Symmetric Coupling of Finite-Element and Boundary-Element Method for Electro-Quasistatic Field Simulations

T. Steinmetz[1], N. Gödel[1], G. Wimmer[1], M. Clemens[1], S. Kurz[1], M. Bebendorf[2] and S. Rjasanow[3]
[1]School of Electrical Engineering, Helmut-Schmidt-University, D-22043 Hamburg, Germany
[2]Mathematical Institute, University of Leipzig, Leipzig D-04109, Germany
[3]Mathematical Institute, Saarland University, Saarbrücken D-66041, Germany

***Abstract*—The electrodynamic simulation of high-voltage technical devices can be performed under the electro-quasistatic assumption. In order to avoid large spatial discretization domains, a Finite-Element-Method (FEM) is coupled to a Boundary-Element-Method (BEM) which implicitly asserts the electrophysical asymptotic attenuation condition. A symmetric FEM-BEM coupled formulation in time domain is presented. First numerical results are shown for the simulation of a three dimensional high-voltage application.**

Keywords—transient electro-quasistatic fields, FEM, BEM, symmetric coupling

I. Introduction

Transient simulations under the electro-quasistatic assumption, where the time derivative of the magnetic flux density in the induction law is omitted, can be performed for the analysis of technical devices for which electromagnetic wave propagation effects are negligible and where the electric energy density of the problem is much greater than the magnetic energy density. Typically, these conditions are valid for applications from high-voltage technology or microelectronics. Electro-quasistatic simulations have already been presented using volume-based discretization schemes e.g. in [1].

II. Transient electro-Quasistatic Fields

Introducing the electro-quasistatic assumption $\partial_t \vec{B} = 0$ into Maxwell's equations, a scalar potential function φ exists which allows to compute the resulting irrotational electric field strengths $\vec{E}$ via $\vec{E} = -\operatorname{grad}\varphi$. As a consequence, the governing differential equation for electro-quasistatic fields reads

$$-\operatorname{div}\left((\kappa + \varepsilon\partial_t)\operatorname{grad}\varphi\right) \;=\; 0.$$

Here, the electric conductivity is denoted by κ, while the electric permittivity is denoted by ε.

III. Domain Decomposition

A. Model Problem

For the mathematical modelling of technical problems, the model problem is defined as follows:

$$-\operatorname{div}(\kappa + \varepsilon\partial_t)\operatorname{grad}\varphi = 0 \quad \text{in} \quad \Omega_{\mathrm{FEM}}, \tag{1}$$

$$-\operatorname{div}(\varepsilon_0\partial_t)\operatorname{grad}\varphi = 0 \quad \text{in} \quad \Omega_{\mathrm{BEM}}, \tag{2}$$

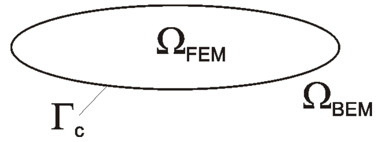

Fig. 1. Geometry of the model problem

in the unbounded domain $\Omega = \Omega_{\mathrm{FEM}} \cup \Gamma_{\mathrm{c}} \cup \Omega_{\mathrm{BEM}}$ with the interface boudary Γ_{c}. Furthermore, $\overline{\Omega}_{\mathrm{FEM}} = \Omega_{\mathrm{FEM}} \cup \Gamma_{\mathrm{c}}$ and $\overline{\Omega}_{\mathrm{BEM}} = \Omega_{\mathrm{BEM}} \cup \Gamma_{\mathrm{c}}$ holds.

B. Finite-Element Formulation

For the closure of Ω_{FEM}, the standard variational formulation of (1) can be achieved by multiplication with a trial function v and application of Green's first integral theorem:

$$\begin{aligned} &\int\limits_{\Omega_{\mathrm{FEM}}} (\operatorname{grad}\varphi)\ (\kappa + \varepsilon\partial_t)\ (\operatorname{grad} v)\mathrm{d}\Omega - \\ &\int\limits_{\Gamma_{\mathrm{c}}} (\kappa + \varepsilon\partial_t)\,\gamma_1^{\mathrm{int}}\varphi\ \gamma_0^{\mathrm{int}} v(\vec{r})\mathrm{d}\Gamma = 0, \end{aligned} \tag{3}$$

with the interior trace operator γ_0^{int} and the operator of the interior normal derivative γ_1^{int}. The second integral term allows the coupling to the boundary element formulation.

