The energy viewpoint in computational electromagnetics

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***Abstract*—In a recent paper, an energy-based theory of electromagnetism was proposed. The fundamental postulate has the form of a diagram of interconnected energy reservoirs and the completely covariant equations stating energy conservation in this diagram are shown to be a combination of Maxwell's equations with the constitutive laws of the material. This formulation clarifies several issues related to the computation of dissipative and coupled phenomena in magnetic materials, dielectrics and conductors.**

Keywords—Electromagnetism, energy, forces, superconductors, magnetostriction, duality, inductances.

I. Introduction

Maxwell's equations are generally presented as the fundamental equations that rule electromagnetic (EM) phenomena. But they address only one side of the question: they do not provide any energy conservation rule and they leave all material aspects aside. Consequently, they need to be complemented by constitutive laws, which are often regarded as *ad hoc* relations to close the system, not directly subjected to physical laws. In order to tackle with multi-physics problems in a consistent way, it is however desirable, if not necessary, to dispose of a theory of electromagnetism where energy aspects are involved from the beginning and throughout. This paper recalls briefly the theory presented in [1] and develops in more details the aspects related to multiphysics computations in the presence of electromagnetic fields. The issues of electromagnetic forces, charges and superconductors, material laws (hysteresis and striction), duality, and the definition of lumped parameters are analysed in the light of the energy approach.

II. Theory

A. Energy diagram

Let us state as a postulate that, in an arbitrary material region M, electromagnetic energy flows according to the diagram depicted in Fig. 1. The diagram consists of four energy reservoirs. Each reservoir is associated with a state variable, resp. the magnetic vector potenrial A, the electric displacement D, the current density J and the electric scalar potential U, from the upper left to the lower right corner. The $A-$reservoir contains the magnetic energy

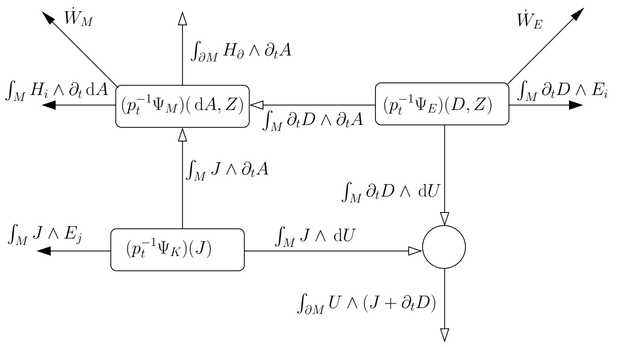

Fig. 1. Electromagnetic energy diagram in the material domain M.

$(p_t^{-1}\Psi_M)(\,\mathrm{d}A, Z)$, a function of the induction $\mathrm{d}A$ (the exterior derivative d play here the role of the curl operator) but possibly also of one or several other fields represented in a generic way by the unspecified variable Z. The $D-$reservoir contains the electric energy $(p_t^{-1}\Psi_E)(D, Z)$, a function of the electric displacement D and possibly of some other fields Z as well. The most obvious interpretation of Z is the mechanical strain ε but there might be dependencies of other origins as well, e.g. chemical (Think of a battery). The $U-$reservoir is always empty. The $J-$reservoir finally, contains the kinetic energy of the charge carriers. If m_c denote the mass of one charge carrier, q_c its charge, ρ_c the density of charge carriers and $\mathbf{V}_c$ their velocity field with respect to the crystal lattice, the current density in E^3 is $\mathbf{j} = q_c\rho_c(\mathbf{v}+p_t\mathbf{V}_c)$, and the kinetic energy density writes $\rho_K^\Psi(\mathbf{j}) = \rho_c m_c(\mathbf{v}+p_t\mathbf{V}_c)^2/2 = \alpha\mathbf{j}^2/2$ in E^3, with $\alpha = m_c/(\rho_c q_c^2)$.

