Adaptive Methods for Transient Noise Analysis

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***Abstract*—Stochastic differential algebraic equations (SDAEs) arise as a mathematical model for electrical network equations that are influenced by additional sources of Gaussian white noise.**
We discuss adaptive linear multi-step methods for their numerical integration, in particular stochastic analogues of the trapezoidal rule and the two-step backward differentiation formula. For the case of small noise we present a strategy for controlling the step-size in the numerical integration. It is based on estimating the mean-square local errors and leads to step-size sequences that are identical for all computed paths.
Test results illustrate the performance of the presented methods.

Keywords—transient noise analysis, stochastic differential algebraic equations, small noise, mean-square numerical methods, step-size control

I. Transient noise analysis in circuit simulation

The increasing scale of integration, high tact frequencies and low supply voltages cause smaller signal-to-noise ratios. In several applications the noise influences the system behavior in an essentially nonlinear way such that linear noise analysis is no longer satisfactory and transient noise analysis, i.e. the simulation of noisy systems in the time domain, becomes necessary.

We deal with the thermal noise of resistors as well as the shot noise of semiconductors that are modelled by additional sources of additive or multiplicative Gaussian white noise currents. Combining Kirchhoff's Current law with the element characteristics and using the charge-oriented formulation yields a stochastic differential algebraic equation (SDAE) of the form

$$A\frac{d}{dt}q(x(t)) + f(x(t),t) + \sum_{r=1}^{m} g_r(x(t),t)\xi_r(t) = 0 \ , \quad (1)$$

where A is a constant singular matrix determined by the topology of the electrical network and ξ is an m-dimensional vector of independent Gaussian white noise sources (see e.g. [1], [2]). One has to deal with a large number of equations as well as of noise sources. Compared to the other quantities the noise intensities $g_r(x,t)$ are small.

We understand (1) as a stochastic integral equation

$$\begin{aligned} Aq(X(s))|_{t_0}^{t} + \int_{t_0}^{t} f(X(s),s)ds \\ + \sum_{r=1}^{m}\int_{t_0}^{t} g_r(X(s),s)dW(s) &= 0 \ , \quad (2) \end{aligned}$$

where the second integral is an Itô-integral, and W denotes an m-dimensional Wiener process (or Brownian motion) given on the probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)_{t\geq t_0}$. The solution is a stochastic process depending on the time t and on the random sample ω. Typical paths are nowhere differentiable. Using techniques from the theory of DAEs as well as of the theory of stochastic differential equations (SDEs) one derives existence and uniqueness for the solutions as well as convergence results for certain drift-implicit methods for systems with DAE-index 1.

II. Numerical Methods

We discuss stochastic analogues of the two-step backward differentiation formula (BDF_2) and the trapezoidal rule, where only the increments of the driving Wiener process are used to discretize the diffusion part. The trapezoidal rule is given by

$$\begin{aligned} A\frac{q(X_\ell)-q(X_{\ell-1})}{h_\ell} + \frac{1}{2}(f(X_\ell,t_\ell)+f(X_{\ell-1},t_{\ell-1})) \\ + \sum_{r=1}^{m} g_r(X_{\ell-1},t_{\ell-1})\frac{\Delta W_r^\ell}{h_\ell} = 0, \quad (3) \end{aligned}$$

$\ell = 1,\ldots,N$, whereas the BDF_2 has the form

$$\begin{aligned} A\frac{\sum_{j=0}^{2}\alpha_{\ell,j}\ q(X_{\ell-j})}{h_\ell} + \beta_{\ell,0}f(X_\ell,t_\ell) \\ + \sum_{j=1}^{2}\gamma_{\ell,j}\sum_{r=1}^{m} g_r(X_{\ell-j},t_{\ell-j})\frac{\Delta W_r^{\ell-j}}{h_\ell} &= 0, \quad (4) \end{aligned}$$

$\ell = 2,\ldots,N$. Here, X_ℓ denotes the approximation to $X(t_\ell)$, $h_\ell = t_\ell - t_{\ell-1}$, and $\Delta W_r^\ell = W_r(t_\ell) - W_r(t_{\ell-1}) \sim N(0,h_\ell)$ on the grid $0 = t_0 < t_1 < \ldots < t_N = T$. The coefficients $\alpha_{\ell,j}, \beta_{\ell,0}, \gamma_{\ell,j}$ actually depend on the step-size ratio $\kappa_\ell = h_\ell/h_{\ell-1}$ and have to satisfy consistency conditions of order one and two (see [3]).