C. Boundary-Element Formulation

A symmetric formulation of the exterior boundary-value problem in eqn. (2) can be gained, if the complete Calderón projector

$$\mathcal{C}^{\mathrm{ext}} \;=\; \begin{pmatrix} \frac{1}{2}\mathcal{I} + \mathcal{K} & -\mathcal{V} \\ -\mathcal{D} & \frac{1}{2}\mathcal{I} - \mathcal{K}' \end{pmatrix}$$

with the factor 1/2 for smooth boundary points is assembled [2]. Here, the single layer potential operator $\mathcal{V}$, the hypersingular potential operator $\mathcal{D}$ and the double layer potential operator $\mathcal{K}$ and its adjoint $\mathcal{K}'$, respectively, are used. The identity operator is denoted by $\mathcal{I}$. Hence, using the exterior trace operator γ_0^{ext} and the operator of the exterior normal derivative γ_1^{ext}, the system of boundary integral equations

$$\begin{pmatrix} \gamma_0^{\mathrm{ext}}\partial_t\varphi \\ \gamma_1^{\mathrm{ext}}\partial_t\varphi \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\mathcal{I} + \mathcal{K} & -\mathcal{V} \\ -\mathcal{D} & \frac{1}{2}\mathcal{I} - \mathcal{K}' \end{pmatrix} \begin{pmatrix} \gamma_0^{\mathrm{ext}}\partial_t\varphi \\ \gamma_1^{\mathrm{ext}}\partial_t\varphi \end{pmatrix} \tag{4}$$

characterizes the solution of the model problem in the domain Ω_{BEM}.

IV. Symmetric Coupling

A. Continuous Formulation

The symmetric BEM formulation can be coupled to the finite-element formulation (3) by expressing $\gamma_1^{\mathrm{int}}\varphi$ by the second boundary integral equation of the system (4) and by applying the interface conditions:

$$\gamma_0^{\mathrm{int}}\varphi = \gamma_0^{\mathrm{ext}}\varphi, \tag{5}$$

$$(\kappa + \varepsilon\partial_t)\,\gamma_1^{\mathrm{int}}\varphi = (\varepsilon_0\partial_t)\,\gamma_1^{\mathrm{ext}}\varphi, \tag{6}$$

for all $\vec{r} \in \Gamma_{\mathrm{c}}$, [2, 3]. (6) expresses the normal continuity of the total (conduction + displacement) current density. According to (6), substituting $(\varepsilon_0\partial_t)\,\gamma_1^{\mathrm{ext}}\varphi$ for $(\kappa + \varepsilon\partial_t)\,\gamma_1^{\mathrm{int}}\varphi$ in (3), and inserting $\varepsilon_0\gamma_1^{\mathrm{ext}}\partial_t\varphi$ from (4) yields the variational equation

$$\int\limits_{\Omega_{\mathrm{FEM}}} (\operatorname{grad}\varphi)\,(\kappa + \varepsilon\partial_t)\,(\operatorname{grad} v)\mathrm{d}\Omega - \int\limits_{\Gamma_{\mathrm{c}}} \varepsilon_0 \left(-\mathcal{D}\gamma_0^{\mathrm{ext}}\partial_t\varphi + \left(\frac{1}{2}\mathcal{I} - \mathcal{K}'\right)\gamma_1^{\mathrm{ext}}\partial_t\varphi\right)\gamma_0^{\mathrm{int}} v\mathrm{d}\Gamma = 0. \tag{7}$$

Another variational equation is obtained from the first equation of (4), with another trial function τ:

$$\int\limits_{\Gamma_{\mathrm{c}}} \varepsilon_0 \left(\left(-\frac{1}{2}\mathcal{I} + \mathcal{K}\right)\gamma_0^{\mathrm{ext}}\partial_t\varphi - \mathcal{V}\gamma_1^{\mathrm{ext}}\partial_t\varphi\right)\tau\mathrm{d}\Gamma = 0. \tag{8}$$