The energy flows presented in Fig. 1 can be classified into four categories. The white-headed arrows represent 4 internal volume flows depending on the state variables only, and 2 surface flows depending on the state variables and on a boundary magnetic field H_∂. The black-headed arrows represent 3 dissipative volume flows involving the state variables (U excepted) and and empirical dissipative generalised forces H_i, E_i and E_j of which the exact physical interpretations are discussed below. Finally, 2 flows, $\dot{W}_M$ and $\dot{W}_E$, connect the electromagnetic energy diagram with the exterior and account respectively for electric and magnetic energy converted into non-electromagnetic forms of energy (e.g. mechanical, chemica...).

The structure of this diagram and the mathematical expression of the flows constitute the foundation of this theory. They tell something fundamental about the Universe and how electromagnetic fields behave and interact with matter and spacetime.

B. Conservation equations

As the fields A, D, J and U are independent variables, they can be varied freely in order to obtain, following a variational line of argument, the conservation equations implied by the structure of the diagram. One obtains for an arbitrary region Ω of which the relative velocity field with respect to the material domain M is $\mathbf{v}$

$$\begin{aligned} \operatorname{curl}\bar{\mathbf{h}} &= \mathbf{j} + \mathcal{L}_{\mathbf{v}}\,\mathbf{d} && (1)\\ \bar{\mathbf{e}} &= -\,\mathcal{L}_{\mathbf{v}}\,\mathbf{a} - \operatorname{grad} u && (2)\\ \mathbf{e}_j + \alpha\,\mathcal{L}_{\mathbf{v}+\mathbf{v}_c}\,\mathbf{j} &= -\,\mathcal{L}_{\mathbf{v}}\,\mathbf{a} - \operatorname{grad} u && (3)\\ 0 &= \operatorname{div}\,(\mathbf{j} + \mathcal{L}_{\mathbf{v}}\,\mathbf{d}) && (4) \end{aligned}$$

where $\mathcal{L}_{\mathbf{v}}$ denotes the co-moving time derivative [4] and with the magnetic and electric fields *defined* by

$$\begin{aligned} \bar{\mathbf{h}} &\equiv \left(\partial_{\mathbf{b}}\,\rho^{\Psi}_{M}\right)(\operatorname{curl}\mathbf{a}, z) + \mathbf{h}_i, && (5)\\ \bar{\mathbf{e}} &\equiv \left(\partial_{\mathbf{d}}\,\rho^{\Psi}_{E}\right)(\mathbf{d}, z) + \mathbf{e}_i. && (6) \end{aligned}$$

The boundary condition writes $\bar{\mathbf{h}} = \mathbf{h}_{\partial}$ on $\partial\Omega$ and the equations

$$\begin{aligned} \dot{W}_M &= \int_{\Omega} \left(\mathcal{L}_{\mathbf{v}}\,\rho^{\Psi}_{M}\right)(\operatorname{curl}\mathbf{a}, z)\\ &+ \int_{\Omega} \left(\partial_z\rho^{\Psi}_{M}\right)(\operatorname{curl}\mathbf{a}, z) \otimes \mathcal{L}_{\mathbf{v}}\,z && (7)\\ \dot{W}_E &= \int_{\Omega} \left(\mathcal{L}_{\mathbf{v}}\,\rho^{\Psi}_{E}\right)(\mathbf{d}, z)\\ &+ \int_{\Omega} \left(\partial_z\rho^{\Psi}_{M}\right)(\operatorname{curl}\mathbf{a}, z) \otimes \mathcal{L}_{\mathbf{v}}\,z && (8) \end{aligned}$$

for getting the balance right.

III. Discussion

We have so gathered all necessary elements to discuss a number of issues related to multiphysics computations in the presence of electromagnetic fields.

