carried out by a comparison to known preconditioning concepts.

## II. PROBLEM FORMULATION AND REGULARISATION

Magnetostatic field problems based on the vector potential ansatz are described by

$$\operatorname{curl} \mu \operatorname{curl} \vec{A} = \vec{j}, \quad (1)$$

where  $\vec{A}$  is the magnetic vector potential and  $\vec{j}$  the electric current density. The material parameter  $\mu$  describes the magnetic permeability and may depend on the magnetic field. The vector potential ansatz contains ambiguity resulting from gradient fields and cohomology fields.

Domain decomposition into inner (FEM-) and outer (BEM-) domain is applied. At the coupling boundary of the domain, the boundary data have to fulfil transmission conditions. The discretisation is based on edge elements. We want to address the construction of a preconditioner for the FE stiffness matrix  $Q \in \mathbf{R}^{k_E \times k_E}$ . This means, we restrict on solving the linear system of equations  $Qx = b$ , with vector of degrees of freedom  $x \in \mathbf{R}^{k_E}$  and right hand side  $b \in \mathbf{R}^{k_E}$ . The dimension  $k_E$  is given by the number of edges in the three-dimensional FE-mesh.

Because of the ambiguity,  $Q$  is singular having a large kernel. The regularisation of the system by constructing a basis of the kernel  $U \in \mathbf{R}^{k_E}$  is proposed, so that

$$K = Q + UU^T, \quad U \in \mathbf{R}^{k_E \times (k_N + l - 1)}, \quad (2)$$

with  $k_N$  the number of nodes and  $l$  the number of holes. The representatives spanning the kernel  $U$  need to be computed. In case of trivial topologies only discrete gradient fields given by an incidence matrix between edges and nodes of the mesh [9] are necessary. In non-trivial domains, discrete representatives of cohomology fields need to be considered in order to find the additional kernel. There are different ways of computing cohomology kernel vectors. An algebraic approach consists of computing a

Cholesky decomposition with pivoting so that zero diagonal entries are accumulated at the end of the process. The resulting decomposition allows a computation of the kernel vectors. If this factorisation is done approximately, we do not only get the kernel but also a preconditioner. Another way is a topological consideration on the discretisation. There are several methods computing cuts of the topology in order to discretise cohomology fields [6]–[8]. We will apply them to construct representatives of the kernel.

### III. HIERARCHICAL PRECONDITIONING

In recent development of numerical linear algebra hierarchical matrices ( $\mathcal{H}$ -matrices) were applied to dense matrices arising from integral equations.  $\mathcal{H}$ -matrices are based on a geometrical clustering of the degrees of freedom so that the matrix can be partitioned into small blocks  $K|_{\Omega'} \in \mathbf{R}^{n \times m}$  where low-rank approximation  $K|_{\Omega'} = AB^T$  with  $A \in \mathbf{R}^{n \times r}$  and  $B \in \mathbf{R}^{m \times r}$ ,  $r \ll n$  can be applied.

In the context of sparse matrices, the idea of hierarchical approximation can be reused in order to approximate the much more populated matrix of the Cholesky decomposition. This was done for general elliptic differential equations by [4].

The theory of exact decomposition of sparse matrices contains several strategies of reordering rows and columns of a matrix in order to reduce memory requirement. One of them, called nested dissection [10], provides a reordering which partitions the matrix so that zero blocks occur. This is done in graph theoretical way.

We will reuse this idea and the information of the geometry corresponding to the degrees of freedom to build a so called interface clustering [5]. The clustering algorithm consists of recursive repeats of the two steps: 1. Geometrical bisection; 2. Construction of the interface cluster. With this, a reordering of the matrix is reached which is schematically shown in Figure 1.

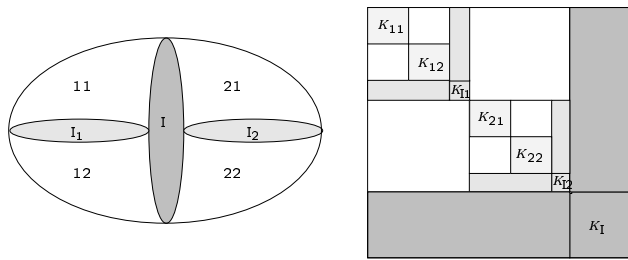


Fig. 1. Hierarchical interface clustering, geometry (left), matrix partitioning (right)

In the  $\mathcal{H}$ -matrix based decomposition  $\mathcal{H}$ -LLT, we apply the interface clustering for computation of the partitioning of the matrix. With the  $\mathcal{H}$ -arithmetic the operations are performed approximately.

The second new way to apply low-rank approximation in order to get an approximate decomposition is the method called HSILLT. It is also based on the interface clustering. The matrix is organised in block rows given by the leaves of the interface clustering. The decomposition

process works block column wise. The approximation is done by two parameters  $\varepsilon_{drop}$  and  $\varepsilon_{appr}$ . The first one controls the zero bound of small sub-diagonal block rows. The second one is the bound for low-rank approximation of the Schur complement.

### IV. NUMERICAL EXAMPLES

In order to evaluate the presented methods we want to compare the performance by means of electromagnetic field problems with non-trivial topologies as they occur in the development process.

In this case, we construct a positive semidefinite matrix  $K'$  considering the kernel of  $Q$  due to the discrete gradients. The regularised matrix  $K = K' + VV^T$  with  $V \in \mathbf{R}^{k_E \times l}$  due to cohomology contribution is given. An approximate Cholesky decomposition  $M(K')$  is constructed by HSILLT or  $\mathcal{H}$ -LLT with a coarse accuracy. An iterative Krylov subspace method is used in order to solve the preconditioned problem

$$[M(K')]^{-1}(K' + VV^T)x = [M(K')]^{-1}b. \quad (3)$$

The application to a physical example profiting from the new algorithms compared to other known preconditioner concepts will be given, so that the efficiency of the method can be seen.

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