Initial-Boundary Value Problems of Warped Multirate Partial Differential Algebraic Equations

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***Abstract*—In radio frequency applications, oscillatory signals with widely separated time rates arise in electric circuits. A multidimensional modelling of these signals enables an alternative approach for a numerical simulation. The representation of frequency modulated signals leads to warped multirate partial differential algebraic equations, where initial-boundary value problems are considered. The selection of an appropriate local frequency function is crucial for the efficiency of the model. We present strategies based on minimisation criteria for determining a suitable local frequency function. Numerical methods using the resulting conditions are discussed.**

Keywords—multirate partial differential algebraic equation, initial-boundary value problem, local frequency, circuit simulation, frequency modulation.

I. Multidimensional Signal Model

To outline the multidimensional model, we consider the frequency modulated (FM) signal

$$x(t) = \left[1 + \alpha \sin\left(\tfrac{2\pi}{T_1} t\right)\right] \cdot \sin\left(\tfrac{2\pi}{T_2} t + \beta \sin\left(\tfrac{2\pi}{T_1} t\right)\right) \quad (1)$$

with $T_1 \gg T_2$. The parameters α and β introduce amplitude and frequency modulation, respectively. Hence a huge number of oscillations arises in the time interval $[0, T_1]$, which limits the size of time steps for resolving the signal. However, we can assign an own variable for each time scale. Consequently, we obtain a *multivariate function* (*MVF*) of the signal (1), namely

$$\begin{aligned} \hat{x}_1(t_1, t_2) &= \left[1 + \alpha \sin\left(\tfrac{2\pi}{T_1} t_1\right)\right] \\ &\quad \cdot \sin\left(2\pi t_2 + \beta \sin\left(\tfrac{2\pi}{T_1} t_1\right)\right), \end{aligned} \quad (2)$$

where the period of the fast scale t_2 is standardised to 1. The original signal (1) can be reconstructed completely via $x(t) = \hat{x}_1(t, t/T_2)$. Fig. 1 illustrates this MVF. Unfortunately, many oscillations arise in the domain of dependence. Thus the straightforward representation (2) is inefficient. Alternatively, we just include the amplitude modulation part of the signal (1) in the MVF and achieve

$$\hat{x}_2(t_1, t_2) = \left[1 + \alpha \sin\left(\tfrac{2\pi}{T_1} t_1\right)\right] \cdot \sin(2\pi t_2). \quad (3)$$

Fig. 2 demonstrates that this representation is appropriate, since just one oscillation proceeds in each coordinate direction. The frequency modulation part of (1) is described by the time-dependent *warping function*

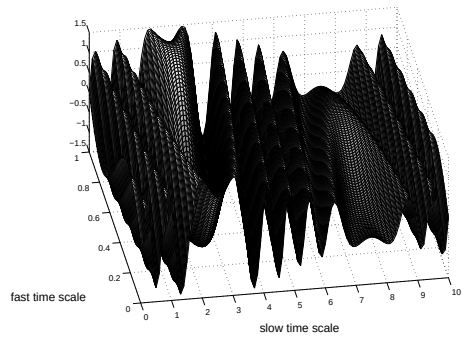

Fig. 1. Naive MVF of FM signal.

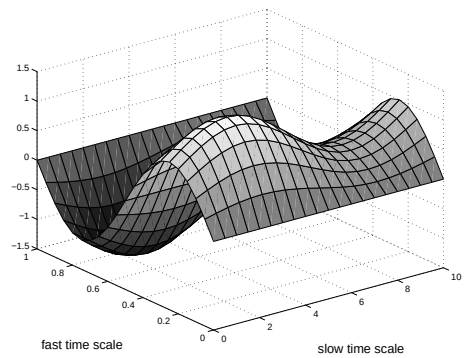

Fig. 2. Adequate MVF of FM signal.

$$\Psi(t) = \tfrac{1}{T_2} t + \tfrac{\beta}{2\pi} \sin\left(\tfrac{2\pi}{T_1} t\right). \quad (4)$$

Now the reconstruction of (1) reads $x(t) = \hat{x}_2(t, \Psi(t))$. The derivative $\nu := \Psi'$ can be seen as a *local frequency* of the signal. The inefficient MVF (2) corresponds to the local frequency $\nu \equiv 1/T_2$, which is not reasonable.

This discussion shows that the multidimensional model of a FM signal is not unique. Different local frequency functions imply different MVFs. Inappropriate choices of the local frequencies produce undesired oscillations in the corresponding MVF. The presented model is also applicable, if only the fast time scale is periodic, whereas the slow time scale is aperiodic.