A. Electromagnetic forces

By setting $\partial_t A = \mathcal{L}_{\mathbf{v}}\,\mathbf{a} = 0$, the $A-$reservoir is isolated from the rest of the electromagnetic energy diagram, Fig. 1. Similar considerations hold for the $D-$reservoir, so that the variation of energy

$$\dot{W}_M + \dot{W}_E = \partial_t\Psi_M|_{\mathcal{L}_{\mathbf{v}}\,\mathbf{a}=0} + \partial_t\Psi_E|_{\mathcal{L}_{\mathbf{v}}\,\mathbf{d}=0} \qquad (9)$$

represents the power converted into non-electromechanical forms of energy (mechanical, chemical...), the conditions $\mathcal{L}_{\mathbf{v}}\,\mathbf{a} = 0$ and $\mathcal{L}_{\mathbf{v}}\,\mathbf{d} = 0$ being the precise mathematical expression of what is usually stated "holding (magnetic or electric) fluxes constants". Formal expressions of $\dot{W}_M$ and $\dot{W}_E$ are given by (7) and (8). In particular, factorising $\nabla\mathbf{v}$, leads to the definition of the Maxwell stress tensor of the material, which is the fundamental quantity representing the electromechanical coupling and the unifying ingredients of virtually all force formulae used in numerical computations [2].

B. Charges and superconductors

Electric charges are not explicitly present in the diagram, nor in the conservation equations. They are *defined* by

$$\rho^{Q} = \mathrm{d}D = \operatorname{div}\mathbf{d}. \qquad (10)$$

Equation (3) is the equilibrium equation for charge carriers, up to a factor q_c. The dynamics of charges is thus made by the energy-based approach an integral part of the theory. The term $-\operatorname{grad} u$ is the applied electrostatic force and the term $\mathbf{e}_j = \sigma^{-1}\mathbf{j}$ is the viscous force opposed by the crystal lattice. When the charge carrier accelerates, a certain amount of energy is given to increase its kinetic energy and another amount of energy to increase the magnetic energy of the system, as the accelerated particle is associated with a larger current, which in turn generates a larger magnetic field. These two energy transfers are respectively represented by the forces (up to the factor q_c again) $\alpha\,\mathcal{L}_{\mathbf{v}+p_t\mathbf{v}_c}\,\mathbf{j}$ and $\partial_t A$ that can be regarded as two inertial forces of different natures.

In practice, the $J-$reservoir can often be considered as being empty as well, because of the very small value of α (negligible inertia of the charge carriers), and the corresponding term in (3) can be disregarded. However, in superconductors, for which σ is infinite ($\mathbf{e}_j = 0$) and $\operatorname{grad} u$ is zero, (3) reads

$$\alpha\,\mathcal{L}_{\mathbf{v}+p_t\mathbf{v}_c}\,\mathbf{j} = -\,\mathcal{L}_{\mathbf{v}}\,\mathbf{a}. \qquad (11)$$

If the cloud of charge is not too much distorted, one has $\mathcal{L}_{\mathbf{v}} \equiv \partial_t$, so that London's equation for superconductors $\mathbf{a} = -\alpha\mathbf{j}$ is found back.

The inertia of the charge carrier is also at the root of the definition of the static charges that are present at the surface of current carrying conductors [5]. Identifying the left hand sides of (2) and (4) and assuming $\mathbf{e}_i = 0$, one has

$$\left(\partial_{\mathbf{d}}\,\rho^{\Psi}_{E}\right)(\mathbf{d}) = \varepsilon_0^{-1}\mathbf{d} = \sigma^{-1}\mathbf{j} + \alpha\,\mathcal{L}_{\mathbf{v}+p_t\mathbf{v}_c}\,\mathbf{j}. \qquad (12)$$

The divergence of the right-hand side is identically zero (div and $\mathcal{L}_{\mathbf{v}}$ commute) inside the conductor, but the term in α has a non-zero contribution on the surface of the conductor, whence the expression $\varepsilon_0\alpha\,\mathcal{L}_{\mathbf{v}+p_t\mathbf{v}_c}\,\mathbf{j}\cdot\mathbf{n}$ of the surface charges.

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