# Coupled FETI/BETI solvers for nonlinear potential problems in bounded and unbounded domains

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***Abstract*—In nonlinear magnetic field computations, one is not only faced with large jumps of coefficients across material interfaces but often also with high variation of coefficients inside homogeneous material. We present an efficient solver for nonlinear potential problems, the core part of which is a special FETI preconditioner for the case of high variation but moderate anisotropy in the coefficients. In electromagnetics, the coupling FEM and BEM can be very useful, for instance in the cases of unbounded domains and small air gaps.**

**Keywords—Nonlinear potential problems, FEM, BEM, domain decomposition, inexact Newton methods**

## I. Introduction

Domain decomoposition (DD) methods like the rather popular Finite Element Tearing and Interconnecting (FETI) methods [1], dual-primal FETI (FETI-DP) methods [2] and Balanced Domain Decomposition by Constraints (BDDC) techniques [3] offer preconditioners that result in a condition number proportional to $(1 + \log(H/h))^2$, where $h$ is the average mesh size and $H$ the maximal diameter of the subdomains. Moreover, the preconditioners are robust with respect to jumps in the coefficients across subdomain interfaces, cf. [4].

Recently, Langer and Steinbach have introduced the Boundary Element Tearing and Interconnecting methods as a boundary element counterpart of the FETI method [5] and the coupled FETI/BETI methods [6].

Coupling finite and boundary elements, one can benefit from the advantages of both techniques, for instance in electromagnetics, the boundary element method (BEM) allows a rather comfortable treatment of unbounded domains and air gaps, whereas source terms and nonlinearities can be modelled with the finite elements.

Applying Newton's method, the spectrum of the Jacobi matrices in the nonlinear subdomains may show high variation, especially due to corner singularities in the solution. A special FETI preconditioner is proposed to overcome these problems.

## II. Coupled FETI/BETI Methods

Let $\Omega \subset \mathbf{R}^d$ (where $d = 2,\, 3$) be a bounded Lipschitz domain with the boundary $\Gamma$ and the outward unit normal vector $n$. We further assume a regular non-overlapping decomposition $\overline{\Omega} = \bigcup_i \overline{\Omega}_i$ with the local boundaries $\Gamma_i = \partial\Omega_i$ and the interfaces $\Gamma_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j$. The outward unit normal vector on $\Gamma_i$ is denoted by $n_i$. We consider the following homogeneous Poisson problem with piecewise constant coefficients: Find $u$ such that

$$\begin{aligned} &-\operatorname{div}(\alpha_i \nabla u) = f \quad \text{in } \Omega_i\,, \\ &u = 0 \quad \text{on } \Gamma\,, \qquad \alpha_i \tfrac{\partial u}{\partial n_i} + \alpha_j \tfrac{\partial u}{\partial n_j} = 0 \quad \text{on } \Gamma_{ij}\,, \end{aligned} \tag{1}$$

with $\alpha_i = \text{const}$. The solution $u_i$ of a local sub-problem

$$-\operatorname{div}(\alpha_i \nabla u_i) = 0 \quad \text{in } \Omega_i, \qquad u = g \quad \text{on } \Gamma_i\,, \tag{2}$$

defines the Steklov-Poincaré operator

$$S_i g := \alpha_i \tfrac{\partial u_i}{\partial n_i}\,, \tag{3}$$

mapping the Dirichlet trace $g$ to the corresponding Neumann trace. This operator can be approximated by the FEM using the Schur complement $S_{i,h}^{FEM}$ of the FEM stiffness matrix eliminating inner unknowns, or using a symmetric approximation $S_{i,h}^{BEM}$ by the BEM, cf. [5], [6].

Introducing separate variables $u_i$ on the local discrete spaces $V_h(\Gamma_i)$ one can re-enforce the continuity over the interaces $\Gamma_{ij}$ by constraints $\sum_i B_i u_i = 0$, where $B_i$ contains only entries $1$, $-1$, $0$. The discrete dual FETI/BETI formulation of problem (1) is of the form: Find the Lagrange parameter $\lambda$ such that $P^T F \lambda = d$, where $P$ is a special projection operator addressing the kernels of the sub-problems and the Dirichlet boundary condition, and $F$ denotes the FETI/BETI operator defined by

$$F = \sum_i B_i \big[S_{i,h}^{FEM/BEM}\big]^\dagger B_i^\top\,. \tag{4}$$

The preconditioner

$$\begin{aligned} M_{S,\alpha}^{-1} = (B C_\alpha^{-1} B^\top)^{-1} \big[ \textstyle\sum_i B_i C_{\alpha,i}^{-1} S_{i,h}^{FEM/BEM}\, C_{\alpha,i}^{-1} B_i^\top \big] \\ m \cdot (B C_\alpha^{-1} B^\top)^{-1} \end{aligned} \tag{5}$$

fulfills the condition estimate

$$\kappa(P M_{S,\alpha}^{-1} P^T P^T F P) \le C(1 + \log(H/h))^2\,, \tag{6}$$

independent of the jumps in the coefficients $\alpha_i$. Essential for this feature is the scaling matrix $C_\alpha$, involving weighted mean values of $\alpha_i$ on corner points, edges and faces between the subdomains, cf. [4], [6].