II. Warped MPDAEs

The mathematical model of electric circuits yields *differential algebraic equations* (*DAEs*), which we write in the form

$$\frac{\mathrm{d}\mathbf{q}(\mathbf{x})}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}(t)) + \mathbf{b}(t). \tag{5}$$

Thereby, the solution $\mathbf{x}$ represents unknown node voltages and branch currents. We assume that the signal $\mathbf{x}$ exhibits a time behaviour as outlined in the previous section. The application of the multidimensional model transforms the system of DAEs into a multiscale system. Brachtendorf et al. [1] introduced the corresponding system of *multirate partial differential algebraic equations* (*MPDAEs*) in case of constant time rates. Considering the model for FM signals from the previous section, Narayan and Roychowdhury [2] constructed the corresponding system of *warped MPDAEs*

$$\frac{\partial \mathbf{q}(\hat{\mathbf{x}})}{\partial t_1} + \nu(t_1)\frac{\partial \mathbf{q}(\hat{\mathbf{x}})}{\partial t_2} = \mathbf{f}(\hat{\mathbf{x}}(t_1,t_2)) + \mathbf{b}(t_1), \tag{6}$$

where $\hat{\mathbf{x}}$ denotes the MVF of $\mathbf{x}$. An appropriate local frequency function ν is unknown a priori, too.

The determination of quasiperiodic solutions of the DAEs (5) yields biperiodic boundary conditions for the MPDAEs (6). Considering the initial value problem $\mathbf{x}(0) = \mathbf{x}_0$ of (5), we obtain the *initial-boundary value problem*

$$\begin{array}{rcll} \hat{\mathbf{x}}(0,t_2) & = & \mathbf{h}(t_2) & \text{for all } t_2, \\ \hat{\mathbf{x}}(t_1,t_2+1) & = & \hat{\mathbf{x}}(t_1,t_2) & \text{for all } t_1 \geq 0,\ t_2, \end{array} \tag{7}$$

where the predetermined function $\mathbf{h}$ fulfills $\mathbf{h}(0) = \mathbf{x}_0$. Consequently, we achieve a solution of the DAEs (5) via the reconstruction $\mathbf{x}(t) = \hat{\mathbf{x}}(t, \int_0^t \nu(s)\,\mathrm{d}s)$.

III. Minimisation Criteria

Solutions of the system (6), which exhibit the same initial values, are connected by a certain transformation. Surprisingly, a MVF $\hat{\mathbf{x}}$ satisfying (7) exists for arbitrary given local frequency function ν. The problem is to find an adequate choice, since inappropiate frequencies produce inefficient MVFs as shown in the first section. In [2], phase conditions are successfully applied to identify suitable local frequencies. However, the efficiency of this strategy can not be guaranteed in general.

Alternatively, Houben [3] proposes a minimisation criterion of the form

$$s(t_1) := \int_0^1 \left\| \frac{\partial \mathbf{q}(\hat{\mathbf{x}})}{\partial t_1} \right\|^2 \mathrm{d}t_2 \quad \to \quad \min. \tag{8}$$

for each $t_1 \geq 0$ using the Euclidean norm, which shall avoid unessential oscillations. This approach produces a necessary condition for an optimal solution, which causes an explicit formula for the local frequency function in dependence on the MVF. However, the minimisation (8) is based on the charge term $\mathbf{q}(\hat{\mathbf{x}})$ instead of the MVF $\hat{\mathbf{x}}$ itself, since a direct application of information from the MPDAE (6) becomes feasible in this case.

We want to achieve more efficiency and flexibility by demanding the minimisation ($\hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_k)^\top$)

$$p(t_1) := \int_0^1 \sum_{l=1}^{k} w_l \left(\frac{\partial \hat{x}_l}{\partial t_1}\right)^2 \mathrm{d}t_2 \quad \to \quad \min. \tag{9}$$

for each $t_1 \geq 0$, where $w_1, \ldots, w_k \geq 0$ represent constant weights. Hence the minimisation is based on a weighted norm of the derivative of the solution $\hat{\mathbf{x}}$ itself. In this case, we do not obtain an explicit formula for the local frequency. Instead, a variational calculus based on the transformation of solutions to the system (6) yields a necessary condition for the MVF. Thus the corresponding local frequency function is determined indirectly.

IV. Numerical Methods

For solving biperiodic boundary value problems of the MPDAE system (6), a specific method of characteristics represents a suitable technique, see [4]. In case of initial-boundary conditions (7), this strategy becomes inefficient. Thus we focus on numerical methods, which are produced by semidiscretisation. In [3], a *method of lines* has been used successfully, where the condition from the criterion (8) has been included. Thus we observe corresponding schemes, too. For example, using asymmetric differences of second order yields the system of DAEs

$$\begin{aligned} \frac{\partial \mathbf{q}(\tilde{\mathbf{x}}_j)}{\partial t_1} &= \mathbf{f}(\tilde{\mathbf{x}}_j) + \mathbf{b}(t_1) \\ &\quad - \frac{\nu(t_1)}{h_2}\left[\frac{3}{2}\mathbf{q}(\tilde{\mathbf{x}}_j) - 2\mathbf{q}(\tilde{\mathbf{x}}_{j-1}) + \frac{1}{2}\mathbf{q}(\tilde{\mathbf{x}}_{j-2})\right] \end{aligned} \tag{10}$$

for $j = 1, \ldots, m$ with approximations $\tilde{\mathbf{x}}_j(t_1) \doteq \hat{\mathbf{x}}(t_1, jh_2)$. The periodic boundary conditions are used to identify some approximations. Thus we obtain a system of mk DAEs for mk unknown functions.

Alternatively, discretisations analogue to the *Rothe method* for parabolic PDEs are feasible. In each case, the condition from the minimisation criterion (8) or (9) has to be included in discretised form.

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