In general, numerical schemes that include only information on the increments of the Wiener process have an asymptotic order of strong convergence of $1/2$, i.e.

$$\max_{\ell=1,\ldots,N}(E|X(t_\ell)-X_\ell|^2)^{1/2}\leq c\cdot h^{1/2},$$

where $h := \max_{\ell=1,\ldots,N} h_\ell$. (For additive noise the order may be 1.) However, when the noise is small, the error behavior is much better. In fact, the errors are dominated by the deterministic terms as long as the step-size is large enough [4]. In more detail, the error of the given methods is bounded by $\mathcal{O}(h^2+\varepsilon h+\varepsilon^2 h^{1/2})$, when ε is used to measure the smallness of the noise ($g_r(x,t)=\varepsilon\hat{g}_r(x,t)$, $r=1,\ldots,m$, $\varepsilon\ll 1$).

In [5] the authors presented a stepsize control for the drift-implicit Euler-scheme in the case of small noise that leads to adaptive step-size sequences that are uniform for all paths, see also [1], [2]. In this talk we present an error estimate and, based on this, a step-size control for the methods with deterministic order 2 given above.

III. Numerical results

We illustrate the potential of the step-size control strategy by simulation results for a nonlinear scalar test-SDE with known explicit solution. In Figure 1 we plotted the tolerance ($\triangle$) and the mean-square norm of the errors for adaptively chosen ($+$) and constant ($\times$) stepsizes for 100 computed paths vs. the number of steps in logarithmic scale. Lines with slopes -2 and -0.5 are provided to enable comparisons with convergence of order 2 or $1/2$. We observe order 2 behavior up to accuracies of 10^{-2}.

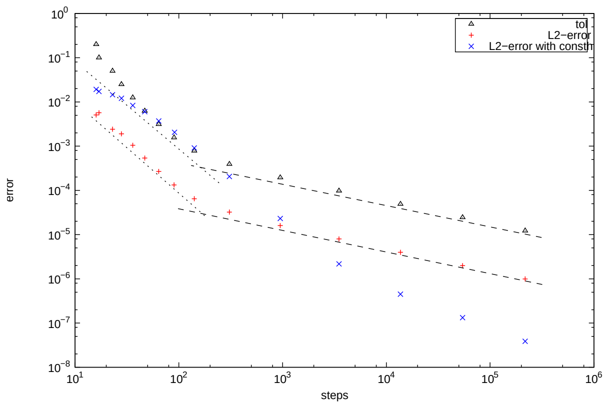

Fig. 1. Tolerance and accuracy versus steps for a test-SDE.

Further, we consider a model of an inverter circuit (see Fig. 2) with a mosfet-transistor under the influence of thermal noise. The mosfet is modelled as a current source from source to drain that is controlled by the nodal potentials at gate, source and drain. The thermal noise of the resistor and of the mosfet is modelled by additional white noise current sources that are shunt in parallel to the original, noise-free elements. To make the effect of the noise more visible we scaled the noise intensities by a factor of 1000. For the simulation we used the BDF$_2$ with adaptively chosen stepsizes using the information of 100 simultaneously computed paths.

In Figure 3 we plotted the input voltage U_{in} and values of the output voltage e_1 versus time. The red lines show

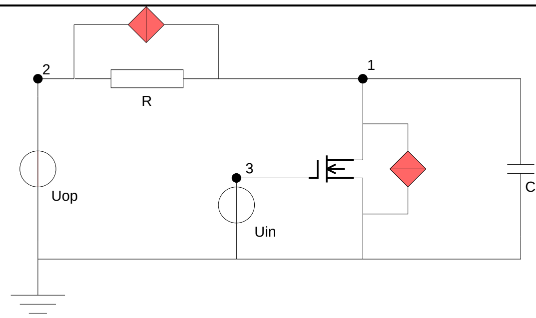

Fig. 2. Thermal noise sources in a mosfet inverter circuit.

the values of two different solution paths, the blue line gives the mean of 100 paths and the black lines the 3σ-confidence interval for the output voltage e_1. Moreover, the applied stepsizes, suitably scaled, are shown by means of single crosses.

Using the information of an ensemble of simultaneously computed solution paths smoothes the step-size sequence and reduces the number of rejected steps considerably, compared to the simulation of a single path.

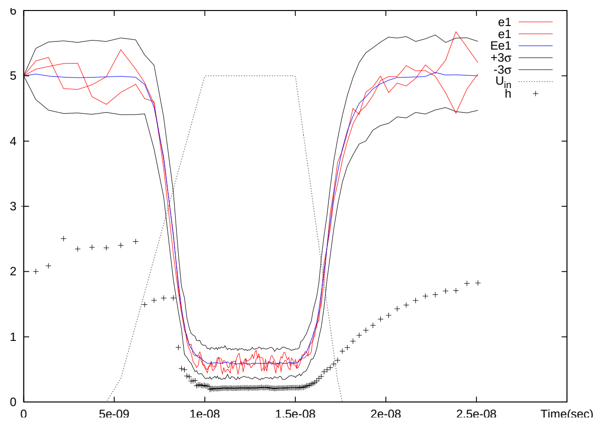

Fig. 3. Simulation results for the noisy inverter circuit.

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