Now, on some subdomains (which are discretized by the FEM), we consider instead of a constant coefficient $\alpha_i$ a varying matrix coefficient $A_i(x)$ which we assume to be constant on the finite elements $T \in \mathcal{T}_{i,h}$. In order to determine the amount of variance, we introduce the spectral variance measure

$$m_{SV}(A_i) := \frac{\sup_{x\in\Omega_i} \overline{\alpha}_i(x)}{\inf_{x\in\Omega_i} \underline{\alpha}_i(x)}, \tag{7}$$

where $\overline{\alpha}_i(x)$ and $\underline{\alpha}_i(x)$ denote the maximal and minimal local eigenvalues of $A_i(x)$, respectively. The application of a preconditioner with Steklov-Poincaré operators corresponding to constant coefficients leads to a condition number being proportional to $\max_i m_{SV}(A_i)$, which is not at all acceptable in magnetostatic applications. We propose a new preconditioner $\widehat{M}_{S,A}^{-1}$ based on a varying scalar coefficient $\widehat{\alpha}_i(x)$ together with a suitable scaling matrix $\widehat{C}_\alpha$. If the local anisotropy

$$m_{anis}(A_i) := \sup_{x\in\Omega_i} \frac{\overline{\alpha}_i(x)}{\underline{\alpha}_i(x)}, \tag{8}$$

is moderate, our new preconditioner works fine, see [7].

## III. Nonlinear Problems

We now consider the following nonlinear magnetostatic model problem: Find $u$ such that

$$\begin{aligned} -\mathrm{div}[\nu_i(|\nabla u|)\nabla u] &= f \quad \text{in } \Omega_i\,, \\ u &= 0 \quad \text{on } \Gamma\,, \\ \nu_i(|\nabla u|)\tfrac{\partial u}{\partial n_i} + \nu_j(|\nabla u|)\tfrac{\partial u}{\partial n_j} &= 0 \quad \text{on } \Gamma_{ij}\,. \end{aligned} \tag{9}$$

Assuming that the reluctivity curves $\nu_i : [0,\infty) \to (0,\infty)$ are strongly monotonic and $\mathcal{C}^2$, (9) is uniquely solvable in the weak sense and the corresponding Newton iteration converges locally at quadratic rate, cf. [8]. We mention, that such material curves can be generated also from noisy measurements in a robust way by a special interproximation technique [9]. In the linearized problems a varying matrix coefficient $\zeta_i(\nabla u_h^{(k)}(x))$ appears. In our numerical experiments it turns out that typically the anisotropy measure $m_{anis}(\zeta_i(\nabla u_h^{(k)}))$ is small, whereas the spectral variance measure $m_{SV}(\zeta_i(\nabla u_h^{(k)}))$ is high. Hence, with our new preconditioner $\widehat{M}_{S,A}^{-1}$ such linearized Newton-problems can be solved satisfactorily. In order to get a good initial guess, it is convenient to set up a hierarchy of nested grids and use coarse grid solutions as initial guesses on finer levels. In Fig. 1 we show a computation on a 2D magnetic valve model problem, cf. [7].

## IV. Unbounded Domains

Instead of the Dirichlet boundary condition in (9), we consider the exterior problem with a suitable radiation condition, i. e.

$$\begin{aligned} -(1/\mu_0)\Delta u &= 0 \quad \text{in } \Omega^{\mathsf{c}}\,, \\ |u(x)| &= \mathcal{O}(|x|^{-1}) \quad \text{for } |x| \to \infty\,, \end{aligned} \tag{10}$$

where $\Omega^{\mathsf{c}} = \mathbf{R}^d \backslash \bar{\Omega}$ denotes the exterior space and $\mu_0$ is the permeability of vacuum. In this case, the Dirichlet to Neumann map on $\Omega^{\mathsf{c}}$ can also be characterized by a Steklov Poincaré operator approximation $S_{\mathsf{c},h}^{BEM}$. The proof of the condition estimate (6) is based on spectral equivalences of $S_{i,h}^{FEM/BEM}$ to the $H^{1/2}(\Gamma_i)$-semi-norm. Since $S_{\mathsf{c},h}^{BEM}$ is equivalent to this semi-norm as well, at least on the corresponding sub-space, an extension of the result to the case of (10) by similar techniques is possible.

We observe, that in Newton's iteration, a coarse solution of a homogeneous Dirichlet problem serves as a good initial guess for the extended problem on finer levels.


Fig. 1. Reluctivity $\nu$ in the ferromagnetic parts of a magnetic valve, logarithmic scale.

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