B. Discrete Formulation

These variational formulations can be discretized using the Galerkin scheme, which results in finite-element stiffness matrices $\mathbf{A}$ for the electrical conductivity and $\mathbf{B}$ for the electrical permittivity. The potential operators of the boundary integral equations are discretized using the Galerkin-scheme, too, resulting in the single layer potential matrix $\mathbf{V}$, the hypersingular potential matrix $\mathbf{D}$ and the double layer potential matrix $\mathbf{K}$. Additionally, a mass matrix $\mathbf{M}$ is needed. This leads to the following discrete form of equations (7-8), which is a system of ordinary differential equations (ODE) in the time domain:

$$\begin{pmatrix} \mathbf{A}_{\mathrm{ff}} & \mathbf{A}_{\mathrm{fc}} & 0 \\ \mathbf{A}_{\mathrm{cf}} & \mathbf{A}_{\mathrm{cc}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{\Phi}_{\mathrm{f}} \\ \mathbf{\Phi}_{\mathrm{c}} \\ \mathbf{t} \end{pmatrix} + \begin{pmatrix} \mathbf{B}_{\mathrm{ff}} & \mathbf{B}_{\mathrm{fc}} & 0 \\ \mathbf{B}_{\mathrm{cf}} & \mathbf{B}_{\mathrm{cc}} + \mathbf{D} & \left(-\frac{1}{2}\mathbf{M}^T + \mathbf{K}^T\right) \\ 0 & \left(-\frac{1}{2}\mathbf{M} + \mathbf{K}\right) & -\mathbf{V} \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \mathbf{\Phi}_{\mathrm{f}} \\ \mathbf{\Phi}_{\mathrm{c}} \\ \mathbf{t} \end{pmatrix} = 0. \tag{9}$$

Herein, the vector of the degrees of freedom (DoF) is divided into three partitions. The first partition, Φ_{f}, represents the DoF inside the domain Ω_{FEM}. The second partition of this vector, Φ_{c}, consists of the DoF of the scalar potential on the interface boundary Γ_{c}. The third partition contains the DoF of the normal derivative values $\mathbf{t}$ of the scalar potential on the interface boundary Γ_{c} as well. The latter two partitions are needed to evaluate the scalar potential in Ω_{BEM} by Kirchhoff's representation formula.

V. Solution of the ODE System

The system (9) is of the form $\mathbf{Hx} + \mathbf{N}\dot{\mathbf{x}} = 0$. Hence, the time discretization can be performed using singly-diagonal-implicit-Runge-Kutta-method (see [1]), resulting in a symmetric but indefinite linear system of equations which can be solved by a preconditioned MinRes iterative solver.

VI. First Numerical Results

First numerical results of electrostatic FEM-BEM computations of an exterior Dirichlet problem are presented. For this purpose, a FEM simulation was performed first in accordance to the problem definition of an high-voltage surge arrester characterized in an IEC norm, [4]. The Dirichlet data on the boundary needed for the BEM simulations are obtained from the preceding FEM simulation. The difference in the scalar electric potential distributions, which are shown in Fig. (2), results substantially in the homogenous Dirichlet boundary condition for the FEM simulation according to the IEC norm in contrast to the asymptotic attenuation condition of the BEM simulation. The boundary for the BEM computation is discretized with 21401 nodes and 42814 boundary elements; therefore, a compression of the BEM matrix blocks is essential which is done by using the Adaptive-Cross-Approximation, which approximates the BEM matrix blocks by low-rank matrices, see [5].

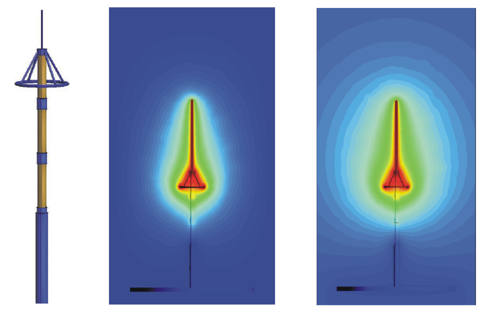

Fig. 2. Electrostatic simulations of a high-voltage surge arrester. From left: geometry, scalar potential computed by FEM with $\varphi = 0$ on the boundary and by FEM-BEM.

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